J. DIFFERENTIAL GEOMETRY
89 (2011) 49-85

MODULI SPACES OF NONNEGATIVE SECTIONAL CURVATURE AND NON-UNIQUE SOULS

Igor Belegradek, Slawomir Kwasik & Reinhard Schultz

Abstract

We apply various topological methods to distinguish connected components of moduli spaces of complete Riemannian metrics of nonnegative sectional curvature on open manifolds. The new geometric ingredient is that souls of nearby nonnegatively curved metrics are ambiently isotopic.

1. Introduction

A fundamental structure result, due to Cheeger-Gromoll [**6**], is that any open complete manifold of sec ≥ 0 is diffeomorphic to the total space of a normal bundle to a compact totally geodesic submanifold, called a soul. A soul is not unique; e.g. in the Riemannian product $M\times\mathbb{R}^k$ of a closed manifold M with sec ≥ 0 and the standard $\mathbb{R}^k$, the souls are of the form $M\times\{x\}$. Yet Sharafutdinov [**44**] proved that any two souls can be moved to each other by a diffeomorphism that induces an isometry of the souls.

The diffeomorphism class of the soul may depend on the metric; e.g. any two homotopy equivalent 3-dimensional lens spaces L, L' become diffeomorphic after multiplying by $\mathbb{R}^3$ [**33**], so taking non-homeomorphic L, L' gives two product metrics on $L\times\mathbb{R}^3 = L'\times\mathbb{R}^3$ with non-homeomorphic souls. It turns out that codimension 3 is optimal; indeed, Kwasik-Schultz proved in [**28**] that if S, S' are linear spherical space forms such that $S\times\mathbb{R}^2$, $S'\times\mathbb{R}^2$ are diffeomorphic, then S, S' are diffeomorphic. Another well-known example is that all homotopy 7-spheres become diffeomorphic after taking product with $\mathbb{R}^3$ (see Remark 5.8); since some homotopy 7-spheres [**16**] have metrics of sec ≥ 0, so do their products with $\mathbb{R}^3$, which therefore have nonnegatively curved metrics with non-diffeomorphic souls. Codimension 3 is again optimal, because any simply-connected manifold S of dimension ≥ 5 can be recovered (up to diffeomorphism) from $S\times\mathbb{R}^2$ (see [**28**] or Remark 5.12).

Belegradek in [**1**] used examples of Grove-Ziller [**16**] to produce first examples of infinitely many nondiffeomorphic souls for metrics on the

Received 12/17/2009.

same manifold, e.g. on $S^3 \times S^4 \times \mathbb{R}^5$. Other examples of simply-connected manifolds with infinitely many nondiffeomorphic souls, and better control on geometry, were constructed by Kapovitch-Petrunin-Tuschmann in [**25**].

One motivation for the present work was to construct non-diffeomorphic souls of the smallest possible codimension; of course, multiplying by a Euclidean space then yields examples in any higher codimension. We sharpen examples in [**1**] by arranging the soul to have codimension 4 and any given dimension ≥ 7.

Theorem 1.1. *For each $k \geq 3$, there are infinitely many complete metrics of* sec ≥ 0 *on $N = S^4 \times S^k \times \mathbb{R}^4$ whose souls are pairwise non-homeomorphic.*

Similarly, in Theorem 3.4 we sharpen theorems B and C in [**25**] to make the souls there of codimension 4; in particular, we prove:

Theorem 1.2. *There exists an open simply-connected manifold N that admits infinitely many complete metrics of* sec $\in [0,1]$ *with pairwise non-homeomorphic codimension 4 souls of diameter 1. Moreover, one can choose N so that each soul has nontrivial normal Euler class.*

We do not know examples of manifolds with infinitely many non-diffeomorphic souls of codimension < 4, and in an effort to find such examples we systematically study vector bundles with diffeomorphic total spaces, and among other things prove the following:

Theorem 1.3. *Suppose there is a manifold N that admits complete nonnegatively curved metrics with souls S_k of codimension < 4 such that the pairs (N, S_k) lie in infinitely many diffeomorphism types. If $\pi_1(N)$ is finite, S_k is orientable, and* $\dim(S_k) \geq 5$*, then*
(1) *$\pi_1(N)$ is nontrivial and* $\dim(S_k)$ *is odd;*
(2) *the products $S_k \times \mathbb{R}^3$ lie in finitely many diffeomorphism types.*

In Example 5.3 we describe two infinite families of closed manifolds with the property that if each manifold in the family admits a metric of sec ≥ 0, then they can be realized as codimension 1 souls in the same open manifold N. In general, if M is a closed oriented smooth manifold of dimension $4r - 1 \geq 7$ whose fundamental group contains a nontrivial finite order element, then there are infinitely many pairwise non-homeomorphic closed manifolds M_i such that $M_i \times \mathbb{R}^3$ is diffeomorphic to $M \times \mathbb{R}^3$ (see [**9**]); thus, if each M_i admits a metric of sec ≥ 0, then $M \times \mathbb{R}^3$ carries infinitely many (product) metrics with nondiffeomorphic souls.

Another goal of this paper is to study moduli spaces of complete metrics of nonnegative sectional curvature on open manifolds. Studying moduli spaces of Riemannian metrics that satisfy various geometric assumptions is largely a topological activity; see e.g. [**13**], [**14**], [**26**], [**36**], [**37**], [**43**] [**53**], and references therein.

Let $\mathfrak{R}^{k,u}(N)$ denote the space of complete Riemannian C^∞ metrics on a smooth manifold N with topology of uniform C^k-convergence, where $0 \leq k \leq \infty$, and let $\mathfrak{R}^{k,c}(N)$ denote the same set of metrics with topology of C^k-convergence on compact subsets. Let $\mathfrak{R}^{k,u}_{\sec\geq 0}(N)$, $\mathfrak{R}^{k,c}_{\sec\geq 0}(N)$ be the subspaces of $\mathfrak{R}^{k,u}(N)$, $\mathfrak{R}^{k,c}(N)$ respectively, consisting of metrics of $\sec \geq 0$, and let $\mathfrak{M}^{k,u}_{\sec\geq 0}(N)$, $\mathfrak{M}^{k,c}_{\sec\geq 0}(N)$, $\mathfrak{M}^{k,u}(N)$, $\mathfrak{M}^{k,c}(N)$ denote the corresponding moduli spaces, i.e. their quotient spaces by the $\mathrm{Diff}(N)$-action via pullback. We adopt:

Convention 1.4. If an assertion about a moduli space or a space of metrics holds for any k, then the superscript k is omitted from the notation, and if N is compact, then c, u are omitted.

The space $\mathfrak{R}^{c}(N)$ is closed under convex combinations [**12**] and hence is contractible; in particular, $\mathfrak{M}^{c}(N)$ is path-connected. By contrast, if N is non-compact, $\mathfrak{M}^{u}(N)$ typically has uncountably many connected components because metrics in the same component of $\mathfrak{M}^{u}(N)$ lie within a finite uniform distance of each other, while the uniform distance is infinite between metrics with different asymptotic geometry (such as rotationally symmetric metrics on $\mathbb{R}^2$ with non-asymptotic warping functions).

It was shown in [**25**] that metrics with non-diffeomorphic souls lie in different components of $\mathfrak{M}^{c}_{\sec\geq 0}(N)$ provided any two metrics of $\sec \geq 0$ on N have souls that intersect, which can be forced by purely topological assumptions on N; e.g. this holds if N has a soul with nontrivial normal Euler class, or if N has a codimension 1 soul.

A simple modification of the proof in [**25**] shows (with no extra assumptions on N) that metrics with non-diffeomorphic souls lie in different components of $\mathfrak{M}^{u}_{\sec\geq 0}(N)$. In fact, this result and the result of [**44**] that any two souls of the same metric can be moved to each other by a diffeomorphism of the ambient nonnegatively curved manifold have the following common generalization.

Theorem 1.5. (i) *If two metrics are sufficiently close in* $\mathfrak{R}^{u}_{\sec\geq 0}(N)$*, their souls are ambiently isotopic in* N*.*
(ii) *The map associating to a metric* $g \in \mathfrak{R}^{u}_{\sec\geq 0}(N)$ *the diffeomorphism type of the pair* $(N, \text{ soul of } g)$ *is locally constant.*
(iii) *The diffeomorphism type of the pair* $(N, \text{ soul of } g)$ *is constant on connected components of* $\mathfrak{M}^{u}_{\sec\geq 0}(N)$*.*

Theorem 1.5 also holds for $\mathfrak{M}^{c}_{\sec\geq 0}(N)$, provided any two metrics of $\sec \geq 0$ on N have souls that intersect.

Thus, to detect different connected components of $\mathfrak{M}^{u}_{\sec\geq 0}(N)$, it is enough to produce nonnegatively curved metrics on N such that no self-diffeomorphism of N can move their souls to each other. From Theorem 1.1 we deduce:

Corollary 1.6. *For any integers $k \geq 3$, $m \geq 4$ the space $\mathfrak{M}^u_{\sec\geq 0}(S^4 \times S^k \times \mathbb{R}^m)$ has infinitely many connected components that lie in the same component of $\mathfrak{M}^u(S^4 \times S^k \times \mathbb{R}^m)$.*

Similarly, Theorem 1.2 yields an infinite sequence of metrics that lie in different connected components of $\mathfrak{M}^c_{\sec\geq 0}(N)$ and in the same component of $\mathfrak{M}^u(N)$.

Even if the souls are diffeomorphic, they need not be ambiently isotopic, as is illustrated by the following theorem exploiting examples of smooth knots due to Levine [**31**].

Theorem 1.7. *If L is a closed manifold of $\sec \geq 0$, then $N := S^7 \times L \times \mathbb{R}^4$ admits metrics that lie in different connected components of $\mathfrak{M}^u_{\sec\geq 0}(N)$ and in the same component of $\mathfrak{M}^u(N)$, and such that their souls are diffeomorphic to $S^7 \times L$ and not ambiently isotopic in N.*

Here L is allowed to have dimension 0 or 1, and in general, throughout the paper we treat S^1, $\mathbb{R}$, and a point as manifolds of $\sec \geq 0$.

Example 1.8. *For $L = S^5$, note that any closed manifold in the homotopy type of $S^7 \times S^5$ is diffeomorphic to $S^7 \times S^5$; in fact, the structure set of $S^7 \times S^5$ fits into the surgery exact sequence between the trivial groups Θ_{12} and $\pi_7(F/O) \oplus \pi_5(F/O)$* [**8**, theorem 1.5]. *Thus, any soul in $S^7 \times S^5 \times \mathbb{R}^4$ is diffeomorphic to $S^7 \times S^5$, while Theorem 1.7 detects different components of the moduli space.*

As mentioned above, there exist exotic 7-spheres with $\sec \geq 0$ that appear as codimension 3 souls in $S^7 \times \mathbb{R}^3$. Examples with non-diffeomorphic simply-connected souls of codimension 2 seem considerably harder to produce, as is suggested by the following:

Theorem 1.9. *If a simply-connected manifold N admits complete nonnegatively curved metrics with souls S, S' of dimension ≥ 5 and codimension 2, then S' is diffeomorphic to the connected sum of S with a homotopy sphere.*

Thus, non-diffeomorphic codimension 2 simply-connected souls are necessarily homeomorphic, while until now non-diffeomorphic homeomorphic closed manifolds of $\sec \geq 0$ have only been known in dimension 7; see e.g., [**16**, **26**]. In the companion paper [**4**], we show that for every integer $r \geq 2$ there is an open $(4r+1)$-dimensional simply-connected manifold that admits two metrics with non-diffeomorphic codimension 2 souls.

Non-diffeomorphic simply-connected souls do not exist in codimension 1, except possibly when the soul has dimension 4; indeed, any two codimension 1 simply-connected souls are h-cobordant, and hence diffeomorphic provided their dimension is $\neq 4$. By contrast, manifolds with nontrivial fundamental group may contain non-homeomorphic codimension 1 souls:

Example 1.10. ([**33**]) *Let L, L' be homotopy equivalent, non-homeomorphic 3-dimensional lens spaces, such as $L(7,1)$, $L(7,2)$. Then $L \times S^{2k}$, $L' \times S^{2k}$ are non-homeomorphic and h-cobordant for $k > 0$; hence they can be realized as non-homeomorphic souls in $N := L \times S^{2k} \times \mathbb{R}$, which is diffeomorphic to $L' \times S^{2k} \times \mathbb{R}$. In particular, $\mathfrak{M}^{c}_{\sec\geq 0}(N)$ is not connected.*

The codimension 1 is special both for geometric and topological reasons. As we show in Proposition 2.8, if a manifold N admits a metric with a codimension 1 soul, then the obvious map $\mathfrak{M}^{\infty,u}_{\sec\geq 0}(N) \to \mathfrak{M}^{\infty,c}_{\sec\geq 0}(N)$ is a homeomorphism, and either space is homeomorphic to the disjoint union of the moduli spaces of all possible pairwise non-diffeomorphic souls of metrics in $\mathfrak{M}^{\infty}_{\sec\geq 0}(N)$.

Kreck-Stolz [**26**] used index-theoretic arguments to construct a closed simply-connected 7-manifold B which carries infinitely many metrics of $\mathrm{Ric} > 0$ that lie in different components of $\mathfrak{M}^{\infty}_{\mathrm{scal}>0}(B)$. It was shown in [**25**] that some other metrics on B have $\sec \geq 0$ and lie in infinitely many different components of $\mathfrak{M}^{\infty}_{\mathrm{scal}\geq 0}(B)$. In particular, we conclude

Corollary 1.11. *$\mathfrak{M}^{\infty,c}_{\sec\geq 0}(B \times \mathbb{R})$ has infinitely many connected components.*

We also give examples of infinitely many isometric metrics that cannot be deformed to each other through complete metrics of $\sec \geq 0$.

Theorem 1.12. *If $n = 4r - 1$ and $3 \leq k \leq 2r + 1$ for some $r \geq 2$, then $\mathfrak{R}^{u}_{\sec\geq 0}(S^n \times \mathbb{R}^k)$ has infinitely many components that lie in the same component of $\mathfrak{R}^{u}(S^n \times \mathbb{R}^k)$.*

Theorem 1.13. *$\mathfrak{R}^{u}_{\sec\geq 0}(N)$ has infinitely many components if*
(i) *$N = L \times L(4r+1, 1) \times S^{2k} \times \mathbb{R}$ where L is any complete manifold of $\sec \geq 0$ and nonzero Euler characteristic, and $k \geq 3$, $r > 0$;*
(ii) *$N = M \times \mathbb{R}$ where M is a closed oriented manifold of even dimension ≥ 5 with $\sec \geq 0$ such that $G = \pi_1(M)$ is finite and $\mathrm{Wh}(G)$ is infinite.*

The proof of (i) relies on a geometric ingredient of independent interest: if S, S' are souls of metrics lying in the same component of $\mathfrak{R}^{u}_{\sec\geq 0}(N)$, then the restriction to S of any deformation retraction $N \to S'$ is homotopic to a diffeomorphism; e.g. this applies to the Sharafutdinov retraction.

Structure of the paper. Section 2 contains various geometric results on the spaces of nonnegatively curved metrics, including Theorem 1.5. Section 3 contains proofs of Theorems 1.1–1.2, giving examples with infinitely many codimension 4 souls. Sections 4–5 are a topological study of vector bundles with diffeomorphic total spaces, especially those of fiber dimension ≤ 3; in particular, there we prove Theorems 1.3 and 1.9. In Section 6, we prove Theorem 1.7, implying that diffeomorphic souls need not be ambiently isotopic; Theorem 1.12 is also proved there. Section 7 contains the proof of Theorem 1.13 and the geometric result stated in the previous paragraph.

On topological prerequisites. This paper employs a variety of topological tools, which we feel are best learned from the following sources. We refer to books by Husemoller [**23**], Milnor-Stasheff [**35**], and Spanier [**47**, chapter 6.10] for bundle theory and characteristic classes; to books by Cohen [**7**], and Oliver [**38**], and Milnor's survey [**34**] for Whitehead torsion and h-cobordisms; and to monographs of Wall [**51**] and Ranicki [**42**] for surgery theory. In the companion paper [**4**, sections 3 and 8], we survey aspects of surgery that are most relevant to [**4**] and to the present paper. We refer to [**42**, section 9.2 and proposition 9.20] for results on classifying spaces for spherical fibrations associated with topological monoids F_k, G_k, F, and their identity components SF_k, SG_k, SF; observe that Ranicki denotes F_k, G_k by $F(k+1)$, $G(k)$, respectively.

