

## THE DIFFERENTIABLE-INVARIANCE OF THE ALGEBRAIC MULTIPLICITY OF A HOLOMORPHIC VECTOR FIELD

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### Abstract

We prove that the algebraic multiplicity of a holomorphic vector field at an isolated singularity is invariant by topological equivalences which are differentiable at the singular point.

### 1. Introduction

Given a holomorphic curve  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ , singular at  $0 \in \mathbb{C}^2$ , we define its *algebraic multiplicity* as the degree of the first nonzero jet of  $f$ , that is,  $\nu(f) = \nu$  where

$$f = f_\nu + f_{\nu+1} + \cdots$$

is the Taylor development of  $f$  and  $f_\nu \neq 0$ . A well known result by Burau [2] and Zariski [15] states that  $\nu$  is a *topological invariant*, that is, given  $\tilde{f} : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  and a homeomorphism  $h : U \rightarrow \tilde{U}$  between neighborhoods of  $0 \in \mathbb{C}^2$  such that  $h(f^{-1}(0) \cap U) = \tilde{f}^{-1}(0) \cap V$  then  $\nu(f) = \nu(\tilde{f})$ . Consider now a holomorphic vector field  $Z$  in  $\mathbb{C}^2$  with a singularity at  $0 \in \mathbb{C}^2$ . If

$$Z = Z_\nu + Z_{\nu+1} + \cdots, \quad Z_\nu \neq 0$$

we define  $\nu = \nu(Z)$  as the *algebraic multiplicity* of  $Z$  at  $0 \in \mathbb{C}^2$ . The vector field  $Z$  defines a holomorphic foliation by curves  $\mathcal{F}$  with isolated singularity in a neighborhood of  $0 \in \mathbb{C}^2$  and the algebraic multiplicity  $\nu(Z)$  depends only on the foliation  $\mathcal{F}$ . A natural question, posed by J.F. Mattei is: is  $\nu(\mathcal{F})$  a topological invariant of  $\mathcal{F}$ ? In [3], the authors give a positive answer if  $\mathcal{F}$  is a *generalized curve*, that is, if the desingularization of  $\mathcal{F}$  does not contain complex saddle-nodes. If  $\mathcal{F}$  is *dicritical*, that is, after a blow up the exceptional divisor is not invariant by the strict transform of  $\mathcal{F}$ , the conjecture is also true: in this case, it is not difficult to show that the algebraic multiplicity of  $\mathcal{F}$  is equal to the index of  $\mathcal{F}$  (as defined in [3]) along a generic separatrix. Then the topological invariance of the algebraic multiplicity of a dicritical singularity is a consequence of the topological invariance of the index

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Received 05/05/2006.

along a curve, which is proved in [3]. Thus, in this paper we always assume the non-dicritical case. Given foliations  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  with isolated singularities at  $0 \in \mathbb{C}^2$ , we say that  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  are *topologically equivalent* (at  $0 \in \mathbb{C}^2$ ) if there is a homeomorphism  $h : U \rightarrow \tilde{U}$ ,  $h(0) = 0$  between neighborhoods of  $0 \in \mathbb{C}^2$ , taking leaves of  $\mathcal{F}$  to leaves of  $\tilde{\mathcal{F}}$ . Such a homeomorphism is a *topological equivalence* between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$ . In this work we impose conditions on the topological equivalence  $h : U \rightarrow \tilde{U}$  and prove the following.

**Theorem 1.1.** *Let  $h : U \rightarrow \tilde{U}$  be a topological equivalence between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  and assume that  $h$  preserves the orientation of  $\mathbb{C}^2$ . Suppose that  $h$  is differentiable at  $0 \in \mathbb{C}^2$  and such that  $dh(0) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is a real isomorphism. Then the algebraic multiplicities of  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  are the same.*

Let  $\pi : \widehat{\mathbb{C}^2} \rightarrow \mathbb{C}^2$  be the blow up at  $0 \in \mathbb{C}^2$ . Given a complex line  $P$  passing through  $0 \in \mathbb{C}^2$ , we say that  $P$  is *regular for  $\mathcal{F}$* , if the strict transform of  $P$  by  $\pi$  intersects the divisor  $E$  at a regular point of the strict transform of  $\mathcal{F}$ . The following theorem is a key step in the proof of Theorem 1.1.

