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KOBAYASHI GEODESICS IN $\mathcal{A}_g$

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Abstract

We consider Kobayashi geodesics in the moduli space of abelian varieties A_g, that is, algebraic curves that are totally geodesic submanifolds for the Kobayashi metric. We show that Kobayashi geodesics can be characterized as those curves whose logarithmic tangent bundle splits as a subbundle of the logarithmic tangent bundle of A_g.

Both Shimura curves and Teichmüller curves are examples of Kobayashi geodesics, but there are other examples. We show moreover that non-compact Kobayashi geodesics always map to the locus of real multiplication and that the $\mathbb{Q}$-irreducibility of the induced variation of Hodge structures implies that they are defined over a number field.

Introduction

Let Y be a non-singular projective curve defined over $\mathbb{C}$, let $Y_0 \subset Y$ be an open dense subscheme, and let $\mathbb{W}_{\mathbb{Z}}$ be a polarized $\mathbb{Z}$-variation of Hodge structures on Y_0 of weight one. So $\mathbb{W}_{\mathbb{Z}}$ gives rise to a morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$, where $g = \frac{\mathrm{rk}(\mathbb{W})}{2}$.

It is our aim to characterize those φ that are totally geodesic submanifolds for the Kobayashi metric, in the sequel referred to as Kobayashi geodesics. Roughly speaking, the characterization will be in terms of the irreducible direct factors of $\mathbb{W} = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ or in terms of the splitting of the natural map

$$\varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \longrightarrow \Omega^1_Y(\log(Y \setminus Y_0)).$$

We now explain this characterization and the above notation in more detail and then why the main theorem, stated as Theorem 1.2, is a unified treatment of the characterization of Shimura curves and Teichmüller curves—and beyond.

All the conditions we are interested in are preserved by étale coverings. Hence we will allow to replace Y_0 by an étale covering and $\mathbb{W}_{\mathbb{Z}}$ by its pullback. By abuse of notation, $\mathcal{A}_g$ will denote a fine moduli scheme of polarized abelian varieties with a suitable level structure. $\overline{\mathcal{A}}_g$

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will always denote a *Mumford compactification*, i.e., one of the toroidal compactifications, constructed by Ash, Mumford, Rapoport, and Tai and studied by Mumford in [**Mu77**]. We write $S_{\overline{\mathcal{A}}_g} = \overline{\mathcal{A}}_g \setminus \mathcal{A}_g$ for the boundary and $\varphi : Y \to \overline{\mathcal{A}}_g$ for the extension of $\varphi_0 : Y_0 \to \overline{\mathcal{A}}_g$ to Y.

One of the characterizations we are heading for uses the logarithmic Higgs bundle of a polarized complex variation of Hodge structures ([**Si90**], recalled in Section 1). Writing $(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta)$ for the logarithmic Higgs bundle of an irreducible direct factor $\mathbb{V}$ of $\mathbb{W}$, we will say that (E, θ) (or $\mathbb{V}$) is *maximal Higgs* if $\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S)$ is an isomorphism.

Then Theorem 1.2 states that $\varphi_0 : Y_0 \to \mathcal{A}_g$ is a Kobayashi geodesic if and only if *at least one* of the direct factors $\mathbb{V}$ of $\mathbb{W}$ is maximal Higgs.

The decomposition of $\mathbb{W}$ in irreducible direct factors is in fact defined over $\bar{\mathbb{Q}}$, and hence induced by a decomposition of $\mathbb{W}_{\bar{\mathbb{Q}}} = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$. If one assumes in addition that the general fiber $f_0 : X_0 \to Y_0$ is simple, then for suitable elements

$$\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_\ell$$

of the Galois group of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ one can write

$$\mathbb{W} = \mathbb{V} \oplus \mathbb{V}^{\sigma_2} \oplus \cdots \oplus \mathbb{V}^{\sigma_\ell},$$

where $\mathbb{V}^{\sigma_i}$ denotes the conjugate of $\mathbb{V}$ under σ_i.

The condition "maximal Higgs" is not compatible with Galois conjugation. In particular, the Higgs field of $\mathbb{V}^{\sigma_i}$ might be zero, or equivalently $\mathbb{V}^{\sigma_i}$ might be unitary. By [**VZ04**] $\varphi_0 : Y_0 \to \overline{\mathcal{A}}_g$ is a Shimura curve, and hence a totally geodesic submanifold for the Bergman-Siegel metric if and only if *all* direct factors of $\mathbb{W}$ are either maximal Higgs or unitary.

Geodesics for the Kobayashi metric have been considered by the first named author in [**Mö06**] under the additional assumption that $f_0 : X_0 \to Y_0$ is a family of Jacobians of a smooth family of curves. In this case $\varphi_0(Y_0)$ is a geodesic for the Kobayashi metric if and only if the image of Y_0 in the moduli scheme M_g of curves of genus g with the right level structure is a geodesic for the Teichmüller metric, and hence if and only if Y_0 is a Teichmüller curve. In particular, Y_0 will be affine.

By [**Mö06**] Y_0 being a Teichmüller curve is equivalent to the existence of one direct factor $\mathbb{V}$ of $\mathbb{W}$ which is maximal Higgs. This implies that it is of rank two and that its conjugates are neither unitary nor maximal Higgs. So $\mathbb{V}$ is the variation $\mathbb{L}$ of Hodge structures, defined in Section 1 starting from logarithmic theta characteristic. The variation $\mathbb{L}$ is defined over a totally real number field, and it looks like the uniformizing variation of Hodge structures on a modular curve, except that it has no $\mathbb{Q}$ structure. Adding the condition that the general fiber of f_0 is simple, the Teichmüller curve is determined by the Weil restriction of $\mathbb{L}$. In a quite sloppy way one could say that the variation of Hodge structures on a Teichmüller curve is like the one on "a modular curve without a $\mathbb{Q}$-structure."

The curves $\varphi_0 : Y_0 \to \overline{\mathcal{A}}_g$ satisfying the equivalent conditions of Theorem 1.2 include Teichmüller curves as well as Shimura curves, and again they could be seen as a generalization of Shimura curves, obtained by dropping the condition that the variations of Hodge structures are defined over $\mathbb{Q}$.

Examples of geodesics for the Kobayashi metric are given by curves Y_0 on Hilbert modular varieties U. By definition, the universal covering $\widetilde{U}$ is a product of disks Δ, and the property "geodesic" just means that one of the projections $\widetilde{U} \to \Delta$ induces an isometry between the universal covering of Y_0 and Δ. In Section 5 we will show that all affine geodesics for the Kobayashi metric are obtained in this way, again a result which is well known for Teichmüller curves.

In Section 6 we consider families of curves $\mathcal{Y}_0 \to U$ in $\mathcal{A}_g$, whose general fiber is a Kobayashi geodesic. Assuming that the corresponding variation of Hodge structures is irreducible over $\mathbb{Q}$, or equivalently that the general fiber of the induced family of abelian varieties is simple, we show that such families have to be locally analytically products $Y_0 \times U$, and hence just the map to $\mathcal{A}_g$ might vary. In the affine case, the latter can not happen, which implies that affine Kobayashi geodesics are defined over number fields. In [**Mö06**] a similar result was stated for Teichmüller curves, but its proof, as pointed out by C. McMullen, was incomplete.

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1. A characterization of Kobayashi geodesics

Let $f_0 : X_0 \to Y_0$ be a non-isotrivial family induced by a finite morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$ and $\mathbb{W}_{\mathbb{Z}}$ be the induced polarized variation of Hodge structures. We denote the complement $Y \setminus Y_0$ by S. If K is a subfield of $\mathbb{C}$, we write $\mathbb{W}_K = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ and usually $\mathbb{W} = \mathbb{W}_{\mathbb{C}}$.

Kobayashi geodesics and their characterization. The Kobayashi pseudo-distance $k_W(\cdot,\cdot)$ is the defined in the following way intrinsically for all complex domains W. We denote by d_Δ the Poincaré metric on the unit disc Δ and for all $x, y \in W$ we call a *chain* from x to y points $x_0, x_1, \ldots, x_n \in \Delta$ together with maps $f_i : \Delta \to W$ such that

$$f_1(x_0) = 0, \quad f_j(x_j) = f_{j+1}(x_j),\ j = 1, \ldots, n-1, \quad f_n(x_n) = y.$$

Then

$$k_W(x,y) = \inf \sum_{i=1}^{n} d_\Delta(x_{i-1}, x_i),$$

where the infimum is over all chains from x to y.

We will rely on two main properties of the Kobayashi pseudo-distance: All holomorphic maps are distance-decreasing, and on the Siegel upper half space $\mathbb{H}_g$ (or its bounded realization $\mathcal{D}_g$) the Kobayashi pseudo-distance is in fact a metric.

Definition 1.1. Let M be a complex domain and W be a subdomain. W is a *totally geodesic submanifold for the Kobayashi metric* if the restriction of the Kobayashi metric on M to W coincides with the Kobayashi metric on W. If $W = \Delta$, we call Δ simply *a (complex) Kobayashi geodesic.*

We call a map $\varphi_0 : Y_0 \to A_g$ *a Kobayashi geodesic* if its universal covering map

$$\tilde{\varphi}_0 : \widetilde{Y_0} \cong \Delta \to \mathbb{H}_g$$

is a Kobayashi geodesic. In particular, a Kobayashi geodesic will always be one-dimensional.

Theorem 1.2. *Keeping the notations and assumptions introduced above, the following conditions are equivalent:*

a. $\varphi_0 : Y_0 \to \mathcal{A}_g$ *is Kobayashi geodesic.*
b. *After replacing* Y *be a finite covering, étale over* Y_0*, the natural map* $\varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \to \Omega^1_Y(\log S)$ *splits.*
c. $\mathbb{W}$ *contains a non-unitary irreducible subvariation of Hodge structures* $\mathbb{V}$ *which satisfies the Arakelov equality as defined below in Lemma* 1.3.

The Arakelov (in)equality. Replacing Y_0 by an étale covering, we may assume that the local monodromy operators around $s \in S$ are unipotent and in addition that $\deg(\Omega^1_Y(\log S))$ is even, and hence that there exists a *logarithmic theta-characteristic*, i.e., an invertible sheaf $\mathcal{L}$ on Y with $\mathcal{L}^2 \cong \Omega^1_Y(\log S)$.

