CURVE COUNTING INVARIANTS
AROUND THE CONIFOLD POINT

Yukinobu Toda

Abstract

In this paper, we investigate the space of certain weak stability conditions on the triangulated category of D0-D2-D6 bound states on a smooth projective Calabi-Yau 3-fold. In the case of a quintic 3-fold, the resulting space is interpreted as a universal covering space of an infinitesimal neighborhood of the conifold point in the stringy Kähler moduli space. We then associate the DT type invariants counting semistable objects, which give new curve counting invariants on Calabi-Yau 3-folds. We also investigate the wall-crossing formula of our invariants and their interplay with the Seidel-Thomas twist.

1. Introduction

1.1. Motivation. Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e.

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

So far, several curve counting theories on X have been introduced and studied:

- **Gromov-Witten (GW) theory [25]:** counting stable maps $f: C \to X$ from projective nodal curves C.
- **Donaldson-Thomas (DT) theory [41]:** counting 1-dimensional subschemes $C \subset X$.
- **Pandharipande-Thomas (PT) theory [37]:** counting stable pairs (F, s). Here F is a 1-dimensional pure sheaf and s is a morphism $s: \mathcal{O}_X \to F$ with 0-dimensional cokernel.

The above theories are conjecturally equivalent in terms of generating functions. The GW/DT correspondence is proved for local toric Calabi-Yau 3-folds [33], local curves [14], [36], and the DT/PT correspondence (including the Euler characteristic version) is available in [40], [7], [49].

The idea of DT/PT correspondence discussed by Pandharipande-Thomas [37] is to use the wall-crossing formula of DT type invariants in the space of Bridgeland's stability conditions [11] on the category $D^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves on X. Namely, it is expected that there are two stability conditions σ, τ on $D^b(\mathrm{Coh}(X))$ such that the moduli space of σ-stable objects and that of τ-stable objects with a certain numerical condition coincide with the moduli spaces which define DT and PT theories, respectively. Then DT/PT correspondence should follow by investigating the behavior of the invariants under the change of stability conditions. A general framework for such a study, known as a *wall-crossing formula of DT type invariants*, is now established by the work of Joyce-Song [24] and Kontsevich-Soibelman [26].

However, there have been difficulties in constructing stability conditions on $D^b(\mathrm{Coh}(X))$, and not even a single example is available yet. Instead of working with Bridgeland's stability conditions, Bayer [1] and the author [47], [49] independently introduce *polynomial stability*, *limit stability* and *weak stability*, respectively. These notions are interpreted as "limiting degenerations" of Bridgeland's stability conditions. By using the above degenerated stability conditions, it turns out in [1], [49] that DT/PT correspondence is realized near a particular point, called the *large volume limit*. By analyzing weak stability conditions near the large volume limit and the relevant wall-crossing formula, the author proves the Euler characteristic version of DT/PT correspondence [49] and the rationality conjecture of the generating series of DT and PT invariants [44].

The space of stability conditions on $D^b(\mathrm{Coh}(X))$ is expected to be related to the stringy Kähler moduli space of X, which is the moduli space of complex structures of a mirror manifold. For instance, if X is a quintic Calabi-Yau 3-fold in $\mathbb{P}^4$, the mirror family is a simultaneous crepant resolution $\widehat{Y}_\psi$ of the following one parameter family,

$$Y_\psi = \left\{ \sum_{i=0}^4 y_i^5 - 5\psi \prod_{i=0}^4 y_i = 0 \right\} / G \subset \mathbb{P}^4/G,$$

where $G = (\mathbb{Z}/5\mathbb{Z})^3$. The stringy Kähler moduli space is a parameter space of ψ^5, and the large volume limit corresponds to $\psi^5 = \infty$. (See Figure 1.)

So far, the above studies on curve counting theories appear near the large volume limit. Now it is natural to address the following question.

Question 1.1. What kinds of curve counting invariants (or DT type invariants) appear at other limiting points?

In this paper, we study the above question for another limiting point, called the *conifold point*. In the case of a quintic 3-fold, this point corresponds to $\psi^5 = 1$. There is a Lagrangian sphere in a mirror manifold $\widehat{Y}_\psi$ which vanishes at the conifold point, and it corresponds to the object $\mathcal{O}_X$ under the mirror symmetry. In physic terminology, the *mass* of the object $\mathcal{O}_X$, denoted by $m(\mathcal{O}_X)$, behaves as

$$(1) \qquad m(\mathcal{O}_X) \to \begin{cases} \infty, & \text{at large volume limit,} \\ 0, & \text{at conifold point.} \end{cases}$$

Namely, the conifold point is a point where the object $\mathcal{O}_X$ becomes massless, and its effect on the stability should be infinitesimally small. On the other hand, there is an autoequivalence associated to $\mathcal{O}_X$, called a *Seidel-Thomas twist* [**38**],

$$\Phi_{\mathcal{O}_X} \colon D^b(\mathrm{Coh}(X)) \xrightarrow{\sim} D^b(\mathrm{Coh}(X)).$$

The above equivalence should correspond to the Dehn twist of $\widehat{Y}_\psi$ along the Lagrangian vanishing cycle under the mirror symmetry, and should be a monodromy on $D^b(\mathrm{Coh}(X))$ around the conifold point. Therefore the Seidel-Thomas twist must be relevant in studying Question 1.1, and it seems interesting to see how the twist functor is related to the wall-crossing.
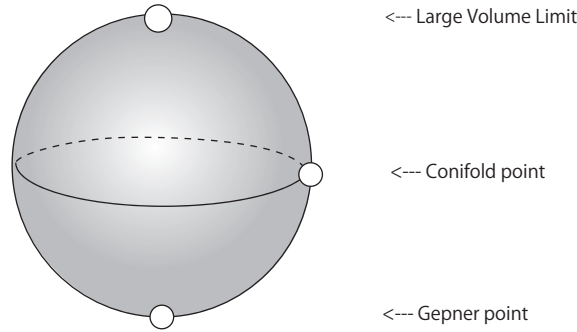

Figure 1. Stringy Kähler moduli space of a quintic 3-fold

1.2. Weak stability conditions on D0-D2-D6 bound states. In this paper, we focus on the triangulated category called *D0-D2-D6 bound states*

$$(2) \qquad \mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)).$$

This is the smallest triangulated subcategory of $D^b(\mathrm{Coh}(X))$ which contains $\mathcal{O}_X$ and the objects in $\mathrm{Coh}_{\leq 1}(X)$,

$$\mathrm{Coh}_{\leq 1}(X) = \{ E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \leq 1 \}.$$

The category $\mathcal{D}_X$ is especially important in studying curve counting invariants on X. For instance, it contains ideal sheaves of curves, and two term complexes associated with stable pairs; DT/PT correspondence is realized there [**49**].

We use the following finitely generated abelian group Γ,

$$\Gamma = H^0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z}),$$

which is the image of the Chern character map from the category $\mathcal{D}_X$. Roughly speaking, a Bridgeland's stability condition on $\mathcal{D}_X$ consists of data $(Z, \mathcal{P})$,

$$Z \colon \Gamma \to \mathbb{C}, \quad \mathcal{P}(\phi) \subset \mathcal{D}_X,$$

where Z is a group homomorphism and $\mathcal{P}(\phi)$ is a full subcategory for $\phi \in \mathbb{R}$, which satisfies some axiom.

The notion of weak stability conditions on $\mathcal{D}_X$ is determined after we specify a filtration $\Gamma_\bullet$ of Γ (cf. Definition 2.1), which is to do with the limiting direction of Bridgeland stability. The set of weak stability conditions is denoted by

$$(3) \qquad\qquad \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X),$$

and it has a natural structure of a complex manifold (cf. Theorem 2.2).

In this paper, we are interested in the following filtration,

$$(4) \qquad\qquad \Gamma_0 = H^0(X, \mathbb{Z}) \subset \Gamma_1 = \Gamma.$$

A point in the space (3) w.r.t. the filtration (4) corresponds to a limiting degeneration of Bridgeland stability $(Z, \mathcal{P})$ on $\mathcal{D}_X$, whose limiting direction is given by the constraint

$$|Z(\mathrm{ch}(\mathcal{O}_X))| \ll |Z(\mathrm{ch}(F))|,$$

for any non-zero object $F \in \mathrm{Coh}_{\leq 1}(X)$. This means that the effect of $\mathcal{O}_X$ is infinitesimally small relative to the objects in $\mathrm{Coh}_{\leq 1}(X)$; hence the space (3) seems to be related to an infinitesimal neighborhood of the conifold point in Figure 1. In fact, we have the following result.

Theorem 1.2. [Theorem 2.19] *Suppose that $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ (e.g. quintic 3-fold). Then there is a connected component*

$$\mathrm{Stab}^\circ_{\Gamma_\bullet}(\mathcal{D}_X) \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$$

such that there is an isomorphism

$$(5) \qquad\qquad \mathrm{Stab}^\circ_{\Gamma_\bullet}(\mathcal{D}_X) \cong \mathbb{C} \times \widetilde{\mathrm{GL}}_+(2, \mathbb{R}),$$

where $\widetilde{\mathrm{GL}}_+(2, \mathbb{R})$ is the universal cover of $\mathrm{GL}_+(2, \mathbb{R})$. The Seidel-Thomas twist $\Phi_{\mathcal{O}_X}$ acts on the space (5), and we have the isomorphism

$$\langle \Phi_{\mathcal{O}_X} \rangle \backslash \mathrm{Stab}^\circ_{\Gamma_\bullet}(\mathcal{D}_X)/\mathbb{C} \cong \mathbb{C}^* \times \mathbb{H}^\circ.$$

Here $\langle \Phi_{\mathcal{O}_X} \rangle$ is the subgroup of the group of autoequivalences of $\mathcal{D}_X$ generated by $\Phi_{\mathcal{O}_X}$, and $\mathbb{H}^\circ = \{ z \in \mathbb{C} : \mathrm{Im}\, z > 0 \}$.

Applying Theorem 1.2, we will construct a commutative diagram (cf. Subsection 2.8),

$$
(6) \qquad
\begin{array}{ccc}
\mathbb{R} & \longrightarrow & \mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X) \\
\exp(\pi i *) \downarrow & & \downarrow \\
S^1 & \xrightarrow{\ \iota\ } & \mathbb{C}^* \times \mathbb{H}^{\circ},
\end{array}
$$

where ι is an embedding of S^1 to (unit circle) $\times \{\sqrt{-1}\}$. When X is a quintic 3-fold, the image of ι may be interpreted as a loop around the conifold point in Figure 1, since the monodromy around it is given by $\Phi_{\mathcal{O}_X}$.

1.3. DT theory around the conifold point. Similarly to [**49**], [**43**], we associate DT type invariants counting semistable objects in $\mathcal{D}_X$, and investigate their wall-crossing phenomena. In order to formulate the result, we denote the top arrow in the diagram (6) by γ,

$$
\gamma \colon \mathbb{R} \ni t \mapsto (Z_t, \mathcal{P}_t) \in \mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X).
$$

For the data,

$$
(r, \beta, n) \in H^0 \oplus H_2 \oplus H_0, \quad t \in \mathbb{R}, \quad \phi \in \mathbb{R},
$$

we will associate the generalized DT invariant (cf. Definition 3.6),

$$
(7) \qquad \mathrm{DT}_t(r, \beta, n, \phi) \in \mathbb{Q},
$$

following the construction by Joyce-Song [**24**]. The invariant (7) counts objects $E \in \mathcal{P}_t(\phi)$ satisfying the numerical condition

$$
(\mathrm{ch}_0(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)) = (r, -\beta, -n).
$$

The generating series $\mathrm{DT}_t(\phi)$ is defined by

$$
(8) \qquad \mathrm{DT}_t(\phi) = \sum_{(r,n,\beta)\in\Gamma} \mathrm{DT}_t(r, \beta, n, \phi) x^r y^\beta z^n.
$$

The wall-crossing formula by Joyce-Song [**24**] and Kontsevich-Soibelman [**26**] enables us to see how $\mathrm{DT}_t(\phi)$ changes under the change of t. Assuming a technical result announced by Behrend-Getzler [**6**] (cf. Conjecture 3.3), we will show the following result.

Theorem 1.3. [Lemma 3.8, Theorem 3.9]
   (i) For a given $k \in \mathbb{Z}$, the series $\mathrm{DT}_t(\phi)$ does not depend on a choice of $t \in (\phi + k, \phi + k + 1)$. In particular, we may write it as $\mathrm{DT}^k(\phi)$.
   (ii) The series $\mathrm{DT}^k(\phi)$ is obtained from $\mathrm{DT}^{k-1}(\phi)$ by the following transformation:

$$
z^n \mapsto
\begin{cases}
(1 - (-1)^n x)^n z^n, & \text{if } k \text{ is even.} \\
x^n z^n / (1 - (-1)^n x)^n, & \text{if } k \text{ is odd.}
\end{cases}
$$

The above theorem implies that the series $\mathrm{DT}^k(\phi)$ is obtained from $\mathrm{DT}^{k-2}(\phi)$ by the variable change $z \mapsto xz$, which coincides with the variable change by the Seidel-Thomas twist $\Phi_{\mathcal{O}_X}$. This means that, unfortunately, the wall-crossing formula does not provide any information on the invariant (7), e.g. modularity (cf. Remark 3.11).

On the other hand, the above theorem can be used to compute the series (8) for a general t if we know it for one point $t \in \mathbb{R}$ with $t \notin \mathbb{Z} + \phi$. For instance, we will see that

$$\mathrm{DT}_t(1) = -\chi(X) \sum_{\substack{n \geq 1, m \geq 1, \\ m | n.}} \frac{1}{m^2} z^n,$$

when $0 < t < 1$ in Subsection 4.1. Applying Theorem 1.3, we can write down the series (8) for $\phi = 1$ and a general t (cf. Theorem 4.2).

1.4. Invariants on a local $(-1, -1)$-curve. In Subsection 4.2, we focus on the invariants on a local $(-1, -1)$ curve, and especially investigate what kinds of objects the invariants (7) count. Let $C \subset X$ be an exceptional locus of a crepant small resolution of an ordinary double point, $f \colon X \to Y$. It satisfies that

$$\mathbb{P}^1 \cong C \subset X, \quad N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

For instance, we will see that the invariant

(9) $\mathrm{DT}_{t=1}(r, m[C], n, \phi) \in \mathbb{Q}, \quad 1/2 < \phi < 1$

is non-zero only if $n = ma$ for some $a \in \mathbb{Z}_{\geq 1}$. In this case, the invariant (9) counts two term complexes,

$$\mathcal{O}_X^{\oplus r} \xrightarrow{s} \mathcal{O}_C(a-1)^{\oplus m},$$

such that the induced morphism,

$$H^0(s) \colon \mathbb{C}^r \to H^0(C, \mathcal{O}_C(a-1))^{\oplus m},$$

is injective. Applying Theorem 1.3, we will compute the generating series of our invariants in this situation (cf. Theorem 4.7). It turns out that there is a curious phenomenon for the rank one generating series: it coincides with the logarithm of the generating series of stable pairs on a local $(-1, -1)$-curve (cf. Equation (75)). Under GW/DT correspondence, this implies that our invariants relate to *connected* GW theory, while stable pair theory is related to non-connected GW theory. It seems interesting to give a geometric understanding of this phenomenon.

1.5. Relation to existing works. Several examples of stability conditions have been studied in the literature, for instance [**12**], [**8**], [**10**], [**42**], [**32**], [**18**], [**2**], [**19**], [**46**], [**48**]. However, global descriptions of the spaces of degenerated stability conditions introduced in [**1**], [**47**], [**49**] have not been studied so far. The result of Theorem 1.2 is the first example of such a study.

The weak stability conditions on $\mathcal{D}_X$ are also studied in [**49**], [**43**] without giving any global descriptions of the spaces of weak stability conditions. We note that the filtrations taken in these works are different from (4), and the resulting spaces should correspond to infinitesimal neighborhoods of the large volume limit. It is worth mentioning that the equivalence $\Phi_{\mathcal{O}_X}$ does *not* act on the spaces discussed in [**49**], [**43**].

In [**49**], [**43**], we also investigate the wall-crossing formula of DT type invariants with respect to certain weak stability conditions on $\mathcal{D}_X$. However, we focus only on the rank one case in these works. Since the Seidel-Thomas twist is relevant in the study of our invariants, and the twist functor changes the rank, it is natural to consider higher rank invariants in our situation. In fact, we observe in Theorem 1.3 that the wall-crossing formula can be described only when we consider the generating series of all rank.

Recently there have been studies on higher rank DT type invariants [**39**], [**50**], [**15**], [**34**], [**30**]. Since our study naturally involves higher rank invariants, it seems interesting to see a relationship to these works.

1.6. Notation and convention. For a variety X, the category of coherent sheaves on X is denoted by $\mathrm{Coh}(X)$, and its derived category is denoted by $D^b(\mathrm{Coh}(X))$. We use the following abelian subcategories of $\mathrm{Coh}(X)$:

$$\mathrm{Coh}_{\leq 1}(X) = \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \leq 1\},$$
$$\mathrm{Coh}_0(X) = \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) = 0\}.$$

For a triangulated category $\mathcal{D}$ and a set of objects $S \subset \mathcal{D}$, the subcategory $\langle S \rangle_{\mathrm{tr}} \subset \mathcal{D}$ is the smallest triangulated subcategory of $\mathcal{D}$ which contains objects in $S \cup \{0\}$. Also the category $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{D}$ is the smallest extension-closed subcategory which contains objects in $S \cup \{0\}$. If S is a set of objects in an abelian category $\mathcal{A}$, the subcategory $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{A}$ is also defined in a similar way. The varieties in this paper are defined over $\mathbb{C}$. For a variety X, we occasionally write $H^i(X, \mathbb{Z})$, $H_i(X, \mathbb{Z})$ as H^i, H_i for simplicity.

2. The Space of Weak Stability Conditions

The notion of weak stability conditions on triangulated categories is introduced in [**49**] to give limiting degenerations of Bridgeland's stability conditions [**11**]. In this section, we investigate the space of weak stability

conditions on the triangulated category of D0-D2-D6 bound states on a smooth projective Calabi-Yau 3-fold.

2.1. Weak stability conditions on triangulated categories. In this subsection, we recall the notion of weak stability conditions on triangulated categories, and collect some results we need in the latter subsections. For the detail, see [**49**, section 2].

Let $\mathcal{D}$ be a triangulated category, and $K(\mathcal{D})$ the Grothendieck group of $\mathcal{D}$. We fix a finitely generated free abelian group Γ together with a group homomorphism,

$$\mathrm{cl} \colon K(\mathcal{D}) \to \Gamma.$$

We also fix a filtration,

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_N = \Gamma,$$

such that each subquotient Γ_i/Γ_{i-1} is a free abelian group.

Definition 2.1. *A weak stability condition on $\mathcal{D}$ consists of data* $(Z = \{Z_i\}_{i=0}^N, \mathcal{A})$,

$$Z_i \colon \Gamma_i/\Gamma_{i-1} \to \mathbb{C}, \quad \mathcal{A} \subset \mathcal{D},$$

where Z_i are group homomorphisms and $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, which satisfies the following.

- *For any non-zero $E \in \mathcal{A}$ with $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$, we have*

(10) $$Z(E) := Z_i([\mathrm{cl}(E)]) \in \mathbb{H}.$$

   Here $[\mathrm{cl}(E)] \in \Gamma_i/\Gamma_{i-1}$ is the class of $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$ and
   $$\mathbb{H} = \{r \exp(i\pi\phi) : r > 0, 0 < \phi \le 1\}.$$

   We say $E \in \mathcal{A}$ is Z-(semi)stable if for any exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$, we have
   $$\arg Z(F) < (\le) \arg Z(G).$$

- *For any $E \in \mathcal{A}$, there is a filtration in $\mathcal{A}$ (Harder-Narasimhan filtration),*
   $$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

   such that each subquotient $F_i = E_i/E_{i-1}$ is Z-semistable with
   $$\arg Z(F_i) > \arg Z(F_{i+1}),$$

   for all i.

Here we remark that for $N = 0$, the pair $(Z, \mathcal{A})$ determines a stability condition by Bridgeland [**11**].