Acknowledgments. Belegradek is grateful to A. Dessai for the idea of Proposition 2.7, V. Kapovitch for helpful conversations about [**25**], and B. Wilking and L. Polterovich for useful comments about uniform topology on the moduli space. We also appreciate referees' comments on exposition. Belegradek was partially supported by the NSF grant #DMS-0804038. Kwasik was partially supported by BOR award LEQSF(2008-2011)-RD-A-24.

2. Moduli Spaces and Souls

In this section we prove Theorem 1.5, Corollary 1.11, and related results. We focus on moduli spaces with uniform topology; Remark 2.2 discusses when the same results hold for moduli spaces with topology of convergence on compact subsets. Here and elsewhere in the paper we follow notational Convention 1.4.

Riemannian metrics are sections of a tensor bundle, so they lie in a continuous function space, which is metrizable; thus $\mathfrak{R}^u_{\sec\geq 0}(N)$ is metrizable, and in particular, a map with domain $\mathfrak{R}^u_{\sec\geq 0}(N)$ is continuous if and only if it sends convergent sequences to convergent sequences.

Theorem 1.5 follows immediately from Lemma 2.1 below. Indeed, Lemma 2.1 implies that the map sending g in $\mathfrak{R}^u_{\sec\geq 0}(N)$ to the diffeomorphism class of the pair $(N,$ soul of $g)$ is locally constant, and hence continuous with respect to the discrete topology on the codomain, which implies that it descends to a continuous map from the quotient space $\mathfrak{M}^u_{\sec\geq 0}(N)$.

Let S_i be a soul of g_i in $\mathfrak{R}^u_{\sec\geq 0}(N)$, let $p_i\colon N\to S_i$ denote the Sharafutdinov retraction, and let $\breve{g}_i$ be the induced metric on S_i. Since S_i is convex, $\breve{g}_i$ and g_i induce the same distance functions on S_i, which is denoted d_i. For brevity, g_0, S_0, p_0, $\breve{g}_0$, d_0 are denoted by g, S, p, $\breve{g}$, d, respectively.

Lemma 2.1. *If g_i converges to g in $\mathfrak{R}^u_{\sec\geq 0}(N)$, then for all large i*
(1) *$p_i|_S\colon S\to S_i$ is a diffeomorphism,*
(2) *the pullback metrics $(p_i|_S)^*\breve{g}_i$ converge to $\breve{g}$ in $\mathfrak{R}^{0,u}_{\sec\geq 0}(S)$,*
(3) *S_i is C^∞ ambiently isotopic to S in N.*

Proof. (1) Arguing by contradiction, pass to a subsequence for which $p_i|_S$ is never a diffeomorphism. Wilking proved in [**52**] that Sharafutdinov retractions are smooth Riemannian submersions onto the soul. Note that $p_i(S)=S_i$ and $p(S_i)=S$ because degree one maps are onto.

Since the convergence $g_i\to g$ is uniform, given any positive ϵ, R and all large enough i, the distance functions d_i are ϵ-close to d on any R-ball in (N,d). Then S has uniformly bounded d_i-diameter, and since p_i are distance nonincreasing, and $p_i(S)=S_i$, we conclude that S_i has uniformly bounded d_i-diameter; thus the metrics d, d_i are close on S, and on S_i. As p_i, p are distance-nonincreasing, with respect to d_i, d, respectively, the self-map $f_i := p\circ p_i|_S$ of (S,d) is almost distance nonincreasing. Then compactness of S implies via Ascoli's theorem that f_i subconverges to a self-map of (S,d), which is distance nonincreasing and surjective, and hence is an isometry.

Any isometry is a diffeomorphism. Diffeomorphisms form an open subset among smooth mappings, so $p\circ p_i|_S$ is a diffeomorphism for large i. It follows that $p_i|_S$ is an injective immersion, and hence a diffeomorphism, as S is a closed manifold, giving a contradiction that proves (1).

(2) Arguing by contradiction, pass to a subsequence for which $p_i^*\breve{g}_i$ lies outside a C^0-neighborhood of $\breve{g}$. Note that $p_i\colon (S,d)\to(S_i,d_i)$ is a Gromov-Hausdorff approximation; indeed, if $x,y\in S$, then $d(x,y)$ is almost equal to $d(f_i(x),f_i(y))\le d(p_i(x),p_i(y))$, where the right hand side is almost equal to $d_i(p_i(x),p_i(y))$ which is $\le d_i(x,y)$, which is almost equal to $d(x,y)$; thus all the inequalities are almost equalities and hence $d(x,y)$ is almost equal to $d_i(p_i(x),p_i(y))$.

By Yamaguchi's fibration theorem [**54**], there is a diffeomorphism $h_i\colon S_i\to S$ such that $h_i^*\breve{g}$ is C^0-close to $\breve{g}_i$. Note that $h_i\circ p_i$ almost preserves $\breve{d}$ so it subconverges to an isometry of $(S,\breve{g})$, and in particular, it pulls $\breve{g}$ back to a metric that is C^0-close to $\breve{g}$. It follows that $p_i^*\breve{g}_i$ is C^0-close to $\breve{g}$, giving a contradiction which proves (2).

(3) Let $E(\nu_i)$ denote the total space of the normal bundle ν_i to S_i. Wilking showed in [**52**, corollary 7] that there exists a diffeomorphism $e_i\colon E(\nu_i) \to N$ such that $p_i \circ e_i$ is the projection of ν_i. Thus (1) implies that the projection of ν_i restricts to a diffeomorphism from $e_i^{-1}(S)$ onto S_i, whose inverse is a section of ν_i with image $e_i^{-1}(S)$. Any two sections of a vector bundle are ambiently isotopic, so applying e_i, we get an ambient isotopy of S and S_i in N. q.e.d.

Remark 2.2. We do not know whether the conclusion of Lemma 2.1 holds for $\mathfrak{R}^c_{\sec\geq 0}(N)$. The proof of Lemma 2.1 works for $\mathfrak{R}^c_{\sec\geq 0}(N)$ as written, provided $\mathrm{dist}(S, S_i)$ is uniformly bounded. This happens if any two metrics in $\mathfrak{R}^c_{\sec\geq 0}(N)$ have souls that intersect, which as noted in [**25**] is true, e.g. when N contains a soul with nontrivial normal Euler class. Note that except for examples discussed in Remark 3.6, all the metrics we construct in this paper have souls with trivial normal Euler class.

Remark 2.3. Let $\{S_i\}_{i\in I}$ be a collection of pairwise nondiffeomorphic manifolds representing the diffeomorphism classes of souls of all possible complete nonnegatively curved metrics on an open manifold N, and for $g \in \mathfrak{R}^u_{\sec\geq 0}(N)$, let $i(g) \in I$ be such that $S_{i(g)}$ is diffeomorphic to a soul of (N, g). By [**44**], one has a well-defined map that associates to g the isometry class of its soul in $\mathfrak{M}_{\sec\geq 0}(S_{i(g)})$, which can be thought of as a map $\mathfrak{R}^u_{\sec\geq 0}(N) \to \coprod_i \mathfrak{M}_{\sec\geq 0}(S_i)$, where the codomain is given the topology of disjoint union of $\mathfrak{M}_{\sec\geq 0}(S_i)$'s. This map descends to a map

$$\mathbf{soul}\colon \mathfrak{M}^{k,u}_{\sec\geq 0}(N) \to \coprod_i \mathfrak{M}^k_{\sec\geq 0}(S_i).$$

If $k = 0$, then part (2) of Lemma 2.1 implies that the map **soul** is continuous (the continuity can be checked on sequences in $\mathfrak{R}^{0,u}_{\sec\geq 0}(N)$ because it is metrizable).

Remark 2.4. Suppose that N has a soul S with trivial normal bundle. Let $\mathfrak{M}^{k,u}_{\sec\geq 0}(N, S)$ denote the union of the components of $\mathfrak{M}^{k,u}_{\sec\geq 0}(N)$ consisting of the isometry classes of metrics with soul diffeomorphic to S. Then the map **soul** restricts to a retraction

$$\mathfrak{M}^{0,u}_{\sec\geq 0}(N, S) \to \mathfrak{M}^0_{\sec\geq 0}(S)$$

where $\mathfrak{M}^0_{\sec\geq 0}(S)$ sits in $\mathfrak{M}^0_{\sec\geq 0}(N, S)$ as the set of isometry classes of Riemannian products of nonnegatively curved metrics on S and the standard $\mathbb{R}^n$. Like any retraction, it induces a surjective map on homotopy and homology, and hence one potentially could get lower bounds on the topology of $\mathfrak{M}^{0,u}_{\sec\geq 0}(S \times \mathbb{R}^n)$ in terms of the topology of $\mathfrak{M}^0_{\sec\geq 0}(S)$. Unfortunately, nothing is known about the topology of $\mathfrak{M}^0_{\sec\geq 0}(S)$, which naturally leads to the following.

Problem 2.5. Find a closed manifold S with non-connected $\mathfrak{M}^0_{\sec\geq 0}(S)$.

Problem 2.6. Is the map **soul**: $\mathfrak{M}^\infty_{\sec\geq 0}(N) \to \coprod_i \mathfrak{M}^\infty_{\sec\geq 0}(S_i)$ continuous?

The only known examples with non-connected $\mathfrak{M}^k_{\sec\geq 0}(S)$ are (modifications of) those in [**26**] where $k=\infty$ (it may suffice to take k sufficiently large but definitely not $k=0$). These examples were modified in [**25**] to yield a closed simply-connected manifold B admitting infinitely many metrics g_i with $\sec\geq 0$ and $\mathrm{Ric}>0$ that lie in different components of $\mathfrak{M}^\infty_{\mathrm{scal}>0}(B)$. It was asserted in [**25**] that g_i lie in different components of $\mathfrak{M}^\infty_{\mathrm{scal}\geq 0}(B)$, but it takes an additional argument which hopefully will be written by the authors of [**25**]. The following shows that g_i lie in different components of $\mathfrak{M}^\infty_{\sec\geq 0}(B)$.

Proposition 2.7. *Metrics of* $\sec\geq 0$ *and* $\mathrm{Ric}>0$ *on a closed manifold* X *that lie in different components of* $\mathfrak{M}^\infty_{\mathrm{Ric}>0}(X)$ *also lie in different components of* $\mathfrak{M}^\infty_{\sec\geq 0}(X)$.

Proof. We abuse terminology by not distinguishing a metric from its isometry class. First we show that each $h\in\mathfrak{M}^\infty_{\sec\geq 0}(X)$ has a neighborhood U_h such that any $h'\in U_h$ can be joined to h by a path of metrics h_s with $h_0=h$, $h_1=h'$, and $\mathrm{Ric}(h_s)>0$ for $0<s<1$. If there is no such U_h, then using Ebin's slice theorem [**11**] one can show that there is a sequence $h_i\in\mathfrak{R}^\infty_{\sec\geq 0}(X)$ converging to h such that h_i cannot be joined to h by a path as above. Böhm-Wilking [**5**] showed that Ricci flow instantly makes a metric of $\sec\geq 0$ on a closed manifold with finite fundamental group into a metric with $\mathrm{Ric}>0$. Thus h_i and h can be flown to nearby metrics $h_i(t)$, $h(t)$ of positive Ricci curvature where $h_i(t)\to h(t)$ for any fixed small t. Since $\mathfrak{M}^\infty_{\mathrm{Ric}>0}(X)$ is open in the space of all metrics, Ebin's slice theorem ensures that if i is large enough, $h_i(t)$, $h(t)$ can be joined by a path in $\mathfrak{M}^\infty_{\mathrm{Ric}>0}(X)$, and concatenating the three paths yields a desired path from h_i to h via $h_i(t)$ and $h(t)$.

Given an open cover $\{U_k\}$ of a connected set for any two g, g' in this set there exists a finite sequence $g_0=g, g_1,\dots,g_n=g'$ such that $g_k\in U_k$ and $U_k\cap U_{k-1}\neq\emptyset$ for every $0<k\leq n$ [**29**, section 46, theorem 8] .

Thus, given two metrics g, g' in a component of $\mathfrak{M}^\infty_{\sec\geq 0}(X)$, we get a finite sequence g_k in this component with $g_0=g, g_1,\dots,g_n=g'$ and such that for each k one can join g_{k-1} to g_k by a path of metrics that have $\mathrm{Ric}>0$, except possibly at endpoints. By assumption, g, g' have $\mathrm{Ric}>0$. By construction, the paths backtrack at g_k, as they are given by Ricci flow $g_k(t)$ near g_k, so the concatenated path from g to g' can be cut short at $g_1,\dots,g_{n-1}$ to entirely consist of metrics of $\mathrm{Ric}>0$.
q.e.d.

Now Corollary 1.11 follows from the proposition below, applied for $k=\infty$.

Proposition 2.8. *For an integer $k \geq 0$, set k_0 to be the maximum of 0 and $k-1$; for $k=\infty$, set $k_0=\infty$. If N admits a complete metric with* $\sec \geq 0$ *and a codimension 1 soul, then the maps* **soul**: $\mathfrak{M}^{k,c}_{\sec\geq 0}(N) \to \coprod_i \mathfrak{M}^{k_0}(S_i)$ *and* $\mathrm{id}\colon \mathfrak{M}^{k,u}_{\sec\geq 0}(N) \to \mathfrak{M}^{k,c}_{\sec\geq 0}(N)$ *are homeomorphisms.*

Proof. It suffices to show that the maps **soul**: $\mathfrak{M}^{k,c}_{\sec\geq 0}(N) \to \coprod_i \mathfrak{M}^{k_0}(S_i)$ and **soul**: $\mathfrak{M}^{k,u}_{\sec\geq 0}(N) \to \coprod_i \mathfrak{M}^{k_0}(S_i)$ are homeomorphisms, and the argument below works for both maps.

By the splitting theorem, any complete metric g of $\sec \geq 0$ on N locally splits off an $\mathbb{R}$-factor that is orthogonal to the soul. The splitting becomes global in the cover of order ≤ 2 that corresponds to the first Stiefel-Whitney class w_1 of the normal bundle to the soul. If $w_1=0$, then g is the product of $\mathbb{R}$ and a closed nonnegatively curved manifold S_g, in which case there is a unique soul $S_g \times \{t\}$ through every point. If $w_1 \neq 0$, and if S_g is a soul of g, then g can be written as $\tilde{S}_g \times_{O(1)} \mathbb{R}$, where $\tilde{S}_g$ is the 2-fold cover of S_g induced by w_1, and $S_g = \tilde{S}_g \times_{O(1)} \{0\}$; in this case, S_g is a unique soul of g because by [**55**] any soul is obtained by exponentiating a parallel normal vector field along S_g, but the existence of a nonzero parallel normal vector field along S_g would imply triviality of the normal line bundle to S_g, contradicting $w_1 \neq 0$.

As follows (e.g. from Section 4) if two real line bundles over closed manifolds S, S' have diffeomorphic total spaces, then the line bundles are either both trivial, or both nontrivial. Thus triviality of w_1 depends only on N, and not on the metric.

If $w_1=0$, then the map **soul** has a continuous inverse induced by the map that sends a metric on the soul to its product with $\mathbb{R}$.

If $w_1 \neq 0$, then each closed nonnegatively curved manifold S that is homotopy equivalent to N has a 2-fold cover $\tilde{S}$ induced by w_1. Thus a metric in $\mathfrak{M}_{\sec\geq 0}(S)$ gives rise to the metric $\tilde{S} \times_{O(1)} \mathbb{R}$, which defines a continuous inverse for **soul**.

Finally, we show that **soul** is continuous. It suffices to do so for the topology of convergence on compact sets, as $\mathrm{id}\colon \mathfrak{M}^{k,u}_{\sec\geq 0}(N) \to \mathfrak{M}^{k,c}_{\sec\geq 0}(N)$ is continuous. Let $g_j \to g$ be a converging sequence in $\mathfrak{R}^{k,c}_{\sec\geq 0}(N)$. By Lemma 2.1, their souls of g_j converge in C^0 topology (as abstract Riemaniann manifolds) to the soul of g. This gives continuity of **soul** for $k \in \{0,1\}$, so we assume $k \geq 2$.