For a $\mathbb{C}$-subvariation of Hodge structures $\mathbb{V} \subset \mathbb{W}$, let $\mathcal{V}$ denote the Deligne extension of $\mathbb{V}\otimes\mathcal{O}_{Y_0}$ to Y and let (E,θ) be the induced logarithmic Higgs bundle, i.e., the graded bundle $E = E^{1,0} \oplus E^{0,1}$ with respect to the $\mathcal{F}$-filtration, together with the Higgs field $\theta : E \to E \otimes \Omega^1_Y(\log S)$ induced by the Gauss-Manin connection. So $\theta|_{E^{0,1}} = 0$ and $\theta(E^{1,0}) \subset E^{0,1} \otimes \Omega^1_Y(\log S)$, and we may consider θ as a morphism

$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S).$$

Writing for a coherent sheaf $\mathcal{E}$ on Y,

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})},$$

the Arakelov inequality, due to Faltings and Deligne (see [**Fa83**] and [**De87**]), says that

$$\mu(\mathbb{V}) := \mu(E^{1,0}) - \mu(E^{0,1}) \leq \mu(\Omega^1_Y(\log S)) = \deg(\Omega^1_Y(\log S)).$$

Lemma 1.3 ([**VZ04**])**.** *The following conditions are equivalent for a non-unitary irreducible subvariation* $\mathbb{V}$ *of Hodge structures in* $\mathbb{W}$*:*

a. (E, θ) *satisfies the* Arakelov equality*, i.e.,* $\mu(\mathbb{V}) = \mu(\Omega^1_Y(\log S))$*.*
b. (E, θ) *is maximal Higgs.*
c. $E^{1,0}$ *and* $E^{0,1}$ *are both stable, and* θ *is a morphism between poly-stable bundles of the same slope.*
d. *There exists a rank-two, weight-one variation of Hodge structures* $\mathbb{L}$ *on* Y_0 *and an irreducible unitary local system* $\mathbb{U}$ *on* Y_0*, regarded as a variation of Hodge structures of bidegree* $(0, 0)$ *such that*
   i. $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$
   ii. *the Higgs bundle of* $\mathbb{L}$ *is of the form*

$$\big(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau : \mathcal{L} \xrightarrow{\cong} \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S)\big),$$

   where $\mathcal{L}$ *is a logarithmic theta characteristic and* τ *the induced isomorphism.*

Towards the proof of Theorem 1.2. Remark that by definition $\varphi_0 : Y_0 \to A_g$ is a Kobayashi geodesic if and only if $Y_0 \to \varphi_0(Y_0)$ is étale and $\varphi_0(Y_0) \subset \mathcal{A}_g$ a Kobayashi geodesic. Since the conditions (b) and (c) in Theorem 1.2 also are invariant under étale coverings we may always assume that φ_0 is an embedding.

The proof that (c) implies (a) is quite easy, and it will be given in Section 3. There, assuming that $Y_0 \to \mathcal{A}_g$ is Kobayashi geodesic, we will also construct a candidate for the splitting in (b) over Y_0. In Section 4 we will show that the splitting extends to a splitting of the sheaves of log-differential forms, finishing the proof that (a) implies (b).

We start in Section 2 with the algebraic part of this note, i.e., by showing that (b) implies (c). In fact, since it hardly requires any additional work, we will show directly that (b) and (c) are equivalent.

A reader with a background in Teichmüller theory might want to compare our characterization of Kobayashi geodesics with the work of Kra ([**Kr81**]). Kra studies when Kobayashi geodesics in the moduli space of curves remain Kobayashi geodesics $\mathcal{A}_g$ after postcomposition with the Torelli map. By Kra this holds if and only if the Kobayashi geodesic in the moduli space of curves is generated (in the sense of Teichmüller's theorems) by a quadratic differential which is a square of an abelian differential. The above Theorem 1.2 makes no reference to the moduli space of curves (or the Schottky locus). Of course there are many Kobayashi geodesics outside the Schottky locus, both Shimura curves and others. We give more comments on these examples in Section 7.

2. Slopes of Higgs bundles

At the end of this section we will prove the implication "(b) $\Longrightarrow$ (c)" of Theorem 1.2. Instead of studying the log differentials on Y_0 we consider the dual sheaves $T_Y(-\log S) = (\Omega^1_Y(\log S))^\vee$ and $T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) = (\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}))^\vee$ in this section, as well as the dual Higgs field

$$\theta' : E^{1,0} \otimes T_Y(-\log S) \longrightarrow E^{0,1}.$$

Recall that $\overline{\mathcal{A}}_g$, being a Mumford compactification of $\mathcal{A}_g$, implies that $\overline{\mathcal{A}}_g$ is non-singular and the boundary $S_{\overline{\mathcal{A}}_g}$ a normal crossing divisor. Moreover, as shown in [**Mu77**, §3], the sheaf $\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ is nef. Furthermore recall that, writing (E', θ') for the logarithmic Higgs bundle of $\mathbb{W} = \mathbb{W}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, the pullback of the logarithmic tangent sheaf of $(\overline{\mathcal{A}}_g, S_{\overline{\mathcal{A}}_g})$ to Y is given by

$$S^2(E'^{0,1}) \subset \mathcal{H}om(E'^{1,0}, E'^{0,1}),$$

where one uses the polarization to identify $E'^{0,1}$ and the dual of $E'^{1,0}$. We write $\mathbb{W} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_\ell$, where $\mathbb{V}_0$ is a unitary subsheaf of $\mathbb{W}$ and where $\mathbb{V}_1, \ldots, \mathbb{V}_\ell$ are irreducible non-unitary direct factors. The logarithmic Higgs bundle of $\mathbb{V}_j$ will be denoted by $(E_j = E_j^{1,0} \oplus E_j^{0,1}, \theta_j)$. The dual Higgs field on $\overline{\mathcal{A}}_g$ induces an identification

$$\begin{aligned} \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) &= S^2(E_0^{0,1} \oplus \cdots \oplus E_\ell^{0,1}) \\ &\cong \Big(\bigoplus_j S^2(E_j^{0,1})\Big) \oplus \Big(\bigoplus_{i<j} E_i^{0,1} \otimes E_j^{0,1}\Big) \qquad (2.1) \end{aligned}$$

$$\subset \mathcal{H}om(E_0^{1,0} \oplus \cdots \oplus E_\ell^{1,0}, E_0^{0,1} \oplus \cdots \oplus E_\ell^{0,1}) \cong \bigoplus_{i,j} E_i^{1,0\vee} \otimes E_j^{0,1}. \qquad (2.2)$$

Since $(E_j = E_j^{1,0} \oplus E_j^{0,1}, \theta_j)$ is a Higgs subbundle, the composition

$$T_Y(-\log S) \longrightarrow \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) \longrightarrow E_i^{1,0\vee} \otimes E_j^{0,1} \qquad (2.3)$$

is zero, except for $i = j \neq 0$. The dual $E_j^{1,0\vee}$ is the complex conjugate of $E_j^{1,0}$ and hence isomorphic to $E_\iota^{0,1}$ if $\mathbb{V}_\iota = \bar{\mathbb{V}}_j$. In this case the right-hand side of (2.3) is identified with $E_\iota^{0,1} \otimes E_j^{0,1}$. If this holds for $\iota = j$, hence if $\mathbb{V}_j$ is defined over $\mathbb{R}$, the map in (2.3) factors through $S^2(E_j^{0,1})$.

In order not to be forced to distinguish different cases, we define for each of the non-unitary irreducible $\mathbb{C}$-subvariation of Hodge structures $\mathbb{V}_j$ in $\mathbb{W} = R^1 f_* \mathbb{C}_{X_0}$, with Higgs bundle (E_j, θ_j),

$$\mathcal{T}_j = \begin{cases} S^2(E_j^{0,1}) & \text{if } \mathbb{V}_j \text{ is defined over } \mathbb{R}, \\ E_j^{1,0\vee} \otimes E_j^{0,1} & \text{otherwise.} \end{cases} \qquad (2.4)$$

The sheaf $\mathcal{T}_j$ is a direct factor of $\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g})$, uniquely determined by $\mathbb{V}_j$. As we have seen above, the image of $T_Y(-\log S)$ lies in $\mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell$. The Higgs bundle (E_j, θ_j) gives rise to a morphism

$$\eta_j = \eta_{(E_j,\theta_j)} : T_Y(-\log S) \longrightarrow \mathcal{T}_j.$$

Since $\mathbb{V}_j$ is non-unitary, hence θ_j non-trivial, one finds η_j to be injective.

As in Theorem 1.2 (c) we start with a non-unitary irreducible direct factors of $\mathbb{W}$, say $\mathbb{V}_1$. For simplicity we will drop the lower index $_1$ in the sequel and write $\mathcal{T}$, η and (E, θ) instead of $\mathcal{T}_1$, η_1 and (E_1, θ_1)

The following lemma is the converse of what we really want.

Lemma 2.1. *Let $\mathbb{V}$ be non-unitary and satisfying the Arakelov equality. Then for the direct factor $\mathcal{T}$ of $\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g})$, induced by $\mathbb{V}$, the natural injection $T_Y(-\log S) \to \mathcal{T}$ splits.*

In particular, the existence of such a direct factor $\mathbb{V}$ of $\mathbb{W}$ implies that

$$T_Y(-\log S) \longrightarrow \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\mathcal{A}_g})$$

splits.

Proof. Using the notation from Lemma 1.3, (d) one finds that $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$, with $\mathbb{U}$ unitary. So the logarithmic Higgs bundle of $\mathbb{U}$ is of the form $(\mathcal{U}, 0)$. The sheaf $T_Y(-\log S) \cong \mathcal{L}^{-2}$ is the subsheaf of

$$E^{1,0^\vee} \otimes E^{0,1} \cong \mathcal{L}^{-1} \otimes \mathcal{U}^\vee \otimes \mathcal{L}^{-1} \otimes \mathcal{U},$$

induced by the homotheties in $\mathcal{H}om(\mathcal{U}, \mathcal{U})$. In particular, one has a splitting of $\eta(T_Y(-\log S))$ as subsheaf of $E^{1,0^\vee} \otimes E^{0,1}$, and hence of $\mathcal{T}$. q.e.d.

As a next step we will show the converse of the first part of Lemma 2.1.