**Theorem 1.2.** *Let  $h : U \rightarrow \tilde{U}$  be a topological equivalence between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  and assume that  $h$  preserves the orientation of  $\mathbb{C}^2$ . Let  $P$  and  $\tilde{P}$  be two complex lines passing through  $0 \in \mathbb{C}^2$  which are regular for  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  respectively. Suppose that  $P \cap U$  is homeomorphic to a disc and  $h(P \cap U) = \tilde{P} \cap \tilde{U}$ . Then the algebraic multiplicities of  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  are equal.*

The paper is organized as follows. In section 2 we prove a weaker version of Theorem 1.2. In section 3 we stay and prove a topological lemma, fundamental for the following sections. We prove Theorem 1.2 in section 4. Finally, in section 5 we prove Theorem 1.1.

**Acknowledgments.** The contents of this paper originally comprised a Ph.D. dissertation at Instituto de Matemática Pura e Aplicada, Rio de Janeiro. The author would like to thank his advisor, César Camacho, for guidance and support. I also thank Alcides Lins Neto, Paulo Sad, Luis Gustavo Mendes and specially Jorge Vitório Pereira for the remarks that helps in the redaction of the present paper.

## 2. A first theorem.

Let  $h : U \rightarrow \tilde{U}$  be a topological equivalence between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$ . Let  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  be the strict transforms of  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  respectively. Let  $W$  and  $\tilde{W}$  be denote the sets  $\pi^{-1}(U)$  and  $\pi^{-1}(\tilde{U})$  respectively. Let

$$h : W \setminus E \rightarrow \tilde{W} \setminus E$$

be the homeomorphism defined by  $h = \pi^{-1} \circ h \circ \pi$ . We have a natural fibration  $\rho$  on  $\widehat{\mathbb{C}^2}$  whose fibers are the strict transforms of the complex lines passing through  $0 \in \mathbb{C}^2$ . Consider  $p, \tilde{p} \in E$  and let  $L_p$  and  $L_{\tilde{p}}$  be the fibers of  $\rho$  passing through  $p$  and  $\tilde{p}$  respectively. This section is devoted to prove the following.

**Theorem 2.1.** *Suppose that  $p$  and  $\tilde{p}$  are regular points of  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  respectively. Let  $\Omega$  be a neighborhood of  $p$  in  $\widehat{\mathbb{C}^2}$ . Suppose that  $h$  extends to  $(W \setminus E) \cup \Omega$  as a homeomorphism onto its image, such that  $h(L_p \cap W) = L_{\tilde{p}} \cap \tilde{W}$ . Then the algebraic multiplicities of  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$  are the same.*

Let  $\nu$  be the algebraic multiplicity of  $\mathcal{F}$  at  $0$  and let  $p_1, \dots, p_k$  be the singularities of  $\mathcal{F}_0$  on  $E$ . We have the following relation due to Ven Den Essen (see [9], appendix I):

$$\sum_{i=1}^k \mu(\mathcal{F}_0, p_i) = \mu(\mathcal{F}, 0) - \nu^2 + \nu + 1,$$

where  $\mu(\mathcal{F}, p)$  is the Milnor number of  $\mathcal{F}$  at  $p$ . Let  $s = \sum_{i=1}^k \mu(\mathcal{F}_0, p_i)$ . In the same way, let  $\tilde{s}$  be the sum of the Milnor numbers of the singularities on  $E$  of  $\tilde{\mathcal{F}}_0$ . Then, since the Milnor number is a topological invariant, it is sufficient to prove that  $s = \tilde{s}$ .

Let  $\mathcal{D} \subset E \cap \Omega$  be a closed disc containing  $p$ , which does not contain singularities of  $\mathcal{F}_0$  and such that  $h(\mathcal{D})$  does not contain singularities of  $\tilde{\mathcal{F}}_0$ . Let  $D$  and  $\tilde{D}$  be the closed discs in  $E$  equal to the closure of  $E \setminus \mathcal{D}$  and  $E \setminus h(\mathcal{D})$  respectively. Then  $h$  maps  $W \setminus D$  homeomorphically onto  $\tilde{W} \setminus \tilde{D}$ , and the interiors of  $D$  and  $\tilde{D}$  contain all the singularities of  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  respectively. Observe that  $h$  is a topological equivalence between  $\mathcal{F}_0|_{W \setminus D}$  and  $\tilde{\mathcal{F}}_0|_{\tilde{W} \setminus \tilde{D}}$ . Since  $h(L_p \cap W) = L_{\tilde{p}} \cap \tilde{W}$ , we have the homeomorphism

$$h : (W \setminus D) \setminus L_p \rightarrow (\tilde{W} \setminus \tilde{D}) \setminus L_{\tilde{p}}.$$