Let $(Z, \mathcal{A})$ be a weak stability condition on $\mathcal{D}$. For $0 < \phi \le 1$, the subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ is defined to be the category of Z-semistable objects $E \in \mathcal{A}$ satisfying

(11) $$Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi).$$

For other $\phi \in \mathbb{R}$, the subcategory $\mathcal{P}(\phi)$ is determined by the rule,

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1].$$

The family of subcategories $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ determines a *slicing* introduced in [**11**, definition 3.3]. As in [**49**, proposition 2.13], giving a weak stability condition is equivalent to giving the data,

$$(12) \qquad \sigma = (Z = \{Z_i\}_{i=0}^{N}, \mathcal{P}),$$

where Z is as above and $\mathcal{P}$ is a slicing, satisfying the condition (11) for any non-zero $E \in \mathcal{P}(\phi)$. In what follows, we occasionally write a weak stability condition as a pair of group homomorphisms $\{Z_i\}_{i=0}^{N}$ and a slicing $\mathcal{P}$, as in (12). The subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ is called the category of *σ-semistable objects of phase ϕ*. The category $\mathcal{P}(\phi)$ is easily seen to be an abelian category, and we denote by $\mathcal{P}_s(\phi) \subset \mathcal{P}(\phi)$ the subcategory of simple objects. An object in $\mathcal{P}_s(\phi)$ is called a *σ-stable object of phase ϕ*.

For an interval $I \subset \mathbb{R}$, we set

$$\mathcal{P}(I) := \langle \mathcal{P}(\phi) : \phi \in I \rangle_{\mathrm{ex}}.$$

We also need the following technical conditions.

- **(Support property):** There is a constant $C > 0$ such that for any $E \in \mathcal{P}(\phi)$ with $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$, we have

$$\|[\mathrm{cl}(E)]\|_i \leq C \cdot |Z(E)|.$$

  Here $\|*\|_i$ is a fixed norm on $(\Gamma_i/\Gamma_{i-1}) \otimes_{\mathbb{Z}} \mathbb{R}$.
- **(Local finiteness):** There is $\varepsilon > 0$ such that the quasi-abelian category $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length for any $\phi \in \mathbb{R}$.

Here we refer to [**11**, Definition 4.1, Definition 5.7] for the detail on the notion of quasi-abelian categories and their finite length property. The set of weak stability conditions satisfying the above two properties is denoted by $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$. The following result is an analogue of [**11**, theorem 7.1] and proved in [**49**, Theorem 2.15].

Theorem 2.2. *There is a natural topology on $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ such that the forgetting map*

$$\Pi \colon \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}) \ni (Z, \mathcal{A}) \mapsto Z \in \prod_{i=0}^{N} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$$

is a local homeomorphism. In particular, each connected component of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ is a complex manifold.

Remark 2.3. As mentioned in [**49**, remark 2.16], the set of $\sigma \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ in which a fixed object $E \in \mathcal{D}$ is σ-semistable is a closed subset.

There is a continuous $\mathbb{C}$-action on the space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ in the following way. For a pair $\sigma = (Z, \mathcal{P})$ as in (12) and $\lambda \in \mathbb{C}$, we set

$$\lambda \cdot \sigma = (\exp(-i\pi\lambda)Z, \mathcal{P}'),$$

where $\mathcal{P}'$ is a slicing given by $\mathcal{P}'(\phi) = \mathcal{P}(\phi + \mathrm{Re}\,\lambda)$ for all $\phi \in \mathbb{R}$.

For the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$, we denote by

$$\mathcal{H}_{\mathcal{A}}^i \colon \mathcal{D} \to \mathcal{A},$$

the i-th cohomology functor with respect to the t-structure with heart $\mathcal{A}$. We will use the following notions of torsion pair and tilting to construct weak stability conditions.

Definition 2.4. *Let $\mathcal{A}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$. A pair of subcategories $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is called a torsion pair if the following conditions hold.*

- *For any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, we have $\mathrm{Hom}(T, F) = 0$.*
- *For any $E \in \mathcal{A}$, there is an exact sequence*

$$0 \to T \to E \to F \to 0,$$

for $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ as above, its *tilting* is defined by

$$\mathcal{A}^\dagger := \left\{ E \in \mathcal{D} : \begin{array}{l} \mathcal{H}_{\mathcal{A}}^0(E) \in \mathcal{F},\ \mathcal{H}_{\mathcal{A}}^1(E) \in \mathcal{T}, \\ \mathcal{H}^i(E) = 0 \text{ for all } i \neq 0, 1. \end{array} \right\},$$
$$= \langle \mathcal{F}, \mathcal{T}[-1] \rangle_{\mathrm{ex}}.$$

The category $\mathcal{A}^\dagger$ is also the heart of a bounded t-structure on $\mathcal{D}$ (cf. [**17**, proposition 2.1]).

2.2. Construction of weak stability conditions. Let X be a smooth projective Calabi-Yau 3-fold satisfying

(13) $$H^1(X, \mathcal{O}_X) = 0.$$

We define the triangulated category $\mathcal{D}_X$ to be

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)).$$

We set the finitely generated abelian group Γ to be

$$\Gamma = H^0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z}).$$

By the Poincaré duality, the Chern characters of E define a group homomorphism $\mathrm{cl} \colon K(\mathcal{D}_X) \to \Gamma$,

$$\mathrm{cl}(E) = (\mathrm{ch}_0(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)).$$

We set the two step filtration of Γ to be

$$\Gamma_0 = H^0(X, \mathbb{Z}) \subset \Gamma_1 = \Gamma.$$

We are going to study the space of weak stability conditions $\mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{D}_X)$. Note that

$$\mathrm{Hom}(\Gamma_0, \mathbb{C}) \cong \mathbb{C},$$

$$\mathrm{Hom}(\Gamma_1/\Gamma_0, \mathbb{C}) \cong H^2(X, \mathbb{C}) \oplus \mathbb{C}.$$

The forgetting map $(Z, \mathcal{A}) \mapsto Z$ is as follows,

(14) $$\Pi \colon \mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{D}_X) \to \mathbb{C} \times H^2(X, \mathbb{C}) \times \mathbb{C}.$$

Remark 2.5. As mentioned in [**49**, remark 2.21], a weak stability condition in this situation may be interpreted to be a limiting point $m \to \infty$ of some sequence of stability conditions,

$$\sigma^{(m)} = (Z^{(m)}, \mathcal{C}^{(m)}),$$

where $\mathcal{C}^{(m)} \subset \mathcal{D}_X$ is the heart of a bounded t-structure and $Z^{(m)} \colon \Gamma \to \mathbb{C}$ is written as

$$Z^{(m)}(r, \beta, n) = Z_0(r) + m Z_1(\beta, n).$$

Here $Z_i \colon \Gamma_i/\Gamma_{i-1} \to \mathbb{C}$ are group homomorphisms for $i = 0, 1$. Note that we have

$$|Z^{(m)}(\mathrm{cl}(\mathcal{O}_X))| \ll |Z^{(m)}(\mathrm{cl}(F))|, \quad m \gg 0,$$

where $F \in \mathcal{D}_X$ satisfies $\mathrm{cl}(F) \in \Gamma_1 \backslash \Gamma_0$. This implies that the mass of the object $\mathcal{O}_X$ is infinitesimally small w.r.t. our weak stability conditions.

Here we construct three types of weak stability conditions on $\mathcal{D}_X$.

Lemma 2.6. *(i) There is the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}_X$, written as*

$$\mathcal{A} = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X)[-1] \rangle_{\mathrm{ex}}.$$

(ii) There is the heart of a bounded t-structure $\mathcal{B}_+ \subset \mathcal{D}_X$, written as

$$\mathcal{B}_+ = \langle \mathcal{A}_+, \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}.$$

Here $\mathcal{A}_+ = \{E \in \mathcal{A} : \mathrm{Hom}(\mathcal{O}_X, E) = 0\}$.
(iii) There is the heart of a bounded t-structure $\mathcal{B}_- \subset \mathcal{D}_X$, written as

$$\mathcal{B}_- = \langle \mathcal{O}_X[1], \mathcal{A}_- \rangle_{\mathrm{ex}}.$$

Here $\mathcal{A}_- = \{E \in \mathcal{A} : \mathrm{Hom}(E, \mathcal{O}_X) = 0\}$.

Proof. The proof of (i) is given in [**49**, lemma 3.5]. For the proof of (ii), note that the pair

$$\langle \langle \mathcal{O}_X \rangle_{\mathrm{ex}}, \mathcal{A}_+ \rangle_{\mathrm{ex}}$$

is a torsion pair. This is easily checked by the fact that $\mathcal{A}$ is a noetherian abelian category [**49**, lemma 6.2]. The tilting with respect to the above torsion pair yields the heart $\mathcal{B}_+$. The proof of (iii) is similar. q.e.d.

For the given data,

$$u = (z, B + i\omega) \in \mathbb{C} \times H^2(X, \mathbb{C}),$$

we associate the element

(15) $$Z_u = \{Z_{u,i}\}_{i=0}^1 \in \prod_{i=0}^1 \mathrm{Hom}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$$

as follows:

(16) $$Z_{u,0} \colon \Gamma_0 = H^0(X, \mathbb{Z}) \ni r \mapsto rz,$$

$$Z_{u,1} \colon \Gamma_1/\Gamma_0 = H_2(X, \mathbb{Z}) \oplus H_0(X, \mathbb{Z})$$

(17) $$\ni (\beta, n) \mapsto n - (B + i\omega) \cdot \beta.$$

Let $A(X)_{\mathbb{C}} \subset H^2(X, \mathbb{C})$ be the complexified ample cone

$$A(X)_{\mathbb{C}} = \{B + i\omega \in H^2(X, \mathbb{C}) : \omega \text{ is ample}\}.$$

We have the following lemma.

Lemma 2.7. *(i) The pairs*

(18) $$\sigma_u = (Z_u, \mathcal{A}), \quad u \in \mathbb{H} \times A(X)_{\mathbb{C}}$$

determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$.
   (ii) The pairs

$$\tau_{u\pm} = (Z_u, \mathcal{B}_{\pm}), \quad u \in (-\mathbb{H}) \times A(X)_{\mathbb{C}}$$

determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$.

Proof. The proofs of some technical conditions (Harder-Narasimhan property, support property, local finiteness) are postponed until Section 5. Here we only check that the condition (10) is satisfied.
   (i) For $E \in \mathcal{A}$, let us write

$$\mathrm{cl}(E) = (r, -\beta, -n) \in H^0 \oplus H_2 \oplus H_0.$$

Suppose that $\mathrm{cl}(E) \in \Gamma_1 \setminus \Gamma_0$. Then the description (18) shows that β is an effective curve class or $\beta = 0, n > 0$. Hence $Z_u(E) \in \mathbb{H}$ follows. If $\mathrm{cl}(E) \in \Gamma_0$, then $E \in \langle \mathcal{O}_X \rangle_{\mathrm{ex}}$, hence we have $Z_u(E) = rz \in \overline{\mathbb{H}}$.
   (ii) For simplicity, we check the case of $(Z_u, \mathcal{B}_+)$. For an object $E \in \mathcal{B}_+$, there is an exact sequence in $\mathcal{B}_+$,

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{A}_+$ and $F \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}$. If $\mathrm{cl}(E) \in \Gamma_1 \setminus \Gamma_0$, then $T \neq 0$ and we have

$$Z_u(E) = Z_u(T) \in \mathbb{H},$$

by the same argument of (i). If $\mathrm{cl}(E) \in \Gamma_0$, then we have $E \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}$; hence we have $Z_u(E) = rz \in \overline{\mathbb{H}}$.   q.e.d.

2.3. Standard regions in the space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$. The constructions of weak stability conditions in the last subsection yield some standard regions in the space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$. We set $\mathcal{U}$ and $\mathcal{U}_{\pm 1}$ to be

(19) $\qquad \mathcal{U} = \{\sigma_u \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) : u \in \mathbb{H} \times A(X)_{\mathbb{C}}\},$

(20) $\qquad \mathcal{U}_{\pm 1} = \{\tau_{u\pm} \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) : u \in (-\mathbb{H}) \times A(X)_{\mathbb{C}}\}.$

The above subspaces lie in the space of normalized weak stability conditions,

$$\mathrm{Stab}_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X) = \{(Z, \mathcal{A}) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) : Z(\mathcal{O}_x) = -1\},$$

where $x \in X$ is a closed point. The forgetting map (14) restricts to the local homeomorphism,

$$\Pi_{\mathrm{n}} \colon \mathrm{Stab}_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X) \to \mathbb{C} \times H^2(X, \mathbb{C}).$$

We have the following lemma.

Lemma 2.8. *(i) The map Π_{n} restricts to the homeomorphisms,*

$$\Pi_{\mathrm{n}} \colon \mathcal{U} \xrightarrow{\sim} \mathbb{H} \times A(X)_{\mathbb{C}},$$
$$\Pi_{\mathrm{n}} \colon \mathcal{U}_{\pm 1} \xrightarrow{\sim} (-\mathbb{H}) \times A(X)_{\mathbb{C}}.$$

(ii) The map Π_{n} restricts to the homeomorphisms,

$$\Pi_{\mathrm{n}} \colon \mathcal{U} \cap \overline{\mathcal{U}}_{+1} \xrightarrow{\sim} \mathbb{R}_{<0} \times A(X)_{\mathbb{C}},$$
$$\Pi_{\mathrm{n}} \colon \overline{\mathcal{U}} \cap \mathcal{U}_{-1} \xrightarrow{\sim} \mathbb{R}_{>0} \times A(X)_{\mathbb{C}}.$$

Proof. Since $\mathcal{B}_{\pm}$ is obtained from $\mathcal{A}$ by tilting, both of (i) and (ii) follow by applying Lemma 2.9 below. $\qquad\qquad$ q.e.d.

We have used the following lemma, whose proof is given in [**49**, lemma 7.1].

Lemma 2.9. [**49**, lemma 7.1] *Let $\mathcal{C}$ be the heart of a bounded t-structure on $\mathcal{D}_X$ and $(\mathcal{T}, \mathcal{F})$ a torsion pair on $\mathcal{C}$. Let $\mathcal{C}' = \langle \mathcal{F}, \mathcal{T}[-1] \rangle_{\mathrm{ex}}$ be the associated tilting. Let*

$$[0, 1) \ni t \mapsto Z_t \in \prod_{i=0}^{1} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i-1}, \mathbb{C}),$$

be a continuous map such that $\sigma_t = (Z_t, \mathcal{C})$ for $0 < t < 1$ and $\sigma_0 = (Z_0, \mathcal{C}')$ determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$. Then we have $\lim_{t \to 0} \sigma_t = \sigma_0$.

By Lemma 2.8, the subspaces $\mathcal{U}, \mathcal{U}_{\pm 1}$ are contained in the same connected component, which we denote by

$$\mathrm{Stab}^{\circ}_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X) \subset \mathrm{Stab}_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X).$$

2.4. Weak stability conditions and Seidel-Thomas twist. By our assumption (13), the object $\mathcal{O}_X$ is a *spherical object*, i.e.

$$\mathrm{Ext}_X^i(\mathcal{O}_X, \mathcal{O}_X) = \begin{cases} \mathbb{C}, & i = 0, 3, \\ 0, & \text{otherwise.} \end{cases}$$

We have the associated derived equivalence, called the *Seidel-Thomas twist* [**38**],

$$(21) \qquad \Phi_{\mathcal{O}_X} \colon D^b(\mathrm{Coh}(X)) \xrightarrow{\sim} D^b(\mathrm{Coh}(X)).$$

The above equivalence has the property that there is a distinguished triangle,

$$(22) \qquad \mathbf{R}\mathrm{Hom}(\mathcal{O}_X, E) \otimes_{\mathbb{C}} \mathcal{O}_X \to E \to \Phi_{\mathcal{O}_X}(E),$$

for any object $E \in D^b(\mathrm{Coh}(X))$.

Lemma 2.10. *The equivalence $\Phi_{\mathcal{O}_X}$ preserves the subcategory $\mathcal{D}_X$, and we have the commutative diagram,*

$$(23) \qquad \begin{array}{ccc} K(\mathcal{D}_X) & \xrightarrow{\Phi_{\mathcal{O}_X}} & K(\mathcal{D}_X) \\ {\scriptstyle \mathrm{cl}}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}} \\ \Gamma & \xrightarrow{\phi_{\mathcal{O}_X}} & \Gamma. \end{array}$$

Here $\phi_{\mathcal{O}_X}$ is given by

$$\phi_{\mathcal{O}_X}(r, \beta, n) = (r - n, \beta, n),$$

for $(r, \beta, n) \in H^0 \oplus H_2 \oplus H_0$.

Proof. By the distinguished triangle (22), it is obvious that the equivalence $\Phi_{\mathcal{O}_X}$ preserves $\mathcal{D}_X$. Since $\mathrm{ch}_1(E) = 0$ for any $E \in \mathcal{D}_X$, the Riemann-Roch theorem yields

$$\sum_i (-1)^i \dim \mathrm{Hom}(\mathcal{O}_X, E[i]) = \mathrm{ch}_3(E).$$

Then the distinguished triangle (22) implies that the diagram (23) is commutative.   q.e.d.

Note that $\phi_{\mathcal{O}_X}$ preserves the filtration $\Gamma_\bullet$ and the induced map on $\mathrm{gr}(\Gamma_\bullet)$ is identity. Hence by Lemma 2.10 and [**43**, lemma 2.9], we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) & \xrightarrow{\Phi_{\mathcal{O}_X*}} & \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X) \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \mathrm{gr}(\Gamma_\bullet)^\vee \otimes \mathbb{C} & \xrightarrow{\mathrm{id}} & \mathrm{gr}(\Gamma_\bullet)^\vee \otimes \mathbb{C}. \end{array}$$

Here $\Phi_{\mathcal{O}_X*}$ is given by

$$\Phi_{\mathcal{O}_X*}(Z, \mathcal{A}) = (Z, \Phi_{\mathcal{O}_X}(\mathcal{A})),$$

where $\mathcal{A} \subset \mathcal{D}_X$ is the heart of a bounded t-structure. It is obvious that $\Phi_{\mathcal{O}_{X*}}$ preserves the normalized weak stability conditions, so there is a commutative diagram,

$$
\begin{CD}
\mathrm{Stab}_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X) @>{\Phi_{\mathcal{O}_{X*}}}>> \mathrm{Stab}_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X) \\
@V{\Pi_\mathrm{n}}VV @VV{\Pi_\mathrm{n}}V \\
\mathbb{C} \times H^2(X)_\mathbb{C} @>{\mathrm{id}}>> \mathbb{C} \times H^2(X)_\mathbb{C}.
\end{CD}
$$

Under the Seidel-Thomas twist (21), the regions (20) are related as follows.

Lemma 2.11. *We have*

$$\Phi_{\mathcal{O}_{X*}}\mathcal{U}_{-1} = \mathcal{U}_{+1}.$$

In particular, $\Phi_{\mathcal{O}_{X*}}$ preserves the connected component $\mathrm{Stab}^\circ_{\Gamma_\bullet,\mathrm{n}}(\mathcal{D}_X)$.

Proof. By the construction of $\mathcal{U}_{\pm 1}$, it is enough to show that

$$\Phi_{\mathcal{O}_X}(\mathcal{B}_-) = \mathcal{B}_+.$$

Since both sides are hearts of bounded t-structures, it is enough to see that the LHS is contained in the RHS. By Lemma 2.6 (ii), this follows by showing that

$$\Phi_{\mathcal{O}_X}(\mathcal{O}_X[1]) \in \mathcal{B}_+, \quad \Phi_{\mathcal{O}_X}(\mathcal{A}_-) \subset \mathcal{B}_+.$$

First it is easy to see that

$$\tag{24} \Phi_{\mathcal{O}_X}(\mathcal{O}_X[1]) = \mathcal{O}_X[-1] \in \mathcal{B}_+,$$

using the distinguished triangle (22). Next let us take $E \in \mathcal{A}_-$, and show that $\Phi_{\mathcal{O}_X}(E) \in \mathcal{B}_+$. We set

$$r_i = \dim \mathrm{Hom}(\mathcal{O}_X, E[i]).$$

The Serre duality implies that $r_3 = 0$. Applying $\mathcal{H}^\bullet_\mathcal{A}$ to the distinguished triangle (22) and noting that $\mathcal{O}_X \in \mathcal{A}$ is a simple object, it is easy to see that

$$\mathcal{H}^i_\mathcal{A}(\Phi_{\mathcal{O}_X}(E)) = 0, \ i \neq 0, 1, \quad \mathcal{H}^1_\mathcal{A}(\Phi_{\mathcal{O}_X}(E)) \cong \mathcal{O}_X^{\oplus r_2}.$$

Also this implies that

$$
\begin{aligned}
\mathrm{Hom}(\mathcal{O}_X, \mathcal{H}^0_\mathcal{A}(\Phi_{\mathcal{O}_X}(E))) &\cong \mathrm{Hom}(\mathcal{O}_X, \Phi_{\mathcal{O}_X}(E)) \\
&\cong \mathrm{Hom}(\mathcal{O}_X[2], E) \\
&\cong 0.
\end{aligned}
$$

Therefore $\mathcal{H}^0_\mathcal{A}(\Phi_{\mathcal{O}_X}(E)) \in \mathcal{A}_+$. By the construction of $\mathcal{B}_+$, we conclude that $\Phi_{\mathcal{O}_X}(E) \in \mathcal{B}_+$. q.e.d.