Proposition 2.2. *Assume that $\eta(T_Y(-\log S))$ is non-zero and a direct factor of $\mathcal{T}$. Then $\mathbb{V}$ satisfies the Arakelov equality.*

Proof. The sheaf $\mathcal{T}$ is a direct factor of $E^{1,0^\vee} \otimes E^{0,1}$, and hence by assumption the invertible sheaf $T_Y(-\log S) \cong \eta(T_Y(-\log S))$ as well.

By Lemma 1.3 the conclusion of Proposition 2.2 would imply the stability of the sheaves $E^{1,0}$ and $E^{0,1}$. The strategy of the proof of the proposition will be to verify first the semistability of both sheaves. This will give the semistability of $E^{1,0^\vee} \otimes E^{0,1}$, which implies in turn that the slope of the direct factor $T_Y(-\log S)$ has to be $\mu(E^{0,1}) - \mu(E^{1,0})$.

Let $0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_\alpha$ be the Harder-Narasimhan (HN) filtration of $E^{1,0^\vee} \otimes E^{0,1}$.

Claim 2.3. For some $\ell > 0$ the sheaf $T_Y(-\log S)$ maps to a direct factor of $\mathcal{G}_\ell / \mathcal{G}_{\ell-1}$.

Proof. Since $T_Y(-\log S)$ is a direct factor of $E^{1,0^\vee} \otimes E^{0,1}$, we can write

$$E^{1,0^\vee} \otimes E^{0,1} = \mathcal{G}' \oplus T_Y(-\log S)$$

for some $\mathcal{G}'$. Let $0 = \mathcal{G}'_0 \subset \mathcal{G}'_1 \subset \cdots \subset \mathcal{G}'_{\alpha'} = \mathcal{G}'$ be the HN-filtration of $\mathcal{G}'$.

If $\mu(T_Y(-\log S)) < \mu(\mathcal{G}'_\nu/\mathcal{G}'_{\nu-1})$, define $\mathcal{G}''_\nu = \mathcal{G}'_\nu$.

If $\mu(T_Y(-\log S)) = \mu(\mathcal{G}'_\nu/\mathcal{G}'_{\nu-1})$, choose $\mathcal{G}''_\beta = \mathcal{G}'_\beta \oplus T_Y(-\log S)$ for all $\beta \geq \nu$.

If $\mu(\mathcal{G}'_{\nu-1}/\mathcal{G}'_{\nu-2}) > \mu(T_Y(-\log S)) > \mu(\mathcal{G}'_\nu/\mathcal{G}'_{\nu-1})$, choose $\mathcal{G}''_\beta = \mathcal{G}'_{\beta-1} \oplus T_Y(-\log S)$ for all $\beta \geq \nu$.

Obviously $\mathcal{G}''_\bullet$ is a second HN-filtration of $E^{1,0^\vee} \otimes E^{0,1}$, and the uniqueness of the HN-filtration implies the claim. q.e.d.

Let $\mathcal{F}^{1,0}_\bullet$ and $\mathcal{F}^{0,1}_\bullet$ be the HN-filtrations of $E^{1,0}$ and $E^{0,1}$, and write

$$\mathfrak{gr}_{\mathcal{F}^{1,0}_\bullet} = \bigoplus H^{1,0}_\eta, \quad \mathfrak{gr}_{\mathcal{F}^{0,1}_\bullet} = \bigoplus H^{0,1}_\eta.$$

So $\mathcal{G}_\nu/\mathcal{G}_{\nu-1}$ is the direct sum of copies of $H^{1,0^\vee}_{\eta_1} \otimes H^{0,1}_{\eta_2}$, all of the same slope. Since those sheaves are semi-stable, we find some pairs (η_1, η_2) such that $T_Y(-\log S)$ is a direct factor of $H^{1,0^\vee}_{\eta_1} \otimes H^{0,1}_{\eta_2}$. In particular,

$$\mu(H^{0,1}_{\eta_2}) = \mu(T_Y(-\log S)) + \mu(H^{1,0}_{\eta_1}). \tag{2.5}$$

If one replaces $\mathbb{V}$ by its complex conjugate, the role of $H^{1,0^\vee}_{\eta_1}$ and $H^{0,1}_{\eta_2}$ is interchanged. So we may assume that

$$\mu(H^{0,1}_{\eta_2}) \geq \mu(H^{1,0^\vee}_{\eta_1}) = -\mu(H^{1,0}_{\eta_1}), \tag{2.6}$$

for at least one pair (η_1, η_2) for which (2.5) holds. Let us fix one of those pairs.

Consider next the restriction of the Higgs field to $\mathcal{F}^{1,0}_{\eta_1}$ with image

$$\mathcal{B} \otimes \Omega^1_Y(\log S) \subset E^{0,1} \otimes \Omega^1_Y(\log S).$$

Writing $\mathcal{K}$ for the kernel of $\mathcal{F}^{1,0}_{\eta_1} \to \mathcal{B} \otimes \Omega^1_Y(\log S)$, one gets

$$\begin{aligned} &\deg(\mathcal{B}) - \mathrm{rk}(\mathcal{B}) \cdot \mu(T_Y(-\log S)) = \deg(\mathcal{B} \otimes \Omega^1_Y(\log S)) \\ &= \deg(\mathcal{F}^{1,0}_{\eta_1}) - \deg(\mathcal{K}) \end{aligned} \tag{2.7}$$

Both, $(\mathcal{F}^{1,0}_{\eta_1} \oplus \mathcal{B}, \theta|_{\mathcal{F}^{1,0}_{\eta_1}})$ and $(\mathcal{K}, 0)$ are Higgs subbundles of (E, θ). So Simpson's correspondence [**Si90**] implies that

$$0 \geq \deg(\mathcal{B}) + \deg(\mathcal{F}^{1,0}_{\eta_1}) \tag{2.8}$$

and that $0 \geq \deg(\mathcal{K})$. By (2.7) the second statement is equivalent to

$$\deg(\mathcal{B}) + \deg(\mathcal{F}^{1,0}_{\eta_1}) \geq 2 \cdot \deg(\mathcal{F}^{1,0}_{\eta_1}) + \mathrm{rk}(\mathcal{B}) \cdot \mu(T_Y(-\log S)). \tag{2.9}$$

Since obviously $\mathrm{rk}(\mathcal{B}) \leq \mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1})$ and since $\mu(T_Y(-\log S)) < 0$, one obtains that

$$\begin{aligned} &2 \cdot \deg(\mathcal{F}^{1,0}_{\eta_1}) + \mathrm{rk}(\mathcal{B}) \cdot \mu(T_Y(-\log S)) \\ &\geq \mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1}) \cdot (2\mu(\mathcal{F}^{1,0}_{\eta_1}) + \mu(T_Y(-\log S))), \end{aligned} \tag{2.10}$$

with equality if and only if $\mathrm{rk}(\mathcal{B}) = \mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1})$. By definition of the HN-filtration, one has

$$\deg(\mathcal{F}^{1,0}_{\eta_1}) \geq \mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1}) \cdot \mu(H^{1,0}_{\eta_1}) \quad \text{or} \quad \mu(\mathcal{F}^{1,0}_{\eta_1}) \geq \mu(H^{1,0}_{\eta_1}), \tag{2.11}$$

and since we assumed (2.6), one finds that $2\mu(\mathcal{F}^{1,0}_{\eta_1}) \geq \mu(H^{1,0}_{\eta_1}) - \mu(H^{0,1}_{\eta_2})$ or

$$\begin{aligned} &\mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1}) \cdot (2\mu(\mathcal{F}^{1,0}_{\eta_1}) + \mu(T_Y(-\log S))) \geq \\ &\qquad \mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1}) \cdot (\mu(H^{1,0}_{\eta_1}) - \mu(H^{0,1}_{\eta_2}) + \mu(T_Y(-\log S))). \end{aligned} \tag{2.12}$$

By (2.5) the right-hand side in (2.12) is zero, and hence (2.12), (2.10), (2.9), and (2.8) must all be equalities.

In particular, one finds $\mathrm{rk}(\mathcal{B}) = \mathrm{rk}(\mathcal{F}^{1,0}_{\eta_1})$ and $\deg(\mathcal{B}) + \deg(\mathcal{F}^{1,0}_{\eta_1}) = 0$. The first equality implies that $\mathcal{K} = 0$. This and the second equality imply that the Higgs bundle $(\mathcal{F}^{1,0}_{\eta_1} \oplus \mathcal{B}, \theta|_{\mathcal{F}^{1,0}_{\eta_1}})$ is induced by a direct factor of $\mathbb{V}$ or, since we assumed $\mathbb{V}$ to be irreducible, by $\mathbb{V}$ itself. So $\mathcal{F}^{1,0}_{\eta_1} = E^{1,0}$ and $\mathcal{B} = E^{0,1}$.

The equality in (2.12) implies that $\mu(\mathcal{F}^{1,0}_{\eta_1}) = \mu(H^{1,0}_{\eta_1})$. By the definition of the Harder-Narasimhan filtration, this can only happen for $\eta_1 = 1$, and hence if $\mathcal{F}^{1,0}_{\eta_1}$ is semistable. So $E^{1,0}$ and hence $E^{0,1} \cong E^{1,0} \otimes T_Y(-\log S)$ are both semistable.

As said already at the beginning of the proof, the semistability implies the Arakelov equality. In fact, we could also refer to the equality in (2.5) since we know that

$$\mu(E^{1,0}) - \mu(E^{0,1}) = \mu(H^{1,0}_{\eta_1}) - \mu(H^{0,1}_{\eta_2}).$$

q.e.d.

Proof of "(b) $\iff$ (c)" in Theorem 1.2. By Lemma 2.1 it only remains to show that (b) implies (c). Recall that one has a factorization

$$T_Y(-\log S) \longrightarrow \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell \longrightarrow \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g})$$

and that the composition with the projection $\mathrm{pr} : \varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) \to E^{1,0\vee}_j \otimes E^{0,1}_j$ is induced by the Higgs field of $\mathbb{V}_j$.