Fix a soul S of g. Then there exists a soul S_j of g_j that intersects S; indeed, given a complete metric g' of $\sec \geq 0$ on N, if $w_1=0$, then g' has a soul through every point of N, and if $w_1 \neq 0$, then souls of g' and

g must intersect because w_1 can be interpreted as the first obstruction to deforming the homotopy equivalence $S_{g'} \to S_g$ away from S_g.

As S_j (abstractly) C^0 converge to S, their diameters are uniformly bounded; in particular, they all lie in a compact domain D of N. Convergence $g_j \to g$ in C^k topology implies convergence $\nabla^l_{g_j} R_{g_j} \to \nabla^l_g R_g$ of covariant derivatives of the curvature tensors for every $l \le k-2$, and in particular, one gets a uniform bound on $\|\nabla^l_{g_j} R_{g_j}\|$ over D. Since S_j is totally geodesic, the restriction of $\nabla^l_{g_j} R_{g_j}$ to S_j is $\nabla^l_{g_j|_{S_j}} R_{g_j|_{S_j}}$ [**24**, proposition 8.6], so $\|\nabla^l_{g_j|_{S_j}} R_{g_j|_{S_j}}\|$ are uniformly bounded for $l \le k-2$. Since the C^0 limit S of S_j has the same dimension, the convergence is without collapse, so there is a common lower injectivity radius bound for S_j. Hence the family S_j is precompact in C^{k-1} topology [**41**, page 192], but since S_j converges to S in C^0 topology, all C^{k-1} limit points of S_j are isometric to S because Gromov-Hausdorff limits are unique up to isometry. Thus S_j converges to S in C^{k-1} topology, as claimed. q.e.d.

Question 2.9. Can k_0 in Proposition 2.8 be replaced by k?

Remark 2.10. An analog of Proposition 2.8 holds for complete n-manifolds of $\text{Ric} \ge 0$ with nontrivial $(n-1)$-homology, because each such manifold is a flat line bundle over a compact totally geodesic submanifold [**48**]. In particular, once it is shown that metrics g_i of [**25**] lie in different components of $\mathfrak{M}^\infty_{\text{scal}\ge 0}(B)$, we can conclude that $\mathfrak{M}^\infty_{\text{Ric}\ge 0}(B \times \mathbb{R})$ has infinitely many connected components.

As mentioned in the introduction, $\mathfrak{M}^u(N)$ need not be connected. It is therefore desirable to arrange our metrics with non-diffeomorphic souls to lie in the same component of $\mathfrak{M}^u(N)$. This can be accomplished under a mild topological assumption:

Proposition 2.11. *Suppose an open manifold N admits two complete metrics of* $\sec \ge 0$ *with souls S, S'. If the normal sphere bundle to S is simply-connected and has dimension ≥ 5, then N admits two complete metrics of* $\sec \ge 0$ *with souls S, S' which lie in the same path-component of $\mathfrak{R}^u(N)$.*

Proof. By Proposition 4.1, the normal sphere bundle to S' is also simply-connected, and by Lemma 4.8, if normal sphere bundles to S, S' are chosen to be disjoint, then the region between them is a (trivial) h-cobordism. Thus closed tubular neighborhoods of S, S' are diffeomorphic. The complement of an open tubular neighborhood of the soul is of course the product of a ray and the boundary of the tubular neighborhood. The diffeomorphism of closed tubular neighborhoods of S, S' extends to a self-diffeomorphism of N, which can be chosen to preserve any given product structures on the complements of tubular neighborhoods, and which is the identity near S and S'.

By [**15**], any complete metric of $\sec \geq 0$ can be modified by changing it outside a sufficiently small tubular neighborhood of the soul so that the new metric has the same soul and, outside a larger tubular neighborhood, it is the Riemannian product of a ray and a metric on the normal sphere bundle. Performing this modification to the metrics at hand, and pulling back one of the metrics via a self-diffeomorphism of N as above, we get nonnegatively curved metrics g, g' with souls S, S' such that outside some of their common tubular neighborhood $D = D'$ the metrics are Riemannian products $\partial D \times \mathbb{R}_+$, $\partial D' \times \mathbb{R}_+$ with the same $\mathbb{R}_+$-factor. Now the convex combination of g, g' defines a path joining g, g' in $\mathfrak{R}^u(N)$. q.e.d.

3. Infinitely Many Souls of Codimension 4

This section contains examples of manifolds that admit metrics with infinitely many non-homeomorphic souls of codimension 4. The examples are obtained by modifying arguments in [**1**, **25**] and invoking the new topological ingredient, Proposition 3.1 below, which is best stated with the following notation.

Given vector bundles α_0, β_0 over a space Z, let $\mathbf{V}(Z, \alpha_0, \beta_0)$ be the set of pairs (α, β) of vector bundles over Z such that α, β are (unstably) fiber homotopy equivalent to α_0, β_0, respectively, and the rational Pontryagin classes of $\alpha \oplus \beta$, $\alpha_0 \oplus \beta_0$ become equal when pullbacked via the sphere bundle projection $b\colon S(\beta) \to Z$. Also denote the fiber dimension of $S(\alpha_0)$, $S(\beta_0)$ by k_{α_0}, k_{β_0}, respectively.

Proposition 3.1. *If $k_{\alpha_0} + k_{\beta_0} + \dim(Z) \geq 5$ and $k_{\alpha_0} \geq 2$, and Z is a closed smooth manifold, then the number of diffeomorphism types of the disk bundles $D(b^{\#}\alpha)$ with (α, β) in $\mathbf{V}(Z, \alpha_0, \beta_0)$ is finite.*

Proof. Denote the sphere bundle projection of α, β, α_0, β_0 by a, b, a_0, b_0, respectively, and fiber homotopy equivalences by $f_\alpha\colon S(\alpha) \to S(\alpha_0)$ and $f_\beta\colon S(\beta) \to S(\beta_0)$.

The fiberwise cone construction yields a homotopy equivalence

$$\hat{f}_\alpha\colon (D(\alpha), S(\alpha)) \to (D(\alpha_0), S(\alpha_0))$$

that extends f_α and satisfies $a_0 \circ \hat{f}_\alpha = a$. Pulling back $\hat{f}_\alpha$ via b gives a homotopy equivalence $b^{\#}\hat{f}_\alpha\colon (D(b^{\#}\alpha), S(b^{\#}\alpha)) \to (D(b^{\#}\alpha_0), S(b^{\#}\alpha_0))$. Since $b = b_0 \circ f_\beta$, the disk bundle $D(b^{\#}\alpha_0)$ is the f_β-pullback of $D(b_0^{\#}\alpha_0)$, so composing $b^{\#}\hat{f}_\alpha$ with the bundle isomorphism induced by f_β gives a homotopy equivalence

$$F_{\alpha,\beta}\colon (D(b^{\#}\alpha), S(b^{\#}\alpha)) \to (D(b_0^{\#}\alpha_0), S(b_0^{\#}\alpha_0)).$$

Now we show that $F_{\alpha,\beta}$ pulls back rational Pontryagin classes. The tangent bundles to $D(b^{\#}\alpha)$ and $D(b_0^{\#}\alpha_0)$ are determined by their restrictions to the zero sections, and these restrictions stably are respectively

$$b^{\#}\alpha \oplus \tau_{S(\beta)} = b^{\#}\alpha \oplus b^{\#}(\beta \oplus \tau_Z) = b^{\#}(\alpha \oplus \beta \oplus \tau_Z)$$

and $b_0^{\#}(\alpha_0 \oplus \beta_0 \oplus \tau_Z)$. The restriction of $F_{\alpha,\beta}$ to the zero section is f_β, so pulling back the latter bundle via f_β gives two bundles over $S(\beta)$, namely, $b^{\#}(\alpha \oplus \beta \oplus \tau_Z)$ and $b^{\#}(\alpha_0 \oplus \beta_0 \oplus \tau_Z)$, which by assumption have the same rational total Pontryagin class.

Arguing by contradiction lets us pass to subsequences; thus, since rational Pontryagin classes determine a stable vector bundle up to finite ambiguity, we may pass to a subsequence in $\mathbf{V}(Z,\alpha_0,\beta_0)$ for which the $F_{\alpha,\beta}^{-1}$-pullbacks of all the bundles $b^{\#}(\alpha \oplus \beta \oplus \tau_Z)$ to $D(b_0^{\#}\alpha_0)$ are isomorphic. Fix (α_1,β_1) in the subsequence so that $G_{\alpha,\beta} := F_{\alpha,\beta} \circ F_{\alpha_1,\beta_1}^{-1}$ is now tangential for any (α,β).

To finish the proof, we need a well-known tangential surgery exact sequence

$$\begin{aligned} L^s_{n+1}(\pi_1(Y),\pi_1(\partial Y)) \longrightarrow\ & \mathbf{S}^{s,t}(Y,\partial Y) \longrightarrow [Y,F] \\ \longrightarrow\ & L^s_n(\pi_1(Y),\pi_1(\partial Y)) \end{aligned}$$

described, e.g., in [**4**, section 8], where $\mathbf{S}^{s,t}(Y,\partial Y)$ is the tangential simple structure set for a smooth manifold with boundary Y of dimension $n \geq 6$.

Set $Y := D(b_1^{\#}\alpha_1)$; then $G_{\alpha,\beta}\colon (D(b^{\#}\alpha), S(b^{\#}\alpha)) \to (Y,\partial Y)$ represents an element in $\mathbf{S}^{s,t}(Y,\partial Y)$. The assumption $k_{\alpha_0} \geq 2$ ensures that $\partial Y \to Y$ is a π_1-isomorphism so that the Wall groups $L^s_*(\pi_1(Y),\pi_1(\partial Y)$ vanish, and the other dimension assumption gives $\dim(Y) = k_{\alpha_0} + k_{\beta_0} + 1 + \dim(Z) \geq 6$. By exactness, $\mathbf{S}^{s,t}(Y,\partial Y)$ is bijective to the set $[Y,F]$, which is a finite [**42**, proposition 9.20(iv)], so that manifolds $D(b^{\#}\alpha)$ fall into finitely many diffeomorphism classes. q.e.d.

Remark 3.2. If in the definition of $\mathbf{V}(Z,\alpha_0,\beta_0)$ we require that $\alpha \oplus \beta$, $\alpha_0 \oplus \beta_0$ are stably isomorphic, then the number of diffeomorphism types of manifolds $D(b^{\#}\alpha)$ is at most the order of the set $[Z,F]$. Indeed, let $\hat{b}\colon D(\beta) \to Z$, $\hat{b}_0\colon D(\beta_0) \to Z$ denote the disk bundle projections, and extend $F_{\alpha,\beta}$ by the fiberwise cone construction to the homotopy equivalence of triads

$$\hat{F}_{\alpha,\beta}\colon D(\hat{b}^{\#}\alpha) \to D(\hat{b}_0^{\#}\alpha_0),$$

which is tangential as $\alpha \oplus \beta$, $\alpha_0 \oplus \beta_0$ are stably isomorphic. Hence $F_{\alpha,\beta}$ is also tangential, as a restriction of $\hat{F}_{\alpha,\beta}$ to submanifolds with trivial normal bundles. The geometric definition of the normal invariant (see [**51**] after lemma 10.6) easily implies that the normal invariant of $F_{\alpha,\beta}$ is the restriction of the normal invariant of $\hat{F}_{\alpha,\beta}$; hence the normal invariant of $F_{\alpha,\beta}$ lies in the image of the restriction $[D(\hat{b}_0^{\#}\alpha_0), F] \to [D(b_0^{\#}\alpha_0), F]$ whose domain is bijective to $[Z, F]$, as $D(\hat{b}_0^{\#}\alpha_0)$ is homotopy equivalent to Z. As in the proof of Proposition 3.1, the tangential surgery exact sequence implies that the number of diffeomorphism types of manifolds $D(b^{\#}\alpha)$ is at most the number of normal invariants of the map $F_{\alpha,\beta}$, proving the claim.

Given $m \in \pi_4(BSO_3) \cong \mathbb{Z}$, let ξ_m^n be the corresponding rank n vector bundle over S^4 with structure group SO_3 sitting in SO_n in the standard way. Let $\eta_{l,m}^{k,n}$ denote the pullback of ξ_m^n via the sphere bundle projection $S(\xi_l^k) \to S^4$. Theorem 1.1 is obtained from the following by setting $l = 0 = m$.

Theorem 3.3. *If $k, n \geq 4$, then $E(\eta_{l,m}^{k,n})$ admits infinitely many complete metrics of* $\sec \geq 0$ *with pairwise non-homeomorphic souls.*

Proof. It is explained in [**1**] that $S(\xi_l^k)$, $S(\xi_i^k)$ are fiber homotopy equivalent if $l - i$ is divisible by 12 for $k \geq 4$. (In fact, up to fiber homotopy equivalence there are only finitely many oriented S^3-fibrations over a finite complex Z that admit a section, because their classifying map in $[Z, BSG_4]$ factors through BSF_3 and $[Z, BSF_3]$ is finite as BSF_3 is rationally contractible; see [**42**, proposition 9.20(i)].)

Also, it is noted in [**1**] that $\xi_l^k \oplus \xi_m^n$ and $\xi_i^k \oplus \xi_j^n$ are equal in $\pi_4(BSO)$ if $l + m = i + j$. Of course, if $j := l - i + m$ and $l - i$ is divisible by 12, then $m - j$ is divisible by 12.

Thus we get an infinite family (ξ_i^k, ξ_{l-i+m}^n) parametrized by i with $l-i$ divisible by 12 such that $\xi_i^k \oplus \xi_{l-i+m}^n = \xi_l^k \oplus \xi_m^n$ in BSO, and $S(\xi_i^k)$, $S(\xi_{l-i+m}^n)$ is fiber homotopy equivalent to $S(\xi_l^k)$, $S(\xi_m^n)$, respectively.

By Proposition 3.1, $D(\eta_{i,l-i+m}^{k,n})$ lie in finitely many diffeomorphism classes, one of which must contain $D(\eta_{l,m}^{k,n})$. A priori, this does not show that there are infinitely many $D(\eta_{i,l-i+m}^{k,n})$'s that are diffeomorphic to $D(\eta_{l,m}^{k,n})$. Yet $\pi_4(F) = 0$, so Remark 3.2 implies that $F_{\xi_i^k, \xi_{l-i+m}^n} : D(\eta_{i,l-i+m}^{k,n}) \to D(\eta_{l,m}^{k,n})$ is homotopic to a diffeomorphism. (Without invoking Remark 3.2, we only get an infinite sequence of $D(\eta_{i,l-i+m}^{k,n})$'s that are diffeomorphic to some $D(\eta_{i_0,l-i_0+m}^{k,n})$.)

As in [**1**], results of Grove-Ziller show that each $E(\eta_{i,l-i+m}^{k,n})$ is nonnegatively curved with zero section $S(\xi_i^k)$ being a soul, and $p_1(S(\xi_i^k))$ is $\pm 4i$-multiple of the generator; so assuming $i \geq 0$, we get that the souls are pairwise non-homeomorphic. q.e.d.

The proof of Theorem 3.4 below is a slight variation of an argument in [**25**]. A major difference is in employing Proposition 3.1, and checking it is applicable, in place of "above metastable range" considerations of [**25**]. Another notable difference is that to satisfy the conditions of Proposition 3.1, we have to vary q, r and keep a, b fixed, while exactly the opposite is done in [**25**]. This requires a number of minor changes, so instead of extracting what we need from [**25**], we find it easier (and more illuminating) to present a self-contained proof below; we stress that all computational tricks in the proof are lifted directly from [**25**].

Recall that for a cell complex Z, each element in $H^2(Z)$ can be realized as the Euler class of a unique SO_2-bundle over Z.

Let $X = S^2 \times S^2 \times S^2$. Fix an obvious basis in $H^2(X)$ whose elements are dual to the S^2-factors. Let γ, ξ, μ be the complex line bundles over X with respective Euler classes $(a, b, 0)$, $(0, q, r)$, $(0, -q, r)$ in this basis, where a, b, q, r are nonzero integers and a, b are coprime. Let $\eta = \xi \oplus \epsilon$ and $\zeta = \mu \oplus \epsilon$, where ϵ is the trivial complex line bundle.