Applying the twist functor $\Phi_{\mathcal{O}_X}$ to the regions (19), (20), we can construct other regions in the space $\mathrm{Stab}_{\Gamma_\bullet,\mathrm{n}}^\circ(\mathcal{D}_X)$. For $k \in \mathbb{Z}$, they are defined in the following way:

$$(25) \qquad\qquad \mathcal{U}_{2k} = \Phi_{\mathcal{O}_X*}^{(k)}(\mathcal{U}),$$

$$(26) \qquad\qquad \mathcal{U}_{2k+1} = \Phi_{\mathcal{O}_X*}^{(k)}(\mathcal{U}_{+1}).$$

We have the following lemma.

Lemma 2.12. *For $\sigma = (Z, \mathcal{P}) \in \mathcal{U}_k$, we have*

$$\mathcal{O}_X \in \mathcal{P}_s(\phi), \quad k < \phi \le k + 1.$$

Proof. It is easy to check that the objects

$$\mathcal{O}_X \in \mathcal{A}, \quad \mathcal{O}_X[-1] \in \mathcal{B}_+$$

are simple objects in $\mathcal{A}$, $\mathcal{B}_+$ respectively. Therefore, the statement follows for $k = 0, 1$. Since $\Phi_{\mathcal{O}_X}(\mathcal{O}_X) = \mathcal{O}_X[-2]$, the result also follows for all $k \in \mathbb{Z}$.   q.e.d.

2.5. Semistable sheaves and semistable objects. In this subsection, we recall the classical notion of (semi)stability on the category $\mathrm{Coh}_{\le 1}(X)$, and compare it with our weak stability conditions. For $B + i\omega \in A(X)_{\mathbb{C}}$ and $F \in \mathrm{Coh}_{\le 1}(X)$, we set

$$\mu_{(B,\omega)}(F) = \frac{\mathrm{ch}_3(E) + B \cdot \mathrm{ch}_2(E)}{\omega \cdot \mathrm{ch}_2(E)}.$$

Definition 2.13. *We say F is a (B, ω)-(semi)stable sheaf if for any non-zero proper subsheaf $F' \subset F$, we have*

$$\mu_{(B,\omega)}(F') < (\le)\mu_{(B,\omega)}(F).$$

If $B = 0$, we simply write $\mu_{(0,\omega)}(*) = \mu_\omega(*)$ and call a $(0, \omega)$-(semi)stable sheaf just an ω-(semi)stable sheaf. Note that we have the inclusions,

$$\mathrm{Coh}_{\le 1}(X)[-1] \subset \mathcal{A}, \ \mathcal{B}_\pm,$$

and hence it is natural to relate (B, ω)-stability with our weak stability conditions. The following lemma will not be needed except in showing Lemma 2.16 below, but it helps us to see what kind of objects appear as semistable objects w.r.t. our weak stability conditions. The proof will be given in Section 5.

Lemma 2.14. *(i) Take $u = (z, B + i\omega) \in \mathbb{H} \times A(X)_{\mathbb{C}}$ and a (B, ω)-semistable sheaf $F \in \mathrm{Coh}_{\le 1}(X)$ Then the object $F[-1] \in \mathcal{A}$ is a Z_u-semistable object.*

(ii) Take $u = (z, B + i\omega) \in (-\mathbb{H}) \times A(X)_{\mathbb{C}}$ and a (B, ω)-semistable sheaf $F \in \mathrm{Coh}_{\le 1}(X)$. Then we have the following.

- *Suppose that $\arg Z_u(F[-1]) > \arg(-z)$. Then the object $F[-1] \in \mathcal{B}_+$ is Z_u-semistable.*

- *Suppose that* $\arg Z_u(F[-1]) < \arg(-z)$. *Then we have* $\Phi_{\mathcal{O}_X}(F[-1]) \in \mathcal{B}_+$ *and it is* Z_u-*semistable.*

2.6. The space of normalized weak stability conditions. Let $\{\mathcal{U}_i\}_{i \in \mathbb{Z}}$ be the family of regions constructed in (25), (26). We have the following description of the space of normalized weak stability conditions.

Theorem 2.15. *Assume that*

$$(27) \qquad\qquad H^2(X, \mathbb{Z}) \cong \mathbb{Z}.$$

Then we have the following:

$$(28) \qquad\qquad \mathrm{Stab}^{\circ}_{\Gamma_\bullet, \mathrm{n}}(\mathcal{D}_X) = \coprod_{i \in \mathbb{Z}} \mathcal{U}_i.$$

In particular, the forgetting map is a universal covering map,

$$\Pi_{\mathrm{n}} \colon \mathrm{Stab}^{\circ}_{\Gamma_\bullet, \mathrm{n}}(\mathcal{D}_X) \to \mathbb{C}^* \times \mathbb{H}^{\circ},$$

with Galois group generated by $\Phi_{\mathcal{O}_X*}$.

Proof. By Lemma 2.8, the RHS of (28) is an open subset in the LHS. Hence it is enough to show that the RHS is closed in the LHS. By Lemma 2.12, the RHS is a locally finite union, i.e. for any compact subset $\mathfrak{B} \subset \mathrm{Stab}^{\circ}_{\Gamma_\bullet, \mathrm{n}}(\mathcal{D}_X)$, the number of $i \in \mathbb{Z}$ satisfying $\mathcal{U}_i \cap \mathfrak{B} \neq \emptyset$ is finite. This implies that

$$\overline{\coprod_{i \in \mathbb{Z}} \mathcal{U}_i} = \coprod_{i \in \mathbb{Z}} \overline{\mathcal{U}_i}.$$

Hence it is enough to show that $\overline{\mathcal{U}_i}$ is contained in the RHS of (28). Furthermore, applying $\Phi_{\mathcal{O}_X}$, we may assume that $i = 0$ or $i = 1$. For simplicity, we show the case of $i = 1$. The other case is similarly discussed.

Let us take a point $\sigma \in \overline{\mathcal{U}_1} \setminus \mathcal{U}_1$. We can write $\sigma = (Z_u, \mathcal{P})$ for $u = (z, B + i\omega) \in (-\mathbb{H}) \times A(X)_{\mathbb{C}}$ and a slicing $\mathcal{P}$. Since $\mathcal{O}_X$ is stable in $\mathcal{U}_1$, it is also semistable in σ; hence we have $z \neq 0$. By the assumption (27), we have the following possibilities.

   (i) $z \in \mathbb{R}_{<0}, \quad \omega \neq 0$.
   (ii) $z \in -\overline{\mathbb{H}} \setminus \{0\}, \quad \omega = 0$.

Suppose that (i) holds. Then $\sigma \in \mathcal{U}_0$ by Lemma 2.8, and σ is contained in the RHS of (28). We show that the case (ii) doesn't happen.

Suppose by contradiction that (ii) holds. We set

$$\phi_0 = \frac{1}{\pi} \arg(-z) \in [0, 1].$$

Since $\omega = 0$, we have $\mathcal{P}(\phi) = \{0\}$ unless $\phi \in \mathbb{Z}$ or $\phi \in \mathbb{Z} + \phi_0$. Therefore we can find $\psi \in (0, 1)$ and $0 < \varepsilon \ll 1$ satisfying

$$(\psi - 2\varepsilon, \psi + 2\varepsilon) \subset (0, 1) \setminus \{\phi_0\}.$$

Since $\sigma \in \overline{\mathcal{U}_1}$, there is $\tau = (Z', \mathcal{P}') \in \mathcal{U}_1$ satisfying $\mathcal{P}'(\phi) \subset \mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ for all $\phi \in \mathbb{R}$. Then we obtain

$$\mathcal{P}'((\psi - \varepsilon, \psi + \varepsilon)) \subset \mathcal{P}((\psi - 2\varepsilon, \psi + 2\varepsilon)) = \{0\}.$$

However, this contradicts Lemma 2.16 below. The result for the forgetting map easily follows from (28), Lemma 2.8, and Lemma 2.12.
q.e.d.

We have used the following lemma, whose proof will be given in Section 5.

Lemma 2.16. *For* $u \in (-\mathbb{H}) \times A(X)_{\mathbb{C}}$, *we write* $\tau_{u+} = (Z_u, \mathcal{P})$ *for a slicing* $\mathcal{P}$. *Then the set*

(29) $$\{\phi \in \mathbb{R} : \mathcal{P}(\phi) \neq \{0\}\} \subset \mathbb{R}$$

is dense in $\mathbb{R}$.

2.7. The space of non-normalized weak stability conditions.
In this subsection, we investigate the space of non-normalized weak stability conditions. Let

$$\mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X) \subset \mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{D}_X)$$

be the connected component which contains $\mathcal{U}$. We show the following lemma.

Lemma 2.17. *For any* $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X)$, *we have* $Z(\mathcal{O}_x) \neq 0$ *for any closed point* $x \in X$.

Proof. Suppose by contradiction that $Z(\mathcal{O}_x) = 0$. We define the set of objects $\mathcal{S}$ to be

$$\mathcal{S} = \{E \in \mathcal{D}_X : \mathrm{cl}(E) = (0, 0, 1)\}.$$

By the condition $Z(\mathcal{O}_x) = 0$, there is no $a, b \in \mathbb{R}$ satisfying

$$0 < b - a \leq 1, \quad \mathcal{P}((a, b]) \cap \mathcal{S} \neq \{0\}.$$

By deforming σ, we can find $\tau = (W, \mathcal{Q}) \in \mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X)$ such that $W(\mathcal{O}_x) \neq 0$ and there is no $a, b \in \mathbb{R}$ satisfying

$$0 < b - a \leq 1/2, \quad \mathcal{Q}((a, b]) \cap \mathcal{S} \neq \{0\}.$$

This in particular implies that there is no τ-semistable object $E \in \mathcal{S}$. Since $W(\mathcal{O}_x) \neq 0$, we can apply $\mathbb{C}$-action on $\mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X)$ to find

$$\sigma' = (Z', \mathcal{P}') \in \mathrm{Stab}^{\circ}_{\Gamma_{\bullet}, \mathrm{n}}(\mathcal{D}_X),$$

such that there is no σ'-semistable object $E \in \mathcal{S}$. However, it is easy to check that the object $\mathcal{O}_x[-1]$ for a closed point $x \in X$ is a simple object in both $\mathcal{A}$ and $\mathcal{B}_-$; hence $\mathcal{O}_x \in \mathcal{S}$ is a stable object in $\mathcal{U}$ and $\mathcal{U}_{-1}$. Applying $\Phi_{\mathcal{O}_X}$-action and Theorem 2.15, we obtain a contradiction.
q.e.d.

The relationship between normalized stability conditions and non-normalized stability conditions is described as follows.

Proposition 2.18. *The $\mathbb{C}$-action on* $\mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X)$ *induces a commutative diagram,*

$$\begin{array}{ccc}
\mathrm{Stab}^{\circ}_{\Gamma_{\bullet},\mathrm{n}}(\mathcal{D}_X) \times \mathbb{C} & \xrightarrow{\ \alpha\ } & \mathrm{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X) \\
{\scriptstyle \Pi_{\mathrm{n}} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \Pi} \\
\mathbb{C}^* \times \mathbb{H}^{\circ} \times \mathbb{C} & \xrightarrow{\ \ e\ \ } & \mathbb{C}^3.
\end{array}$$

Here α is an isomorphism and e is a map defined by

$$e(s,t,u) = (\exp(-i\pi u)s, \exp(-i\pi u)t, \exp(-i\pi u)).$$

Proof. The diagram is obviously commutative by the construction, so it is enough to show that α is an isomorphism. By Lemma 2.17, the map α is surjective, and it remains to check that α is injective. Take two elements,

$$(\sigma_i, \lambda_i) \in \mathrm{Stab}^{\circ}_{\Gamma_{\bullet},\mathrm{n}}(\mathcal{D}_X) \times \mathbb{C}, \quad i = 1, 2,$$

which are mapped to the same element under α. We may assume that $\lambda_1 = 0$, and $\sigma_1 \in \mathcal{U}_0 \cup \mathcal{U}_1$ by Theorem 2.15. For simplicity, we show the case of $\sigma_1 \in \mathcal{U}_0$. The other case is similarly discussed.

Let us write $\sigma_1 = (Z_1, \mathcal{A})$ and $\sigma_2 = (Z_2, \mathcal{A}_2)$ for the heart of a bounded t-structure $\mathcal{A}_2 \subset \mathcal{D}_X$. Since $\lambda_1 = 0$ and $Z_i(\mathcal{O}_x) = -1$ for $i = 1, 2$, we obtain $\exp(-i\pi\lambda_2) = 1$. Hence we may write $\lambda_2 = 2m$ for some $m \in \mathbb{Z}$. By Theorem 2.15 and Lemma 2.12, we can write the heart $\mathcal{A}_2$ in two ways,

$$\mathcal{A}_2 = \mathcal{A}[2m] = \Phi^{(m)}_{\mathcal{O}_X}(\mathcal{A}).$$

Therefore we have the autoequivalence,

(30)
$$(\Phi_{\mathcal{O}_X}[-2])^{(m)} \colon \mathcal{A} \xrightarrow{\sim} \mathcal{A},$$

which takes $\mathcal{O}_X$ to $\mathcal{O}_X$. Since the equivalence (30) takes simple objects to simple objects, it takes an object $\mathcal{O}_x[-1]$ for $x \in X$ to an object of the form $\mathcal{O}_{x'}[-1]$ for some $x' \in X$. Then a standard argument (cf. [**13**, theorem 2.5]) shows that

(31)
$$(\Phi_{\mathcal{O}_X}[-2])^{(m)} \cong f^*,$$

for an automorphism $f \colon X \xrightarrow{\sim} X$. However, by Lemma 2.10, the isomorphisms on Γ induced by both sides of (31) are equal only if $m = 0$. Therefore $\lambda_2 = 0$, and $\sigma_1 = \sigma_2$ follows.   q.e.d.

Note that we have

$$\mathrm{Im}\, e = \mathbb{C}^* \times \mathrm{GL}_+(2, \mathbb{R}).$$

Here $\mathrm{GL}_+(2, \mathbb{R})$ is the subgroup of $\mathrm{GL}(2, \mathbb{R})$ preserving the orientation of $\mathbb{R}^2$, and it is embedded into $\mathbb{C}^2$ via

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto (a + ci, b + di).$$

Therefore we obtain the following theorem.

Theorem 2.19. *We have the isomorphism,*

$$(32) \qquad \mathrm{Stab}_{\Gamma_\bullet}^\circ(\mathcal{D}_X) \cong \mathbb{C} \times \widetilde{\mathrm{GL}}_+(2, \mathbb{R}),$$

and the isomorphism of the double quotient space,

$$\langle \Phi_{\mathcal{O}_X} \rangle \backslash \mathrm{Stab}_{\Gamma_\bullet}^\circ(\mathcal{D}_X) / \mathbb{C} \cong \mathbb{C}^* \times \mathbb{H}^\circ.$$

2.8. A loop around the conifold point. Let $S^1 \subset \mathbb{C}^*$ be the unit circle, and ι be the embedding,

$$\iota = (\mathrm{id}, \sqrt{-1}) \colon S^1 \hookrightarrow \mathbb{C}^* \times \mathbb{H}^\circ.$$

The embedding ι lifts to a map from the universal cover $\mathbb{R} \to S^1$, i.e. there is a commutative diagram,

$$(33) \qquad \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathrm{Stab}_{\Gamma_\bullet}^\circ(\mathcal{D}_X) \\ {\scriptstyle \exp(i\pi*)} \downarrow & & \downarrow \\ S^1 & \xrightarrow{\;\iota\;} & \mathbb{C}^* \times \mathbb{H}^\circ. \end{array}$$

The top arrow of (33) is denoted by γ,

$$(34) \qquad \gamma \colon \mathbb{R} \ni t \mapsto \gamma(t) = (Z_t, \mathcal{P}_t) \in \mathrm{Stab}_{\Gamma_\bullet}^\circ(\mathcal{D}_X),$$

where $\mathcal{P}_t$ is a slicing of $\mathcal{D}_X$ and the commutative diagram (33) implies

$$Z_t = Z_{(\exp(i\pi t), i\omega)},$$

for an ample generator $\omega \in H^2(X, \mathbb{Z})$. Here the RHS is defined by (15) with $u = (\exp(i\pi t), i\omega)$. The map γ is uniquely determined by requiring that

$$\gamma((k, k+1]) \subset \mathcal{U}_k, \quad \text{for all } k \in \mathbb{Z}.$$

Namely, we have $\mathcal{P}_t((0,1]) = \mathcal{A}_k$ for $t \in (k, k+1]$ with $k \in \mathbb{Z}$, where $\mathcal{A}_k$ are hearts of bounded t-structures given as

$$(35) \qquad \mathcal{A}_k = \left\{ \begin{array}{ll} \Phi_{\mathcal{O}_X}^{(k')}(\mathcal{A}), & k = 2k', \\ \Phi_{\mathcal{O}_X}^{(k')}(\mathcal{B}_{+1}), & k = 2k' + 1. \end{array} \right.$$

In what follows, we fix the continuous map (34). We will use the following lemma.

Lemma 2.20. *For $E \in \mathcal{A}_k$, we have $\mathrm{cl}(E) \in \Gamma_0$ if and only if $E \in \langle \mathcal{O}_X[-k] \rangle_{\mathrm{ex}}$. In this case, E is a Z_t-semistable object of phase t.*

Proof. Since $\Phi_{\mathcal{O}_X}$ preserves the filtration $\Gamma_\bullet$ and we have (24), we may assume that $k = 0$ or $k = 1$. In both cases, the assertion is easily checked by the construction of weak stability conditions and Lemma 2.12. q.e.d.

For $t \in \mathbb{R}$, the slicing $\mathcal{P}_t$ defined in (34) satisfies the following.

Lemma 2.21. *For fixed $\phi \in \mathbb{R}$ and $k \in \mathbb{Z}$, the subcategory $\mathcal{P}_t(\phi) \subset \mathcal{D}_X$ does not depend on $t \in (\phi + k, \phi + k + 1)$.*

Proof. Let us take

$$\phi + k < t_1 < t_2 < \phi + k + 1,$$

and show that $\mathcal{P}_{t_1}(\phi) \subset \mathcal{P}_{t_2}(\phi)$. The other inclusion $\mathcal{P}_{t_2}(\phi) \subset \mathcal{P}_{t_1}(\phi)$ is similarly discussed. We take an object $E \in \mathcal{P}_{t_1}(\phi)$ and set

$$I = \{t \in [t_1, t_2] : E \in \mathcal{P}_t(\phi)\}.$$

Since I is a closed subset (cf. Remark 2.3), it is enough to see that $t_0 := \sup I$ satisfies $t_0 = t_2$. Suppose by contradiction that $t_0 < t_2$. Then there is a distinguished triangle,

$$(36) \qquad\qquad\qquad E' \to E \to E'',$$

which destabilizes E w.r.t. the weak stability condition $\gamma(t)$ for $0 < t - t_0 \ll 1$. If both $\mathrm{cl}(E')$ and $\mathrm{cl}(E'')$ are not contained in Γ_0, we have

$$\arg Z_{t_0}(E') = \arg Z_t(E') > \arg Z_t(E'') = \arg Z_{t_0}(E'').$$

This contradicts that $E \in \mathcal{P}_{t_0}(\phi)$; therefore either $\mathrm{cl}(E')$ or $\mathrm{cl}(E'')$ is contained in Γ_0. Then by Lemma 2.20, there is $k' \in \mathbb{Z}$ such that E_1 or E_2 is contained in $\langle \mathcal{O}_X[k'] \rangle_{\mathrm{ex}}$. This implies that

$$t + k' > \phi \geq t_0 + k' \quad \text{or} \quad t_0 + k' \geq \phi > t + k',$$

respectively. Obviously both cases do not happen. q.e.d.