The splitting $\varphi^* T_{\overline{\mathcal{A}}_g}(-\log S_{\overline{\mathcal{A}}_g}) \to T_Y(-\log S)$ of $T_Y(-\log S)$ defines a splitting $\hat{\eta} : \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell \to T_Y(-\log S)$, and the the composition

$$T_Y(-\log S) \xrightarrow{\subset} \mathcal{T}_j \xrightarrow{\subset} \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_\ell \xrightarrow{\hat{\eta}} T_Y(-\log S)$$

is either zero or an isomorphism. So one finds some $j > 0$ for which $T_Y(-\log S)$ is a direct factor of $\mathcal{T}_j$ and, by Proposition 2.2, $\mathbb{V}_j$ satisfies the Arakelov equality. q.e.d.

3. Geodesics and multidiscs

The proof of the implication "(c) $\Longrightarrow$ (a)" of Theorem 1.2 relies on a normal form of Kobayashi geodesics in the Siegel half-space given by Proposition 3.1 (c). We reproduce the proof of this normal form by [**Ab93**]. Since we are dealing at the end with Kobayashi geodesics in $\mathcal{A}_g$ we need to take care of how the fundamental group of $\mathbb{Y}_0$ acts on this normal form. Besides the proof of "(c) $\Longrightarrow$ (a)" we will deduce from this in Construction 3.3 the splitting needed for the implication "(a) $\Longrightarrow$ (b)" of Theorem 1.2 over the open $\mathcal{A}_g \subset \overline{\mathcal{A}_g}$. The extension of this splitting to all of $\overline{\mathcal{A}_g}$ will be done in the next section.

We recall some facts needed for a characterization of Kobayashi geodesics in $\mathcal{A}_g$. The universal covering of $\mathcal{A}_g$ is isomorphic to the Siegel half space. For $Z \in M^{g\times g}(\mathbb{C})$, we denote bz $||\cdot||$ the operator matrix norm. Via the Cayley transformation, it also has a realization as a bounded symmetric domain

$$(3.1) \qquad \begin{aligned} \mathcal{D}_g &= \{Z \in M^{g\times g}(\mathbb{C}) \,|\, I_g - ZZ^* > 0\} \\ &= \{Z \in M^{g\times g}(\mathbb{C}) \,|\, ||Z|| < 1\} \subset \Delta^{g^2} \subset \mathbb{C}^{g^2} \end{aligned}$$

that will be more convenient to work with in the next section when we discuss boundary components. The intersection of $\mathcal{D}_g$ with the diagonal in $\mathbb{C}^{g^2}$ is isomorphic to Δ^g, a totally geodesic submanifold for the Bergman metric on $\mathcal{D}_g$. Submanifolds of $\mathcal{D}_g$ that are isomorphic to Δ^r for some r and totally geodesic for the Bergman metric are called *multidiscs*. It is well known that $r \leq g$, and the multidiscs of maximal dimension, and hence isomorphic to Δ^g, will be called *polydiscs*. Note that the Cayley transformation maps diagonal matrices to diagonal matrices and that hence polydiscs are given by diagonal matrices in $\mathbb{H}_g$ as well.

Fix a base point $0 \in \mathcal{D}_g$. The fixgroup of 0 in the isometry group of $\mathcal{D}_g$ is a maximal compact subgroup K. It acts on the set of polydiscs and

$$(3.2) \qquad \mathcal{D}_g = \bigcup_{k\in K} k(\Delta^g).$$

In particular, for any $v \in T_{0,\mathcal{D}_g}$ there is a polydisc in $\mathcal{D}_g$ tangent to v, and for a second point $p \in \mathcal{D}_g$, one can find some k with $0, p \in k(\Delta^g)$.

The *Carathéodory metric* c_M on M is defined by

$$c_M(x,y) = \sup_{p:M\to\Delta} d_\Delta(p(x),p(y)),$$

where d_Δ is the Poincaré metric on Δ. One shows using the Arzela-Ascoli theorem that this supremum is attained for some map p.

In the following proposition, we summarize a structure theorem for Kobayashi geodesics in $\mathcal{D}_g$. Part (a) and (b) are well known, and part (c) is the special case of a structure result for Kobayashi geodesic in Hermitian symmetric spaces by Abate ([**Ab93**]). For the convenience of the reader, we repeat his proof.

Proposition 3.1. *Suppose we are given a holomorphic map $\tilde{\varphi}_0 : \Delta \to \mathcal{D}_g$.*

a. *If there is a map $p : \mathcal{D}_g \to \Delta$ such that $p \circ \tilde{\varphi}_0$ is an isometry for the Kobayashi metric on Δ, then $\tilde{\varphi}_0$ is a Kobayashi geodesic.*
b. *Conversely, if $\tilde{\varphi}_0$ is a Kobayashi geodesic, then there is a holomorphic map $p : \mathcal{D}_g \to \Delta$ such that $p \circ \tilde{\varphi}_0 = \mathrm{id}$.*
c. *More precisely, if $\tilde{\varphi}_0$ is a Kobayashi geodesic and $0 \in \Delta$ a base point with $\tilde{\varphi}_0(0) = 0 \in \mathcal{D}_g$, then there exists an integer $1 \leq \ell \leq g$, a matrix $U \in U(g)$, a morphism $\tilde{\psi}_0 : \Delta \to \mathcal{D}_{g-\ell}$, and a totally geodesic embedding*

$$i : \Delta^\ell \times \mathcal{D}_{g-\ell} \longrightarrow \mathcal{D}_g$$

for the Bergman metric with

i. $i_0 = i|_{\Delta^\ell \times \{0\}} : \Delta^\ell \longrightarrow \mathcal{D}_g$ *is a multidisc.*
ii. $U \cdot \tilde{\varphi}_0 \cdot {}^tU = i \circ (\mathrm{diag}, \tilde{\psi}_0)$, *where* diag *denotes the diagonal embedding.*
iii. $\tilde{\psi}_0(\tau)$ *is a matrix $Z_{g-\ell}(\tau)$ with $||Z_{g-\ell}(\tau)|| < |\tau|$.*

Proof. Part (a) is immediate from the distance-decreasing property of the Kobayashi metric.

For (c) fix some point $\tau_0 \in \Delta \setminus \{0\}$. By (3.2) there exists some $U \in U(g)$ such that $M = U\tilde{\varphi}_0(\tau_0)^tU$ is a diagonal matrix. Let m_{ii} denote the diagonal entries of M and let ℓ be the number of entries with $d_\Delta(m_{ii}, 0) = d_\Delta(\tau_0, 0)$. Since φ_0 is a Kobayashi geodesic, ℓ is not zero.

Modifying U, we may assume that $d_\Delta(m_{ii}, 0) = d_\Delta(\tau_0, 0)$ for $i = 1, \dots, \ell$, and multiplying U with a product of matrices in $U(1) \times \cdots \times U(1)$, we may moreover suppose that $m_{ii} = \tau_0$ for $i = 1, \dots, \ell$. Since we will need this later, let us state to which extent U is unique:

iv. *If for some point $\tau_0 \in \Delta \setminus \{0\}$, some $\ell' \leq \ell$ and some U the map $U \cdot \tilde{\varphi}_0 \cdot {}^tU$ has the block form*

$$(U \cdot \tilde{\varphi}_0(\tau_0) \cdot {}^tU) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

with $A = (a_{ij})$ an $\ell' \times \ell'$ diagonal matrix with $d_\Delta(a_{ii}, 0) = d_\Delta(\tau_0, 0)$, then

$$U = \begin{pmatrix} U_{\ell'} & 0 \\ 0 & U_{g-\ell'} \end{pmatrix},$$

with $U_{g-\ell'} \in U(g - \ell')$ and with $U'_\ell \in U(1) \times \cdots \times U(1)$.

There are $Z_{11} \in M^{\ell\times\ell}(\mathbb{C})$, $Z_{12} \in M^{\ell\times(g-\ell)}(\mathbb{C})$, $Z_{22} \in M^{(g-\ell)\times(g-\ell)}(\mathbb{C})$, and $Z_{21} \in M^{(g-\ell)\times\ell)}(\mathbb{C})$, so that we can write

$$U\tilde{\varphi}_0(\tau)^t U = \begin{pmatrix} Z_{11}(\tau) & Z_{12}(\tau) \\ Z_{21}(\tau) & Z_{22}(\tau) \end{pmatrix}.$$

We now check that

$$\psi : \tau \mapsto (Z_{11}(\tau), Z_{12}(\tau))$$

maps to B_ℓ, the ball of radius ℓ in $M^{\ell\times g}(\mathbb{C})$. In fact, $1_g - \tilde{\varphi}_0(\tau)\tilde{\varphi}_0(\tau)^*$ is positive definite, and thus $1_\ell - Z_{11}(\tau)Z^*_{11}(\tau) - Z_{12}(\tau)Z^*_{12}(\tau)$ is positive definite. This implies that the trace of this matrix is positive, or equivalently, writing $(Z_{11}(\tau), Z_{12}(\tau)) = (z_{ij})$ that

$$\sum_{i=1}^{g} \sum_{j=1}^{\ell} |z_{ij}(\tau)| < \ell,$$

which implies the claim.

By construction, $d_{B_\ell}(\psi(\tau_0), 0) = d_\Delta(\tau_0, 0)$. Since B_ℓ is a strictly convex bounded domain, $\psi : \Delta \to B_\ell$ is a Kobayashi geodesic by [**Ve81**, Proposition 3.3]. Moroever, by [**Le81**, Theorem 2] there is a unique Kobayashi geodesic in B_ℓ with $\psi(\tau_0) = (\tau_0 \cdot I_d, 0)$, and by [**Ve81**, example on p. 386], it is linear, and hence given by $\psi(\tau) = (\tau \cdot 1_\ell, 0)$. In particular, $Z_{12}(\tau) = 0$, and by the symmetry of the period matrix, one finds that $Z_{21}(\tau) = 0$ as well.

The maps p claimed to exist in part b) are just the projection maps to $Z \mapsto z_{ii}$, for one $i \in \{1, \ldots, \ell\}$. q.e.d.

Let us return to the situation considered in Section 1. So the map $\varphi_0 : Y_0 \to \mathcal{A}_g$ is induced by the variation of Hodge structures $\mathbb{W}_\mathbb{Z}$, and the image $\tilde{\varphi}_0(\widetilde{Y}_0)$ of the universal covering map is isomorphic to a disc.