Denote the pullback of η, ζ via $\pi_\gamma \colon S(\gamma) \to X$ by $\hat{\eta}$, $\hat{\zeta}$, respectively, and the pullback of γ via $\pi_\eta \colon S(\eta) \to X$ by $\hat{\gamma}$. By definition of pullback, $S(\hat{\eta})$ and $S(\hat{\gamma})$ have the same total space, which we denote $M_{\gamma,\eta}$. Denote by $\pi_{\hat{\gamma}}$, $\pi_{\hat{\eta}}$ the respective sphere bundle projections $S(\hat{\eta}) \to S(\eta)$, $S(\hat{\gamma}) \to S(\gamma)$; note that $\pi_{\hat{\gamma}} \circ \pi_\eta = \pi_{\hat{\eta}} \circ \pi_\gamma$. Let $\tilde{\zeta}$ be the pullback of $\hat{\zeta}$ via $\pi_{\hat{\eta}} \colon M_{\gamma,\eta} \to S(\gamma)$. With these notations we prove:

Theorem 3.4. (i) *For a universal* $c > 0$*, the manifold* $E(\tilde{\zeta})$ *admits a complete metric with* $\sec(E(\tilde{\zeta})) \in [0, c]$ *such that the zero section* $M_{\gamma,\eta}$ *of* $\tilde{\zeta}$ *is a soul of diameter* 1*.*
(ii) *For fixed* γ *and variable* η*,* ζ*, the manifolds* $D(\tilde{\zeta})$ *lie in finitely many diffeomorphism classes, while the manifolds* $M_{\gamma,\eta}$ *lie in infinitely many homeomorphism classes.*

Proof. (i) Recall that any principal S^1-bundle P over $(S^2)^n$ can be represented as $(S^3)^n \times_\rho S^1$ where $\rho \colon T^n \to S^1$ is a homomorphism and T^n acts on $(S^3)^n$ as the product of standard S^1-actions on S^3. (Indeed, the pullback of the S^1-bundle to $(S^3)^n$ can be trivialized as $H^2((S^3)^n) = 0$, and ρ comes from the T^n-action on the S^1-factor.) Therefore, $P \to (S^2)^n$ can be identified with $(S^3)^n / \ker(\rho) \to (S^3)^n / T^n$.

Specializing to our situation, let ρ_γ, ρ_η, ρ_ζ be the homomorphisms $T^3 \to S^1$ corresponding to the principal circle bundles that are (uniquely) determined by γ, η, ζ, respectively. Thus the principal circle bundle $S(\gamma)$ equals $(S^3)^3 / \ker(\rho_\gamma)$, and the fiber product $S(\eta) \oplus \zeta$ can be written as the associated bundle $(S^3)^3 \times_{\rho_\eta} S^3 \times_{\rho_\zeta} \mathbb{R}^4$; this is an $S^3 \times \mathbb{R}^4$-bundle over B. The pullback of this latter bundle to $S(\gamma)$ has total space $E(\tilde{\zeta})$, and it can then be written as $(S^3)^3 \times_{\rho_\eta|_{\ker(\rho_\gamma)}} S^3 \times_{\rho_\gamma|_{\ker(\rho_\gamma)}} \mathbb{R}^4$.

All the actions are isometric, so giving $\mathbb{R}^4$ a rotationally symmetric metric isometric to $S^3 \times \mathbb{R}_+$ outside a compact subset, we see that $E(\tilde{\zeta})$ gets a Riemannian submersion metric of $\sec \in [0, c]$ for a universal c. By a standard argument involving a rotationally symmetric exhaustion function on $\mathbb{R}^4$, the zero section $M_{\gamma,\eta}$ is a soul. Since $M_{\gamma,\eta}$ is a quotient of $(S^3)^3 \times_{\rho_\eta} S^3$ that can be further Riemannian submersed onto a fixed manifold $S(\gamma)$, the diameter of $M_{\gamma,\eta}$ is uniformly bounded above and below. So the diameter can be rescaled to 1, while keeping universal curvature bounds on $E(\tilde{\zeta})$.

(ii) First, we show that $M_{\gamma,\eta}$ fall into infinitely many homeomorphism types. Since τ_X is stably trivial, computing the first Pontryagin class gives

$$p_1(M_{\gamma,\eta}) = \pi_{\hat{\eta}}^* p_1(\hat{\eta} \oplus \tau_{S(\gamma)}) = \pi_{\hat{\eta}}^* \pi_\gamma^* p_1(\eta \oplus \gamma \oplus \tau_X) = \pi_{\hat{\eta}}^* \pi_\gamma^* p_1(\xi \oplus \gamma).$$

Now $p_1(\gamma \oplus \xi) = p_1(\gamma) + p_1(\xi) = e(\gamma)^2 + e(\xi)^2$ and the Gysin sequence for γ gives $\pi_\gamma^* e(\gamma)^2 = 0$ because the kernel of $\pi_\gamma^* \colon H^4(X) \to H^4(S(\gamma))$ is the image of the (cup) multiplication by $e(\gamma)$.

We compute $\pi_{\hat{\eta}}^* \pi_\gamma^* p_1(\xi)$ from the commutative diagram below, whose rows are Gysin sequences for γ, $\hat{\gamma}$, while all vertical arrows are isomorphisms for $i \leq 2$ because they fit into the Gysin sequences for η, $\hat{\eta}$ where injectivity follows as $e(\eta)$, $e(\hat{\eta})$ vanish and surjectivity holds for $i \leq 2$ as X, $S(\eta)$, $M_{\gamma,\eta}$ are simply-connected, as a, b are coprime.

$$\begin{array}{ccccccc}
H^i(X) & \xrightarrow{\cup e(\gamma)} & H^{i+2}(X) & \xrightarrow{\pi_\gamma^*} & H^{i+2}(S(\gamma)) & \longrightarrow & H^{i+1}(X) = 0 \\
\downarrow{\scriptstyle \pi_\eta^*} & & \downarrow{\scriptstyle \pi_\eta^*} & & \downarrow{\scriptstyle \pi_{\hat{\eta}}{}^*} & & \\
H^i(S(\eta)) & \xrightarrow{\cup e(\hat{\gamma})} & H^{i+2}(S(\eta)) & \xrightarrow{\pi_{\hat{\gamma}}{}^*} & H^{i+2}(M_{\gamma,\eta}) & \longrightarrow & 0
\end{array}$$

Also, the commutativity of the rightmost square implies that $\pi_{\hat{\gamma}}{}^*$ is onto.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be the basis in $H^2(S(\eta))$ corresponding to the chosen basis in $H^2(X) = \mathbb{Z}^3$; thus $\pi_\eta^* e(\xi) = q\mathbf{y} + r\mathbf{z}$, and $e(\hat{\gamma}) = a\mathbf{x} + b\mathbf{y}$, which is primitive as a, b are coprime. Another basis in $H^2(S(\eta))$ is $a\mathbf{x} + b\mathbf{y}$, $-m\mathbf{x} + n\mathbf{y}$, $\mathbf{z}$ where n, m are integers with $an + bm = 1$. Thus $H^2(M_{\gamma,\eta})$ is isomorphic to $\mathbb{Z}^2$ generated by $\mathbf{u} := \pi_{\hat{\gamma}}^*(\mathbf{z})$ and $\mathbf{w} := \pi_{\hat{\gamma}}^*(-m\mathbf{x} + n\mathbf{y})$. In particular, $\pi_{\hat{\gamma}}^*$ maps $\mathbf{y}$ to $a\mathbf{w}$ because $-am\mathbf{x} + an\mathbf{y} = \mathbf{y} - m(a\mathbf{x} + b\mathbf{y})$, and similarly $\pi_{\hat{\gamma}}^*(\mathbf{x}) = -b\mathbf{w}$, even though we do not use it.

The cup squares of $\mathbf{x}^2$, $\mathbf{y}^2$, $\mathbf{z}^2$ vanish because the S^2-factors of X have trivial self-intersection numbers when computed in some $S^2 \times S^2$-factor of X. Now $\pi_\eta^* e(\xi) = q\mathbf{y} + r\mathbf{z}$ implies $\pi_\eta^* p_1(\xi) = \pi_\eta^* e(\xi)^2 = (2qr)\mathbf{yz}$; hence

$$p_1(M_{\gamma,\eta}) = \pi_{\hat{\eta}}^* \pi_\gamma^* p_1(\xi) = \pi_{\hat{\gamma}}^* \pi_\eta^* p_1(\xi) = (2qra)\mathbf{wu}.$$

The basis $\mathbf{z}(a\mathbf{x} + b\mathbf{y})$, $\mathbf{z}(-m\mathbf{x} + n\mathbf{y})$, $\mathbf{xy} = (a\mathbf{x} + b\mathbf{y})(n\mathbf{y} + m\mathbf{x})$ in $H^4(S(\eta))$ is projected to 0, $\mathbf{wu}$, 0 by $\hat{\pi}_\gamma^*$; in particular, $\mathbf{wu}$ generates $H^4(M_{\gamma,\eta})$. It follows that for any fixed a, b by varying q, r, we get (by the topological invariance of rational Pontryagin classes) that the manifolds $M_{\gamma,\eta}$ lie in infinitely many homeomorphism types.

We show that the manifolds $D(\tilde{\zeta})$ lie in finitely many diffeomorphism types by applying Proposition 3.1 for $(\alpha,\beta)=(\hat{\zeta},\hat{\eta})$. To see it applies, note that $p_1(\zeta\oplus\eta)=p_1(\mu)+p_1(\xi)=e(\mu)^2+e(\xi)^2$, so $p_1(\hat{\zeta}\oplus\hat{\eta})=(-2qr+2qr)\mathbf{yz}=0$. It remains to check that $S(\hat{\eta})$, $S(\hat{\zeta})$ lie in finitely many fiber homotopy types.

If an oriented S^3-fibration over a finite complex Z has a section, which is true for $S(\hat{\eta})$, $S(\hat{\zeta})$, then it is classified by a map $[Z,BSG_4]$ that factors through BSF_3 [**42**, proposition 9.20(i)]. Since SF_3 is a component of Ω^3S^3, the space BSF_3 is rationally contractible, so $[Z,BSF_3]$ is finite. Thus, for all choices of parameters a,b,q,r the S^3-fibrations $S(\hat{\eta})$, $S(\hat{\zeta})$ lie in finitely many fiber homotopy types; in particular, $M_{\gamma,\eta}$ lie in finitely many homotopy types. q.e.d.

Remark 3.5. It is instructive to see why the argument at the end of the proof fails for oriented S^2-fibrations with a section: the classifying map in $[Z,BSG_3]$ only factors through BSF_2 and the inclusion $BSF_2\to BSG_3$ is rationally equivalent to $BSO_2\to BSO_3$ [**19**], while $[Z,BSO_2]\to[Z,BSO_3]$ has infinite image for $Z=S^2\times S^2$ corresponding to classifying maps in $[Z,BSO_2]$ of circle bundle with nonzero e and p_1. This is the reason we have to assume ζ has rank ≥ 4. Similarly, in Theorem 3.3 we assume ξ^n_m has rank $n\geq 4$ because S^2-bundles over S^4 with structure group SO_3 lie in infinitely many fiber homotopy classes; indeed, the inclusion $BSO_3\to BSG_3$ is a rational isomorphism [**19**], and $\pi_4(BSO_3)=\mathbb{Z}$.

Remark 3.6. In view of Remark 2.2, one wants to have a version of Theorem 3.4 for which the normal Euler class to the soul is nontrivial. As in [**25**], this is achieved by modifying the above proof to work for ζ equal to the Whitney sum of the line bundles over X with Euler classes $(0,-q,r)$ and $(0,c,c)$ where c,q,r are nonzero integers, c is fixed, and $r=q+1$. Indeed,

$$e(\hat{\zeta})=(-q\mathbf{y}+r\mathbf{z})(c\mathbf{y}+c\mathbf{z})=c(r-q)\mathbf{yz}=c\,\mathbf{yz},$$

so since the Euler class determines an oriented spherical fibration up to finite ambiguity, there are finitely many fiber homotopy possibilities for $S(\hat{\zeta})$. Now

$$p_1(\hat{\zeta}\oplus\hat{\eta})=p_1(\hat{\zeta})+p_1(\hat{\eta})=(2c-2qr+2qr)\mathbf{yz}=2c\,\mathbf{yz},$$

so $\pi^*_{\hat{\eta}}(p_1(\hat{\zeta}\oplus\hat{\eta}))$ is constant; hence $D(\tilde{\zeta})$ lie in finitely many diffeomorphism types. The rest of the proof is the same. Finally, note that the normal bundle to the soul has nonzero Euler class: $e(\tilde{\zeta})=\pi^*_{\hat{\eta}}(c\,\mathbf{yz})=ca\,\mathbf{wu}$.

Remark 3.7. More examples of manifolds with infinitely many souls can be obtained from Theorems 3.3–3.4 by taking products with suitable complete nonnegatively curved manifold L. The only point we have to check is that the souls in the product are pairwise non-homeomorphic, which is true, e.g., if the soul of L has trivial first Pontryagin class; then the souls in the product are not homeomorphic because their p_1's are different integer multiples of primitive elements, and this property is preserved under any isomorphism of their 4th cohomology groups.

Problem 3.8. Find a manifold N with an infinite sequence of complete metrics g_k of $\sec \geq 0$ satisfying one of the following:
(i) souls of (N, g_k) are pairwise non-diffeomorphic and have codimension ≤ 3;
(ii) souls of (N, g_k) are all diffeomorphic while the pairs $(N$, soul of $g_k)$ are pairwise non-diffeomorphic.

Added in proof: Problems 4.8(i) and 4.9 were solved by Sadeeb Ottenburger in [**39**] and [**40**], respectively.

Examples as in (ii), only without nonnegatively curved metrics, can be found in [**2**, appendix A].

Problem 3.9. Find a manifold N with two complete metrics of $\sec \geq 0$ whose souls S, S' are diffeomorphic and have codimension ≤ 3, while the pairs (N, S), (N, S') are not diffeomorphic.

4. Vector Bundles with Diffeomorphic Total Spaces

One of the things we are unable to do in this paper is construct a manifold that admits metrics with infinitely many nondiffeomorphic souls of codimension ≤ 3. To get an idea what such a manifold could look like, in this section we systematically study vector bundles with diffeomorphic total spaces, especially those of rank ≤ 3.

Throughout this section, N is the total space of vector bundles ξ, η over closed manifolds B_ξ, B_η, respectively. Composing the zero section of ξ with the projection of η gives a canonical homotopy equivalence $f_{\xi,\eta}\colon B_\xi \to B_\eta$.

The map $f_{\xi,\eta}$ pulls $TN|_{B_\eta}$ to $TN|_{B_\xi}$ because the projection of $N \to B_\eta \hookrightarrow N$ is homotopic to $\mathbf{id}(N)$.

Any homotopy equivalence of closed manifold preserves Stiefel-Whitney classes, as follows from their definition via Steenrod squares, so $f^*_{\xi,\eta} w(TB_\eta) = w(TB_\xi)$. Therefore, the Whitney sum formula implies that $f_{\xi,\eta}$ also pulls back the normal total Stiefel-Whitney class w, i.e. $f^*_{\xi,\eta} w(\eta) \cong w(\xi)$. In fact, Stiefel-Whitney classes of a vector bundle depend on the fiber homotopy type of its sphere bundle. To this end we show:

Proposition 4.1. *There is a fiber homotopy equivalence* $S(f^{\#}_{\xi,\eta}\eta) \cong S(\xi)$.

It follows that $f_{\xi,\eta}$ pulls back the normal Euler classes (with any local coefficients).