Let us take $0 < \phi < 1$, $k \in \mathbb{Z}$ and set $t_0 = \phi + k$. We take $t_- < t_0 < t_+$ satisfying

$$(37) \qquad\qquad\qquad [t_-, t_+] \subset (k, k + 1).$$

Note that $\gamma(t) \in \mathcal{U}_k$ for $t \in [t_-, t_+]$. We have the following proposition.

Proposition 2.22. *For an object $E \in \mathcal{P}_{t_0}(\phi)$, the HN filtrations with respect to $\gamma(t_\pm)$ yield short exact sequences in $\mathcal{A}_k$ respectively,*

$$(38) \qquad\qquad 0 \to E_1 \to E \to E_2 \to 0,$$

$$(39) \qquad\qquad 0 \to E_1' \to E \to E_2' \to 0,$$

satisfying the following.

- *$E_1 \in \langle \mathcal{O}_X[-k] \rangle_{\mathrm{ex}}$ and $E_2' \in \langle \mathcal{O}_X[-k] \rangle_{\mathrm{ex}}$.*
- *$E_2 \in \mathcal{P}_{t_+}(\phi)$ and $E_1' \in \mathcal{P}_{t_-}(\phi)$.*

Conversely, if an object $E \in \mathcal{A}_k$ fits into an exact sequence (38) or (39) satisfying the above conditions, then $E \in \mathcal{P}_{t_0}(\phi)$.

Proof. For simplicity, we only see the sequence (38). The result follows from Lemma 2.20 if $\mathrm{cl}(E) \in \Gamma_0$; therefore we assume that $\mathrm{cl}(E) \in \Gamma \setminus \Gamma_0$. Also we may assume that E is not Z_{t_+}-semistable; hence there is an exact sequence in $\mathcal{A}_k$,

$$0 \to E' \to E \to E'' \to 0,$$

such that $\arg Z_{t_+}(E') > \arg Z_{t_+}(E'')$. Then the same argument in the proof of Lemma 2.21 shows that $\mathrm{cl}(E')$ or $\mathrm{cl}(E'')$ is contained in $\langle \mathcal{O}_X[-k]\rangle_{\mathrm{ex}}$. This implies that the HN filtration w.r.t. $\gamma(t_+)$ consists of a two step filtration in $\mathcal{A}_k$, which we denote by (38), and either E_1 or E_2 is contained in $\langle \mathcal{O}_X[-k]\rangle_{\mathrm{ex}}$. Since $t_+ > t_0$, we have

$$
\begin{aligned}
\arg Z_{t_+}(\mathcal{O}_X[-k]) &= \pi(t_+ - k) \\
&> \pi(t_0 - k) \\
&= \arg Z_{t_0}(E) \\
&= \arg Z_{t_+}(E).
\end{aligned}
$$

Therefore we must have $\mathrm{cl}(E_1) \in \langle \mathcal{O}_X[-k]\rangle_{\mathrm{ex}}$ and $E_2 \in \mathcal{P}_{t_+}(\phi)$.

Conversely, suppose that $E \in \mathcal{A}_k$ fits into an exact sequence (38). By Lemma 2.20, Lemma 2.21, and noting Remark 2.3, we have

$$\mathcal{P}_{t_+}(\phi) \subset \mathcal{P}_{t_0}(\phi), \quad \mathcal{O}_X[-k] \in \mathcal{P}_{t_0}(\phi).$$

Therefore we have $E \in \mathcal{P}_{t_0}(\phi)$.   q.e.d.

Remark 2.23. As we discussed in the introduction, the image of ι in the diagram (33) may be interpreted as a loop around the conifold point in Figure 1 if X is a quintic 3-fold, since the covering transformation of the left arrow of (33) is induced by the action of $\Phi_{\mathcal{O}_X}$.

Remark 2.24. Although the result of Theorem 2.19 holds under the assumption (27), the continuous map γ exists without that assumption once we fix an ample divisor ω. The results of Lemma 2.21 and Proposition 2.22 hold as well without (27).

3. Donaldson-Thomas Theory

In this section, we introduce generalized DT invariants counting semistable objects in $\mathcal{D}_X$ with respect to our weak stability conditions, and establish their wall-crossing formula. Originally DT theory is introduced in [**41**] as counting stable coherent sheaves on Calabi-Yau 3-folds, and defined only when semistable sheaves and stable sheaves coincide. The generalized DT theory introduced by Joyce-Song [**24**] is also defined when there is a semistable but not stable sheaf, and the notion of Hall algebra is used for the definition. The same construction is also applied in our situation, and we first give some notions needed for the definition.

3.1. Hall algebra. In this subsection, we recall the notion of Hall algebra via moduli stacks. See [**27**] for the introduction of stacks and [**20**] for the detail on the Hall algebra.

Let $\mathcal{M}$ be the moduli stack of objects $E \in D^b(\mathrm{Coh}(X))$ satisfying

(40) $$\mathrm{Ext}^i_X(E, E) = 0, \quad i < 0.$$

By the result of Lieblich [**31**], $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$. For each element

$$\sigma = (Z, \mathcal{C}) \in \mathrm{Stab}^\circ_{\Gamma_\bullet}(\mathcal{D}_X),$$

where $\mathcal{C} \subset \mathcal{D}_X$ is the heart of a bounded t-structure, we have the (abstract) substack,

$$\mathcal{O}bj(\mathcal{C}) \subset \mathcal{M},$$

which parameterizes objects $E \in \mathcal{C}$. The stack $\mathcal{O}bj(\mathcal{C})$ decomposes as follows:

$$\mathcal{O}bj(\mathcal{C}) = \coprod_{v \in \Gamma} \mathcal{O}bj^v(\mathcal{C}),$$

where $\mathcal{O}bj^v(\mathcal{C})$ is the stack of objects $E \in \mathcal{C}$ with $\mathrm{cl}(E) = v$.

Suppose for instance that $\mathcal{O}bj(\mathcal{C})$ is an algebraic stack locally of finite type over $\mathbb{C}$. The $\mathbb{Q}$-vector space $\mathcal{H}(\mathcal{C})$ is generated by the isomorphism classes of symbols,

$$[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{C})],$$

where $\mathcal{X}$ is an algebraic stack of finite type over $\mathbb{C}$, and f is a 1-morphism of stacks. Here two symbols $[\mathcal{X}_i \xrightarrow{f_i} \mathcal{O}bj(\mathcal{C})]$ for $i = 1, 2$ are *isomorphic* if there is an isomorphism of stacks,

$$g \colon \mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2,$$

such that $f_2 \circ g \cong f_1$. The relations are generated by

$$[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{C})] \sim [\mathcal{U} \xrightarrow{f|_{\mathcal{U}}} \mathcal{O}bj(\mathcal{C})] + [\mathcal{Z} \xrightarrow{f|_{\mathcal{Z}}} \mathcal{O}bj(\mathcal{C})],$$

where $\mathcal{U} \subset \mathcal{X}$ is an open substack and $\mathcal{Z} = \mathcal{X} \setminus \mathcal{U}$.

Let $\mathcal{E}x(\mathcal{C})$ be the stack of short exact sequence in $\mathcal{C}$,

(41) $$0 \to A_1 \to A_2 \to A_3 \to 0.$$

By sending the exact sequence (41) to the object A_i, we obtain morphisms,

$$p_i \colon \mathcal{E}x(\mathcal{C}) \to \mathcal{O}bj(\mathcal{C}), \quad i = 1, 2, 3.$$

For two elements

$$\rho_i = [\mathcal{X}_i \xrightarrow{f_i} \mathcal{O}bj(\mathcal{C})] \in \mathcal{H}(\mathcal{C}), \quad i = 1, 2,$$

we have the diagram

$$\begin{array}{ccccc}
\mathcal{Z} & \xrightarrow{\ h\ } & \mathcal{E}x(\mathcal{C}) & \xrightarrow{\ p_2\ } & \mathcal{O}bj(\mathcal{C}) \\
\downarrow & & \downarrow{\scriptstyle (p_1,p_3)} & & \\
\mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{(f_1,f_2)} & \mathcal{O}bj(\mathcal{C})^{\times 2}. & &
\end{array}$$

Here the left diagram is a Cartesian square. We define the $*$-product $\rho_1 * \rho_2$ to be

$$\rho_1 * \rho_2 = [\mathcal{Z} \xrightarrow{p_2 \circ h} \mathcal{O}bj(\mathcal{C})] \in \mathcal{H}(\mathcal{C}).$$

It is proved in [**20**, theorem 5.2] that $*$ is an associative product on $\mathcal{H}(\mathcal{C})$ with unit given by

$$1 = [\operatorname{Spec}\mathbb{C} \to \mathcal{O}bj(\mathcal{C})],$$

whose image corresponds to $0 \in \mathcal{C}$. The algebra $\mathcal{H}(\mathcal{C})$ is Γ-graded,

$$\mathcal{H}(\mathcal{C}) = \bigoplus_{v \in \Gamma} \mathcal{H}^v(\mathcal{C}),$$

where $\mathcal{H}^v(\mathcal{C})$ is spanned by symbols $[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{C})]$ such that f factors through the substack $\mathcal{O}bj^v(\mathcal{C}) \subset \mathcal{O}bj(\mathcal{C})$.

We will use certain completions of the algebra $\mathcal{H}(\mathcal{C})$. Let $V \subset \mathbb{H}$ be a subset written as

(42) $$V = \{r \exp(i\pi\phi) : r > 0, \ \phi_1 \le \phi \le \phi_2\}$$

for some $\phi_1, \phi_2 \in \mathbb{R}$ with $0 \le \phi_2 - \phi_1 < 1$. We define $\widehat{\mathcal{H}}(\mathcal{C})_{Z,V}$ to be

$$\widehat{\mathcal{H}}(\mathcal{C})_{Z,V} = \prod_{v \in \Gamma, \ Z(v) \in V} \mathcal{H}^v(\mathcal{C}).$$

3.2. Elements $\delta^v(Z)$ and $\epsilon^v(Z)$. For $\sigma = (Z, \mathcal{C}) \in \operatorname{Stab}^{\circ}_{\Gamma_{\bullet}}(\mathcal{D}_X)$ and $v \in \Gamma$, the stack of Z-semistable objects $E \in \mathcal{C}$ with $\operatorname{cl}(E) = v$ is denoted by

$$\mathcal{M}^v(Z) \subset \mathcal{O}bj^v(\mathcal{C}).$$

Suppose for instance that $\mathcal{M}^v(Z)$ is an algebraic stack of finite type over $\mathbb{C}$. Then the above stack defines the element,

$$\delta^v(Z) = [\mathcal{M}^v(Z) \hookrightarrow \mathcal{O}bj^v(\mathcal{C})] \in \mathcal{H}^v(\mathcal{C}).$$

We say a subset $l \subset \mathbb{H}$ is a *ray* if there is $\phi \in (0,1]$ such that $l = \mathbb{R}_{>0} \exp(i\pi\phi)$. For each ray l, we define $\delta^l(Z)$ to be

(43) $$\delta^l(Z) = 1 + \sum_{Z(v) \in l} \delta^v(Z) \in \widehat{\mathcal{H}}(\mathcal{C})_{Z,l}.$$

Then we define the element $\epsilon^l(Z)$ to be

$$\epsilon^l(Z) = \log \delta^l(Z) \in \widehat{\mathcal{H}}(\mathcal{C})_{Z,l}.$$

Namely, $\epsilon^l(Z)$ is given by

$$\epsilon^l(Z) = \sum_{Z(v) \in l} \epsilon^v(Z),$$

where $\epsilon^v(Z)$ is written as

$$(44) \qquad \epsilon^v(Z) = \sum_{\substack{m \geq 1, \ v_1, \ldots, v_m \in \Gamma, \\ Z(v_i) \in \mathbb{R}_{>0} Z(v), \\ v_1 + \cdots + v_m = v.}} \frac{(-1)^{m-1}}{l} \delta^{v_1}(Z) * \cdots * \delta^{v_m}(Z).$$

The above definition makes sense by the following lemma, whose proof will be given in Section 5.

Lemma 3.1. *The sum (44) is a finite sum.*

So far we have assumed that the stacks $\mathcal{O}bj(\mathcal{C})$ and $\mathcal{M}^v(Z)$ are algebraic stacks locally of finite type, finite type, respectively. However, these conditions are too strong for the applications. In fact, it is enough to show the following lemma, by discussing with the framework of Kontsevich-Soibelman [**26**, section 3]. The proof will be given in Section 5.

Lemma 3.2. *For any $\sigma = (Z, \mathcal{C}) \in \mathrm{Stab}_{\Gamma_\bullet}^\circ(\mathcal{D}_X)$ and $v \in \Gamma$, we have the following.*
(i) The $\mathbb{C}$-valued points of the substack $\mathcal{O}bj^v(\mathcal{C}) \subset \mathcal{M}$ are a countable union of constructible subsets in $\mathcal{M}$.
(ii) The $\mathbb{C}$-valued points of the substack $\mathcal{M}^v(Z) \subset \mathcal{M}$ form a constructible subset in $\mathcal{M}$.

3.3. Generalized DT invariants. For a quasi-projective variety Y, we define

$$\Upsilon(Y) = \sum_{j,k \geq 0} (-1)^k \dim W_j(H_c^k(Y, \mathbb{Q})) q^{j/2} \in \mathbb{Q}[q^{1/2}],$$

where $W_\bullet$ is the weight filtration on the compact support cohomology group $H_c^*(Y, \mathbb{Q})$. The assignment $Y \mapsto \Upsilon(Y)$ extends to the Hall algebra $\mathcal{H}(\mathcal{C})$,

$$\Upsilon \colon \mathcal{H}(\mathcal{C}) \to \mathbb{Q}(q^{1/2}),$$

such that we have

$$\Upsilon([[Y/G] \to \mathcal{O}bj(\mathcal{C})]) = \Upsilon(Y)/\Upsilon(G),$$

where Y is a quasi-projective variety, G is a special algebraic group acting on G, and $[Y/G]$ is the quotient stack with respect to the G-action (cf. [**22**, theorem 4.9]). Here an algebraic group G is *special* if any principal G-bundle is Zariski locally trivial (cf. [**22**, definition 2.1]).

Recall that for any variety Y, there is a canonical constructible function by Behrend [3],

$$\nu_Y \colon Y \to \mathbb{Z}.$$

The above function satisfies the following properties.

- For $p \in Y$, suppose that there is an analytic open neighborhood $p \in U \subset Y$, and a holomorphic function $f \colon V \to \mathbb{C}$ on a complex manifold V such that $U \cong \{df = 0\}$. Then we have

(45) $$\nu_Y(p) = (-1)^{\dim V}(1 - \chi(M_f(p))).$$

  Here $M_f(p)$ is a Milnor fiber of f at p.
- If there is a symmetric perfect obstruction theory on Y, we have

$$\int_{[Y]^{\mathrm{vir}}} 1 = \sum_{m \in \mathbb{Z}} m \cdot \chi(\nu_Y^{-1}(m)).$$

The Behrend's constructible function can be generalized to any algebraic stack $\mathcal{Y}$,

$$\nu_{\mathcal{Y}} \colon \mathcal{Y} \to \mathbb{Z}.$$

Namely, if $\mathcal{Y}$ is written as a global quotient stack $\mathcal{Y} = [Y/G]$, then $\nu_{\mathcal{Y}} = (-1)^{\dim G}\nu_Y$. For a general case, the existence of $\nu_{\mathcal{Y}}$ is proved in [24, proposition 4.4].

Let $\mathcal{M}$ be the moduli stack of objects $E \in D^b(\mathrm{Coh}(X))$ satisfying (40). By the above argument, there is Behrend's constructible function $\nu_{\mathcal{M}}$ on $\mathcal{M}$. The function $\nu_{\mathcal{M}}$ should be calculated by the Euler characteristic of some holomorphic function as in (45). In fact, the following conjecture, which is a derived category version of [24, theorem 5.5], should be true.

Conjecture 3.3. *For any $[E] \in \mathcal{M}(\mathbb{C})$, let G be a maximal reductive subgroup in $\mathrm{Aut}(E)$. Then there exists a G-invariant analytic open neighborhood V of 0 in $\mathrm{Ext}^1(E, E)$, a G-invariant holomorphic function $f \colon V \to \mathbb{C}$ with $f(0) = df|_0 = 0$, and a smooth morphism of complex analytic stacks $\Phi \colon [\{df = 0\}/G] \to \mathcal{M}$ of relative dimension $\dim \mathrm{Aut}(E) - \dim G$.*

The above conjecture is proved in [24, theorem 5.5] if $E \in \mathrm{Coh}(X)$. We believe that similar arguments show the above conjecture for any $[E] \in \mathcal{M}(\mathbb{C})$, although several details have to be checked. Also Behrend-Getzler [6] have announced a similar result, so in what follows we assume that the above conjecture is true.

The Behrend function on $\mathcal{M}$ defines the map $\nu \cdot \colon \mathcal{H}(\mathcal{C}) \to \mathcal{H}(\mathcal{C})$,

$$\nu \cdot ([\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{C})]) = \sum_{m \in \mathbb{Z}} m \cdot [\mathcal{X} \times_{\mathcal{O}bj(\mathcal{C})} \nu_{\mathcal{M}}|_{\mathcal{O}bj(\mathcal{C})}^{-1}(m) \to \mathcal{O}bj(\mathcal{C})].$$

The generalized DT invariant is defined as follows.

Definition 3.4. [**24**, definition 5.13] *We define* $\mathrm{DT}_Z(v)$ *as follows:*

$$\mathrm{DT}_Z(v) := - \lim_{q^{1/2} \to 1} (q-1) \Upsilon(\nu \cdot \epsilon^v(Z)) \in \mathbb{Q}.$$

The existence of the limit is essentially proved in [**22**, section 6.2].

3.4. Lie algebra homomorphism. Let $\chi \colon \Gamma \times \Gamma \to \mathbb{Z}$ be an anti-symmetric bilinear form on Γ, defined by

$$\chi((r, \beta, n), (r', \beta', n')) = rn' - r'n.$$

By the Riemann-Roch theorem and the Serre duality, we have

$$
\begin{aligned}
\chi(\mathrm{cl}(E), \mathrm{cl}(F)) = {} & \dim \mathrm{Hom}(E, F) - \dim \mathrm{Ext}^1(E, F) \\
& + \dim \mathrm{Ext}^1(F, E) - \dim \mathrm{Hom}(F, E),
\end{aligned}
$$

(46)

for $E, F \in \mathcal{D}_X$.

Let $\mathfrak{g}$ be the $\mathbb{Q}$-vector space spanned by symbols c_v for $v \in \Gamma$,

$$\mathfrak{g} = \bigoplus_{v \in \Gamma} \mathbb{Q}c_v.$$

There is a Lie-algebra structure on $\mathfrak{g}$ with bracket given by

$$[c_v, c_{v'}] = (-1)^{\chi(v,v')} \chi(v, v') c_{v+v'}.$$

For a weak stability condition $\sigma = (Z, \mathcal{C}) \in \mathrm{Stab}^\circ_{\Gamma_\bullet}(\mathcal{D}_X)$, we can define the Lie algebra of *virtual indecomposable objects*,

$$\mathfrak{H}(\mathcal{C}) \subset \mathcal{H}(\mathcal{C}),$$

in the same way as [**20**, definition 5.14]. The definitions of virtual indecomposable objects and the Lie algebra $\mathfrak{H}(\mathcal{C})$ are complicated, and we omit the detail. The Lie algebra $\mathfrak{H}(\mathcal{C})$ is also Γ-graded,

$$\mathfrak{H}(\mathcal{C}) = \bigoplus_{v \in \Gamma} \mathfrak{H}^v(\mathcal{C}), \quad \mathfrak{H}^v(\mathcal{C}) = \mathcal{H}^v(\mathcal{C}) \cap \mathfrak{H}(\mathcal{C}).$$

For $v \in \Gamma$, the element $\delta^v(Z)$ is not necessary virtual indecomposable, but $\epsilon^v(Z)$ is always virtual indecomposable (cf. [**21**, theorem 8.7]). Assuming Conjecture 3.3, the following result can be proved along the same lines as [**24**, theorem 5.14].