The variation of Hodge structures $\mathbb{W}$ is given by a representation of $\pi(Y_0, *)$, and hence by a homomorphism $\rho : \pi(Y_0, *) \to \mathrm{Sp}(2g, \mathbb{R})$. Its image lies in the subgroup

$$H = \{\sigma \in \mathrm{Sp}(2g, \mathbb{R});\ \sigma(\widetilde{Y}_0) = \widetilde{Y}_0\}.$$

Corollary 3.2. *The multidisc $i_0 : \Delta^\ell \to D_g$ is H invariant, and the block decomposition of $\tilde{\varphi}_0$ is preserved by the action of H. Morover, there is a subgroup $H_0 \subset H$ of finite index which respects the factors of $i_0 : \Delta^\ell \to D_g$.*

Proof. We may assume that in Proposition 3.1 the matrix U is the identity matrix. The block decomposition $i : \Delta^\ell \times \mathcal{D}_{g-\ell} \to \mathcal{D}_g$ in Proposition 3.1(c) induces an embedding of isometry groups

$$\hat{\imath} : \mathrm{Sl}(2,\mathbb{R}) \times \cdots \times \mathrm{Sl}(2,\mathbb{R}) \times \mathrm{Sp}(2(g-\ell),\mathbb{R}) \longrightarrow \mathrm{Sp}(2g,\mathbb{R}).$$

Let G be the subgroup of $\mathrm{Sp}(2g,\mathbb{R})$ generated by the image of $\hat{\imath}$ and by the permutation of the factors of $i_0(\Delta^\ell)$. So we have to show that H is a subgroup of G.

Consider for $h \in H$ the map $h \circ \tilde{\varphi}_0$. Since h leaves $\widetilde{Y}_0$ invariant,

$$h \circ \tilde{\varphi}_0(0) = p \in i(\Delta^\ell \times \mathcal{D}_{g-\ell}),$$

and there exists some element g in the image of $\hat{\imath}$ such that $g(p) = 0$. So it is sufficient to show that $g \circ h$ lies in the image of G. To this aim, we apply Proposition 3.1(c) to the Kobayashi geodesic curve

$$\tilde{\varphi}_0' = g \circ h \circ \tilde{\varphi}_0 : \Delta \longrightarrow \mathcal{D}_g.$$

Since h leaves $\widetilde{Y}_0$ invariant, for a point $\tau_0 \in \Delta \setminus \{0\}$ the image $h \circ \tilde{\varphi}_0(\tau_0)$ is of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A = (a_{ij})$ is an $\ell \times \ell$ diagonal matrix. By the choice of g the same holds for $\tilde{\varphi}_0'(\tau_0)$. Moreover, since h and g are isometries, $d_\Delta(a_{ii}, 0) = d_\Delta(\tau_0, 0)$,l we may choose by condition (iv) stated in the proof of Proposition 3.1, a matrix

$$U = \begin{pmatrix} U_\ell & 0 \\ 0 & U_{g-\ell} \end{pmatrix},$$

with $U_{g-\ell} \in U(g-\ell)$ and with $U_\ell \in U(1) \times \cdots \times U(1)$, for which the conclusion of Proposition 3.1(c) holds with $\tilde{\varphi}_0$ replaced by $\tilde{\varphi}_0'$. So for some $\ell' \leq \ell$, one has

$$U \cdot \tilde{\varphi}_0 \cdot {}^tU = i' \circ (\mathrm{diag}, \tilde{\psi}_0') \quad \text{for} \quad i' : \Delta^{\ell'} \times \mathcal{D}_{g-\ell'} \longrightarrow \mathcal{D}_g.$$

Thus, writing by abuse of notations the conjugation on the other side and $Z_{g-\ell'}(\tau)$ for $\tilde{\psi}_0'(\tau)$, we obtain

$$\tilde{\varphi}_0'(\tau) = \begin{pmatrix} \tau \cdot 1_\ell' & 0 \\ 0 & U_{g-\ell'} Z_{g-\ell'}(\tau) {}^t U_{g-\ell'} \end{pmatrix} = g \circ h \begin{pmatrix} \tau \cdot 1_\ell & 0 \\ 0 & Z_{g-\ell}(\tau) \end{pmatrix}.$$

Since $g \circ h$ is an isometry and since $||Z_{g-\ell}(h^{-1}(\tau))|| < |\tau|$, this implies first of all that $\ell = \ell'$ and that we can choose $i = i' : \Delta^{\ell'} \times \mathcal{D}_{g-\ell'} \to \mathcal{D}_g$. Secondly, $g \circ h$ and hence h leave the image of Δ^ℓ invariant, as well as the decomposition $\Delta^\ell \times \mathcal{D}_{g-\ell} \subset \mathcal{D}_g$. So $h \in G$, as claimed. The permutation of the different factors of $i_0(\Delta^\ell)$ defines a homomorphism $G \to \mathcal{S}_g$ whose kernel is the image of $\hat{\imath}$. We let H_0 denote the intersection of H with the image of $\hat{\imath}$. It is a subgroup of H of finite index with the claimed properties. q.e.d.

Next, we will use that Y_0 is a Kobayashi geodesic to construct a splitting of the map $\varphi_0^*\Omega^1_{\mathcal{A}_g} \to \Omega^1_{Y_0}$. Equivalently, writing $\pi : \widetilde{Y}_0 \to Y_0$ for the universal covering and identifying $\widetilde{Y}_0$ with its image in $\mathcal{D}_g$, we will construct a $\pi_1(Y_0, *)$-invariant invertible subsheaf of $\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0} = \pi^*\Omega^1_{\mathcal{A}_g}$ which splits the restriction map $\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0} \to \Omega^1_{\widetilde{Y}_0}$.

Construction 3.3. We keep the notation from Corollary 3.2 and Proposition 3.1(c). Choose coordinates τ_{ij} on $\mathcal{D}_g$ and hence $\tau_{11}, \ldots, \tau_{\ell\ell}$ for the disc Δ^ℓ. Then $\Omega^1_{\Delta^\ell} = \langle d\tau_{11}, \ldots, dx_{\ell\ell}\rangle_{\mathcal{O}_{\Delta^\ell}}$ is a quotient sheaf of $\Omega^1_{\mathcal{D}_g}|_{\Delta^\ell}$. At the same time, the projections p_i allow us to consider $\Omega^1_{\Delta^\ell}$ as a subsheaf of $\Omega^1_{\mathcal{D}_g}|_{\Delta^\ell}$.

On $\widetilde{Y}_0$, the diagonal subsheaf $\Omega' = \langle d\tau_{11} + \cdots + d\tau_{\ell\ell}\rangle_{\mathcal{O}_{\widetilde{Y}_0}}$ of $\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0}$ satisfies:

1. Ω' is invariant under the action of H and hence $\pi(Y_0, *)$ on $\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0}$.
2. The composition with the natural restriction map
$$\Omega^1_{\mathcal{D}_g}|_{\widetilde{Y}_0} \longrightarrow \Omega^1_{\widetilde{Y}_0}$$
is an isomorphism, compatible with the action of H and hence of $\pi(Y_0, *)$.

The first condition implies that Ω' descends to an invertible subsheaf Ω_0 of $\Omega_{\mathcal{A}_g}|_{Y_0}$. Together we obtain:

3. $\Omega' = \pi^*\Omega_0$ for an invertible subsheaf Ω_0 of $\varphi_0^*\Omega_{\mathcal{A}_g}$ and the composition
$$\Omega_0 \subset \varphi_0^*\Omega^1_{\mathcal{A}_g} \longrightarrow \Omega^1_{Y_0}$$
is an isomorphism.

This completes the proof of the implication "(a) $\Longrightarrow$ (b)" in Theorem 1.2 in the case $Y = Y_0$.

Proof of "(c) $\Longrightarrow$ (a)" in Theorem 1.2.
By Lemma 1.3(d) we know that $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$. At some point $y \in Y_0$ fix a basis $\{a, b\}$ of the fiber $\mathbb{L}_y$, a basis $\{e_1, \ldots, e_s\}$ of the fiber $\mathbb{U}$. Write $a_i = a \otimes e_i$, $b_i = b \otimes e_i$, and choose a symplectic basis $\{a_{s+1}, b_{s+1}, \ldots, a_g, b_g\}$ of the orthogonal complement $\mathbb{V}^\perp$ in $\mathbb{W}$. We identify the universal cover of $\mathcal{A}_g$ in this proof with $\mathbb{H}_g$. There, the basis extends uniquely to a basis of sections of $R^1\tilde{f}_*\mathbb{Q}_A$, where $\tilde{f} : A \to \mathbb{H}_g$ is the universal family of abelian varieties over $\mathbb{H}_g$.

As usual in the construction of a period matrix for an abelian variety, there is a unique basis $\{s_1, \ldots, s_g\}$ of sections of $\tilde{f}_*\Omega^1_{A/\mathbb{H}}$ such that
$$\int_{b_i} s_j(\tau) = \delta_{ij} \quad \text{for} \quad \tau \in \mathbb{H}, \quad i, j = 1, \ldots, g.$$

Then the matrix Z with entries

$$Z_{ij} = \int_{a_i} s_j(\tau) \quad \text{for} \quad \tau \in \mathbb{H}$$

is the period matrix for the fiber A_τ. The point is that $\mathrm{pr}_{11} : Z \mapsto z_{11}$ is the period mapping for $\mathbb{L}$. Consequently, $\mathrm{pr}_{11} \circ \tilde{\varphi}_0$ is an isometry and by Proposition 3.1 the map φ_0 is a Kobayashi geodesic. q.e.d.

So we finished the proof of Theorem 1.2 for $Y = Y_0$. For the general case it remains to extend the Construction 3.3 to the boundary.

4. Splitting of the log-differentials

In this section we show how to extend the splitting produced in Construction 3.3 to a compactification of $\mathcal{A}_g$ and thus complete the proof of Theorem 1.2 by showing "(a) $\Longrightarrow$ (b)". We summarize what we need on toroidal compactifications of $\mathcal{A}_g$ as used in [**Mu77**], following the notation therein whenever possible.