Proof of Proposition 4.1. Use some metric on the fibers to choose tubular neighborhoods $D_r(\eta)$, $D_\rho(\xi)$, $D_R(\eta)$ of the zero sections of η, ξ, η, respectively, such that $D_r(\eta) \Subset D_\rho(\xi) \Subset D_R(\eta)$. In the commutative diagram below, unlabeled arrows are either inclusions or sphere/disk bundle projections, p is the obvious retraction along radial lines, and $p(S_\rho(\xi)) \subset S_r(\eta)$ because of the above inclusions of disk bundles.

$$\begin{array}{ccccccccc} B_\xi & \longrightarrow & D_\rho(\xi) & \longrightarrow & D_R(\eta) & \xrightarrow[p]{} & D_r(\eta) & \longrightarrow & B_\eta \\ & \nwarrow & \uparrow & & & & \uparrow & \nearrow & \\ & & S_\rho(\xi) & & \xrightarrow{\quad p|_{S(\xi)} \quad} & & S_r(\eta) & & \end{array}$$

The composition of top arrows is $f_{\xi,\eta}$, which by commutativity is covered by $p|_{S_\rho(\xi)}$. By a criterion in [**10**, theorem 6.1], to show that $p|_{S_\rho(\xi)}$ induces a fiber homotopy equivalence of $S_\rho(\xi)$ and the pullback of $S_r(\eta)$ via $f_{\xi,\eta}$, it is enough to check that $p|_{S_\rho(\xi)}$ is a homotopy equivalence. Lemma 4.8 below implies that $W_R := D_R(\eta) \setminus \mathring{D}_\rho(\xi)$ and $W_r := D_\rho(\xi) \setminus \mathring{D}_r(\eta)$ are h-cobordisms with ends $S_R(\eta)$, $S_\rho(\xi)$ and $S_\rho(\xi)$, $S_r(\eta)$, respectively. Therefore, the inclusion of $S_\rho(\xi)$ into the trivial h-cobordism $W := W_R \cup W_r = D_R(\eta) \setminus \mathring{D}_r(\eta)$ is a homotopy equivalence, and so is $p|_W \colon W \to S_r(\eta)$; hence $p|_{W_r}$ defines a deformation retraction $D_\rho(\xi) \to D_r(\eta)$ that restricts to the homotopy equivalence $p|_{S_\rho(\xi)} \colon S_\rho(\xi) \to S_r(\eta)$. q.e.d.

Corollary 4.2. *If ξ has rank $i \in \{1,2\}$, then $f^{\#}_{\xi,\eta}\eta \cong \xi$, and $f_{\xi,\eta}$ is tangential.*

Proof. Since $O_i \to G_i$ is a homotopy equivalence, the fiber homotopy equivalence of $f^{\#}_{\xi,\eta}S(\eta)$ and $S(\xi)$ is induced by an isomorphism of $f^{\#}_{\xi,\eta}\eta \cong \xi$. Thus $\xi \oplus TB_\xi = TN|_{B_\xi} = f^{\#}_{\xi,\eta}TN|_{B_\eta} = f^{\#}_{\xi,\eta}(\eta \oplus TB_\eta) \cong \xi \oplus f^{\#}_{\xi,\eta}TB_\eta$. Subtracting ξ, we see that $f_{\xi,\eta}$ pulls back stable tangent bundles. q.e.d.

In codimension 3, all we can say is that $f_{\xi,\eta}$ pulls back rational Pontryagin classes of normal and tangent bundles; recall that a stable vector bundle is determined by its rational Pontryagin classes up to finite ambiguity.

Proposition 4.3. *If ξ has rank 3, and p denotes the rational total Pontryagin class, then $f^*_{\xi,\eta}p(\eta) \cong p(\xi)$ and $f^*_{\xi,\eta}p(TB_\eta) \cong p(TB_\xi)$.*

Proof. By Proposition 4.1, and Lemma 4.7 below, $f^*_{\xi,\eta}p_1(\eta) \cong p_1(\xi)$, while the higher Pontryagin classes vanish as $H^*(BSO_3;\mathbb{Q}) \cong \mathbb{Q}[p_1]$. Now $f^{\#}_{\xi,\eta}TN|_{B_\eta} \cong TN|_{B_\xi}$ and the Whitney sum formula gives $f^*_{\xi,\eta}p(TB_\eta) \cong p(TB_\xi)$. q.e.d.

Proposition 4.4. *If* $f^{\#}_{\xi,\eta}\eta \cong \xi$*, then* $f_{\xi,\eta}$ *has trivial normal invariant; in particular,* $f_{\xi,\eta}$ *is tangential.*

Proof. Use metrics on ξ, η to find their disk bundles that satisfy $D(\xi) \Supset D(\eta) \Supset B_\xi$. Lemma 4.8 below implies that $D(\xi) \setminus \mathring{D}(\eta)$ is an h-cobordism, so there exists a deformation retraction $r\colon D(\xi) \to D(\eta)$. Note that r has trivial normal invariant, because $D(\xi) \times I$ can be thought of as an h-cobordism with boundaries $D(\xi)$, $D(\eta)$ (cf. [**51**] before theorem 1.3), and moreover, the map $D(\xi) \times I \to D(\eta)$ given by composing the coordinate projection with r defines a normal bordism of r and $\mathbf{id}(D(\eta))$.

Since $f^{\#}_{\xi,\eta}\eta \cong \xi$, there is a diffeomorphism $h\colon D(f^{\#}_{\xi,\eta}\eta) \to D(\xi)$ that is identity on the base B_ξ. Let $\hat{f}_{\xi,\eta}\colon D(f^{\#}_{\xi,\eta}\eta) \to D(\eta)$ be the map of disk bundles induced by $f_{\xi,\eta}$. Next note that $r \circ h$, $\hat{f}_{\xi,\eta}$ are homotopic. Indeed, restricting both maps to B_ξ and postcomposing with the projection $p_\eta\colon D(\eta) \to B_\eta$ gives $f_{\xi,\eta}$, so $r \circ h$ and $\hat{f}_{\xi,\eta}$ glue along $B_\xi \times I$ to form a continuous map

$$F\colon (B_\xi \times I) \cup (D(\xi) \times \{0,1\}) \to B_\eta.$$

Since $D(\xi) \times I$ deformation retracts to the union of $B_\xi \times I$ and $D(\xi) \times \{0,1\}$, precomposing F with the retraction defines a homotopy of $p_\eta \circ r \circ h$ and $p_\eta \circ \hat{f}_{\xi,\eta}$, and hence a homotopy of $r \circ h$ and $\hat{f}_{\xi,\eta}$, because p_η is homotopic $\mathbf{id}(D(\eta))$.

Homotopic maps have equal normal invariants, so $\mathfrak{q}(\hat{f}_{\xi,\eta}) = \mathfrak{q}(r \circ h) = \mathfrak{q}(r)$ is trivial. Then Lemma 4.9 below implies that $\mathfrak{q}(f_{\xi,\eta})$ is trivial, because the zero section of $D(\eta)$ pulls $\mathfrak{q}(\hat{f}_{\xi,\eta})$ back to $\mathfrak{q}(f_{\xi,\eta})$. q.e.d.

Remark 4.5. Surgery theory implies (see, e.g., [**42**, theorem 13.2]) that if $f\colon N \to M$ is a homotopy equivalence of closed smooth simply-connected manifolds of dimension $n \geq 5$, then f has trivial normal invariant if and only if N is diffeomorphic to the connected sum of M and a homotopy sphere Σ^n and f is homotopic to the homeomorphism $N \cong M \# \Sigma^n \to M \# S^n \cong M$ where the middle map is the connected sum of $\mathbf{id}(M)$ with a homeomorphism $\Sigma^n \to S^n$. Thus Corollary 4.2 implies Theorem 1.9.

Remark 4.6. Proposition 4.4 is optimal for bundles of rank ≥ 3. Indeed, if a homotopy equivalence of closed manifolds $f\colon N \to M$ has trivial normal invariant, and if α is a vector bundle over M, then by Lemma 4.9 the induced map $\hat{f}\colon D(f^{\#}\alpha) \to D(\alpha)$ of disk bundles has trivial normal invariant. So by Wall's $\pi - \pi$-theorem, $\hat{f}$ is homotopic to a diffeomorphism, provided $\dim(D(\alpha)) \geq 6$ and the inclusion $S(\alpha) \to D(\alpha)$ is a π_1-isomorphism. The latter holds if the bundle α has rank ≥ 3. If the rank of α is 2, then things are a bit more complicated, and we have partial answers when M is simply-connected of dimension ≥ 5.

Namely, if α is trivial, and $\dim(M) \geq 5$, then $N \times \mathbb{R}^2$ is diffeomorphic to $M \times \mathbb{R}^2$ if and only if N is diffeomorphic to M (see Remark 5.12). If α is nontrivial, then $\pi_1(S(\alpha))$ is a finite cyclic group $\mathbb{Z}_d$, and a surgery-theoretic argument in [**4**, section 14] shows that $\hat{f}$ is homotopic to a diffeomorphism, except possibly when d is even and $\dim(M) \equiv 1 \bmod 4$.

The lemmas below are surely known, yet they do not seem to be recorded in the literature in the precise form we need.

Lemma 4.7. *For SO_3-bundles over finite complexes, the first rational Pontryagin class p_1 depends only on the fiber homotopy type of the associated 2-sphere bundles.*

Proof. Denote the natural inclusions $O_3 \subset G_3$ and $SO_3 \subset SG_3$ by j and j_1, respectively. The fiber homotopy invariance of p_1 will follow, once we show that p_1 lies in the image of $Bj^*\colon H^4(BG_3;\mathbb{Q}) \to H^4(BO_3;\mathbb{Q})$. Look at the map induced on rational cohomology by the commutative diagram, whose rows are projections of the 2-fold coverings corresponding to the first Stiefel-Whitney class.

$$\begin{array}{ccc} H^*(BSO_3;\mathbb{Q}) & \longleftarrow & H^*(BO_3;\mathbb{Q}) \\ \uparrow{\scriptstyle Bj_1^*} & & {\scriptstyle Bj^*}\uparrow \\ H^*(BSG_3;\mathbb{Q}) & \longleftarrow & H^*(BG_3;\mathbb{Q}) \end{array}$$

The horizontal arrows are induced by covering projections; hence by a standard argument they are monomorphisms onto the subspace fixed by the covering involution. Now Bj_1^* is an isomorphism [**19**] which is $\mathbb{Z}_2$-equivariant because Bj_1 is $\mathbb{Z}_2$-equivariant. So Bj^* is an isomorphism as well. q.e.d.

Lemma 4.8. *Suppose a manifold N is the total space of two vector bundles over closed manifolds M_1, M_2. If the normal sphere bundles to M_1, M_2 are chosen to be disjoint in N, then the region between these sphere bundles is an h-cobordism.*

Proof. Denote by $S_k(r)$, $D_k(r)$ the normal r-sphere, r-disk bundles determined by some metric on the normal bundle to M_k; denote by p_k the line bundle projection $N \setminus M_k \to S_k(r)$. Since $D_k(r)$ exhaust N, there are positive numbers $r < t < R < T$ such that $D_1(r) \Subset D_2(t) \Subset D_1(R) \Subset D_2(T)$. We are to show that $W := D_1(R) \setminus \mathring{D}_2(t)$ is an h-cobordism.

To see that $S_2(t) \hookrightarrow W$ is a homotopy equivalence, it suffices to note that $N \setminus \mathring{D}_2(t)$ deformation retracts both to W and to $S_2(t)$ along the fibers of p_1, p_2, respectively.

To show that $S_1(R) \hookrightarrow W$ is a homotopy equivalence, we first observe that $S_1(R) \hookrightarrow W$ is π_1-injective, for if a loop in $S_1(R)$ is null-homotopic in W, then it would be null-homotopic in the larger region $D_1(R) \setminus \mathring{D}_1(r)$

which deformation retracts to $S_1(R)$ along the fibers of p_1, so the null-homotopy can be pushed to $S_1(R)$.

To see that $S_1(R) \hookrightarrow W$ is π_1-surjective, start with an arbitrary loop α in W, and since W lies in $D_2(T) \setminus \mathring{D}_2(t)$ which deformation retracts to $S_2(T)$ along the fibers of p_2, the loop α can be homotoped inside $D_2(T) \setminus \mathring{D}_2(t)$ to some β in $S_2(T)$. Since $N \setminus \mathring{D}_1(R)$ deformation retracts to $S_1(R)$, the homotopy can be pushed to W, where β gets mapped into $S_1(R)$. Thus α is homotopic in W to a loop in $S_1(R)$, as claimed.

Since $S_2(t) \hookrightarrow W$ is a homotopy equivalence, the pair $(W, S_2(t))$ has trivial cohomology for any system of local coefficients on W; hence by Poincaré Duality, the pair $(W, S_1(R))$ has trivial homology for any system of local coefficients on W. So by the non-simply-connected version of Whitehead's theorem, which is applicable since $S_1(R) \hookrightarrow W$ induces a π_1-isomorphism, we conclude that $S_1(R) \hookrightarrow W$ is a homotopy equivalence. q.e.d.

Lemma 4.9. *For a homotopy equivalence $f\colon N \to M$ of closed smooth manifolds, and a vector bundle α over M with projection p, let $\hat{f}\colon \hat{N} \to \hat{M}$ be the induced map of disk bundles $\hat{N} := D(f^{\#}\alpha)$, $\hat{M} := D(\alpha)$. Then the normal invariants of f and $\hat{f}$ satisfy $\mathfrak{q}(\hat{f}) = p^{*}\mathfrak{q}(f)$.*

Proof. Let g be a homotopy inverse of f, and denote by $\hat{g}\colon \hat{M} \to \hat{N}$ the corresponding map of disk bundles. Fix a large m such that g postcomposed with the inclusion $N \to N \times \mathbb{R}^m$ is homotopic to a smooth embedding $e\colon M \to N \times \mathbb{R}^m$, where we may assume that its image is disjoint from $N \times \{0\}$; let ν_e denote the normal bundle of e. By [**45**, theorem 2.2], the complement of a tubular neighborhood of $e(M)$ is an open collar; hence $N \times \mathbb{R}^m$ can be identified with the total space of ν_e. By the proof of Proposition 4.1, there are disjoint normal sphere bundles $N \times S^{m-1}$, $S(\nu_e)$ to $N \times \{0\}$, $e(M)$, respectively, such that the region between them is an h-cobordism, and the radial projection $N \times \mathbb{R}^m \setminus \{0\} \to N \times S^{m-1}$ restricted to $S(\nu_e)$ is a fiber homotopy equivalence. Projecting on the S^{m-1}-factor gives a fiber homotopy trivialization $t\colon S(\nu_e) \to S^{m-1}$. As indicated in [**51**] (after lemma 10.6), the pair (ν_e, t) represents $\mathfrak{q}(f)$, and moreover, there is a relative version of the above argument which can be applied to the embedding of disk bundles $\hat{e}\colon \hat{M} \to \hat{N} \times \mathbb{R}^m$ obtained as the pullback of e. Again, the radial projection restricts to a fiber homotopy equivalence $S(\nu_{\hat{e}}) \to \hat{N} \times S^{m-1}$ of normal sphere bundles, which gives rise to a fiber homotopy trivialization $\hat{t}\colon S(\nu_{\hat{e}}) \to S^{m-1}$ such that $(\nu_{\hat{e}}, \hat{t})$ represents $\mathfrak{q}(\hat{f})$.

That $p^{\#}\nu_e$ is stably isomorphic to $\nu_{\hat{e}}$ follows by a straightforward computation showing that $g^{\#}\nu_{N} \oplus \tau_{M} \oplus \nu_e$ and $\hat{g}^{\#}\nu_{\hat{N}} \oplus \tau_{\hat{M}} \oplus \nu_{\hat{e}}$ are stably trivial, and that p pulls $g^{\#}\nu_{N} \oplus \tau_{M}$ back to $\hat{g}^{\#}\nu_{\hat{N}} \oplus \tau_{\hat{M}}$, where ν_{X}, τ_{X} denote the stable normal and tangent bundles of X. That $\hat{t}$ is homotopic to $t\circ p$ follows because by construction $t = \hat{t}|_{S(\nu_e)}$ and p is a deformation retraction. q.e.d.

5. Finiteness Results for Vector Bundles of Rank ≤ 3

In this section we prove Theorem 1.3 and other related results. Let $\{B_\eta\}$ be the set of closed manifolds such that N is the total space of a vector bundle η of rank ≤ 3 over some B_η. All B_η's are homotopy equivalent to N and hence to each other, so $n := \dim(B_\eta)$ is constant; thus all η's have rank equal to $\dim(N) - n$.