Theorem 3.5. [**24**, theorem 5.14] *There is a homomorphism as Γ-graded Lie algebras,*

(47)
$$\Psi \colon \mathfrak{H}(\mathcal{C}) \to \mathfrak{g},$$

which takes $\epsilon^v(Z)$ to $- \mathrm{DT}_Z(v)c_v$.

Let $V \subset \mathbb{H}$ be a subset defined by (42). We can similarly define the completions of the Lie algebras,

$$\widehat{\mathfrak{H}}(\mathcal{C})_{Z,V} = \prod_{v \in \Gamma,\ Z(v) \in V} \mathfrak{H}^v(\mathcal{C}),$$

$$\widehat{\mathfrak{g}}(\mathcal{C})_{Z,V} = \prod_{v \in \Gamma,\ Z(v) \in V} \mathfrak{g}^v(\mathcal{C}),$$

and (47) induces the Lie algebra homomorphism,

$$(48) \qquad\qquad \Psi \colon \widehat{\mathfrak{H}}(\mathcal{C})_{Z,V} \to \widehat{\mathfrak{g}}(\mathcal{C})_{Z,V}.$$

3.5. DT invariants around the conifold point. For $t \in (k, k+1]$ with $k \in \mathbb{Z}$, let

$$\gamma(t) = (Z_t, \mathcal{A}_k) = (Z_t, \mathcal{P}_t) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$$

be the weak stability condition defined in (34). Here $\mathcal{A}_k$ is the heart of a bounded t-structure given by (35) and $\mathcal{P}_t$ is the associated slicing. For an element $v \in \Gamma$, the associated element, $\epsilon^v(Z_t) \in \mathcal{H}(\mathcal{A}_k)$, defines the invariant,

$$\mathrm{DT}_{Z_t}(v) = -\lim_{q^{1/2} \to 1} (q-1) \Upsilon(\nu \cdot \epsilon^v(Z_t)),$$

in the same way as Definition 3.4.

Definition 3.6. *For data*

$$(r, \beta, n) \in \Gamma, \quad t \in \mathbb{R}, \quad \phi \in \mathbb{R},$$

we define the invariant $\mathrm{DT}_t(r, \beta, n, \phi) \in \mathbb{Q}$ *as follows.*
When $0 < \phi \le 1$, *suppose that the following holds:*

$$(49) \qquad\qquad Z_t(r, -\beta, -n) \in \mathbb{R}_{>0} \exp(i\pi\phi).$$

Then we define

$$\mathrm{DT}_t(r, \beta, n, \phi) := \mathrm{DT}_{Z_t}(r, -\beta, -n).$$

If (49) is not satisfied, we set $\mathrm{DT}_t(r, \beta, n, \phi) = 0$.
In a general case, writing $\phi = m + \phi_0$ *with* $m \in \mathbb{Z}$ *and* $0 < \phi_0 \le 1$, *we define*

$$(50) \qquad \mathrm{DT}_t(r, \beta, n, \phi) := \mathrm{DT}_t((-1)^m r, (-1)^m \beta, (-1)^m n, \phi_0).$$

Note that $\mathrm{DT}_t(r, \beta, n, \phi)$ is a counting invariant of objects $E \in \mathcal{D}_X$ satisfying

$$E \in \mathcal{P}_t(\phi), \quad \mathrm{cl}(E) = (r, -\beta, -n).$$

In case of $\beta = n = 0$, the invariant is already computed.

Lemma 3.7. *For* $0 < \phi \le 1$ *and* $t \in (k, k+1]$ *with* $k \in \mathbb{Z}$, *we have*

$$\mathrm{DT}_t(r, 0, 0, \phi) = \begin{cases} \frac{1}{r^2}, & \text{if } t = \phi + k, (-1)^k r > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As in the previous subsection, let $\mathcal{M}^{(r,0,0)}(Z_t)$ be the substack of $\mathcal{O}bj(\mathcal{A}_k)$, which parameterizes Z_t-semistable objects $E \in \mathcal{A}_k$ with $\mathrm{cl}(E) = (r, 0, 0)$. By Lemma 2.20 and the assumption (13), we have the isomorphism of stacks,

$$\mathcal{M}^{(r,0,0)}(Z_t) \cong [\mathrm{Spec}\,\mathbb{C}/\,\mathrm{GL}_{r'}(\mathbb{C})],$$

where $r' = (-1)^k r$. The unique $\mathbb{C}$-valued point of the RHS corresponds to the object $\mathcal{O}_X[-k]^{\oplus r'} \in \mathcal{A}_k$, and it has phase $t - k$ by Lemma 2.20. Hence $\mathrm{DT}_t(r, 0, 0, \phi)$ is non-zero only if $t = \phi + k$, and the contribution of the object $\mathcal{O}_X[-k]^{\oplus r'}$ is computed in the same way as [**24**, example 6.2].

q.e.d.

We set the following generating series:

$$\mathrm{DT}_t(\phi) = \sum_{(r,\beta,n)\in\Gamma} \mathrm{DT}_t(r, \beta, n, \phi)x^r y^\beta z^n.$$

We have the following lemma.

Lemma 3.8. *For a fixed $\phi \in \mathbb{R}$ and $k \in \mathbb{Z}$, the generating series $\mathrm{DT}_t(\phi)$ does not depend on $t \in (\phi + k, \phi + k + 1)$.*

Proof. The result immediately follows from Lemma 2.21.   q.e.d.

In what follows, we set

$$\mathrm{DT}^k(r, \beta, n, \phi) := \mathrm{DT}_t(r, \beta, n, \phi),$$
$$\mathrm{DT}^k(\phi) := \mathrm{DT}_t(\phi),$$

if $t \in (\phi + k, \phi + k + 1)$ with $k \in \mathbb{Z}$. The above notation makes sense by Lemma 3.8.

3.6. Wall-crossing formula. In this subsection, we give a proof of the following theorem, assuming Conjecture 3.3.

Theorem 3.9. *For $\phi \in \mathbb{R}$ and $k \in \mathbb{Z}$, the series $\mathrm{DT}^k(\phi)$ is obtained from $\mathrm{DT}^{k-1}(\phi)$ by the following transformation:*

$$z^n \mapsto \begin{cases} (1 - (-1)^n x)^n z^n, & \text{if } k \text{ is even.} \\ x^n z^n/(1 - (-1)^n x)^n, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Since $\mathrm{DT}_t(\phi) = \mathrm{DT}_t(\phi + 2)$, we may assume that $0 < \phi \leq 2$. First we discuss the case of $0 < \phi < 1$. Let us take $k \in \mathbb{Z}$ and set $t_0 = \phi + k$. We take $0 < \varepsilon \ll 1$ so that $(t_0 - \varepsilon, t_0 + \varepsilon) \subset (k, k+1)$ holds. We set $t_\pm = t_0 \pm \varepsilon$, and $V \subset \mathbb{H}$ to be

$$V = \{r\exp(i\pi\theta) : r > 0,\ \theta \in [\phi - \varepsilon, \phi + \varepsilon]\}.$$

For each $t \in (k, k+1]$, the proof of Lemma 3.7 shows that

$$\delta^{((-1)^k r,0,0)}(Z_t) = [[\mathrm{Spec}\,\mathbb{C}/\,\mathrm{GL}_r(\mathbb{C})] \to \mathcal{O}bj(\mathcal{A}_k)] \in \mathcal{H}(\mathcal{A}_k),$$

which corresponds to the object $\mathcal{O}_X[-k]^{\oplus r}$. The above element of $\mathcal{H}(\mathcal{A}_k)$ does not depend on $t \in (k, k+1]$, and we denote it by $\delta^{((-1)^k r, 0, 0)}$ for simplicity. We define the elements δ^k and ϵ^k to be

$$\delta^k := 1 + \sum_{r \geq 1} \delta^{((-1)^k r, 0, 0)} \in \widehat{\mathcal{H}}(\mathcal{A}_k)_{Z_{t_0}, V},$$

$$\epsilon^k := \log \delta^k \in \widehat{\mathcal{H}}(\mathcal{A}_k)_{Z_{t_0}, V}.$$

The element ϵ^k is shown to be well-defined in the same way as Lemma 3.1. Let us set the ray $l = \mathbb{R}_{>0} \exp(i\pi\phi)$. By the same argument as [**23**, theorem 5.11], the result of Proposition 2.22 can be expressed in terms of a relationship in the completed Hall algebra $\widehat{\mathcal{H}}(\mathcal{A}_k)_{Z_{t_0}, V}$,

(51) $$\delta^k * \delta^l(Z_{t_+}) = \delta^l(Z_{t_0}) = \delta^l(Z_{t_-}) * \delta^k.$$

Therefore we obtain the formula in $\widehat{\mathcal{H}}(\mathcal{A}_k)_{Z_{t_0}, V}$,

$$\exp(\epsilon^l(Z_{t_+})) = \exp(\epsilon^k)^{-1} * \exp(\epsilon^l(Z_{t_-})) * \exp(\epsilon^k).$$

Here $\delta^l(Z_t)$ is defined in (43). Now we can apply a version of the Baker-Campbell-Hausdorff formula, and the RHS coincides with

$$\exp\left(\epsilon^l(Z_{t_-}) + \sum_{m \geq 1} \frac{(-1)^m}{m!} \mathrm{Ad}_{\epsilon^k}^m(\epsilon^l(Z_{t_-}))\right).$$

Here we have set

$$\mathrm{Ad}_{\epsilon^k}^m(\epsilon^l(Z_{t_-})) = \overbrace{[\epsilon^k[\epsilon^k[\cdots[\epsilon^k}^{m}, \epsilon^l(Z_{t_-})]\cdots]].$$

By taking the logarithms of both sides, we obtain the equality in $\widehat{\mathcal{H}}(\mathcal{A}_k)_{Z_{t_0}, V}$,

$$\epsilon^l(Z_{t_+}) = \epsilon^l(Z_{t_-}) + \sum_{m \geq 1} \frac{(-1)^m}{m!} \mathrm{Ad}_{\epsilon^k}^m(\epsilon^l(Z_{t_-})).$$

Let us set

$$\mathfrak{DT}_t(\phi) = \sum_{\substack{(r, -\beta, -n) \in \Gamma, \\ Z_t(r, -\beta, -n) \in l}} \mathrm{DT}_t(r, \beta, n, \phi) c_{(r, -\beta, -n)} \in \widehat{\mathfrak{g}}_{Z_{t_0}, V},$$

$$\mathfrak{E}^k = \sum_{r \geq 1} \frac{1}{r^2} c_{((-1)^k r, 0, 0)} \in \widehat{\mathfrak{g}}_{Z_{t_0}, V}.$$

Applying the Lie algebra homomorphism (48) and using Lemma 3.7, we obtain the equality in $\widehat{\mathfrak{g}}_{Z_{t_0}, V}$,

$$\mathfrak{DT}_{t_+}(\phi) = \mathfrak{DT}_{t_-}(\phi) + \sum_{m \geq 1} \frac{1}{m!} \mathrm{Ad}_{\mathfrak{E}^k}^m(\mathfrak{DT}_{t_-}(\phi)).$$

By expanding the RHS, we can easily obtain the following:

$$\mathrm{DT}_{t_+}(r, \beta, n, \phi) = \mathrm{DT}_{t_-}(r, \beta, n, \phi) +$$

$$(52) \qquad \sum_{\substack{m \geq 1, \\ r_0 \in \mathbb{Z}, r_1, \ldots, r_m \geq 1, \\ r_0 + \sum_{i=1}^m (-1)^k r_i = r.}} \frac{(-1)^{n(\sum_{i=1}^m r_i) + m(k+1)} n^m}{m! \prod_{i=1}^m r_i} \mathrm{DT}_{t_-}(r_0, \beta, n, \phi).$$

For a fixed $(\beta, n) \in H_2 \oplus H_0$, we set

$$\mathrm{DT}_t(\beta, n, \phi) = \sum_{r \in \mathbb{Z}} \mathrm{DT}_t(r, \beta, n, \phi) x^r.$$

Then the equality (52) implies that

$$\mathrm{DT}_{t_+}(\beta, n, \phi) =$$

$$\left(\sum_{m \geq 0} \frac{1}{m!} \sum_{r_1, \cdots, r_m \geq 1} \prod_{i=1}^m \frac{(-1)^{k+1} n}{r_i} \{(-1)^n x\}^{(-1)^k r_i} \right) \mathrm{DT}_{t_-}(\beta, n, \phi).$$

Then the assertion follows from Lemma 3.10 below.

When $\phi = 1$, we consider the rotated weak stability condition,

$$\frac{1}{2} \cdot \gamma(t_0) = (-i Z_{t_0}, \mathcal{P}_{t_0}((1/2, 3/2])).$$

Then the equality similar to (51) holds in the Hall algebra of $\mathcal{P}_{t_0}((1/2, 3/2])$. The argument for $0 < \phi < 1$ is also applied in this situation, and we obtain the same wall-crossing formula.

Finally when $1 < \phi \leq 2$, then the assertion holds, noting that

$$\mathrm{DT}^k(r, \beta, n, \phi) = \mathrm{DT}^{k+1}(-r, -\beta, -n, \phi - 1).$$

$$\text{q.e.d.}$$

We have used the following lemma.

Lemma 3.10.

$$\sum_{m \geq 0} \frac{1}{m!} \sum_{r_1, \ldots, r_m \geq 1} \prod_{i=1}^m \frac{(-1)^{k+1} n}{r_i} \{(-1)^n x\}^{(-1)^k r_i}$$

$$(53) \qquad = \begin{cases} (1 - (-1)^n x)^n, & \text{if } k \text{ is even,} \\ x^n / (1 - (-1)^n x)^n, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. We can calculate as follows.

$$\sum_{m \geq 0} \frac{1}{m!} \sum_{r_1,\ldots,r_m \geq 1} \prod_{i=1}^m \frac{(-1)^{k+1}n}{r_i} \{(-1)^n x\}^{(-1)^k r_i}$$

$$= \exp\left(\sum_{r \geq 1} \frac{(-1)^{k+1}n}{r} \{(-1)^n x\}^{(-1)^k r}\right)$$

$$= \exp\log\left(1 - \{(-1)^n x\}^{(-1)^k}\right)^{(-1)^k n}$$

$$= \left(1 - \{(-1)^n x\}^{(-1)^k}\right)^{(-1)^k n}.$$

The last one is written as the RHS of (53).   q.e.d.

Remark 3.11. By Theorem 3.9, the series $\mathrm{DT}^{k+2}(\phi)$ is obtained from $\mathrm{DT}^k(\phi)$ by the variable change $z \mapsto xz$. On the other hand, $\mathrm{DT}^{k+2}(\phi)$ and $\mathrm{DT}^k(\phi)$ are related by the equivalence $\Phi_{\mathcal{O}_X}$ by our construction of γ, and the variable change by $\Phi_{\mathcal{O}_X}$ is given by $z \mapsto xz$ by Lemma 2.10. This indicates that the wall-crossing formula does not indicate any modularity of our invariants. This is unfortunate in some sense, since there are situations in which the wall-crossing formula indicates some modularity of the invariants, e.g. the invariants on K3 surfaces [**45**].

3.7. Euler characteristic version. We can also investigate the Euler characteristic version of our invariants, which are defined in a similar way to $\mathrm{DT}_t(r, \beta, n, \phi)$ without the Behrend function. For $(r, \beta, n) \in \Gamma$ and $t \in \mathbb{R}$, suppose that (49) holds. When $0 < \phi \leq 1$, we define

$$\mathrm{Eu}_t(r, \beta, n, \phi) := \lim_{q^{1/2} \to 1} (q-1)\Upsilon(\epsilon^{(r,-\beta,-n)}(Z_t)) \in \mathbb{Q},$$

and set $\mathrm{Eu}_t(r, \beta, n, \phi) = 0$ if (49) is not satisfied. Here recall that we defined $\epsilon^{(r,-\beta,-n)}(Z_t)$ as an element of $\mathcal{H}(\mathcal{A}_k)$ if $t \in (k, k+1]$ for $k \in \mathbb{Z}$. For a general $\phi \in \mathbb{R}$, writing $\phi = m + \phi_0$ with $0 < \phi_0 \leq 1$, we define

$$\mathrm{Eu}_t(r, \beta, n, \phi) := \mathrm{Eu}_t((-1)^m r, (-1)^m \beta, (-1)^m n, \phi_0).$$

The generating series is also defined as well:

$$\mathrm{Eu}_t(\phi) = \sum_{(r,\beta,n) \in \Gamma} \mathrm{Eu}_t(r, \beta, n)x^r y^\beta z^n.$$

The following theorem can be proved in a similar way to Theorem 3.9, using a version of [**20**, theorem 6.12] instead of Theorem 3.5.

Theorem 3.12. *(i) For a given $k \in \mathbb{Z}$, the series $\mathrm{Eu}_t(\phi)$ does not depend on a choice of $t \in (\phi + k, \phi + k + 1)$. In particular, we may write it as $\mathrm{Eu}^k(\phi)$.*

(ii) The series $\mathrm{Eu}^k(\phi)$ *is obtained from* $\mathrm{Eu}^{k-1}(\phi)$ *by the following transformation:*

$$z \mapsto \begin{cases} (1+x)z, & \text{if } k \text{ is even,} \\ xz/(1+x), & \text{if } k \text{ is odd.} \end{cases}$$

Remark 3.13. Since the Behrend function is not involved in the definition of $\mathrm{Eu}_t(r, \beta, n, \phi)$, we do not rely on Conjecture 3.3 to show Theorem 3.12.

4. Examples

In this section, we explicitly compute the generating series $\mathrm{DT}^k(\phi)$ in some concrete examples. We also classify semistable objects in $\mathcal{D}_X$ w.r.t. our weak stability conditions in these examples.

4.1. D0-D6 states. In this subsection, we investigate the family of generating series,

(54) $$\mathrm{DT}^k(\phi), \quad k \in \mathbb{Z}, \quad \phi \in \mathbb{Z}.$$

Note that a series in (54) does not contain the variable y, since we have

$$\mathrm{DT}_t(r, \beta, n, 1) = 0, \quad \text{if } \beta \neq 0,$$

which follows from $\mathrm{Im}\, Z_t(r, -\beta, -n) \neq 0$ if $\beta \neq 0$.

By Theorem 3.9 and the relation (50), it is enough to study the series $\mathrm{DT}^{-1}(1)$ to know all of the series (54). By definition, $\mathrm{DT}^{-1}(1) = \mathrm{DT}_t(1)$ for $0 < t < 1$, and $\mathrm{DT}_t(1)$ is a generating series of invariants counting objects $E \in \mathcal{P}_t(1)$. Such objects can be described in the following way.

Lemma 4.1. *For* $0 < t < 1$*, we have*

$$\mathcal{P}_t(1) = \mathrm{Coh}_0(X)[-1].$$

Proof. Let us take an object $E \in \mathcal{P}_t(1) \subset \mathcal{A}$. Since $\mathrm{ch}_2(E) = 0$, we have

$$E \in \mathcal{A} \cap \langle \mathcal{O}_X, \mathrm{Coh}_0(X) \rangle_{\mathrm{tr}},$$
(55) $$= \langle \mathcal{O}_X, \mathrm{Coh}_0(X)[-1] \rangle_{\mathrm{ex}}.$$

By [**50**, proposition 2.2], any object in the category (55) is isomorphic to a two term complex,

$$\cdots \to 0 \to \mathcal{O}_X^{\oplus r} \to F \to 0 \to \cdots,$$

where $r \in \mathbb{Z}_{\geq 0}$, $F \in \mathrm{Coh}_0(X)$ and $\mathcal{O}_X^{\oplus r}$ is located in degree zero. In particular, there is an exact sequence in $\mathcal{A}$,

$$0 \to F[-1] \to E \to \mathcal{O}_X^{\oplus r} \to 0,$$

for some $F \in \mathrm{Coh}_0(X)$. If $r \neq 0$ and $F \neq 0$, then we have

$$\pi = \arg Z_t(F[-1]) > \arg Z_t(\mathcal{O}_X^{\oplus r}) = \pi t,$$

which contradicts the Z_t-semistability of E. Therefore we have $r = 0$ or $F = 0$. If $F = 0$, then $E \in \langle \mathcal{O}_X \rangle_{\mathrm{ex}} \subset \mathcal{P}_t(t)$ by Lemma 2.20, which contradicts $E \in \mathcal{P}_t(1)$. Hence $r = 0$ and $E \in \mathrm{Coh}_0(X)[-1]$ follows.