The degree g' of a point Z on the boundary of $\mathcal{D}_g$ is defined as the rank of $1 - ZZ^*$. Up to the action of $\mathrm{Sp}(2g, \mathbb{R})$ a point of degree g' is equivalent to a point in a standard boundary component

$$F_{g'} = \left\{ \begin{pmatrix} 1_{g-g'} & 0 \\ 0 & Z \end{pmatrix}, \quad Z \in \mathcal{D}_{g'} \right\} \subset \mathcal{D}_{g'}.$$

The sets $M \cdot F_{g'}$ for $0 \leq g' < g$ and $M \in \mathrm{Sp}(2g, \mathbb{R})$ are called *boundary components*. A boundary component is called *rational* if $M \in \mathrm{Sp}(2g, \mathbb{Q})$, provided that the identification of the universal covering $\widetilde{\mathcal{A}_g} \cong \mathcal{D}_g$ has been done using a rational basis of the (relative) homology. The stabilizer of $F_{g'}$ in $\mathrm{Sp}(2g, \mathbb{R})$ is the group

(4.1)

$$N(F_{g'}) = \{M \in \mathrm{Sp}(2g, \mathbb{R}), MF_{g'} = F_{g'}\} = \left\{ \begin{pmatrix} u & * & * & * \\ 0 & A' & * & B' \\ 0 & 0 & (u^t)^{-1} & 0 \\ 0 & C' & * & D' \end{pmatrix} \right\}.$$

We denote the unipotent radical of $N(F)$ by $W(F)$ and denote the center of $W(F)$ by $U(F)$. See [**Na80**] for explicit matrix realizations and examples.

A neighborhood of a boundary component F,

$$D_F = \bigcup_{M \in U(F)_{\mathbb{C}}} M \cdot F,$$

admits Siegel domain coordinates, given for a standard boundary component $F = F_{g'}$ by

$$D_{F_{g'}} \cong U(F_{g'})_{\mathbb{C}} \times (W(F_{g'})/U(F_{g'}))_{\mathbb{C}} \times F_{g'}$$
$$\begin{pmatrix} Z'' & Z''' \\ (Z''')^t & Z' \end{pmatrix} \mapsto (Z'', Z''', Z').$$

A toroidal compactification of $\mathcal{A}_g$ is constructed by gluing $\mathcal{A}_g$ with toroidal compactifications of the quotients of rational boundary components F by $U(F)$. We won't need the details.

Write x_{ij} for the coordinate functions of $U(F)_{\mathbb{C}} \cong M^{(g-g')\times(g-g')}(\mathbb{C})$, write y_{ij} for those of $(W(F)/U(F))_{\mathbb{C}} \cong M^{g'\times(g-g')}$, and write t_{ij} $(i \leq j)$ for the coordinate functions of $F_{g'} \subset M^{g'\times g'}(\mathbb{C})$. By [**Mu77**, Proposition 3.4] the pullback of sections of $\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ to $\mathcal{D}_g$ in the neighborhood a boundary component of degree g' to are generated by dx_{ij}, dy_{ij} and dt_{ij} as $\mathbb{C}[x_{ij}, y_{ij}, t_{ij}]$-module.

Proof of "(a) $\Longrightarrow$ (b)" in Theorem 1.2*.*
It remains to show that the sheaf Ω_0 in the Construction 3.3 extends to a subsheaf $\Omega^1 \subset \varphi^*\Omega_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$, isomorphic to $\Omega^1_Y(\log S)$.

Fix a point $s \in S \subset Y$. The universal covering factors as

$$\widetilde{Y_0} \cong \Delta \to \Delta^* \to Y_0,$$

where the first map is $z \mapsto q = e^{2\pi i}$ and s corresponds to $q = 0 \in \Delta^*$. A local generator of $\Omega^1_Y(\log S)$ around s is thus given by dq/q or equivalently by dz near in a rational boundary point, say, 1, of Δ.

We thus need to check that the local generator $d\tau_{11} + \ldots d\tau_{\ell\ell}$ of Ω' is in the pullback of $\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ to $\mathcal{D}_g$ near $\varphi(s)$. This is a matter of understanding a base change.

Any two identifications of $\widetilde{\mathcal{A}_g} \cong \mathcal{D}_g$ differ by postcomposition with the action of some $M \in \mathrm{Sp}(2g, \mathbb{R})$. This applies in particular to the normalization of the multidisc in Proposition 3.1 and the one where the $F_{g'}$ are rational boundary components.

Since $\tilde{\varphi}$ is the diagonal embedding onto the multidisc of dimension ℓ times a map $Z_{g-\ell}(\tau)$ with $||Z_{g-\ell}(\tau)|| < |\tau|$, the point $\varphi(s)$ lies on a boundary component of degree $g' = g - \ell$, i.e.,

$$\widetilde{\varphi}(1) \ = \begin{pmatrix} 1_\ell & 0 \\ 0 & Z_{g-\ell}(1) \end{pmatrix} \in \overline{D_g},$$

still with respect to the identification $\widetilde{\mathcal{A}_g} \cong \mathcal{D}_g$ using conjugation by the matrix U of Proposition 3.1.

This implies that the base change $M \in N(F_{g'})$. Recall that the action of a symplectic matrix on D_g is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = \frac{(A - IC)(Z + 1) + (B - ID)I(Z - 1)}{(A + IC)(Z + 1) + (B + ID)I(Z - 1)}.$$

Using this, one quickly calculates, writing $M \in N(F_{g'})$ in blocks as in equation (4.1),

$$(dM^*)\begin{pmatrix}(d\tau_{ij}) & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}u\cdot(d\tau_{ij}) & 0\\ 0 & 0\end{pmatrix}.$$

In particular, $(dM^*)(d\tau_{11} + \ldots d\tau_{\ell\ell})$ is a $\mathbb{C}$-linear combination of dx_{ij}, what we needed to show. q.e.d.

5. Affine geodesics for the Kobayashi metric

In this section we will assume that the Kobayashi geodesic Y_0 is affine, and hence that $S \neq \emptyset$, and we will show that the image of Y_0 in $\mathcal{A}_g$ lies in a Hilbert modular surface U. If the general fiber of f_0 is simple, U will be indecomposable, in the sense that no finite étale cover can be a product.

Theorem 5.1. *Let Y_0 be an affine curve, and let $\varphi_0 : Y_0 \to \mathcal{A}_g$ be a geodesic for the Kobayashi metric induced by a family $f_0 : X_0 \to Y_0$ of polarized abelian varieties. Assume that the induced variation of Hodge structures $\mathbb{W}_{\mathbb{Q}} = R^1 f_* \mathbb{Q}_{X_0}$ is irreducible, or equivalently that the general fiber of f_0 is simple.*

Then there exists a totally real number field $\sigma : K \subset \mathbb{C}$ of degree g over $\mathbb{Q}$ and a rank-two variation of Hodge structures $\mathbb{L}_K$ of weight one, such that:

1) *$\mathbb{L} = \mathbb{L}_K \otimes_K \mathbb{C}$ is maximal Higgs. In particular, the Higgs bundle of $\mathbb{L}$ is of the form*
$$\big(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau : \mathcal{L} \xrightarrow{\ \cong\ } \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S)\big),$$
where $\mathcal{L}$ is a logarithmic theta characteristic and τ the induced isomorphism.
2) *If $\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_g : K \to \mathbb{C}$ are the different embeddings of K in $\mathbb{C}$, and if $\mathbb{L}_i = \mathbb{L}_K^{\sigma_i}$ is the corresponding conjugate of $\mathbb{L}$, then*
$$\mathbb{W} = \mathbb{W}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_g.$$
3) *None of the $\mathbb{L}_i$ is unitary.*

Corollary 5.2. *In Theorem* 5.1 *there exists a g-dimensional Hilbert modular variety $Z_0 \subset \mathcal{A}_g$ such that $\varphi_0 : Y_0 \to \mathcal{A}_g$ factors through $Z_0 \subset \mathcal{A}_g$. We have $\ell = g$ in the notation of Proposition* 3.1*, and the corresponding polydisc is the universal covering of Z.*

Proof of Theorem 5.1. Several of the arguments used in the proof are taken from [**VZ04**, Section 3 and 4].

By Theorem 1.2 and by Lemma 1.3, $\mathbb{W}$ contains a direct factor $\mathbb{V}$ which is of the form $\mathbb{L} \otimes \mathbb{U}$ with $\mathbb{U}$ unitary and with $\mathbb{L}$ maximal Higgs of rank two, and hence induced by a logarithmic theta characteristic $\mathcal{L}$.

The local system $\mathbb{U}^{\otimes 2}$ is a subsystem of $\mathbb{W}^{\otimes 2}$. So on one hand, the local monodromy in $s \in S$ is unipotent, and on the other hand it is unitary, of finite order. Then the local residues of $\mathbb{U}^{\otimes 2}$ and hence of $\mathbb{U}$ in $s \in S$ are trivial and $\mathbb{U}$ extends to a unitary local system on Y, again denoted by $\mathbb{U}$. Writing $\mathcal{U} = \mathbb{U} \otimes \mathcal{O}_Y$, the residues of $\mathbb{L} \otimes \mathbb{U}$ in $s \in S$ are given by

$$\mathcal{L} \otimes \mathcal{U} \xrightarrow{\ \tau\ } \mathcal{L}^{-1} \otimes \mathcal{U} \otimes \Omega^1_Y(\log S)\big) \xrightarrow{\text{res}} \mathcal{L}^{-1} \otimes \mathcal{U}|_s,$$

and hence one finds:

Claim 5.3. The residues of $\mathbb{L}\otimes\mathbb{U}$ in $s \in S$ are isomorphisms $\mathcal{L}\otimes\mathcal{U}|_s \to \mathcal{L}^{-1} \otimes \mathcal{U}|_s$.

Since the rank-two variations of Hodge structures, given by a theta characteristic, are unique up to the tensor product with local systems induced by two-division points, we can write

$$\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{T} \quad \text{with} \quad \mathbb{W}_1 = \mathbb{L} \otimes \mathbb{U} \oplus \mathbb{V}',$$

where $\mathbb{T}$ is the maximal unitary local subsystem and where none of the direct factors of $\mathbb{V}'$ is maximal Higgs. By abuse of notations we write $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$, allowing $\mathbb{V}$ to be reducible.

Claim 5.4. The unitary part $\mathbb{T}$ is zero.

Proof. Assume the contrary. By [**VZ04**, Lemma 3.3] the first decomposition is defined over a number field. As above, the residues of $\mathbb{T}$ in $s \in S$ are zero. This being invariant under Galois conjugation, the same holds true for all conjugates of $\mathbb{T}$ in $\mathbb{W}$. Since we assumed that $\mathbb{W}$ is irreducible over $\mathbb{Q}$, the same holds for $\mathbb{W}$, contradicting Claim 5.3. q.e.d.