Proposition 5.1. *$\{B_\eta\}$ can be partitioned into finitely many subsets such that if B_η, B_ξ lie in the same subset, then there is a tangential homotopy equivalence $g_{\xi,\eta}\colon B_\xi \to B_\eta$ such that $g^*_{\xi,\eta}\eta \cong \xi$, and furthermore, $g_{\xi,\eta}$ has trivial normal invariant in $[B_\eta, F/O]$.*

Proof. Fix one base manifold $B_{\eta_0} \in \{B_\eta\}$ with normal bundle η_0 of rank 3, and pullback each η to B_{η_0} via $f_{\eta_0,\eta}$. Since $f^*_{\xi,\eta}p(\eta) \cong p(\eta)$, and since an $O(k)$-bundle is determined by its rational Pontryagin classes up to finite ambiguity, there are finitely many possibilities for $f^{\#}_{\eta_0,\eta}\eta$. Each possibility corresponds to a subset in the partition, for if $f^{\#}_{\eta_0,\eta}\eta = f^{\#}_{\eta_0,\xi}\xi$, then $h_{\xi,\eta} := f^{-1}_{\eta_0,\xi}\circ f_{\eta_0,\eta}$ pulls back η to ξ, and also is tangential as it preserves TN restricted to the zero sections. If the rank is ≤ 2, then by Corollary 4.2, the partition consists of one subset $\{B_\eta\}$; for the sake of uniformity, we set $h_{\xi,\eta} := f_{\xi,\eta}$.

Now fix an arbitrary B_η in a subset S of the partition. The homotopy $h_{\xi,\eta}$ equivalences represent elements $(h_{\xi,\eta}, B_\xi)$ in the structure set $\mathbf{S}^h(B_\eta)$ of homotopy structures. Since $h_{\xi,\eta}$ are tangential, their normal invariants are in the image of $[B_\eta, F] \to [B_\eta, F/O]$, which is finite, so by partitioning S further (into finitely many subsets), we get a subset $\{B_{\xi_i}\}$ such that all $f_i := h_{\xi_i,\eta}$ have equal normal invariants. It is for this subset we find the homotopy equivalencies with desired properties. Fix any $\xi_0 \in \{\xi_i\}$. If g_0 denotes the homotopy inverse of f_0, then by the composition formula for normal invariants, recalled in [**4**, section 3], we get $\mathfrak{q}(g_0\circ f_i) = f_0^*\mathfrak{q}(f_i) + \mathfrak{q}(g_0)$, and $0 = \mathfrak{q}(g_0\circ f_0) = f_0^*\mathfrak{q}(f_0) + \mathfrak{q}(g_0)$. So $\mathfrak{q}(g_0\circ f_i) = f_0^*(\mathfrak{q}(f_i) - \mathfrak{q}(f_0)) = 0$ as $\mathfrak{q}(f_i) = \mathfrak{q}(f_0)$ for all i. Now $g_{\xi,\eta} := g_0\circ f_i = h^{-1}_{\xi_0,\eta}\circ h_{\xi_i,\eta}$ is a tangential homotopy equivalence from B_{ξ_i} to B_{ξ_0} that pulls back ξ_i to ξ_0, and has trivial normal invariant. q.e.d.

Theorem 5.2. *If $n = \dim(B_\eta) \geq 5$, then the pairs (N, B_η) lie in finitely many diffeomorphism types if either* (i) *B_ϵ is simply-connected, or* (ii) *B_η is orientable, n is even, and $G := \pi_1(B_\eta)$ is finite.*

Proof. Fix B_η in a subset of the partition given by Proposition 5.1.

(i) Since B_η is simply-connected, each homotopy equivalence $g_{\xi,\eta}$ is simple. By Proposition 5.1 and exactness of the surgery sequence, the element represented by $g_{\xi,\eta}$ in the simple structure set $\mathbf{S}^s(B_\eta)$ lies in the orbit of $L^s_{n+1}(1)$ of the identity, which is finite because $L^s_{n+1}(1)$-action factors through the finite group bP_{n+1}. If $\{B_\eta\}$ is infinite, which is the only interesting case, then there is a subsequence in which all $g_{\xi,\eta}$ represent the same element in the structure set. So there are diffeomorphisms $\phi_{\xi,\zeta}$ such that $g_{\zeta,\eta}\circ\phi_{\xi,\zeta}$ is homotopic to $g_{\xi,\eta}$. But diffeomorphisms pull back normal bundles, so $\phi_{\xi,\zeta}$ induces a diffeomorphism $(N, B_\xi) \to (N, B_\eta)$.

(ii) The homotopy equivalence $g_{\xi,\eta}$ represents an element in the structure set $\mathbf{S}^h(B_\eta)$ that, because of exactness of the surgery sequence and Proposition 5.1, lies in the orbit of $L^h_{n+1}(G)$ of the identity. Since n is even and G is finite, $L^h_{n+1}(G)$ is a finite group (see, e.g., [**22**]). If $\{B_\eta\}$ is infinite, then there is a subsequence in which all $g_{\xi,\eta}$ represent the same element $\mathbf{S}^h(B_\eta)$, and hence they are pairwise h-cobordant. In particular, there are homotopy equivalences $\phi_{\xi,\zeta}$ that identify the boundaries B_ξ and B_ζ of the h-cobordism $W_{\xi,\zeta}$ such that $g_{\zeta,\eta}\circ\phi_{\xi,\zeta}$ is homotopic to $g_{\xi,\eta}$. Fix orientations on all B_η that are preserved by $f_{\zeta,\eta}$; then by a well-known computation (recalled in Lemma 7.1), the torsion $\tau(\phi_{\xi,\zeta})$ is of the form $(-1)^n\sigma^* - \sigma \in \mathrm{Wh}(G)$ where $\sigma = \tau(W_{\xi,\zeta}, B_\zeta)$, the torsion of the pair $(W_{\xi,\zeta}, B_\zeta)$. By a result of Wall [**38**, 7.4, 7.5], finiteness of G implies that the standard involution $*$ acts trivially on the quotient of $\mathrm{Wh}(G)$ by its maximal torsion subgroup $\mathrm{S}K_1(\mathbb{Z}G)$, which is finite, so $\sigma^* - \sigma$ lies in $\mathrm{S}K_1(\mathbb{Z}G)$, and hence passing to a subsequence we may assume that $\tau(\phi_{\xi,\zeta})$ is constant.

Fix ζ and vary ξ; i.e. let $\xi = \xi_i$. By the composition formula for torsion, we see that $(\phi_{\xi_0,\zeta})^{-1}\circ\phi_{\xi_i,\zeta}\colon B_{\xi_i} \to B_{\xi_0}$ has trivial torsion, which by the above-mentioned computation equals the torsion of the h-cobordism W_{ξ,ξ_0} obtained by concatenating the corresponding h-cobordisms $W_{\xi,\zeta}\cup W_{\xi_0,\zeta}$; thus W_{ξ,ξ_0} is trivial by the s-cobordism theorem. It follows that $(\phi_{\xi_0,\zeta})^{-1}\circ\phi_{\xi_i,\zeta}$ is homotopic to a diffeomorphism, which then pulls back normal bundles, and hence induces a diffeomorphism $(N, B_{\xi_i}) \to (N, B_{\xi_0})$. q.e.d.

Example 5.3. Part (ii) fails for n odd (even though it is unclear how to realize any of the following examples in the nonnegative curvature setting):

(1) If G is finite cyclic of order 5 or of order ≥ 7, and M is a homotopy lens space with fundamental group G and dimension ≥ 5, then the h-cobordism class of M contains infinitely many non-homeomorphic manifolds [**34**, corollary 12.9] distinguished by Reidemeister torsion. Thus these manifolds are not even simply homotopy equivalent, while

their products with $\mathbb{R}$ are diffeomorphic. Also see [**27**] for a similar result for fake spherical space forms.

(2) By a result of López de Medrano [**30**], there are infinitely many homotopy $\mathbb{RP}^{4k-1}$'s with $k > 1$ distinguished by Browder-Livesay invariants, and such that their canonical line bundles are diffeomorphic to the canonical line bundle over the standard $\mathbb{RP}^{4k-1}$.

(3) More generally, Chang-Weinberger showed in [**9**] that for any compact oriented smooth $(4r-1)$-manifold M, with $r \geq 2$ and $\pi_1(M)$ not torsion-free, there exist infinitely many pairwise non-homeomorphic closed smooth manifolds M_i that are simple homotopy equivalent and tangentially homotopy equivalent to M. As in the proof of Proposition 5.7 below, we see that the manifolds $M_i \times D^3$ lie in finitely many diffeomorphism types.

Remark 5.4. If the closed manifolds in (1)–(3) admit metrics of $\sec \geq 0$, then they can be realized as souls of codimension 3 with trivial normal bundle because they lie in finitely many tangential homotopy types so Proposition 5.7 applies; in fact, examples in (1)–(2) could then be realized as codimension 1 souls because any real line bundle over a closed nonnegatively curved manifold admits a complete metric of $\sec \geq 0$ with zero section being a soul.

Remark 5.5. It is an (obvious) implication of Theorem 5.2(ii) that examples (1)–(3) disappear after multiplying by a suitable closed manifold; i.e. if B_η, B are orientable, odd-dimensional, closed manifolds with finite fundamental groups, then the pairs $\{(N \times B, B_\eta \times B)\}$ lie in finitely many diffeomorphism types.

Remark 5.6. It seems the assumption in (ii) that B_η is orientable could be removed by working with the proper surgery exact sequence, but we choose not to do this here.

The case of trivial normal bundles deserves special attention, e.g. because the total space always admits a metric of $\sec \geq 0$, provided the base does.

Proposition 5.7. *Suppose that $\{M_i\}$ is a sequence of closed manifolds of dimension ≥ 5. Then $\{M_i\}$ lies in finitely many tangential homotopy types if and only if $\{M_i \times \mathbb{R}^3\}$ lies in finitely many diffeomorphism types.*

Proof. The "if" direction follows from Proposition 5.1, so we focus on the other direction. As in the proof of Proposition 5.1, after passing to a subsequence we may assume there are homotopy equivalences $h_i \colon M_i \to M_0$ with trivial normal invariants. Then $H_i := h_i \times \mathbf{id}(D^3)$ is normally cobordant to the identity via the product of D^3 with the normal cobordism from h_i to the identity.

The homotopy equivalence H_i need not be simple, so we replace it with a simple homotopy equivalence as follows. Attaching a suitable h-cobordism on the boundary of $M_i \times D^3$ turns $M_i \times D^3$ into a manifold Q_i with $\mathring{Q}_i = M_i \times \mathbb{R}^3$, and precomposing H_i with a deformation retraction $Q_i \to M_i \times D^3$ yields a simple homotopy structure equivalence $F_i \colon Q_i \to N_0 \times D^3$. (Indeed, if W is a cobordism with a boundary component M, then $\tau(W, M) = -\tau(r)$ where $r\colon W \to M$ is a deformation retraction. By the composition formula for torsion [**7**], $\tau(f \circ r) = \tau(f) + f_*\tau(r)$, so we need to find W with $\tau(f) = f_*\tau(W, M)$, or equivalently, since f_* is an isomorphism we need $(f_*)^{-1}\tau(f) = \tau(W, M)$, which can be arranged as any element in $\mathrm{Wh}(\pi_1(M)$ can be realized as $\tau(W, M)$ for some W.)

Note that F_i still has trivial normal invariant, because $Q_i \times I$ can be thought of as an h-cobordism with boundaries Q_i, $M_i \times D^3$, so the maps F_i, H_i are normally cobordant (as explained in [**51**] before theorem 1.3). By Wall's $\pi - \pi$-theorem, Q_i is diffeomorphic to $M_0 \times D^3$. Restricting to interiors gives a desired diffeomorphism $M_i \times \mathbb{R}^3 \to M_0 \times \mathbb{R}^3$. q.e.d.

Remark 5.8. The proof shows that if a homotopy equivalence has trivial normal invariant, then its product with $\mathbf{id}(D^3)$ is homotopic to a diffeomorphism; e.g., this applies to the standard homeomorphism $\Sigma^k \to S^k$ where Σ_k is a homotopy sphere that bounds a parallelizable manifold, so that $\Sigma^k \times\ D^3$ and $S^k \times D^3$ are diffeomorphic.

Remark 5.9. Propositions 5.1, 5.7 imply that if there exists a manifold N with infinitely many codimension ≤ 3 souls S_i, then after passing to a subsequence $S_i \times \mathbb{R}^3$ are all diffeomorphic, so one also gets infinitely many codimension 3 souls with trivial normal bundles, which proves part (2) of Theorem 1.3. By contrast, Proposition 5.10 below shows that an analogous statement fails in codimension 2: indeed, for $r > 1$ there are infinitely many homotopy $\mathbb{RP}^{4r-1}$ with diffeomorphic canonical line bundles; hence the products of the line bundles with $\mathbb{R}$ are also diffeomorphic, yet no two non-diffeomorphic homotopy $\mathbb{RP}^{4r-1}$ are h-cobordant because $\mathrm{Wh}(\mathbb{Z}_2) = 0$.

Proposition 5.10. *Suppose $\{M_i\}$ is a sequence of closed manifolds of dimension ≥ 5 with finite fundamental group G. Then $\{M_i\}$ lies in finitely many h-cobordism types if and only if $\{M_i \times \mathbb{R}^2\}$ lies in finitely many diffeomorphism types.*

Proof. The "only if" direction follows because if M_i, M_j are h-cobordant, then their products with $\mathbb{R}$ are diffeomorphic (e.g., by the weak h-cobordism theorem). For the "if" direction, we pass to a subsequence so that all $M_i \times \mathbb{R}^2$ are diffeomorphic.

By Lemma 4.8, each $M_i \times S^1$ is h-cobordant to $M_0 \times S^1$, and the proof of Proposition 4.1 shows that the fiber homotopy equivalence $M_i \times S^1 \to M_0 \times S^1$ covers the canonical homotopy equivalence $M_i \to M_0$, so the circle factor is preserved up to homotopy. Passing to the infinite cyclic cover corresponding to the circle factor, we get a proper h-cobordism between $M_i \times \mathbb{R}$ and $M_0 \times \mathbb{R}$. The proper h-cobordisms with one end $M_0 \times \mathbb{R}$ are classified by $\tilde{K}_0(\mathbb{Z}G)$ [**46**]. The group G is finite, and hence so is $\tilde{K}_0(\mathbb{Z}G)$ [**49**]. Thus $M_i \times \mathbb{R}$ fall in finitely many diffeomorphism classes, and hence $\{M_i\}$ is finite up to h-cobordism because closed manifolds are h-cobordant if their products with $\mathbb{R}$ are diffeomorphic (this is well known and follows from Lemma 4.8). q.e.d.

Remark 5.11. Milnor noted in [**34**, theorem 11.5] that if B is a closed orientable manifold with finite fundamental group and even dimension ≥ 5, then the h-cobordism class of B contains only finitely many diffeomorphism classes.

Remark 5.12. It is well known that a closed simply-connected manifold of dimension ≥ 5 can be recovered from its product with $\mathbb{R}^2$. The proof of Proposition 5.10 immediately gives a slight generalization: if M_0, M_1 are closed n-manifolds with $n \geq 5$ and $\pi_1(M_0) = G = \pi_1(M_1)$ such that $M_1 \times \mathbb{R}^2$ and $M_0 \times \mathbb{R}^2$ are diffeomorphic, and $\mathrm{Wh}(G) = 0 = \tilde{K}_0(\mathbb{Z}G)$, then M_1 and M_0 are diffeomorphic. This applies, e.g., if G is $\mathbb{Z}_n$ for $n = 2, 3, 4, 6$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, D_{2m} for $m = 3, 4, 6$, as well as if G is torsion-free and virtually abelian (see references in [**32**]).

6. Smooth Knots and Disconnectedness of Moduli Spaces

In this section we use results of Haefliger and Levine on smooth knots to investigate how many components of $\mathfrak{R}^u_{\mathrm{sec}\geq 0}(N)$ can be visited by the $\mathrm{Diff}(N)$-orbit of a given metric, and to give an example of disconnected $\mathfrak{M}^u_{\mathrm{sec}\geq 0}(N)$.

We start by recalling some results on smooth knots. According to Haefliger [**17**, **18**], for $n \geq 5$ and $k \geq 3$, isotopy classes of (smooth) embeddings of S^n into S^{n+k} form an abelian group $\Sigma^{n+k,n}$ under connected sum, which vanishes in metastable range (i.e. for $2k > n+3$), and equals to $\mathbb{Z}$ for $n = 4r-1$ and $k = 2r+1$. In general, Levine [**31**] showed that $\Sigma^{n+k,n}$ is either finite, or virtually cyclic, and the latter occurs if and only if $n = 4r-1$ and $3 \leq k \leq 2r+1$.