Conversely, if $E \in \mathrm{Coh}_0(X)[-1]$, the Z_t-semistability of E follows from the fact that $\mathrm{Coh}_0(X)[-1] \subset \mathcal{A}$ is closed under subobjects and quotients.   q.e.d.

The above lemma and the computations in [**24**, paragraph 6.3], [**26**, paragraph 6.5], [**49**, remark 8.13] show that

$$\mathrm{DT}^{-1}(0,0,n,1) = -\chi(X) \sum_{m \geq 1, m|n} \frac{1}{m^2},$$

and $\mathrm{DT}^{-1}(r,\beta,n,1) = 0$ if $(r,\beta) \neq (0,0)$. Therefore we have

$$\mathrm{DT}^{-1}(1) = -\chi(X) \sum_{\substack{n \geq 1, m \geq 1, \\ m|n}} \frac{1}{m^2} z^n.$$

Applying Theorem 3.9, we obtain the following.

Theorem 4.2. *For $k \in \mathbb{Z}$, we have*

$$\mathrm{DT}^{2k-1}(1) = -\chi(X) \sum_{\substack{n \geq 1, m \geq 1, \\ m|n}} \frac{1}{m^2} x^{kn} z^n,$$

$$\mathrm{DT}^{2k}(1) = -\chi(X) \sum_{\substack{n \geq 1, m \geq 1, \\ m|n}} \frac{1}{m^2} (1 - (-1)^n x)^n x^{kn} z^n.$$

Let us consider the series

$$\mathrm{DT}^0(1) = -\chi(X) \sum_{\substack{n \geq 1, m \geq 1, \\ m|n}} \frac{1}{m^2} (1 - (-1)^n x)^n z^n.$$

This is a generating series of invariants counting $E \in \mathcal{P}_t(1)$ for $1 < t < 2$. The following lemma shows that such objects are certain two term complexes $(\mathcal{O}_X^{\oplus r} \xrightarrow{s} F)$ with $F \in \mathrm{Coh}_0(X)$.

Lemma 4.3. *For $1 < t < 2$, an object $E \in \mathcal{D}_X$ is contained in $\mathcal{P}_t(1)$ if and only if E is isomorphic to a two term complex,*

(56) $$\cdots \to 0 \to \mathcal{O}_X^{\oplus r} \xrightarrow{s} F \to 0 \to \cdots,$$

where $\mathcal{O}_X^{\oplus r}$ is located in degree zero and $F \in \mathrm{Coh}_0(X)$, such that the induced morphism

(57) $$H^0(s) \colon \mathbb{C}^r \to H^0(X, F)$$

is injective.

Proof. Let us take $E \in \mathcal{P}_t(1)$ for $1 < t < 2$. We see that E is isomorphic to a two term complex (56) such that (57) is injective. Since $\mathcal{P}_t(1) \subset \mathcal{B}_+$, there is an exact sequence in $\mathcal{B}_+$,

$$0 \to E_1 \to E \to E_2 \to 0,$$

such that $E_1 \in \mathcal{A}_+$ and $E_2 \in \langle \mathcal{O}_X[-1] \rangle_{\text{ex}}$. If $E_2 \neq 0$, then $E_1 \neq 0$ and we have

$$\pi = \arg Z_t(E_1) > \arg Z_t(E_2) = \pi(t - 1),$$

which contradicts the Z_t-semistability of E. Therefore $E_2 = 0$ and $E \in \mathcal{A}_+ \subset \mathcal{A}$ follows. Since $\text{ch}_2(E) = 0$, the same argument in the proof of Lemma 4.1 shows that the E is isomorphic to a two term complex of the form (56), and we need to see that $H^0(s)$ is injective. There is an exact sequence in $\mathcal{A}$,

(58) $$0 \to F[-1] \to E \to \mathcal{O}_X^{\oplus r} \to 0.$$

Applying $\text{Hom}(\mathcal{O}_X, *)$, we obtain the exact sequence,

(59) $$0 \to \text{Hom}(\mathcal{O}_X, E) \to \mathbb{C}^r \overset{H^0(s)}{\to} H^0(X, F).$$

Since $E \in \mathcal{A}_+$, we have $\text{Hom}(\mathcal{O}_X, E) = 0$ and the morphism $H^0(s)$ is injective.

Conversely, suppose that $E \in \mathcal{D}_X$ is isomorphic to a two term complex (56) such that $H^0(s)$ is injective. Then $E \in \mathcal{A}$ and we have the same exact sequences (58), (59). Then we have $\text{Hom}(\mathcal{O}_X, E) = 0$ since $H^0(s)$ is injective, and we have $E \in \mathcal{A}_+ \subset \mathcal{B}_+$. It remains to check that E is Z_t-semistable in $\mathcal{B}_+$. Let us take an exact sequence in $\mathcal{B}_+$,

(60) $$0 \to E_1 \to E \to E_2 \to 0,$$

with non-zero $E_1, E_2 \in \mathcal{B}_+$. Since $\text{ch}_2(E) = 0$, we have

(61) $$\text{ch}_2(E_1) = \text{ch}_2(E_2) = 0.$$

Applying $\mathcal{H}_{\mathcal{A}}^\bullet(*)$ to (60), we have the long exact sequence in $\mathcal{A}$,

$$0 \to \mathcal{H}_{\mathcal{A}}^0(E_1) \to E \to \mathcal{H}_{\mathcal{A}}^0(E_2) \to \mathcal{H}_{\mathcal{A}}^1(E_1) \to 0,$$

and $\mathcal{H}_{\mathcal{A}}^1(E_2) = 0$. Therefore $\mathcal{H}_{\mathcal{A}}^0(E_2) \neq 0$, and hence

$$\text{ch}_3(\mathcal{H}_{\mathcal{A}}^0(E_2)) = \text{ch}_3(E_2) \neq 0.$$

This together with (61) implies that

$$\arg Z_t(E_1) \le \arg Z_t(E_2) = \pi,$$

which shows the Z_t-semistability of E. q.e.d.

Let us consider the rank one generating series. The above lemma shows that the invariant $\text{DT}^0(1, 0, n, 1)$ counts two term complexes,

$$\cdots \to 0 \to \mathcal{O}_X \overset{s}{\to} F \to 0 \to \cdots,$$

such that $F \in \mathrm{Coh}_0(X)$ is of length n and s is non-zero. By Theorem 4.2, the rank one generating series satisfies the following curious equality:

$$\sum_{n \geq 0} \mathrm{DT}^0(1,0,n,1)z^n = -\chi(X) \sum_{\substack{n \geq 1, m \geq 1, \\ m|n}} \frac{1}{m^2}(-1)^{n-1}nz^n,$$

(62)
$$= \log M(-z)^{\chi(X)}.$$

Here $M(z)$ is the MacMahon function,

$$M(z) = \prod_{n \geq 1} \frac{1}{(1-z^n)^n}.$$

Recall that $M(-z)^{\chi(X)}$ is the generating series of DT invariants counting ideal sheaves of points (cf. [**29**], [**5**], [**28**]).

4.2. D0-D2-D6 states on a $(-1,-1)$-curve. Let

$$f \colon X \to Y$$

be a crepant small resolution of an ordinary double point $p \in Y$,

$$\widehat{\mathcal{O}}_{Y,p} \cong \mathbb{C}[\![x_1, x_2, x_3, x_4]\!]/(x_1 x_2 + x_3 x_4).$$

The exceptional locus $C \subset X$ satisfies that

$$C \cong \mathbb{P}^1, \quad N_{C/X} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1).$$

Let $\mathrm{Coh}_C(X)$ be

$$\mathrm{Coh}_C(X) := \{E \in \mathrm{Coh}(X) : \mathrm{Supp}(E) \subset C\},$$

and we define the triangulated category $\mathcal{D}_{X/Y}$ to be

$$\mathcal{D}_{X/Y} := \langle \mathcal{O}_X, \mathrm{Coh}_C(X)\rangle_{\mathrm{tr}} \subset \mathcal{D}_X.$$

Similarly to the case of $\mathcal{D}_X$, we can consider the space of weak stability conditions on $\mathcal{D}_{X/Y}$. The required data is as follows. We set Γ' to be

$$\Gamma' = H^0(X, \mathbb{Z}) \oplus \mathbb{Z}[C] \oplus H_0(X, \mathbb{Z}),$$

and define $\mathrm{cl} \colon K(\mathcal{D}_{X/Y}) \to \Gamma'$ to be

$$\mathrm{cl}(E) = (\mathrm{ch}_0(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)),$$

for $E \in \mathcal{D}_{X/Y}$. Also we choose the filtration $\Gamma'_\bullet$ as

$$\Gamma'_0 = H^0(X, \mathbb{Z}) \subset \Gamma'_1 = \Gamma'.$$

The associated space of weak stability conditions is denoted by $\mathrm{Stab}_{\Gamma'_\bullet}(\mathcal{D}_{X/Y})$. Similarly to the map (34), we can construct a continuous map,

$$\gamma' \colon \mathbb{R} \to \mathrm{Stab}_{\Gamma'_\bullet}(\mathcal{D}_{X/Y}),$$

such that if $t \in (k, k+1]$ for $k \in \mathbb{Z}$, we have

$$\gamma'(t) = (Z'_{(\exp(\pi it), i\omega)}, \mathcal{A}'_k).$$

Here

$$Z'_{(\exp(\pi it), i\omega)} \in \prod_{i=0}^{1} \mathrm{Hom}_{\mathbb{Z}}(\Gamma'_i/\Gamma'_{i-1}, \mathbb{C})$$

is defined in a similar way to (16), (17) for the fixed ample divisor ω on X, and $\mathcal{A}'_k$ are hearts of bounded t-structures, satisfying the following.

- For $k = 0$, we have

$$\mathcal{A}'_0 = \langle \mathcal{O}_X, \mathrm{Coh}_C(X)[-1] \rangle_{\mathrm{ex}}.$$

- For $k = 1$, we have

$$\mathcal{A}'_1 = \langle \mathcal{A}'_{0,+}, \mathcal{O}_X[-1] \rangle_{\mathrm{ex}},$$

  where $\mathcal{A}'_{0,+} = \{E \in \mathcal{A}'_0 : \mathrm{Hom}(\mathcal{O}_X, E) = 0\}$.
- The other $\mathcal{A}'_k$ are determined by the rule,

$$\mathcal{A}'_{k+2} = \Phi_{\mathcal{O}_X} \mathcal{A}'_k.$$

Let us write

$$\gamma'(t) = (Z'_t, \mathcal{P}'_t), \quad t \in \mathbb{R},$$

for a slicing $\mathcal{P}'_t$ on $\mathcal{D}_{X/Y}$. Similarly to Definition 3.6, we can define the DT type invariants,

$$(63) \qquad \mathrm{DT}_t(r, m, n, \phi) \in \mathbb{Q},$$

counting objects $E \in \mathcal{D}_{X/Y}$ satisfying

$$E \in \mathcal{P}'_t(\phi), \quad \mathrm{cl}(E) = (r, -m[C], -n).$$

By abuse of notation, we define the generating series as well:

$$\mathrm{DT}_t(\phi) = \sum_{(r, m[C], n) \in \Gamma'} \mathrm{DT}_t(r, m, n, \phi) x^r y^m z^n.$$

A result similar to Theorem 3.9 also holds as follows. The proof is similar and we omit the proof.

Theorem 4.4. *(i) For a given $k \in \mathbb{Z}$, the series $\mathrm{DT}_t(\phi)$ does not depend on a choice of $t \in (\phi + k, \phi + k + 1)$. In particular, we may write it as $\mathrm{DT}^k(\phi)$.*

(ii) The series $\mathrm{DT}^k(\phi)$ is obtained from $\mathrm{DT}^{k-1}(\phi)$ by the following transformation:

$$z^n \mapsto \begin{cases} (1 - (-1)^n x)^n z^n, & \text{if } k \text{ is even.} \\ x^n z^n / (1 - (-1)^n x)^n, & \text{if } k \text{ is odd.} \end{cases}$$

Let us investigate what kinds of objects the invariants $\mathrm{DT}_t(r, m, n, \phi)$ count. Recall that there is the heart of a bounded t-structure, called the *perverse t-structure* [**9**], [**16**],

$$\mathrm{Per}(X/Y) \subset D^b(\mathrm{Coh}_C(X)).$$

Its generator is given in [**16**, proposition 3.5.7],

$$(64) \qquad \mathrm{Per}(X/Y) = \langle \mathcal{O}_C(-1)[1], \mathcal{O}_C \rangle_{\mathrm{ex}}.$$

We have the following lemma.

Lemma 4.5. *For $1/2 < t \le 3/2$, we have*

$$(65) \qquad \mathcal{P}'_t((1/2, 3/2]) = \langle \mathcal{O}_X, \mathrm{Per}(X/Y)[-1] \rangle_{\mathrm{ex}}.$$

Proof. The argument of [**43**, lemma 3.2 (ii)] shows that the RHS side is the heart of a bounded t-structure on $\mathcal{D}_{X/Y}$. Hence it is enough to check that the RHS is contained in the LHS. It is straightforward to check that

$$\{\mathcal{O}_X, \mathcal{O}_C(-1), \mathcal{O}_C[-1]\} \subset \mathcal{P}'_t((1/2, 3/2]).$$

Since $\mathcal{O}_X$, $\mathcal{O}_C(-1)$ and $\mathcal{O}_C[-1]$ generates the RHS of (65), we obtain the result.   q.e.d.

By the description (64), we can easily see that

$$\mathrm{Hom}(\mathcal{O}_X, E[i]) = 0, \quad i > 0,$$

for any $E \in \mathrm{Per}(X/Y)$. Then the same argument of [**50**, proposition 2.2] shows that the RHS of (65) is equivalent to the abelian category of triples,

$$(66) \qquad (\mathcal{O}_X^{\oplus r}, F, s),$$

where $r \in \mathbb{Z}_{\ge 0}$, $F \in \mathrm{Per}(X/Y)$ and s is a morphism, $s \colon \mathcal{O}_X^{\oplus} \to F$. The set of morphisms is given by the commutative diagram,

$$
\begin{array}{ccc}
\mathcal{O}_X^{\oplus r_1} & \xrightarrow{s_1} & F_1 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
\mathcal{O}_X^{\oplus r_2} & \xrightarrow{s_2} & F_2,
\end{array}
$$

and the equivalence is given by sending the triple (66) to the total complex of the double complex $(\mathcal{O}_X^{\oplus r} \xrightarrow{s} F)$. The above category of triples (66) is nothing but the category of *perverse coherent systems* considered in [**35**]. Noting this, we have the following lemma.

Lemma 4.6. *For $1/2 < t < \phi \le 3/2$ with $\phi \ne 1$, we have $\mathcal{P}'_t(\phi) \ne \{0\}$ if and only if there is $a \in \mathbb{Z}$ satisfying*

$$(67) \qquad -a + (\omega \cdot C)\sqrt{-1} \in \mathbb{R}\exp(i\pi\phi).$$

In this case, we have

$$(68) \qquad \mathcal{P}'_t(\phi) = \begin{cases} \langle \mathcal{O}_C(a-1)[-1] \rangle_{\mathrm{ex}}, & \text{if } 1/2 < \phi < 1, \\ \langle \mathcal{O}_C(a-1) \rangle_{\mathrm{ex}}, & \text{if } 1 < \phi < 3/2. \end{cases}$$

Proof. Let us take a non-zero object $E \in \mathcal{P}'_t(\phi)$. Note that we have

$$\mathcal{P}'_t(\phi) \subset \mathcal{P}'_t((1/2, 3/2]),$$

and E is a semistable object in $\mathcal{P}'_t((1/2, 3/2])$ with respect to the rotated weak stability condition,

$$\frac{1}{2} \cdot \gamma'(t) = (-iZ'_t, \mathcal{P}'_t((1/2, 3/2])).$$

By Lemma 4.5 and the subsequent argument, any object in $\mathcal{P}'_t((1/2, 3/2])$ is isomorphic to the total complex associated to a triple (66). Hence there is an exact sequence in $\mathcal{P}'_t((1/2, 3/2])$,

$$0 \to F[-1] \to E \to \mathcal{O}_X^{\oplus r} \to 0,$$

for $r \in \mathbb{Z}_{\geq 0}$ and $F \in \mathrm{Per}(X/Y)$. Suppose that $r \neq 0$. Then we have

$$\pi\phi = \arg Z'_t(F[-1]) > \arg Z'_t(\mathcal{O}_X^{\oplus r}) = \pi t,$$

which contradicts the $-iZ'_t$-semistability of E. Here we have taken the arguments in $(\pi/2, 3\pi/2]$. Therefore we have $r = 0$ or $F = 0$. If $F = 0$, then $E \in \langle \mathcal{O}_X \rangle_{\mathrm{ex}} \subset \mathcal{P}'_t(t)$, which contradicts $E \in \mathcal{P}'_t(\phi)$. Therefore $r = 0$, and $E \in \mathrm{Per}(X/Y)[-1]$ follows.

Let $W \colon K(\mathcal{D}_{X/Y}) \to \mathbb{C}$ be the group homomorphism defined by

(69) $$W(E) = -\mathrm{ch}_3(E) + i\omega \cdot \mathrm{ch}_2(E).$$

Then the pair

(70) $$(W, \mathrm{Coh}_C(X))$$

is a Bridgeland's stability condition on $D^b(\mathrm{Coh}_C(X))$, and the set of W-(semi)stable objects in $\mathrm{Coh}_C(X)$ coincides with the set of ω-(semi)stable sheaves in $\mathrm{Coh}_C(X)$. Let us write the stability condition (70) as the pair $(W, \mathcal{Q})$ for a slicing $\mathcal{Q}$ in $D^b(\mathrm{Coh}_C(X))$, and consider the rotated stability condition

(71) $$\left(-\frac{1}{2}\right) \cdot (W, \mathrm{Coh}_C(X)) = (iW, \mathcal{Q}((-1/2, 1/2])).$$

It is easy to see that $\mathcal{O}_C(-1)$ and $\mathcal{O}_C[-1]$ are contained in $\mathcal{Q}((-1/2, 1/2])$; hence we have

$$\mathcal{Q}((-1/2, 1/2]) = \mathrm{Per}(X/Y)[-1].$$

Under the above identification, the set of iW-(semi)stable objects in $\mathcal{Q}((-1/2, 1/2])$ coincides with that of $-iZ'_t$-(semi)stable objects in $\mathrm{Per}(X/Y)[-1]$. Since an ω-stable sheaf in $\mathrm{Coh}_C(X)$ is of the form $\mathcal{O}_C(a)$ or $\mathcal{O}_x$ for $x \in C$, the set of $-iZ'_t$-stable objects in $\mathrm{Per}(X/Y)[-1]$ is given as follows:

$$\{\mathcal{O}_C(a-1) : a \leq 0\} \cup \{\mathcal{O}_C(a-1)[-1] : a \geq 1\} \cup \{\mathcal{O}_x[-1] : x \in C\}.$$

Since we have

$$Z'_t(\mathcal{O}_C(a-1)) = a - (\omega \cdot C)\sqrt{-1},$$

there is non-zero $E \in \mathcal{P}'_t(\phi)$ only if the condition (67) is satisfied, and in this case $\mathcal{P}'_t(\phi)$ is given by (68).   q.e.d.