We know that  $W \setminus L_p$  and  $\tilde{W} \setminus L_{\tilde{p}}$  are isomorphic to  $\mathbb{C}^2$ , where the divisor can be represented by the vertical line  $\{z_1 = 0\}$  and the sets  $W \setminus L_p$  and  $\tilde{W} \setminus L_{\tilde{p}}$  give neighborhoods  $V$  and  $\tilde{V}$  of  $\{z_1 = 0\}$ . Thus, we may think that the foliations  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  are defined on the sets  $V$  and  $\tilde{V}$  in  $\mathbb{C}^2$ , and that

$$h : V \setminus D \subset \mathbb{C}^2 \rightarrow \tilde{V} \setminus \tilde{D} \subset \mathbb{C}^2$$

is a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$ . Observe that  $\mathcal{F}_0$  is globally defined by a holomorphic vector field on  $V$  and the same holds for  $\tilde{\mathcal{F}}_0$  on  $\tilde{V}$ . The disc  $D$  is contained in  $\{z_1 = 0\}$  and we may assume that  $D = \{(0, z_2) : |z_2| \leq r\}$ , where  $r > 0$ .

We proceed now to compute  $s$ . Let  $Z$  be a holomorphic vector field which generates the foliation  $\mathcal{F}_0$  on  $V$ . Let  $B$  be a neighborhood of  $D$  homeomorphic to a ball, such that  $\partial B$  is homeomorphic to  $S^3$  and  $\overline{B} \subset V$ . It is well known that the Milnor number is just the Poincaré-Hopf index (considering the holomorphic vector field as a real vector field). Then, since all the singularities of  $\mathcal{F}_0$  are contained in  $B$ , we have ([10], p. 36) that the sum of the Milnor numbers of the singularities of  $\mathcal{F}_0$  is equal to the degree of the map

$$\frac{Z}{\|Z\|} : \partial B \rightarrow \mathbb{S}^3,$$

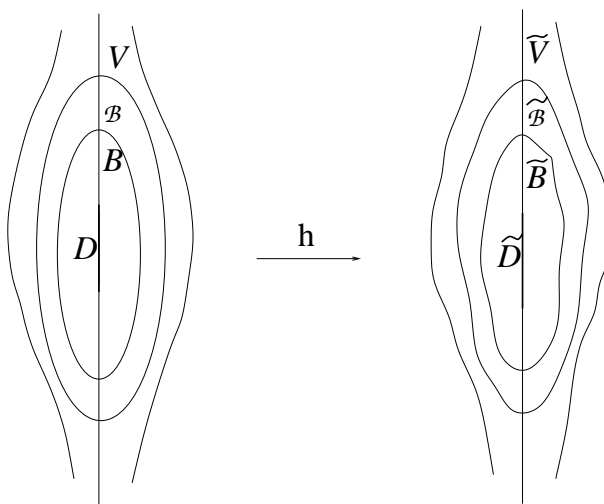
$$\frac{Z}{\|Z\|}(z) = \frac{Z(z)}{\|Z(z)\|}.$$

Let  $\mathcal{B}$  be a neighborhood of  $\overline{B}$  homeomorphic to a ball and such that  $\overline{\mathcal{B}} \subset V$ . Since  $V$  is a neighborhood of  $\{z_1 = 0\}$ , for  $\varepsilon > 0$  small enough, the set  $\{|z_1| < 2\varepsilon, |z_2| < 4r\}$ , which contains  $D$ , is contained in  $V$ . Then, we may choose  $B$  and  $\mathcal{B}$  such that

$$\mathcal{B} \subset \{|z_1| < \varepsilon, |z_2| < 2r\}.$$

The last hypothesis will be only used in the proof of Lemma 2.5.

Consider the sets  $\tilde{B} = h(B \setminus D) \cup \tilde{D}$ ,  $\tilde{\mathcal{B}} = h(\mathcal{B} \setminus D) \cup \tilde{D}$  and  $\tilde{V} = h(V \setminus D) \cup \tilde{D}$ . It is easy to see that  $\tilde{B}$ ,  $\tilde{\mathcal{B}}$  and  $\tilde{V}$  are neighborhoods of  $\tilde{D}$  in  $\mathbb{C}^2$ .



Let

$$\varphi : \overline{\mathbb{D}}_\varepsilon \times \overline{\mathcal{B}} \rightarrow V \subset \mathbb{C}^2$$

and

$$\tilde{\varphi} : \overline{\mathbb{D}}_\varepsilon \times \tilde{\mathcal{B}} \rightarrow \tilde{V} \subset \mathbb{C}^2$$

be the local complex flows of  $Z$  and  $\tilde{Z}$  respectively, where  $\mathbb{D}_\varepsilon = \{T \in \mathbb{C} : \|T\| < \varepsilon\}$  with  $\varepsilon$  small enough. Now, we follow the ideas used in [3] to prove the topological invariance of the Milnor number.