For $k \geq 3$, Hirsch [**21**, theorem 8] showed that any smooth embedding of S^n into S^{n+k} can be ambiently isotoped to have a closed tubular neighborhood equal to the closed tubular neighborhood $S^n \times D^k$ of a standard $S^n \subset S^{n+k}$. Of course, different elements of $\Sigma^{n+k,n}$ define non-isotopic embeddings of S^n into $S^n \times \mathrm{Int}(D^k)$, because any isotopy of S^n inside $S^n \times \mathrm{Int}(D^k)$ is also an isotopy in S^{n+k}.

Levine [**31**] showed that assigning to each element of $\Sigma^{n+k,n}$ the isomorphism class of its normal bundle defines a homomorphism $\Sigma^{n+k,n} \to \pi_n(BSO_k)$. The image $N_0(n,k)$ of this homomorphism is described in [**31**, theorem 6.9] as the image of the kernel of the stabilization homomorphism $\pi_n(SG_k, SO_k) \to \pi_n(SG, SO)$ under the boundary map $\pi_n(SG_k, SO_k) \to \pi_{n-1}(SO_k)$ in the long exact sequence of the pair.

The group $N_0(n,k)$ is finite for any k, n. Indeed, since tubular neighborhoods of the embeddings $e_i\colon S^n \to S^{n+k}$ can be chosen to equal the same $S^n \times D^k$, results of Section 4 imply that the canonical homotopy equivalence $e_i(S^n) \to S^n \times \{0\}$ pulls back the normal Euler class, which is trivial. Also, any homotopy equivalence $e_i(S^n) \to S^n \times \{0\}$ pulls back the stable normal bundles because they are trivial. It follows that the normal bundle to e_i has trivial Euler and Pontryagin classes, which determine an oriented vector bundle up to finite ambiguity.

Theorem 6.1. *Let g be any complete metric of* $\sec \geq 0$ *on $N := S^n \times \mathbb{R}^k$ with soul $S \times \{0\}$. If $r \geq 2$ is an integer and $n = 4r-1$ and $3 \leq k \leq 2r+1$, then metrics that are isometric to g lie in infinitely many components $\mathfrak{R}^u_{\sec\geq 0}(N)$ and in the same component of $\mathfrak{R}^u(N)$.*

Proof. As $\Sigma^{n+k,n}$ is infinite, and the above-mentioned homomorphism $\Sigma^{n+k,n} \to \pi_n(BSO_k)$ has finite image, its kernel contains infinitely many isotopy classes of embeddings of S^n into S^{n+k} with trivial normal bundle. By the above-mentioned result in [**21**], we ambiently isotope the embeddings into infinitely many pairwise non-isotopic embeddings from S^n to $S^n \times \mathrm{Int}(D^k)$, which is a closed tubular neighborhood for all the zero sections. Equipping their normal bundles with the metric g, we get infinitely many metrics on $S^n \times \mathbb{R}^k$ with pairwise non-isotopic souls, and the metrics lie in different components of $\mathfrak{R}^u_{\sec\geq 0}(S^n \times \mathbb{R}^k)$ by Lemma 2.1, and modifying the metrics as in Proposition 2.11, they can be arranged to lie in the same component of $\mathfrak{R}^u(N)$. q.e.d.

By convention we treat S^1, $\mathbb{R}$, and a point as having $\sec \geq 0$.

Theorem 6.2. *If L is any closed manifold of* $\sec \geq 0$*, then the moduli space $\mathfrak{M}^u_{\sec\geq 0}(S^7 \times \mathbb{R}^4 \times L)$ has more than one component with metrics whose souls are diffeomorphic to $S^7 \times L$, and which lie in the same component of $\mathfrak{M}^u(N)$.*

Proof. A key ingredient of the proof is that the group $N_0(7,4)$ is nontrivial. In table 7.2 of [**31**], Levine stated (without proof) that $N_0(7,4)$ has order 4; for completeness, we justify Levine's assertion using results in [**18**] and [**50**].

Since $\pi_7(SG, SO)$ vanishes, $N_0(7,4)$ is the image of $\pi_7(SG_4, SO_4) \to \pi_6(SO_4)$, or equivalently, the kernel of $\pi_6(SO_4) \to \pi_6(SG_4)$. To compute the latter, look at the following commutative diagram in which the rows are exact sequences of the fibrations, and vertical arrows are induced by inclusions.

$$\begin{array}{ccccccc}
\pi_7(S^3) & \xrightarrow[\text{zero}]{} & \pi_6(SO_3) & \xrightarrow[1-1]{} & \pi_6(SO_4) & \xrightarrow[\text{splits}]{} & \pi_6(S^3) \\
\Vert & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \Vert \\
\pi_7(S^3) & \xrightarrow[\text{zero}]{} & \pi_6(SF_3) & \xrightarrow[1-1]{} & \pi_6(SG_4) & \xrightarrow[\text{splits}]{} & \pi_6(S^3)
\end{array}$$

The principal SO_3-bundle $SO_4 \to S^3$ is trivial as $\pi_2(SO_3) = 0$. It follows that the fibration $SG_4 \to S^3$ has a section, which explains how the horizontal arrows are labeled. Furthermore, i is the sum of j and the identify of $\pi_6(S^3)$; in particular, i, j have isomorphic kernels and cokernels, and again, the kernels are isomorphic to $N_0(n,k)$. The groups $\pi_6(SO_3)$, $\pi_6(SF_3) = \pi_9(S^3)$ equal to $\mathbb{Z}_{12}$, $\mathbb{Z}_3$, respectively, so i is either trivial, or onto. In the latter case, the kernel of j is $\mathbb{Z}_4$ as desired, so it remains to show that i cannot be trivial, or equivalently, that the cokernel of j is not $\mathbb{Z}_3$. The cokernel of j lies in $\pi_6(SG_4, SO_4)$, which fits in an exact sequence of homotopy groups of the triad (see [**18**, 4.11])

$$\pi_7(SG; SO, SG_4) \to \pi_6(SG_4, SO_4) \to \pi_6(SG, SO).$$

Now $\pi_6(SG, SO) = \mathbb{Z}_2$ and by [**18**, theorem 8.15] $\pi_7(SG; SO, SG_4) = 0$, which means that $\pi_6(SG_4, SO_4)$ has no subgroup isomorphic to $\mathbb{Z}_3$, as promised.

Actually, Haefliger omits the proof that $\pi_7(SG; SO, SG_4) = 0$, so we fill in the details. Consider the exact sequence given by [**18**, theorem 6.4]:

$$\pi_7(SF_4, SG_4) \to \pi_7(SG; SO, SG_4) \to \pi_7(SG; SO, SG_5) \to \pi_6(SF_4, SG_4).$$

Here $\pi_7(SG; SO, SG_5) = 0$ by [**18**, corollary 6.6], so it remains to see that $\pi_7(SF_4, SG_4) = 0$. To this end, consider the exact sequence of the pair:

$$\pi_7(G_4) \to \pi_7(F_4) \to \pi_7(F_4, G_4) \to \pi_6(G_4) \to \pi_6(F_4) \to \pi_6(F_4, G_4).$$

As mentioned above, the fibration $G_4 \to S^3$ has a section so $\pi_i(G_4)$ splits as $\pi_i(F_3) \oplus \pi_i(S^3) = \pi_{i+3}(S^3) \oplus \pi_i(S^3)$. In particular, $\pi_6(G_4) = \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$ and also $\pi_6(F_4) = \pi_{10}(S^4) = \mathbb{Z}_3 \oplus \mathbb{Z}_{24}$. By [**18**, theorem 8.11], $\pi_6(F_4, G_4) \cong \pi_3(SO, SO_3)$ which equals to $\mathbb{Z}_2$, so exactness at $\pi_6(F_4)$ implies that $\pi_6(G_4) \to \pi_6(F_4)$ is one-to-one. On the other hand, the inclusion $k\colon F_3 \to F_4$ factors through G_3, so it suffices to show that $k_*\colon \pi_7(F_3) \to \pi_7(F_4)$ is onto. As $\pi_7(F_3) = \pi_{10}(S^3) = \mathbb{Z}_{15} \cong \pi_{11}(S^4) = \pi_7(F_4)$, it is enough to see that k_* is one-to-one. In fact, the inclusion $F_3 \to F$, which factors through F_4, is an isomorphism on π_7, because with the above identifications it corresponds to the iterated suspension homomorphism $\pi_{10}(S^3) \to \pi_7^{\mathbf{S}}$, and the latter homomorphism is an isomorphism at primes 3, 5 as shown in [**50**, page 177].

Thus $N_0(7,4) \cong \mathbb{Z}_4$, and hence there are 4 different oriented vector bundles over S^7 with total space diffeomorphic to $S^7 \times \mathbb{R}^4$.

A result of Grove-Ziller [**16**] implies that their total spaces admit complete metrics of $\sec \geq 0$ with souls equal to the zero sections, because by the discussion preceding corollary 3.13 of [**16**], all vector bundles in $\pi_7(BSO_4) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$ classified by elements of orders 1, 2, 4 admit complete metrics with $\sec \geq 0$ and souls equal to the zero sections, which includes all elements of $N_0(7,4)$. In particular, there exists a nontrivial bundle ξ with these properties.

Thus $S^7 \times L \times \mathbb{R}^4$ is the total space of the vector bundle $p^{\#}\xi$, where $p\colon S^7 \times L \to S^7$ is the projection on the first factor. The bundle $p^{\#}\xi$ is nontrivial because its pullback via an inclusion $i\colon S^7 \to S^7 \times L$ is ξ, which is nontrivial. If some self-diffeomorphism of $S^7 \times L \times \mathbb{R}^4$ could take $S^7 \times L \times \{0\}$ to the zero section of $p^{\#}\xi$, then since trivial bundles are preserved by pullback, it would follow that $p^{\#}\xi$ is trivial. Hence by Lemma 2.1, the two metrics lie in different components of the moduli space. Modifying the metrics as in Proposition 2.11, they can be arranged to lie in the same component of $\mathfrak{R}^u(N)$. q.e.d.

Remark 6.3. The above proof shows that $N_0(7,4)$ lies in $\pi_6(SO_3)$-factor of $\pi_6(SO_4) \cong \pi_3(SO_3) \oplus \pi_6(S^3)$, so we can explicitly write $N_0(7,4) = \{0,3,6,9\} \subset \mathbb{Z}_{12} = \pi_7(BSO_3)$. Pulling back the bundle represented by $m \in \pi_7(BSO_3)$ by a degree -1 self-map of S^4 yields the bundle represented by $-m$, so up to the action of homotopy self-equivalences on the base S^7, we get only 3 different bundles, namely $0, 3, 6$. It follows that $\mathfrak{M}^u_{\sec\geq 0}(S^7 \times \mathbb{R}^4)$ has at least 3 components with metrics whose souls are diffeomorphic to S^7. Note that since all homotopy 7-spheres become diffeomorphic after multiplying by $\mathbb{R}^k$ with $k \geq 3$, the moduli space $\mathfrak{M}^u_{\sec\geq 0}(S^7 \times \mathbb{R}^4)$ also has other components with souls diffeomorphic to those homotopy 7-spheres that admit metrics of $\sec \geq 0$.

7. Spaces of Metrics and h-cobordisms

In this section we study components of $\mathfrak{R}^u_{\sec\geq 0}(N)$ via various techniques related to h-cobordisms.

Here is a basic idea. If W is an h-cobordism of dimension ≥ 5 with boundary components M, M', then by the weak h-cobordism theorem there is a diffeomorphism from $\mathring{W} := \mathrm{Int}(W)$ onto $M \times \mathbb{R}$, taking M to $M \times \{0\}$, and similarly for M'. In particular, if M, M' admit metrics with $\sec \geq 0$, then $\mathring{W}$ has two complete metrics of $\sec \geq 0$ with souls isometric to M, M'. Therefore, if M, M' are not diffeomorphic, then $\mathfrak{M}^u_{\sec\geq 0}(\mathring{W})$ is not connected by Lemma 2.1; this implies Example 1.10. If M, M' are diffeomorphic, but the h-cobordism W is nontrivial, then

$\mathfrak{R}^u_{\sec\geq 0}(\mathrm{Int}(W))$ is not connected because the boundaries of a nontrivial h-cobordism are not isotopic (as explained, e.g., in [**3**, lemma 7.3]).

We use [**7**, **34**] as basic references on the Whitehead torsion and h-cobordisms. The following lemma summarizes the standard formulas that we need.

Lemma 7.1. (i) *If W is an oriented h-cobordism with boundaries M, M', and $r\colon W \to M$ is a deformation retraction, then the homotopy equivalence $r|_{M'}\colon M' \to M$ has torsion $-\tau(W,M)+(-1)^{\dim(M)}\tau^*(W,M)$.*
(ii) *Let W_1, W_2 be two oriented h-cobordisms attached along their common boundary component to form an oriented h-cobordism W. Denote the common component of ∂W and ∂W_k by M_k, and let $i_k\colon M_k \to W$ be the inclusion. Then $i_{1*}\tau(W,M_1) = i_{1*}\tau(W_1,M_1) + (-1)^{\dim(M)} i_{2*}\tau^*(W_2,M_2)$.*

Proof. (i) First recall that if $r\colon X \to Y$ is a deformation retraction of finite cell complexes, then $\tau(r) = -\tau(X,Y)$, indeed; if $i\colon Y \to X$ is the inclusion, then $\tau(r) = -r_*\tau(i) = -r_*i_*\tau(W,M) = -\tau(W,M)$ [**7**, 22.3, 22.5]. Now to prove (i), let i, i' be the inclusions of M, M' into W so that [**7**, 22.3, 22.4, 22.5] implies that

$$\tau(r\circ i') = \tau(r) + r_*\tau(i') = \tau(r) + r_*i'_*\tau(W,M')$$

where $i'_*\tau(W,M') = (-1)^{\dim(M)} i_*\tau^*(W,M)$ by duality [**34**, page 394] where $*$ is the standard involution of $\mathrm{Wh}(G)$ induced by $g \to g^{-1}$ in G.

(ii) Fix deformation retractions $R\colon W \to W_1$, $r\colon W_1 \to M_1$. Below we slightly abuse notations by using i_k to also denote the inclusion $M_k \to W_k$. The proof of (i) and the composition formula for torsion [**7**, 22.4] gives

$$\tau(W,M_1) = -\tau(r\circ R) = -\tau(r) - r_*\tau(R) = \tau(W_1,M_1) + r_*\tau(W,W_1).$$

By excision [**7**, 20.3] and duality $\tau(W,W_1) = (-1)^{\dim(M)} i_{2*}\tau^*(W_2,M_2)$. Applying i_{1*} to both sides of the equation and recalling that $i_1\circ r$ is homotopic to $\mathbf{id}(W_1)$, we get the desired formula. q.e.d.

If G is a finite group, then its Whitehead group $\mathrm{Wh}(G)$ is fairly well-understood; in particular, results of Bass show that $\mathrm{Wh}(G)$ is a finitely generated abelian group whose rank equals the difference between the number of conjugacy classes of subsets $\{g, g^{-1}\} \subset G$ and the number of conjugacy classes of cyclic subgroup of G. There is a body of work computing $\mathrm{SK}_1(\mathbb{Z}G)$, the (finite) torsion subgroup of $\mathrm{Wh}(G)$; see [**38**] for details.