In the situation of Lemma 4.6, any non-zero object $E \in \mathcal{P}'_t(\phi)$ is written as

$$E \cong \begin{cases} \mathcal{O}_C(a-1)^{\oplus m}[-1], & \text{if } 1/2 < \phi < 1, \\ \mathcal{O}_C(a-1)^{\oplus m}, & \text{if } 1 < \phi < 3/2, \end{cases}$$

for some $m \in \mathbb{Z}_{\geq 1}$, noting that

$$\text{Ext}^1_X(\mathcal{O}_C, \mathcal{O}_C) = 0.$$

Then a computation similar to Lemma 3.7 shows that

$$\text{DT}^{-1}(r, m, n, \phi) = \begin{cases} \frac{1}{m^2}, & \text{if } r = 0, n = ma, m \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$

for $1/2 < \phi < 1$ and

$$\text{DT}^0(r, m, n, \phi) = \text{DT}^{-1}(-r, -m, -n, \phi+1)$$
$$= \begin{cases} \frac{1}{m^2}, & \text{if } r = 0, n = ma, m \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$

for $0 < \phi \leq 1/2$. Applying Theorem 4.4, we obtain the following.

Theorem 4.7. *For $0 < \phi < 1$, suppose that there is $a \in \mathbb{Z}$ satisfying (67). For $k \in \mathbb{Z}$, we obtain the following.*
(i) If $0 < \phi < 1/2$, we have

$$\text{DT}^{2k-1}(\phi) = \sum_{m \geq 1} \frac{1}{m^2} x^{kma} y^m z^{ma},$$

$$\text{DT}^{2k}(\phi) = \sum_{m \geq 1} \frac{1}{m^2} (1 - (-1)^{ma} x)^{ma} x^{kma} y^m z^{ma}.$$

(ii) If $1/2 \leq \phi < 1$, we have

$$\text{DT}^{2k}(\phi) = \sum_{m \geq 1} \frac{1}{m^2} x^{kma} y^m z^{ma},$$

$$\text{DT}^{2k+1}(\phi) = \sum_{m \geq 1} \frac{1}{m^2} (1 - (-1)^{ma} x)^{ma} x^{kma} y^m z^{ma}.$$

Similarly to Lemma 4.3, we can investigate what kinds of objects the invariants (63) count. The case of $\phi - 1 \ll t < \phi$ is already studied in Lemma 4.6. The case after crossing the wall $t = \phi$, i.e. the case of $\phi < t \ll \phi + 1$, is given as follows.

Lemma 4.8. *For $1/2 < \phi < t \leq 3/2$ with $\phi \neq 1$, we have $\mathcal{P}_t(\phi) \neq \{0\}$ if and only if there is $a \in \mathbb{Z}$ satisfying (67). In this case, an object $E \in \mathcal{D}_X$ is contained in $\mathcal{P}_t(\phi)$ if and only if the following holds.*

- If $1/2 < \phi < 1$, then E is quasi-isomorphic to a two term complex,

(72) $$\cdots \to 0 \to \mathcal{O}_X^{\oplus r} \xrightarrow{s} \mathcal{O}_C(a-1)^{\oplus m} \to 0 \to \cdots ,$$

where $\mathcal{O}_X^{\oplus r}$ is located in degree zero, such that the induced morphism,

$$H^0(s) \colon \mathbb{C}^r \to H^0(C, \mathcal{O}_C(a-1))^{\oplus m},$$

is injective.
- If $1 < \phi < 3/2$, then E fits into the exact sequence of sheaves,

(73) $$0 \to \mathcal{O}_C(a-1)^{\oplus m} \to E \to \mathcal{O}_X^{\oplus r} \to 0,$$

such that the induced morphism,

$$\mathbb{C}^r \to H^1(C, \mathcal{O}_C(a-1))^{\oplus m},$$

is injective.

Proof. Let us take an object $E \in \mathcal{P}_t'(\phi)$. By Lemma 4.5 and the subsequent argument, E is isomorphic to the total complex of a double complex,

(74) $$\mathcal{O}_X^{\oplus r} \xrightarrow{s} F,$$

for some $r \geq 0$ and $F \in \mathrm{Per}(X/Y)$. In particular, we have the exact sequence in $\mathcal{P}_t'((1/2, 3/2])$,

$$0 \to F[-1] \to E \to \mathcal{O}_X^{\oplus r} \to 0.$$

Applying $\mathrm{Hom}(\mathcal{O}_X, *)$, we obtain the exact sequence,

$$0 \to \mathrm{Hom}(\mathcal{O}_X, E) \to \mathbb{C}^r \xrightarrow{H^0(s)} \mathrm{Hom}(\mathcal{O}_X, F).$$

Since $\mathcal{O}_X \in \mathcal{P}_t'(t)$, $E \in \mathcal{P}_t'(\phi)$ and $t > \phi$, we have $\mathrm{Hom}(\mathcal{O}_X, E) = 0$; hence the map $H^0(s)$ is injective.

Next we classify the objects $F \in \mathrm{Per}(X/Y)$ which appear in (74). Let $0 \neq F' \subset F$ be a subobject in $\mathrm{Per}(X/Y)$. Then we have injections

$$F'[-1] \hookrightarrow F[-1] \hookrightarrow E,$$

in $\mathcal{P}_t'((1/2, 3/2])$. As in the proof of Lemma 4.6, we consider rotated weak stability condition on $\mathcal{D}_{X/Y}$, stability condition on $D^b(\mathrm{Coh}_C(X))$, respectively,

$$(-iZ_t', \mathcal{P}_t'((1/2, 3/2])),$$
$$(iW, \mathrm{Per}(X/Y)[-1]).$$

Here W is given by (69). By the $-iZ_t'$-semistability of E, we have the inequality in $(\pi/2, 3\pi/2]$,

$$\begin{aligned}
\arg W(F') = \arg Z_t'(F'[-1]) \\
\leq \arg Z_t'(E) \\
= \arg Z_t'(F[-1]) \\
= \arg W(F).
\end{aligned}$$

Therefore $F[-1]$ is an iW-semistable object in $\mathrm{Per}(X/Y)[-1]$. As in the proof of Lemma 4.6 and the subsequent argument, there is $m \geq 1$ such that

$$F \cong \left\{ \begin{array}{ll}
\mathcal{O}_C(a-1)^{\oplus m}, & \text{if } 1/2 < \phi < 1, \\
\mathcal{O}_C(a-1)^{\oplus m}[1], & \text{if } 1 < \phi < 3/2.
\end{array} \right.$$

Therefore E is isomorphic to a two term complex (72) when $1/2 < \phi < 1$, and isomorphic to a sheaf which fits into the exact sequence (73) when $1 < \phi < 3/2$.

Conversely, suppose that $E \in \mathcal{D}_X$ is an object given by (72) or (73). Then E is the total complex of a double complex (74) for some $F \in \mathrm{Per}(X/Y)$, which satisfies the property that $F[-1] \in \mathrm{Per}(X/Y)[-1]$ is iW-semistable and $H^0(s)$ is injective. In particular, E is an object in $\mathcal{P}_t'((1/2, 3/2])$ by Lemma 4.5. Let us take an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0,$$

in $\mathcal{P}_t'((1/2, 3/2])$ with non-zero E_1 and E_2. The above exact sequence corresponds to an exact sequence of triples (66),

$$0 \to (\mathcal{O}_X^{\oplus r_1} \xrightarrow{s_1} F_1) \to (\mathcal{O}_X^{\oplus r} \xrightarrow{s} F) \to (\mathcal{O}_X^{\oplus r_2} \xrightarrow{s_2} F_2) \to 0.$$

Since $H^0(s)$ is injective, we have $F_1 \neq 0$. If we also have $F_2 \neq 0$, then the iW-semistability of $F[-1]$ implies the inequality in $(\pi/2, 3\pi/2]$,

$$\begin{aligned}
\arg Z_t'(E_1) = \arg W(F_1) \\
\leq \arg W(F_2) \\
= \arg Z_t'(E_2).
\end{aligned}$$

If $F_2 = 0$, we have

$$\pi\phi = \arg Z_t'(E_1) < \arg Z_t'(E_2) = \pi t.$$

The above inequalities show that E is $-iZ_t'$-semistable in $\mathcal{P}_t'((1/2, 2/3])$, and $E \in \mathcal{P}_t'(\phi)$ follows.   q.e.d.

Remark 4.9. In the case of $\phi = 1$, the generating series and the relevant semistable objects are described in a way similar to the results in Subsection 4.1.

In a similar way to (62), Theorem 4.7 can be used to write down the rank one generating series,

$$\sum_{\substack{\phi \in (0,1), \\ (m,n) \in \mathbb{Z}^{\oplus 2}.}} \mathrm{DT}^0(1,m,n,\phi) y^m z^n = \sum_{\substack{a \geq 1, \\ m \geq 1.}} \frac{1}{m^2}(-1)^{ma-1} may^m z^{ma},$$

(75)
$$= \log \prod_{m \geq 1}(1 - (-1)^m y z^m)^m.$$

By Lemma 4.8, the above series is a generating series of invariants counting two term complexes of the form

$$\mathcal{O}_X \xrightarrow{s} \mathcal{O}_C(a-1)^{\oplus m},$$

such that s is non-zero. Here we have again observed a curious phenomenon, since the series

(76)
$$\prod_{m \geq 1}(1 - (-1)^m y z^m)^m$$

coincides with the generating series of stable pairs on a local $(-1,-1)$-curve (cf. [**37**]). Under the GW/DT correspondence [**33**], the series (75) corresponds to the connected GW theory, while the series (76) corresponds to the non-connected GW theory [**4**].

Remark 4.10. Although we rely on Conjecture 3.3 to prove Theorem 3.9, we can check a version of Conjecture 3.3 needed in showing Theorem 4.2 and Theorem 4.7 by hand, following the same strategy of [**50**, proposition 2.12]. Therefore, the results in this subsection are completely rigorous.

5. Some Technical Lemmas

5.1. Proof of Lemma 2.7. For simplicity, we give a proof for the pair $(Z_u, \mathcal{B}_+)$ with $u = (z, B + i\omega) \in (-\mathbb{H}) \times A(X)_{\mathbb{C}}$. We divide the proof into 3 steps.

Step 1. *The pair $(Z_u, \mathcal{B}_+)$ satisfies the Harder-Narasimhan property.*

Proof. By [**49**, proposition 2.12], it is enough to check that the following conditions are satisfied.

- The abelian category $\mathcal{B}_+$ is noetherian.
- There are no infinite sequences of subobjects in $\mathcal{B}_+$,

(77)
$$\cdots E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1,$$

with the following inequality for all j:

(78)
$$\arg Z_u(E_{j+1}) > \arg Z_u(E_j/E_{j+1}).$$

First we show that the abelian category $\mathcal{B}_+$ is noetherian. For an object $E \in \mathcal{B}_+$, suppose that there is an infinite sequence of inclusions in $\mathcal{B}_+$,

(79) $F_1 \subset F_2 \subset \cdots \subset E.$

Applying $\mathcal{H}_\mathcal{A}^\bullet$, we obtain the sequence of inclusions in $\mathcal{A}$,

$$\mathcal{H}_\mathcal{A}^0(F_1) \subset \mathcal{H}_\mathcal{A}^0(F_2) \subset \cdots \subset \mathcal{H}_\mathcal{A}^0(E).$$

Since $\mathcal{A}$ is noetherian by [**50**, lemma 6.2], we may assume that $\mathcal{H}_\mathcal{A}^0(F_i) \overset{\cong}{\to} \mathcal{H}_\mathcal{A}^0(F_{i+1})$ for all i. Then taking the quotients of (79) by $\mathcal{H}_\mathcal{A}^0(F_1)$, we may assume that $\mathcal{H}_\mathcal{A}^0(F_i) = 0$ for all i. This means that we have

$$F_i \cong \mathcal{O}_X^{\oplus r_i}[-1], \quad r_i \in \mathbb{Z}_{\geq 0}.$$

Since $r_1 \leq r_2 \leq \cdots$ and $\mathrm{Hom}(\mathcal{O}_X[-1], E)$ is finite dimensional, the sequence (79) terminates.

Next suppose that there is a sequence (77) satisfying (78). Note that for any object $E \in \mathcal{B}_+$, we have $\mathrm{ch}_2(E) \cdot \omega \leq 0$ by the description of $\mathcal{B}_+$ in Lemma 2.6. Therefore we have

$$\mathrm{ch}_2(E_1) \cdot \omega \leq \cdots \leq \mathrm{ch}_2(E_j) \cdot \omega \leq \mathrm{ch}_2(E_{j+1}) \cdot \omega \leq \cdots \leq 0.$$

Hence we may assume that $\mathrm{ch}_2(E_j) \cdot \omega = \mathrm{ch}_2(E_{j+1}) \cdot \omega$ for all j. This implies that $\mathrm{ch}_2(E_j/E_{j+1}) = 0$, and we have either

$$Z_u(E_j/E_{j+1}) \in \mathbb{R}_{<0}, \text{ or } E_j/E_{j+1} \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}.$$

Since we have the inequality (78), we have $Z_u(E_j/E_{j+1}) \notin \mathbb{R}_{<0}$. Therefore we have $E_j/E_{j+1} \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}$, and hence E_1/E_j is written as $\mathcal{O}_X[-1]^{\oplus r_j}$ for some $r_j \in \mathbb{Z}_{\geq 0}$. There is a sequence of surjections,

$$E_1 \twoheadrightarrow \cdots \twoheadrightarrow E_1/E_3 \twoheadrightarrow E_1/E_2,$$

and hence we have $r_2 \leq r_3 \leq \cdots$. Since $\mathrm{Hom}(E_1, \mathcal{O}_X[-1])$ is finite dimensional, the above sequence must terminate.   q.e.d.

Step 2. *The weak stability condition $(Z_u, \mathcal{B}_+)$ satisfies the local finiteness property.*

Proof. Let $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ be the slicing determined by the pair $(Z_u, \mathcal{B}_+)$. It is enough to check that the following quasi-abelian categories,

$$\mathcal{P}((0,1)), \quad \mathcal{P}((1/2, 3/2)),$$

are of finite length. The category $\mathcal{P}((0,1))$ is contained in $\mathcal{B}_+$, and the same argument of Step 1 shows that $\mathcal{P}((0,1))$ is of finite length. Let us check that $\mathcal{P}((1/2, 3/2))$ is of finite length. We take a sequence of strict epimorphisms in $\mathcal{P}((1/2, 3/2))$ (see [**11**, section 4] for the notion of strict epimorphisms and strict monomorphisms),

(80) $E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots,$

and exact sequences in $\mathcal{P}((1/2, 3/2))$,

$$0 \to F_j \to E_j \to E_{j+1} \to 0.$$

Note that $\mathrm{ch}_3(E) \le 0$ for any $E \in \mathcal{P}((1/2, 3/2))$, and the inequality is strict if $\mathrm{cl}(E) \notin \Gamma \setminus \Gamma_0$. Therefore we have the inequalities

$$\mathrm{ch}_3(E_1) \le \cdots \le \mathrm{ch}_3(E_j) \le \mathrm{ch}_3(E_{j+1}) \le \cdots \le 0.$$

Hence we may assume that $\mathrm{ch}_3(E_1) = \mathrm{ch}_3(E_j)$, which implies that $\mathrm{ch}_3(F_j) = 0$ for all j. Then we have $\mathrm{cl}(F_j) \in \Gamma_0$, and Lemma 2.20 shows that

$$F_j \cong \begin{cases} \mathcal{O}_X^{\oplus r_j}, & \text{if } \mathrm{Re}\, z < 0, \\ 0, & \text{if } \mathrm{Re}\, z = 0, \\ \mathcal{O}_X[-1]^{\oplus r_j}, & \text{if } \mathrm{Re}\, z > 0, \end{cases}$$

for some $r_j \in \mathbb{Z}_{\ge 0}$. Then the same argument of Step 1 shows that the sequence (80) terminates, i.e. $\mathcal{P}((1/2, 3/2))$ is noetherian. A similar argument also shows that $\mathcal{P}((1/2, 3/2))$ is artinian with respect to the strict monomorphisms; hence $\mathcal{P}((1/2, 3/2))$ is of finite length.   q.e.d.

Step 3. *The pair $(Z_u, \mathcal{B}_+)$ satisfies the support property.*

Proof. Let us take a non-zero object $\mathcal{B}_+$, and we set $\mathrm{cl}(E) = (r, -\beta, -n)$. If $(\beta, n) = (0, 0)$, we have

$$\frac{\|[\mathrm{cl}(E)]\|_0}{|Z(E)|} = \frac{1}{|z|}.$$

Suppose that $(\beta, n) \neq (0, 0)$. Then we have

(81) $$\frac{\|[\mathrm{cl}(E)]\|_1}{|Z(E)|} = \sqrt{\frac{\|\beta\|^2 + n^2}{(n - B \cdot \beta)^2 + (\omega \cdot \beta)^2}}.$$

Here $\|*\|$ is a fixed norm on $H_2 \otimes \mathbb{R}$. If $\beta = 0$, then (81) equals to 1. If $\beta \neq 0$, then (81) coincides with

(82) $$\sqrt{\frac{1 + \mu^2}{(\mu - B_0)^2 + \omega_0^2}}.$$

Here we have set

$$\mu = \frac{n}{\|\beta\|}, \quad B_0 = B \cdot \frac{\beta}{\|\beta\|}, \quad \omega_0 = \omega \cdot \frac{\beta}{\|\beta\|}.$$

The values B_0 and $\omega_0 > 0$ are bounded w.r.t. non-zero $\beta \in H_2$. Also for fixed B_0 and ω_0, the value (82) is bounded w.r.t. all $\mu \in \mathbb{Q}$. Therefore (82) is bounded w.r.t. all B_0, ω_0 and μ.   q.e.d.

5.2. Proof of Lemma 2.14.

Proof. (i) It is easy to see that $\mathrm{Coh}_{\leq 1}(X)[-1]$ is closed under subobjects and quotients in the abelian category $\mathcal{A}$. Then it is easy to see that the (B, ω)-semistability of $F \in \mathrm{Coh}_{\leq 1}(X)$ yields the Z_u-semistability of $F[-1] \in \mathcal{A}$.

(ii) First we suppose that

$$\tag{83} \arg Z_u(F[-1]) > \arg(-z).$$

Let us take an exact sequence in $\mathcal{B}_+$,

$$\tag{84} 0 \to M \to F[-1] \to N \to 0,$$

with $M, N \neq 0$. We want to show the inequality,

$$\tag{85} \arg Z_u(M) \leq \arg Z_u(N),$$

to show the Z_u-semistability of $F[-1]$. Applying $\mathcal{H}_{\mathcal{A}}^{\bullet}$ to the sequence (84), we obtain the long exact sequence in $\mathcal{A}$,

$$\tag{86} 0 \to \mathcal{H}_{\mathcal{A}}^0(M) \to F[-1] \xrightarrow{s} \mathcal{H}_{\mathcal{A}}^0(N) \to \mathcal{H}_{\mathcal{A}}^1(M) \to 0,$$

and the vanishing $\mathcal{H}_{\mathcal{A}}^1(N) = 0$. Note that $\mathcal{H}_{\mathcal{A}}^1(M) \in \langle \mathcal{O}_X \rangle_{\mathrm{ex}}$ by the construction of $\mathcal{B}_+$.

Suppose that $\mathcal{H}_{\mathcal{A}}^0(M) = 0$. Then we have

$$\arg Z_u(M) = \arg Z_u(\mathcal{O}_X[-1])$$
$$= \arg(-z).$$

By our assumption (83), this implies the inequality (85).

Suppose that $\mathcal{H}_{\mathcal{A}}^0(M) \neq 0$. Then $\mathcal{H}_{\mathcal{A}}^0(M)$ and the image of s are written as $F'[-1]$, $F''[-1]$ for some $F', F'' \in \mathrm{Coh}_{\leq 1}(X)$, respectively. Note that we have

$$Z_u(M) = Z_u(F'[-1]), \quad Z_u(N) = Z_u(F''[-1]).$$

The exact sequence (86) and the (B, ω)-semistability of F yield

$$\arg Z_u(F'[-1]) \leq \arg Z_u(F''[-1]).$$

Therefore the inequality (85) holds and $F[-1]$ is Z_u-semistable.