Claim 5.5. The direct sum decomposition $\mathbb{W} = \mathbb{V} \oplus \mathbb{V}'$ is defined over a number field.

Proof. If not, as in the proof of [**VZ04**, Lemma 3.3], one obtains a non-trivial family $\mathbb{V}'_t$ of local systems over the disk Δ with $\mathbb{V}'_0 = \mathbb{V}'$. The non-triviality of the family forces the composition

$$\mathbb{V}'_t \to \mathbb{W} \longrightarrow \mathbb{L} \otimes \mathbb{U}$$

to be non-zero, for some t arbitrarily close to zero. The complete reducibility of local systems, coming from variations of Hodge structures, implies that some local system $\mathbb{V}''$ lies in both, $\mathbb{V}'_t$ and $\mathbb{L} \otimes \mathbb{U}$. Since for the second one the Higgs fields are isomorphism, the same holds true for $\mathbb{V}''$. On the other hand, for t sufficiently small, $\mathbb{V}'_t$ will not contain any local subsystem with isomorphisms as Higgs fields. q.e.d.

Since $\mathbb{V}$ is defined over a number field, we can apply [**VZ04**, Corollary 3.7] and we find that $\mathbb{L}$ and $\mathbb{U}$ can be defined over a number field as well.

Consider next the sublocal system $\mathbb{V}^{\otimes 2}$ of $\mathbb{W}^{\otimes 2}$. Since $\mathbb{L}$ is self-dual, one can identify $\mathbb{L}^{\otimes 2}$ with $\mathbb{E}\mathrm{nd}(\mathbb{L})$, and hence one obtains a trivial direct factor and its complement, denoted by $\mathbb{L}_0^{\otimes 2}$ in the sequel. So $\mathbb{V}^{\otimes 2}$ decomposes as a direct sum of $\mathbb{L}_0^{\otimes 2} \otimes \mathbb{U}^{\otimes 2}$ and $\mathbb{U}^{\otimes 2}$. For the first one, the residues are again given by isomorphisms, i.e., there is a single Jordan block of maximal size, whereas for the second one the residues are zero.

These properties are preserved under Galois conjugation. Since for $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ one has $\mathbb{V}^\sigma = \mathbb{L}^\sigma \otimes \mathbb{U}^\sigma$, and since the weight of V^σ is one, either $\mathbb{L}^\sigma$ or $\mathbb{U}^\sigma$ must carry a weight zero Hodge structure and hence be unitary. The first case cannot occur, since the residues are isomorphisms. So the maximal unitary subbundle $\mathbb{T}'$ of $\mathbb{W}^{\otimes 2}$ is the Weil restriction of $\mathbb{U}^{\otimes 2}$. So it is defined over $\mathbb{Q}$ and inherits a $\mathbb{Z}$ structure from the one of $\mathbb{W}^{\otimes 2}$. This implies that $\mathbb{T}'$ trivializes after replacing Y_0 by an étale covering, and hence $\mathbb{U}^{\otimes 2}$ as well. In particular, writing $\mathbb{U} = \mathbb{U}_1 \oplus \cdots \oplus \mathbb{U}_s$ for the decomposition of $\mathbb{U}$ in $\mathbb{C}$-irreducible direct factors, one finds

$$\mathbb{U}^{\otimes 2} = \bigoplus_{i,j} \mathbb{U}_i \otimes \mathbb{U}_j \cong \bigoplus \mathbb{C},$$

which is only possible if all $\mathbb{U}_j$ are isomorphic and of rank one, and hence trivial after passing to a finite covering of Y, étale over Y_0.

Up to now we verified the existence of a direct sum decomposition of $\mathbb{W}$, satisfying the conditions (1), (2), and (3), over some number field L. For the next step, we will not need that Y_0 is affine.

Claim 5.6. Assume that over some curve Y_0 and for some number field L there exists a direct sum decomposition satisfying the conditions (1), (2), and (3) stated in Theorem 5.1, and assume that L is minimal with this property. Then $[L : \mathbb{Q}] = g$, and L is either totally real or an imaginary quadratic extension of a totally real number field K.

Proof. Let L be the field of moduli of $\mathbb{L}_1$, i.e., L is the fixed field of all $\sigma \in \mathrm{Gal}_{\overline{\mathbb{Q}}/Q}$ such that $\mathbb{L}_1^\sigma \cong \mathbb{L}_1$. Obviously one finds that $[L : \mathbb{Q}]$ is equal to the number of conjugates, and hence equal to g.

Let K be the trace field of $\mathbb{L}_1$. Since $\mathbb{L}_1$ is isomorphic to a Fuchsian representation, K is real. It is well known (e.g., [**Ta69**]) that a rank-two local system can be defined over an extension of degree at most two over its trace field. In particular, $[L : K] \leq 2$.

We claim that L injects into the endomorphism ring of the family f_0. In fact, let F be the Galois closure of $L/\mathbb{Q}$ and choose coset representatives τ_i of $\mathrm{Gal}(F/\mathbb{Q})/\mathrm{Gal}(F/L)$. Then for $x \in F$,

$$\sum_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} \sigma(x) \cdot id_{\mathbb{L}^\sigma} = \sum_{\tau_i} \tau_i(\mathrm{tr}_L^F(x)) \cdot id_{\mathbb{L}^{\tau_i}}$$

is in $\mathrm{End}_{\mathbb{Q}}(\mathbb{W}_Q)$ of bidegree $(0,0)$, and hence an endomorphism of f_0. Specializing to $x \in L$ proves the claim.

By Albert's classification ([**BL04**] §5.5) of endomorphism rings and since the general fiber of f_0 is simple, L is totally real or an imaginary quadratic extension of a totally real field K. q.e.d.

To finish the proof of Theorem 5.1 it remains to show that the minimal field L in Claim 5.6 has to be totally real, if Y_0 is affine. If $K = L$, there is nothing left to show.

We distinguish cases according to what the full endomorphism ring $W_{\mathbb{Q}}$ is, following notations in [**BL04**] §5.5. First, suppose the endomophism ring is of the second kind, i.e., the Rosati involution fixes L. Then since $[K : \mathbb{Q}] = \mathrm{rk}(\mathbb{W})$ and since all the $\mathbb{L}_i$ have non-trivial Higgs field, the endomorphism ring is in fact an indefinite quaternion algebra B, containing L ([**BL04**] 9.10.Ex. (4), [**Sh63**] Proposition 18).

Second, suppose that the endomorphism ring is a totally definite quaternion algebra. Then (by [**BL04**] 9.10.Ex (1) or [**Sh63**] Proposition 15) the general fiber of f_0 is not simple. This contradicts the irreducibility hypothesis of the theorem.

Thus it remains only to discuss the third case, a totally indefinite quaternion algebra B. Let Z be the ("PEL"-) Shimura variety of abelian varieties with $B \subset \mathrm{End}(\mathbb{W}_{\mathbb{Q}})$. To construct Z, let G be the $\mathbb{Q}$-algebraic group with $G(\mathbb{Q}) = B^*$, fix a maximal compact subgroup K, and form the quotient of G/K by an arithmetic lattice Γ. For the universal family A over $\Gamma\backslash G/K$ we have, by construction, a homomorphism $B \to \mathrm{End}_Q(A)$, which is injective since B is simple. By [**Sh71**] Proposition 9.3 we conclude that $Z = \Gamma\backslash G/K$ is compact. Thus the map $Y_0 \to Z$ would extend to $Y \to Z$, contradicting the maximality of the Higgs field. q.e.d.

Remark 5.7. If in Claim 5.6 $L \neq K$, then Y_0 is projective. For $K = \mathbb{Q}$ examples are the moduli schemes of *false elliptic curves*, i.e., of polarized abelian surfaces Z with $\mathrm{End}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$ a totally indefinite quaternion algebra over $\mathbb{Q}$.

In a similar way, there should exist compact *false Teichmüller curves* in $\mathcal{A}_g$. The corresponding variation of Hodge structures $\mathbb{W}_{\mathbb{Q}}$ should have a K irreducible direct factor $\mathbb{T}_K$ of rank four, for some totally real number field K, and $\mathbb{T} = \mathbb{T}_K \otimes \mathbb{C}$ should be the direct sum of two complex variations of Hodge structures of rank two.

6. Families of Kobayashi geodesics

Let k be an algebraically closed subfield of $\mathbb{C}$, let U be an irreducible non-singular variety, defined over k, and let $h : \mathcal{Y} \to U$ be a smooth family of projective curves. We consider a relative normal crossing divisor $\mathcal{R} \in \mathcal{Y}$ and write $\mathcal{Y}_0 = \mathcal{Y} \setminus \mathcal{R}$. Recall that $\overline{\mathcal{A}}_g$ denotes the boundary divisor of a Mumford compactification of $\mathcal{A}_g$.

We assume in the sequel that $\tilde{\varphi}_0 : \mathcal{Y}_0 \to \mathcal{A}_g$ is a generically finite morphism, defined over k, which extends to a morphism $\tilde{\varphi} : \mathcal{Y} \to \overline{\mathcal{A}}_g$. Here we consider $\overline{\mathcal{A}}_g$ as a k-variety. So $\tilde{\varphi}$ and h define a morphism

$$\varphi' : \mathcal{Y} \to \overline{\mathcal{A}}_{g\,U} := \overline{\mathcal{A}}_g \times U.$$

We want that all fibers of $h|_{\mathcal{Y}_0}$ are Kobayashi geodesics. However, since we do not want to study the behavior of this property in families, we use instead an extension of condition (b) in Theorem 1.2 to the relative case; hence, the natural morphism

$$\Psi : \varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{R}), \tag{6.1}$$

where $S_{\overline{\mathcal{A}}_{g\,U}} = S_{\overline{\mathcal{A}}_g} \times U = \overline{\mathcal{A}}_{g\,U} \setminus \mathcal{A}_g \times U$.