Theorem 7.2. *Let M be a closed oriented manifold of even dimension ≥ 5 with $\sec \geq 0$ such that $G = \pi_1(M)$ is finite and $\mathrm{Wh}(G)$ is infinite. Then $\mathfrak{R}^c_{\sec\geq 0}(M \times \mathbb{R})$ has infinitely many components.*

Proof. Since $\mathrm{Wh}(G)$ is infinite, G is finite, and $\dim(M) \geq 5$, there is an oriented h-cobordism W_0 with one boundary diffeomorphic to M such that $\tau(W_0, M)$ has infinite order in $\mathrm{Wh}(G)$. We double W_0 along the other boundary, and denote the double by W; thus both boundary components of W are diffeomorphic to M. By Lemma 7.1, the torsion of the double W is given by $\tau(W, M) = \tau(W_0, M) + (-1)^{\dim(M)}\tau^*(W_0, M)$, where we suppress inclusions. By a result of Wall [**38**, 7.4, 7.5], the involution $*$ acts trivially on $\mathrm{Wh}(G)/\mathrm{SK}_1(\mathbb{Z}G)$, so since $\dim(M)$ is even, $\tau(W, M) - 2\tau(W_0, M)$ has finite order. Since the order of $\tau(W_0, M)$ is infinite, stacking k copies of W on top of each other gives a non-trivial h-cobordism for every k. Stacking countably many copies of W on top of each other, we get a manifold W_∞ diffeomorphic to $M \times \mathbb{R}$ and countably many pairwise non-isotopic embeddings $e_k \colon M \to W_\infty$. Since W_∞ is diffeomorphic to a tubular neighborhood of every e_k, we conclude that for each k, the manifold W_∞ carries a metric isometric to $M \times \mathbb{R}$ with soul $e_k(M)$. So by Lemma 2.1, the $\mathrm{Diff}(N)$-orbit of the metric $M \times \mathbb{R}$ visits infinitely many components of $\mathfrak{R}^c_{\sec\geq 0}(\mathring{W})$. q.e.d.

Remark 7.3. The class of groups with infinite Whitehead group is closed under products with any group. Examples of finite G with $\mathrm{Wh}(G)$ infinite include $\mathbb{Z}_m$ with $m = 5$ or $m \geq 7$, and the dihedral group D_{2p} of order $2p$, where $p \geq 5$ is a prime. See [**38**] for more information.

For odd-dimensional souls, doubling produces h-cobordisms with finite torsion, so we use a different idea based on the following addendum to Lemma 2.1.

Proposition 7.4. *Let S, S' be souls for the metrics g, g' that lie in the same component of $\mathfrak{R}^u_{\sec\geq 0}(N)$. If $P\colon N \to S'$ is a deformation retraction, then $P|_S\colon S \to S'$ is homotopic to a diffeomorphism. The same holds for $\mathfrak{R}^c_{\sec\geq 0}(N)$ if any two metrics in the space have souls that intersect.*

Proof. The proof below holds for any k, so according to Convention 1.4 we omit k from notations. By Lemma 2.1, any metric $g_i \in \mathfrak{R}_{\sec\geq 0}(N)$ has an open neighborhood U_i such that for any $g \in U_i$ the Sharafutdinov retraction onto a soul of g_i restricted to any soul of g is a diffeomorphism.

Fix an arbitrary connected component C of $\mathfrak{R}_{\sec\geq 0}(N)$; thus $\{U_i\}$ is an open cover of C. By a basic property of connected sets [**29**, section 46, theorem 8], for any two $g, g' \in C$ there exists a finite sequence $g_0 = g, g_1, \dots, g_n = g'$ with $g_i \in C \cap U_i$ such that $U_i \cap U_{i-1} \neq \emptyset$ for every $0 < i \leq n$.

Denote souls of g_i by S_i and the corresponding Sharafutdinov retractions by $p_i\colon N \to S_i$, where $S_0 = S$ and $S_n = S'$. By construction $p_i\colon N \to S_i$ restricted to S_{i-1} is a diffeomorphism for every $0 < i \leq n$.

Since each $p_i\colon N \to S_i \subset N$ is homotopic to the identity of N, the composition

$$P_n := p_n \circ \ldots \circ p_1\colon N \to S_n \subset N$$

has the same property, and furthermore, P_n maps S_0 diffeomorphically onto S_n. Both P and P_n are homotopic to the identity of N, so if F denotes the homotopy joining P_n and P through maps $N \to N$, then $P \circ F$ is a homotopy of P_n and P through maps $N \to S_n$. Restricting the homotopy to S_0, we conclude that $P|_{S_0}$ is homotopic to a diffeomorphism. q.e.d.

In the following proposition we can, e.g., take W to be Milnor's h-cobordism from Example 1.10.

Proposition 7.5. *Suppose W is an oriented h-cobordism with boundaries M, M' such that $G := \pi_1(W)$ is finite, $\tau(W, M)$ has infinite order in* $\mathrm{Wh}(G)$*, and* $\dim(M)$ *is odd and ≥ 5. Suppose L is a manifold with nonzero Euler characteristic. If M, M', L admit complete metrics with* $\sec \geq 0$*, then $\mathfrak{R}^u_{\sec\geq 0}(L \times \mathring{W})$ is not connected.*

Proof. Let $f\colon M' \to M$ be the homotopy equivalence between the boundary components of W considered in Lemma 7.1; so

$$\tau(f) = -\tau(W, M) + (-1)^{\dim(M)}\tau^*(W, M),$$

which equals to $-2\tau(W, M)$ plus an element of finite order, as G is finite and $\dim(M)$ is odd. Set S to be L when L is compact, and to be a soul of L if L is non-compact. The product formula for torsion [**7**, 23.2b] implies that $\tau(f \times \mathbf{id}(S))$ is mapped to $\chi(S)\tau(g) = -2\chi(S)\tau(W, M)$ by the projection $M \times S \to M$, and $-2\chi(S)\tau(W, M)$ is nonzero because $\tau(W, M)$ has infinite order and $\chi(S) = \chi(L) \neq 0$. Thus $f \times \mathbf{id}(S)$ is not a simple homotopy equivalence, and hence it is not homotopic to a homeomorphism. Now thinking of $\mathring{W}$ as the result of attaching to W an open collar along ∂W, we apply Proposition 7.4 to $L \times \mathring{W}$. q.e.d.

Corollary 7.6. *For $k \geq 3$, $r > 0$, let $M = L(4r+1, 1) \times S^{2k}$ and let L be a complete manifold of* $\sec \geq 0$ *and $\chi(L) \neq 0$. Then $\mathfrak{R}^u_{\sec\geq 0}(L \times M \times \mathbb{R})$ has infinitely many components.*

Proof. Hausmann [**20**] showed that there is a nontrivial h-cobordism W with both boundaries diffeomorphic to $M = L(4r+1, 1) \times S^{2k}$. Stacking infinitely many copies of W on top of each other, we get a manifold diffeomorphic to $M \times \mathbb{R}$, and infinitely many embeddings $M \to M \times \mathbb{R}$ such that the h-cobordism between any two distinct embedded copies of M is obtained by stacking k copies of W on top of each other for some positive integer k. By Lemma 7.1, the homotopy equivalence between its boundaries has torsion $-2k\tau(W, M)$ and hence it is not homotopic to a diffeomorphism, so that Proposition 7.4 applies, yielding the special case when L is a point. The general case follows as in the proof of Proposition 7.5. q.e.d.

References

[1] I. Belegradek, *Vector bundles with infinitely many souls*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2217–2221 (electronic), MR1963770, Zbl 1027.53033.

[2] I. Belegradek & V. Kapovitch, *Topological obstructions to nonnegative curvature*, Math. Ann. **320** (2001), no. 1, 167–190, MR1835067, Zbl 0997.53027.

[3] ———, *Classification of negatively pinched manifolds with amenable fundamental groups*, Acta Math. **196** (2006), no. 2, 229–260, MR2275833, Zbl 1123.53016.

[4] I. Belegradek, S. Kwasik, & R. Schultz, *Codimension two souls and cancellation phenomena*, arXiv:0912.4874v1 [math.DG].

[5] C. Böhm & B. Wilking, *Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature*, Geom. Funct. Anal. **17** (2007), no. 3, 665–681, MR2346271, Zbl 1132.53035.

[6] J. Cheeger & D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443, MR0309010, Zbl 0246.53049.

[7] M. M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, New York, 1973, Graduate Texts in Mathematics, Vol. 10, MR0362320, Zbl 0261.57009.

[8] D. Crowley, *The smooth structure set of $S^p \times S^q$*, Geom. Dedicata **148** (2010), 15–33, MR2721618, Zbl 1207.57043.

[9] S. Chang & S. Weinberger, *On invariants of Hirzebruch and Cheeger-Gromov*, Geom. Topol. **7** (2003), 311–319 (electronic), MR1988288, Zbl 1037.57028.

[10] A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. (2) **78** (1963), 223–255, MR0155330, Zbl 0203.25402.

[11] D. G. Ebin, *The manifold of Riemannian metrics*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 11–40, MR0267604, Zbl 0205.53702.

[12] H. D. Fegan & R. S. Millman, *Quadrants of Riemannian metrics*, Michigan Math. J. **25** (1978), no. 1, 3–7, MR0487890, Zbl 0365.53015.

[13] F. T. Farrell & P. Ontaneda, *The moduli space of negatively curved metrics of a hyperbolic manifold*, J. Topol. **3** (2010), no. 3, 561–577, MR2684513, Zbl 1201.58007.

[14] T. Farrell & P. Ontaneda, *Teichmüller spaces and negatively curved fiber bundles*, Geom. Funct. Anal. **20** (2010), no. 6, 1397–1430, MR2738998, Zbl 1207.58013.

[15] L. Guijarro, *Improving the metric in an open manifold with nonnegative curvature*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1541–1545, MR1443388, Zbl 1006.53028.

[16] K. Grove & W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) **152** (2000), no. 1, 331–367, MR1792298, Zbl 0991.53016.

[17] A. Haefliger, *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math. (2) **75** (1962), 452–466, MR0145539, Zbl 0105.17407.

[18] ———, *Differentiable embeddings of* S^n *in* S^{n+q} *for* $q > 2$, Ann. of Math. (2) **83** (1966), 402–436, MR0202151, Zbl 0151.32502.

[19] V. L. Hansen, *The homotopy groups of a space of maps between oriented closed surfaces*, Bull. London Math. Soc. **15** (1983), no. 4, 360–364, MR0703761, Zbl 0521.55010.

[20] J.-C. Hausmann, *Open books and* h*-cobordisms*, Comment. Math. Helv. **55** (1980), no. 3, 330–346, MR0593050, Zbl 0455.57018.

[21] M. W. Hirsch, *Smooth regular neighborhoods*, Ann. of Math. (2) **76** (1962), 524–530, MR0149492, Zbl 0151.32604.

[22] I. Hambleton & L. R. Taylor, *A guide to the calculation of the surgery obstruction groups for finite groups*, Surveys on surgery theory, Vol. 1, Ann. of Math. Stud., vol. 145, Princeton Univ. Press, Princeton, NJ, 2000, pp. 225–274, MR1747537, Zbl 0952.57009.

[23] D. Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994, MR1249482, Zbl 0794.55001.

[24] S. Kobayashi & K. Nomizu, *Foundations of differential geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969, MR1393941, Zbl 0526.53001.

[25] V. Kapovitch, A. Petrunin, & W. Tuschmann, *Non-negative pinching, moduli spaces and bundles with infinitely many souls*, J. Differential Geom. **71** (2005), no. 3, 365–383, MR2198806, Zbl 1102.53020.

[26] M. Kreck & S. Stolz, *Nonconnected moduli spaces of positive sectional curvature metrics*, J. Amer. Math. Soc. **6** (1993), no. 4, 825–850, MR1205446, Zbl 0793.53041.

[27] S. Kwasik & R. Schultz, *On* h*-cobordisms of spherical space forms*, Proc. Amer. Math. Soc. **127** (1999), no. 5, 1525–1532, MR1473672, Zbl 0923.57009.

[28] ———, *Toral and exponential stabilization for homotopy spherical spaceforms*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 3, 571–593, MR2103917, Zbl 1077.55005.

[29] K. Kuratowski, *Topology. Vol. II*, New edition, revised and augmented. Translated from the French by A. Kirkor, Academic Press, New York, 1968, MR0259835, Zbl 0849.01044.

[30] S. López de Medrano, *Involutions on manifolds*, Springer-Verlag, New York, 1971, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 59, MR0298698, Zbl 0214.22501.

[31] J. Levine, *A classification of differentiable knots*, Ann. of Math. (2) **82** (1965), 15–50, MR0180981, Zbl 0136.21102.

[32] W. Lück & R. Stamm, *Computations of* K*- and* L*-theory of cocompact planar groups*, K-Theory **21** (2000), no. 3, 249–292, MR1803230, Zbl 0979.19003.

[33] J. Milnor, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math. (2) **74** (1961), 575–590, MR0133127, Zbl 0102.38103.

[34] ———, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426, MR0196736, Zbl 0147.23104.

[35] J. W. Milnor & J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 76, MR0440554, Zbl 1079.57504.

[36] A. Nabutovsky & S. Weinberger, *Variational problems for Riemannian functionals and arithmetic groups*, Inst. Hautes Études Sci. Publ. Math. (2000), no. 92, 5–62 (2001), MR1839486, Zbl 1003.58007.

[37] ———, *The fractal nature of Riem/Diff. I*, Geom. Dedicata **101** (2003), 1–54, MR2017894, Zbl 1039.58009.

[38] R. Oliver, *Whitehead groups of finite groups*, London Mathematical Society Lecture Note Series, vol. 132, Cambridge University Press, Cambridge, 1988, MR0933091, Zbl 0636.18001.

[39] S. Ottenburger, *Simply and tangentially homotopy equivalent but non-homeomorphic homogeneous manifolds,* arXiv:1102.5708v2.

[40] S. Ottenburder, *A classification of 5-dimensional manifolds, souls of codimension two and non-diffeomorphic pairs,* arXiv:1103.0099.

[41] P. Petersen, *Convergence theorems in Riemannian geometry*, Comparison geometry (Berkeley, CA, 1993–94), Math. Sci. Res. Inst. Publ., vol. 30, Cambridge Univ. Press, Cambridge, 1997, pp. 167–202, MR1452874, Zbl 0898.53035.

[42] A. Ranicki, *Algebraic and geometric surgery*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2002, Oxford Science Publications, MR2061749, Zbl 1003.57001.

[43] J. Rosenberg, *Manifolds of positive scalar curvature: a progress report*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 259–294, MR2408269, Zbl 1171.53028.

[44] V. A. Sharafutdinov, *Convex sets in a manifold of nonnegative curvature*, Mat. Zametki **26** (1979), no. 1, 129–136, 159, MR0553571, Zbl 0439.53052.

[45] L. C. Siebenmann, *On detecting open collars*, Trans. Amer. Math. Soc. **142** (1969), 201–227, MR0246301, Zbl 0195.53802.

[46] ———, *Infinite simple homotopy types*, Nederl. Akad. Wetensch. Proc. Ser. A 73 = Indag. Math. **32** (1970), 479–495, MR0287542, Zbl 0203.56002.

[47] E. H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint, MR0666554, Zbl 0810.55001.

[48] Z. Shen & C. Sormani, *The codimension one homology of a complete manifold with nonnegative Ricci curvature*, Amer. J. Math. **123** (2001), no. 3, 515–524, MR1833151, Zbl 1024.53027.

[49] R. G. Swan, *Induced representations and projective modules*, Ann. of Math. (2) **71** (1960), 552–578, MR0138688, Zbl 0104.25102.

[50] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962, MR0143217, Zbl 0101.40703.

[51] C. T. C. Wall, *Surgery on compact manifolds*, second ed., Mathematical Surveys and Monographs, vol. 69, American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by A. A. Ranicki, MR0431216, Zbl 0935.57003.

[52] B. Wilking, *A duality theorem for Riemannian foliations in nonnegative sectional curvature*, Geom. Funct. Anal. **17** (2007), no. 4, 1297–1320, MR2373019, Zbl 1139.53014.

[53] M. Y. Wang & W. Ziller, *Einstein metrics on principal torus bundles*, J. Differential Geom. **31** (1990), no. 1, 215–248, MR1030671, Zbl 0691.53036.

[54] T. Yamaguchi, *Collapsing and pinching under a lower curvature bound*, Ann. of Math. (2) **133** (1991), no. 2, 317–357, MR1097241, Zbl 0737.53041.

[55] J.-W. Yim, *Space of souls in a complete open manifold of nonnegative curvature*, J. Differential Geom. **32** (1990), no. 2, 429–455, MR1072913, Zbl 0733.53017.

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
E-mail address: ib@math.gatech.edu

Mathematics Department
Tulane University
6823 St. Charles Ave.
New Orleans, LA 70118
E-mail address: kwasik@tulane.edu

Department of Mathematics
University of California Riverside
900 Big Springs Drive
Riverside, CA 92521
E-mail address: schultz@math.ucr.edu