Next suppose that

$$\arg Z_u(F[-1]) < \arg(-z).$$

In this case, a similar argument to that above shows that $F[-1]$ is a Z_u-semistable object in $\mathcal{B}_-$. Applying the twist functor $\Phi_{\mathcal{O}_X}$ and using Lemma 2.11, we conclude that $\Phi_{\mathcal{O}_X}(F[-1]) \in \mathcal{B}_+$, and it is Z_u-semistable.   q.e.d.

5.3. Proof of Lemma 2.16.

Proof. For a non-zero object $F \in \mathrm{Coh}_{\leq 1}(X)$, we have

$$Z_u(\Phi_{\mathcal{O}_X}(F[-1])) = Z_u(F[-1]).$$

Therefore, by Lemma 2.14, it is enough to show the density of the slope of (B, ω)-semistable sheaves.

Let $D \subset X$ be a sufficiently ample divisor. For each $l \geq 1$ and $k \in \mathbb{Z}$, we choose the following:

$$C_l \in |\mathcal{O}_D(lD)|, \quad L_{l,k} \in \mathrm{Pic}(C_l).$$

Here C_l is a smooth member and the degree of $L_{l,k}$ is equal to k. Note that $L_{l,k}$ is a (B, ω)-stable sheaf on X. Setting $C = C_1$ and $d = \int_X D^3 \in \mathbb{Z}$, we have

$$\mathrm{cl}(L_{l,k}) = (0, l[C], k - dl(l+1)/2).$$

Therefore we obtain

$$Z_u(L_{l,k}[-1]) = -k + (B \cdot C)l + \frac{dl(l+1)}{2} + (\omega \cdot C)l\sqrt{-1}.$$

Then it is easy to see that

$$\left\{ \frac{\mathrm{Re}\, Z_u(L_{l,k}[-1])}{\mathrm{Im}\, Z_u(L_{l,k}[-1])} : l \geq 1, \ k \in \mathbb{Z} \right\} = \mathbb{Q} + \left\{ \frac{B \cdot C}{\omega \cdot C} \right\}.$$

This implies the density of (29).   q.e.d.

5.4. Proof of Lemma 3.1.

Proof. By Theorem 2.15 and Proposition 2.18, we may assume that $\sigma \in \mathcal{U}_0$ or $\sigma \in \mathcal{U}_1$. For simplicity, we show the case of $\sigma \in \mathcal{U}_1$. In this case, we can write σ as a pair,

$$(87) \qquad \sigma = (Z_u, \mathcal{B}_+), \quad u = (z, B + i\omega) \in (-\mathbb{H}) \times A(X)_{\mathbb{C}},$$

as in Lemma 2.7 (ii). We fix $v = (r, -\beta, -n) \in \Gamma$ and take

$$m \geq 1, \quad v_i = (r_i, -\beta_i, -n_i), \ 1 \leq i \leq m,$$

which appears in a non-zero term of the RHS of (44). We set

$$m' := \sharp\{1 \leq i \leq m : (\beta_i, n_i) \neq (0, 0)\}.$$

Note that if $(\beta_i, n_i) \neq (0, 0)$, then we have $\beta_i \cdot \omega > 0$ or $\beta_i = 0$, $n_i > 0$. Also we have

$$Z_u(v_i) = -n_i + (B + i\omega) \cdot \beta_i,$$

if $(\beta_i, n_i) \neq (0, 0)$, and they are contained in the same line. This implies that m' is bounded above and the possibilities of (β_i, n_i) are finite. By Lemma 5.1 below, the r_i is also bounded above, depending only on (β, n). Therefore the possibilities of m and v_i are finite.   q.e.d.

We have used the following lemma.

Lemma 5.1. *For a fixed $(r, -\beta, -n) \in \Gamma$, the following set of objects is bounded:*

$$\left\{ E \in \mathcal{B}_+ : \begin{array}{l} E \text{ is } Z_u\text{-semistable satisfying} \\ \mathrm{cl}(E) = (r', -\beta, -n), \ r' \geq r. \end{array} \right\}.$$

Proof. Let us take a Z_u-semistable object $E \in \mathcal{B}_+$ with $\mathrm{cl}(E) = (r', -\beta, -n)$ for some $r' \geq r$. If $(\beta, n) = 0$, then $E \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}$ and the result is obvious. We assume that $(\beta, n) \neq (0,0)$. By the construction of $\mathcal{B}_+$ in Lemma 2.6 (ii), there is an exact sequence in $\mathcal{B}_+$,

(88) $$0 \to E' \to E \to E'' \to 0,$$

such that $E' \in \mathcal{A}_+$ and $E'' \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}$. Moreover, by the construction of $\mathcal{A}$ and $\mathcal{A}_+$ in Lemma 2.6, there is a filtration of E' in $\mathcal{A}$,

(89) $$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_N = E',$$

such that we have

$$E_i/E_{i-1} \cong \left\{ \begin{array}{ll} F_i[-1], & \text{if } i \text{ is odd,} \\ \mathcal{O}_X^{\oplus r_i}, & \text{if } i \text{ is even,} \end{array} \right.$$

for some $F_i \in \mathrm{Coh}_{\leq 1}(X)$ and $r_i \in \mathbb{Z}_{\geq 1}$. Note that $\mathrm{ch}_2(F_i) \cdot \omega \geq 0$; hence $\mathrm{ch}_2(F_i)$ and the length of the filtration N have a finite number of possibilities. Let us take the exact sequence in $\mathcal{A}$,

$$0 \to E_i \to E' \to E'/E_i \to 0.$$

Applying $\mathcal{H}_{\mathcal{B}_+}^\bullet$, we obtain the exact sequence in $\mathcal{B}_+$,

$$0 \to \mathcal{H}_{\mathcal{B}_+}^{-1}(E'/E_i) \to \mathcal{H}_{\mathcal{B}_+}^0(E_i) \overset{\iota}{\to} E'.$$

Since $\mathcal{H}_{\mathcal{B}_+}^{-1}(E/E_i) \in \langle \mathcal{O}_X \rangle_{\mathrm{ex}}$, the Z_u-semistability of E implies that

$$\begin{aligned} \arg Z_u(\oplus_{j \leq i} F_j[-1]) &= \arg Z_u(E_i), \\ &= \arg Z_u(\mathcal{H}_{\mathcal{B}_+}^0(E_i)), \\ &= \arg Z_u(\mathrm{Im}\,\iota), \\ &\leq \arg Z_u(E'), \\ &= \arg Z_u((0, -\beta, -n)), \end{aligned}$$

for all i. The above inequality implies that the pairs $(\mathrm{ch}_2(F_i), \mathrm{ch}_3(F_i))$ have a finite number of possibilities. By taking Harder-Narasimhan filtrations of F_i with respect to ω-stability and applying the same argument, we can also show that the Chern characters of Harder-Narasimhan factors of each F_i have a finite number of possibilities. Since ω-semistable sheaves with a fixed numerical class are bounded, we conclude that possible F_i which appear in the filtration (89) are contained in a bounded family.

We show by induction on i that the possible E_i in the filtration (89) are contained in a bounded family. Note that we have already proved

the boundedness for $i = 1$. Suppose that the claim holds for $i - 1$. We have the exact sequence in $\mathcal{A}$,

$$0 \to E_{i-1} \to E_i \to E_i/E_{i-1} \to 0.$$

If i is odd, then E_i/E_{i-1} is isomorphic to $F_i[-1]$, which is contained in a bounded family. The object E_{i-1} is also contained in a bounded family by the inductive assumption; hence E_i is contained in a bounded family. If i is even, then E_i/E_{i-1} is written as $\mathcal{O}_X^{\oplus r_i}$ for some $r_i \in \mathbb{Z}_{\geq 0}$. The inductive assumption implies that there is $R > 0$ such that we have

$$\dim \operatorname{Hom}(\mathcal{O}_X, E_{i-1}[1]) \leq R,$$

for any possible E_{i-1} which appears in (89). Therefore if $r_i > R$, then there is a non-trivial morphism $\mathcal{O}_X \to E_i$ which contradicts $E \in \mathcal{A}_+$. Hence $r_i \leq R$ and E_i is contained in a bounded family.

The above argument shows that the object E' in (88) is contained in a bounded family. Since $r' \geq r$ and $E'' \in \langle \mathcal{O}_X[-1] \rangle_{\mathrm{ex}}$, the boundedness of E' implies the boundedness of E''. Therefore the object E is contained in a bounded family.   q.e.d.

5.5. Proof of Lemma 3.2.

Proof. As in the proof of Lemma 3.1, we may assume that $\sigma \in \mathcal{U}_1$; hence it is written as (87). First we show that $\mathcal{M}^v(Z_u)$ is a constructible subset in $\mathcal{M}$. For $n \geq 1$, let $\mathcal{F}il^n(\mathcal{B}_+)$ be the stack of n-step filtrations in $\mathcal{B}_+$. Namely, a $\mathbb{C}$-valued point of $\mathcal{F}il^n(\mathcal{B}_+)$ corresponds to a filtration in $\mathcal{B}_+$,

$$E_1 \subset E_2 \subset \cdots \subset E_n.$$

We have the morphisms of stacks,

$$p_i \colon \mathcal{F}il^n(\mathcal{B}_+) \ni E_\bullet \mapsto E_i/E_{i-1} \in \mathcal{O}bj(\mathcal{B}_+),$$

and the diagram,

$$
\begin{array}{ccc}
\mathcal{F}il^n(\mathcal{B}_+) & \xrightarrow{\;q_n\;} & \mathcal{O}bj(\mathcal{B}_+), \\
{\scriptstyle r_n}\big\downarrow & & \\
\mathcal{O}bj(\mathcal{B}_+)^{\times n}, & &
\end{array}
$$

where $q_n(E_\bullet) = E_n$ and $r_n = (p_1, \ldots, p_n)$. Note that q_n and r_n are piecewise constructible bundles. The proof of Lemma 5.1 shows that for each $v \in \Gamma$, there are $N \geq 1$, finite subset,

$$\mathcal{S}_n \subset \overbrace{\Gamma \times \cdots \times \Gamma}^{n},$$

for each $1 \leq n \leq N$ and algebraic substacks of finite type,

$$\mathcal{M}^{(v_1,\ldots,v_n)} \subset \mathcal{O}bj(\mathcal{B}_+)^{\times n},$$

such that we have

$$\mathcal{M}^v(Z_u) = \bigcup_{\substack{1 \leq n \leq N, \\ (v_1,...,v_n) \in \mathcal{S}_n.}} q_n r_n^{-1}(\mathcal{M}^{(v_1,...,v_n)}).$$

Since the RHS is a finite union of constructible subsets, $\mathcal{M}^v(Z_u)$ is a constructible set in $\mathcal{M}$.

Next the existence of the Harder-Narasimhan filtration implies that

$$\mathcal{O}bj(\mathcal{B}_+) = \bigcup_{\substack{n \geq 1, \\ v_1,...,v_n \in \Gamma.}} q_n r_n^{-1}(\mathcal{M}^{v_1}(Z_u) \times \cdots \times \mathcal{M}^{v_n}(Z_u)).$$

Since $\mathcal{M}^{v_i}(Z_u)$ is a constructible subset in $\mathcal{M}$, the RHS is a countable union of constructible subsets in $\mathcal{M}$.   q.e.d.

References

[1] A. Bayer, *Polynomial Bridgeland stability conditions and the large volume limit,* Geom. Topol., vol. **13**, pp. 2389–2425, 2009, MR 2515708, Zbl 1171.14011.

[2] A. Bayer & E. Macri, *The space of stability conditions on the local projective plane,* preprint. arXiv:0912.0043.

[3] K. Behrend, *Donaldson-Thomas invariants via microlocal geometry,* Ann. of Math., vol. **170**, pp. 1307–1338, 2009, MR 2600874, Zbl 1191.14050.

[4] K. Behrend & J. Bryan, *Super-rigid Donaldson-Thomas invariants,* Math. Res. Lett., vol. **14**, pp. 559–571, 2007, MR 2335983, Zbl 1137.14041.

[5] K. Behrend & B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds,* Algebra Number Theory, vol. **2**, pp. 313–345, 2008, MR 2407118, Zbl 1170.14004.

[6] K. Behrend & E. Getzler, *Chern-Simons functional,* in preparation.

[7] T. Bridgeland, *Hall algebras and curve-counting invariants,* preprint. arXiv:1002.4374.

[8] T. Bridgeland, *Stability conditions and Kleinian singularities,* Int. Math. Res. Not., no. 21, pp. 4142–4157, 2009, MR 2549952, Zbl pre05629294.

[9] T. Bridgeland, *Flops and derived categories,* Invent. Math, vol. **147**, pp. 613–632, 2002, MR 1893007, Zbl 1085.14017.

[10] T. Bridgeland, *Stability conditions on a non-compact Calabi-Yau threefold,* Comm. Math. Phys., vol. **266**, pp. 715–733, 2006, MR 2238896, Zbl 1118.14045.

[11] T. Bridgeland, *Stability conditions on triangulated categories,* Ann. of Math, vol. **166**, pp. 317–345, 2007, MR 2373143, Zbl 1137.18008.

[12] T. Bridgeland, *Stability conditions on K3 surfaces,* Duke Math. J., vol. **141**, pp. 241–291, 2008, MR 2376815, Zbl 1138.14022.

[13] T. Bridgeland & A. Maciocia, *Complex surfaces with equivalent derived categories,* Math. Z, vol. **236**, pp. 677–697, 2001, MR 1827500, Zbl 1081.14023.

[14] J. Bryan & R. Pandharipande, *The local Gromov-Witten theory of curves,* J. Amer. Math. Soc., vol. **21**, pp. 101–136, 2008, MR 2350052, Zbl 1126.14062.

[15] W.C. Chuang, D.E. Diaconescu, & G. Pan, *Rank Two ADHM Invariants and Wallcrossing,* preprint. arXiv:1002.0579.

[16] M. Van den Bergh, *Three dimensional flops and noncommutative rings,* Duke Math. J., vol. **122**, pp. 423–455, 2004, MR 2057015, Zbl 1074.14013.

[17] D. Happel, I. Reiten, & S.O. Smalø, *Tilting in abelian categories and quasitilted algebras*, vol. **120** of Mem. Amer. Math. Soc., 1996, MR 1327209, Zbl 0849.16011.

[18] D. Huybrechts, E. Macri, & P. Stellari, *Stability conditions for generic K3 categories,* Compositio Math, vol. **144**, pp. 134–162, 2008, MR 2388559, Zbl 1152.14037.

[19] A. Ishii, K. Ueda, & H. Uehara, *Stability Conditions on A_n-Singularities,* J. Differ. Geom., vol. **84**, pp. 87–126, 2010, MR 2629510, Zbl 1198.14020.

[20] D. Joyce, *Configurations in abelian categories II. Ringel-Hall algebras,* Advances in Math, vol. **210**, pp. 635–706, 2007, MR 2303235, Zbl 1119.14005.

[21] D. Joyce, *Configurations in abelian categories III. Stability conditions and identities,* Advances in Math, vol. **215**, pp. 153–219, 2007, MR 2354988, Zbl 1134.14007.

[22] D. Joyce, *Motivic invariants of Artin stacks and 'stack functions',* Quarterly Journal of Mathematics, vol. **58**, pp. 345–392, 2007, MR 2354923, Zbl 1131.14005.

[23] D. Joyce, *Configurations in abelian categories IV. Invariants and changing stability conditions,* Advances in Math, vol. **217**, pp. 125–204, 2008, MR 2357325, Zbl 1134.14008.

[24] D. Joyce & Y. Song, *A theory of generalized Donaldson-Thomas invariants,* preprint. arXiv:0810.5645.

[25] M. Kontsevich & Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry,* Comm. Math. Phys., vol. **164**, pp. 525–562, 1994, MR 1291244, Zbl 0853.14020.

[26] M. Kontsevich & Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations,* preprint. arXiv:0811.2435.

[27] G. Laumon & L. Moret-Bailly, *Champs algébriques,* vol. **39** of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer Verlag, Berlin, 2000, MR 1771927, Zbl 0945.14005.

[28] M. Levine & R. Pandharipande, *Algebraic cobordism revisited,* Invent. Math., vol. **176**, pp. 63–130, 2009, MR 2485880, Zbl 1210.14025.

[29] J. Li, *Zero dimensional Donaldson-Thomas invariants of threefolds,* Geom. Topol., vol. **10**, pp. 2117–2171, 2006, MR 2284053, Zbl 1140.14012.

[30] W.P. Li & Z. Qin, *Donaldson-Thomas invariants of certain Calabi-Yau 3-folds,* preprint. arXiv:1002.4080.

[31] M. Lieblich, *Moduli of complexes on a proper morphism,* J. Algebraic Geom., vol. **15**, pp. 175–206, 2006, MR 2177199, Zbl 1085.14015.

[32] E. Macri, *Some examples of moduli spaces of stability conditions on derived categories,* preprint. arXiv:0411613.

[33] D. Maulik, N. Nekrasov, A. Okounkov, & R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory,* I. Compositio Math, vol. **142**, pp. 1263–1285, 2006, MR 2264664, Zbl 1108.14046.

[34] K. Nagao *On higher rank Donaldson-Thomas invariants,* preprint. arXiv:1002.3608.

[35] K. Nagao & H. Nakajima, *Counting invariant of perverse coherent sheaves and its wall-crossing,* preprint. arXiv:0809.2992.

[36] A. Okounkov & R. Pandharipande, *The local Donaldson-Thomas theory of curves,* Geometry and Topology, vol. **14**, pp. 1503–1567, 2010, MR 2679579, Zbl 1205.14067.

[37] R. Pandharipande & R.P. Thomas, *Curve counting via stable pairs in the derived category,* Invent. Math., vol. **178**, pp. 407–447, 2009, MR 2545686, Zbl 1204.14026.

[38] P. Seidel & R.P. Thomas, *Braid group actions on derived categories of coherent sheaves,* Duke Math. J., vol. **108**, pp. 37–107, 2001, MR 1831820, Zbl 1092.14025.

[39] J. Stoppa, *D0-D6 states counting and GW invariants,* preprint, arXiv:0912.2923.

[40] J. Stoppa & R.P. Thomas, *Hilbert schemes and stable pairs: GIT and derived category wall crossings,* preprint, arXiv:0903.1444.

[41] R.P. Thomas. A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on $K3$-fibrations. J. Differential. Geom, Vol. 54, pp. 367–438, 2000, MR 1818182, Zbl 1034.14015.

[42] R.P. Thomas, *Stability conditions and the braid groups,* Comm. Anal. Geom., vol. **14**, pp. 135–161, 2006, MR 2230573, Zbl 1179.53084.

[43] Y. Toda, *Curve counting theories via stable objects II,* DT/ncDT/flop formula, preprint. arXiv:0909.5129.

[44] Y. Toda, *Generating functions of stable pair invariants via wall-crossings in derived categories,* Adv. Stud. Pure Math., vol. **59**, pp. 389–434, 2010. New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008), MR 2683216, Zbl pre05872340.

[45] Y. Toda, *Moduli stacks and invariants of semistable objects on K3 surfaces,* Advances in Math, vol. **217**, pp. 2736–2781, 2008, MR 2397465, Zbl 1136.14007.

[46] Y. Toda, *Stability conditions and crepant small resolutions,* Trans. Amer. Math. Soc., vol. **360**, pp. 6149–6178, 2008, MR 2425708, Zbl pre05358283.

[47] Y. Toda, *Limit stable objects on Calabi-Yau 3-folds,* Duke Math. J., vol. **149**, pp. 157–208, 2009, MR 2541209, Zbl 1172.14007.

[48] Y. Toda, *Stability conditions and Calabi-Yau fibrations,* J. Algebraic Geom., vol. **18**, pp. 101–133, 2009, MR 2448280, Zbl 1157.14025.

[49] Y. Toda, *Curve counting theories via stable objects I: DT/PT correspondence,* J. Amer. Math. Soc., vol. **23**, pp. 1119–1157, 2010, MR 2669709, Zbl 1207.14020.

[50] Y. Toda, *On a computation of rank two Donaldson-Thomas invariants,* Communications in Number Theory and Physics, vol. **4**, pp. 49–102, 2010, MR 2679377, Zbl pre05788451.

Institute for the Physics and Mathematics of the Universe
University of Tokyo
5-1-5, Kashiwanoha, Kashiwa-shi, Chiba, Japan
E-mail address: yukinobu.toda@ipmu.jp