Lemma 6.1. *We keep the notations introduced above and assume that the morphism Ψ in* (6.1) *splits, that $\tilde{\varphi} : \mathcal{Y} \to \overline{\mathcal{A}}_g$ is generically finite, and that the general fiber of the pullback to $\mathcal{Y}$ of the universal family of abelian varieties is simple. Then:*

i. *The family $\mathcal{Y}_0 \to U$ is locally trivial, i.e., locally in the analytic topology a product.*
ii. *If for $u \in U$ in general position the fiber $Y = h^{-1}(u)$ is affine, then U is a point.*

Proof. The splitting of the map Ψ in (6.1) implies that on all fibers Y of h one has a splitting of

$$\varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \longrightarrow \Omega^1_Y(\log S). \tag{6.2}$$

So by Theorem 1.2 all fibers of h are Kobayashi geodesics. Let Ω denote a subsheaf of $\varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}}))$ for which

$$\Psi|_{\Omega} : \Omega \to \Omega^1_{\mathcal{Y}/U}(\log \mathcal{R})$$

an isomorphism. Since

$$\varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) = \varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \oplus h^* \Omega^1_U,$$

the subsheaf $\Omega \oplus \{0\}$ defines a splitting of the composition

$$\varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}/U}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{R}).$$

This composition factors through

$$\varphi'^* \Omega^1_{\overline{\mathcal{A}}_{g\,U}}(\log(S_{\overline{\mathcal{A}}_{g\,U}})) \longrightarrow \Omega^1_{\mathcal{Y}}(\log S_{\mathcal{Y}}) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{R}).$$

Hence the exact sequence

$$0 \to h^* \Omega^1_U \to \Omega^1_{\mathcal{Y}}(\log S_{\mathcal{Y}}) \to \Omega^1_{\mathcal{Y}/U}(\log \mathcal{R}) \to 0 \tag{6.3}$$

splits as well. Recall that the Kodaira-Spencer map

$$\rho_u : T_{u,U} \longrightarrow H^1(Y, T_Y(-\log R)),$$

controlling the infinitesimal deformations of $Y \setminus R$ for $R = \mathcal{R}_u$ and $Y = h^{-1}(u)$, is given by the edge morphism of the dual exact sequence of (6.3), restricted to $u \in U$ (see [**Ka78**], for example). So the splitting of (6.3) implies that ρ_u is zero, for all $u \in U$. By [**Ka78**, Corollary 4] this implies that the family $\mathcal{Y}_0 \to U$ is locally a product, as claimed in (i).

From now on we will use the analytic topology and hence assume by abuse of notations that $\mathcal{Y} = Y_0 \times U$. Part (ii) of the Lemma will follow if we show that the morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$ is rigid. Using the description of the infinitesimal deformations of subvarieties of $\mathcal{A}_g$ in [**Fa83**], one has to verify that $\mathrm{End}(R^1 f_* \mathbb{Q}_X) = \mathrm{End}(\mathbb{W}_{\mathbb{Q}})$ is concentrated in bidegree $(0,0)$.

Part (ii) of the conclustion of Theorem 5.1 allows us to write

$$\mathbb{W} = \mathbb{L}_1^{\oplus \nu_1} \oplus \cdots \oplus \mathbb{L}_\ell^{\oplus \nu_\ell},$$

where $\mathbb{L}_1 = \mathbb{L}$ satisfies the Arakelov equality, where $\mathbb{L}_1, \ldots, \mathbb{L}_\ell$ are pairwise non-isomorphic, and where all the $\mathbb{L}_j$ are irreducible of rank two. Then there are no global homomorphisms $\mathbb{L}_i \to \mathbb{L}_j$, except for $i = j$, and $\mathrm{End}(\mathbb{L}_j)^{0,0} = \mathrm{End}(\mathbb{L}_j) \cong \mathbb{C}$. Hence

$$\mathrm{End}(\mathbb{W}) = \bigoplus_{j=i}^{\ell} M(\nu_j, \mathrm{End}(\mathbb{L}_j)) = \bigoplus_{j=i}^{\ell} M(\nu_j, \mathbb{C})$$

is equal to $\mathrm{End}(\mathbb{W})^{0,0}$. q.e.d.

Corollary 6.2. *Let $\varphi_0 : Y_0 \to \mathcal{A}_g$ be an affine Kobayashi geodesic such that the induced variation of Hodge structures $\mathbb{W}_{\mathbb{Q}}$ is $\mathbb{Q}$-irreducible. Then $\varphi_0 : Y_0 \to \mathcal{A}_g$ can be defined over a number field.*

Proof. Let $\overline{\mathcal{A}}$ be a Mumford compactification, defined over $\bar{\mathbb{Q}}$. Recall that we assumed $\mathcal{A}_g$ to be a fine moduli scheme, and that Y_0 is a Kobayashi geodesic if its image in $\mathcal{A}_g$ has this property. So we may assume that $\varphi_0 : Y_0 \to \mathcal{A}_g$ is injective.

The $\mathbb{C}$-morphism $\varphi_0 : Y_0 \to \mathcal{A}_g$ extends to a morphism $\varphi : Y \to \overline{\mathcal{A}}_g$, and by Theorem 1.2 there exists a subsheaf $\Omega \subset \varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ splitting the natural projection

$$\varphi^* \Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) \to \Omega^1_Y(\log S).$$

Choose a field $K \subset \mathbb{C}$, finitely generated over $\bar{\mathbb{Q}}$ such that Y, S, $\varphi : Y \to \overline{\mathcal{A}}_g$, and Ω are defined over K. By abuse of notation we will use the same letters for the objects, as schemes, morphisms, or sheaves over K.

Let U denote a non-singular quasi-projective $\bar{\mathbb{Q}}$-variety with $K = \bar{\mathbb{Q}}(U)$. Choosing U small enough, we can assume (step by step):

- $Y \to \mathrm{Spec}(K)$ is the general fiber of a projective morphism $h : \mathcal{Y} \to U$.

- $\varphi : Y \to \overline{\mathcal{A}}_g$ extends to a morphism $\tilde{\varphi} : \mathcal{Y} \to \overline{\mathcal{A}}_g$.
- h is smooth and $\mathcal{S} := \tilde{\varphi}^{-1}(S_{\overline{\mathcal{A}}_g})_{\rm red}$ consists of disjoint sections of h.
- $\Omega \subset \varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$ is induced by a subsheaf $\tilde{\Omega} \subset \tilde{\varphi}^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g})$.
- $\tilde{\Omega}$ defines a splitting of the natural map

$$\tilde{\varphi}^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}_g}) = (\tilde{\varphi}\times h)^*\Omega^1_{\overline{\mathcal{A}}_g\times U/U}(\log(S_{\overline{\mathcal{A}}_g}\times U)) \longrightarrow \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}).$$

Writing $\mathcal{Y}_0 = \mathcal{Y}\setminus\mathcal{S}$, we claim that $\dim(\tilde{\varphi}(\mathcal{Y}_0)) = 1$. Otherwise we may replace U by the intersection of $\dim(\tilde{\varphi}(\mathcal{Y}_0))-2$ general hyperplane sections, and hence assume that in addition to the properties stated above U is a curve and the morphism $\tilde{\varphi}$ generically finite, contradicting the Lemma 6.1.

If $\dim(\tilde{\varphi}(\mathcal{Y}_0)) = 1$, choose any $\bar{\mathbb{Q}}$-valued point in U. Then

$$\tilde{\varphi}(h^{-1}(u)\cap\mathcal{Y}_0) = \tilde{\varphi}(\mathcal{Y}_0),$$

and Y_0 is obtained by base extension from $h^{-1}(u)\cap\mathcal{Y}_0$. q.e.d.

Remark 6.3. As stated in Lemma 6.1 the proof of Corollary 6.2 is based on two observations. First, the family $\mathcal{Y}_0 \to U$ constructed in the proof of the Corollary is locally a product, and second one has rigidity of the embedding of a fixed curve as a Kobayashi geodesic.

In [**Mö06**] Corollary 6.2 was formulated for Teichmüller curves, but the first part of the argument was omitted.

Remark 6.4. Without a bit of classification of the possible variations of Hodge structures we are not able to give a criterion for rigidity of compact Kobayashi geodesics. As for Shimura curves, the existence of a large unitary subbundle in $\mathbb{W}$ implies non-rigidity. However, there might be other criteria.

7. Examples and comments

Let $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma\in\Gamma)$ be the trace field of the uniformizing group $Y_0 = \mathbb{H}/\Gamma$. Moreover, let $d = [K:\mathbb{Q}]$ and $B = K\cdot\Gamma$ be the subalgebra of $\mathrm{GL}_2(\mathbb{C})$ generated by Γ.

Among the curves, satisfying the conditions in Theorem 1.2, one finds:

i) Shimura curves ([**VZ04**]). In this case, the $\mathbb{C}$ local system $\mathbb{W}$ decomposes as a direct sum $\mathbb{V}_1\oplus\cdots\oplus\mathbb{V}_\ell$, and each $\mathbb{V}_i$ is either unitary or maximal Higgs. The algebra B is unramified at all but one of the infinite places of K. The curve Y_0 can be both compact and non-compact in this case.

ii) Teichmüller curves. By [**VZ06**] and [**Mö06**] these are the Kobayashi geodesic curves $Y_0 \subset \mathcal{A}_g$ that lie in the image of M_g. In this case, $\ell = 1$ by [**Mö06**], and if $d>1$ then $\mathbb{V}^\perp$ is not unitary. B is ramified at all the infinite places of K. The curve Y_0 is not compact in this case.

iii) There are examples of compact Kobayashi geodesics Y_0 that are not Shimura curves. In fact, by [**DM86**] (see also [**CW90**]) all triangle groups $\Gamma = \Delta(l, m, n)$ arise as uniformizing groups of Kobayashi geodesic curves $Y_0 \subset \mathcal{A}_g$. If $l, m, n < \infty$ are chosen not to be in Takeuchis's finite list of arithmetic triangle groups ([**Ta75**]), then Y_0 is neither a Shimura curve nor a Teichmüller curve.

iv) Not all Kobayashi geodesics in $\mathcal{A}_g$ with $d > 1$, with Y_0 not compact, and B ramified at all the infinite places of K are Teichmüller curves in M_g. In fact, the Prym construction of [**McM06**] gives Teichmüller curves in M_4, whose family of Jacobian splits. The Prym-antiinvariant part together with 1/2 the pullback polarization gives such a Kobayashi geodesic in A_2.

Given this list of examples, the following questions remain open:

1) Are there compact Kobayashi geodesics in $\mathcal{A}_g$ that are neither Shimura curves nor uniformized by a group commensurable to a triangle group?
2) Can one classify all non-Shimura Kobayshi geodesics on the image of a Hilbert modular surface in A_2 for a given discriminant?

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