**Lemma 2.2.** *There exists continuous functions  $\tau : \mathcal{B} \setminus D \rightarrow (0, \varepsilon)$  and  $\tilde{\tau} : h(\mathcal{B} \setminus D) \rightarrow \mathbb{D}_\varepsilon \setminus \{0\}$  such that for all  $z \in \mathcal{B} \setminus D$  we have:*

- (i)  $\varphi(\tau(z), z) \in \mathcal{B} \setminus D$ .
- (ii)  $\varphi(t\tau(z), z) \neq z$ , for any  $t \in (0, 1]$ .
- (iii)  $h(\varphi(\tau(z), z)) = \tilde{\varphi}(\tilde{\tau}(h(z)), h(z))$ .

We say that a function  $f : U \rightarrow \mathbb{R}$  is *lower(upper) semi-continuous* if given  $\epsilon > 0$  and  $x_0 \in U$ , there is a neighborhood  $\Omega$  of  $x_0$  in  $U$  such that  $f(x) \geq f(x_0) - \epsilon$  ( $f(x) \leq f(x_0) + \epsilon$ ) for all  $x \in \Omega$ . We need the following lemma.

**Lemma 2.3.** *Let  $U$  be an open set in  $\mathbb{R}^n$  and let  $f : U \rightarrow \mathbb{R}$  and  $g : U \rightarrow \mathbb{R}$  be an upper and a lower semicontinuous function respectively. Suppose that  $f < g$ . Then there exists a continuous function  $h : U \rightarrow \mathbb{R}$  such that  $f < h < g$ . In particular, if  $g$  is a strictly positive lower semicontinuous function, then there exists a continuous function  $h$  such that  $0 < h < g$ .*

*Proof of Lemma 2.2.* Clearly, given  $z \in \mathcal{B} \setminus D$  there exists  $\delta > 0$  such that  $\varphi(*, z)$  is injective on  $\mathbb{D}_\delta$ . Then define  $\delta(z) > 0$  as the supremum of  $\delta' \leq \varepsilon$  such that  $\varphi(*, z)$  is injective on  $\mathbb{D}_{\delta'}$ .

*Assertion 1.* *The function  $\delta : \mathcal{B} \setminus D \rightarrow (0, \varepsilon]$  is lower semicontinuous.*

*Proof.* Fix  $z_0 \in \mathcal{B} \setminus D$  and let  $\epsilon > 0$ . We will prove that for  $z$  close enough to  $z_0$  we have  $\delta(z) \geq \delta(z_0) - \epsilon$ . Suppose by contradiction that for  $z_k \rightarrow z_0$  we have that  $\varphi(*, z_k)$  is not injective on  $\mathbb{D}_{\delta(z_0) - \epsilon}$ . Then there are points  $t_k, t'_k$  in  $\mathbb{D}_{\delta(z_0) - \epsilon}$ , with  $t_k \neq t'_k$  and such that  $\varphi(t_k, z_k) = \varphi(t'_k, z_k)$  for all  $k$ . By taking a subsequence we may assume that  $t_k \rightarrow a$  and  $t'_k \rightarrow a'$  with  $a, a' \in \overline{\mathbb{D}_{\delta(z_0) - \epsilon}} \subset \mathbb{D}_{\delta(z_0)}$ . By continuity we have

$$\varphi(a, z_0) = \lim_{k \rightarrow \infty} \varphi(t_k, z_k) = \lim_{k \rightarrow \infty} \varphi(t'_k, z_k) = \varphi(a', z_0)$$

and, since  $\varphi(*, z_0)$  is injective on  $\mathbb{D}_{\delta(z_0)}$ , we deduce that  $a = a'$ . Let  $z' = \varphi(a, z_0)$  and take a neighborhood  $\Omega$  of  $z'$  and  $\delta_0 > 0$  such that  $\varphi(*, z)$  is injective on  $\mathbb{D}_{\delta_0}$  for all  $z \in \Omega$ . For  $k$  big enough we have that  $\varphi(a, z_k) \in \Omega$  and  $(t_k - a), (t'_k - a') \in \mathbb{D}_{\delta_0}$ . Then, since

$$\varphi(t_k - a, \varphi(a, z_k)) = \varphi(t_k, z_k) = \varphi(t'_k, z_k) = \varphi(t'_k - a', \varphi(a', z_k)),$$

we have that  $t_k - a = t'_k - a'$ , hence  $t_k = t'_k$ , which is a contradiction.

*Assertion 2.* *Consider  $\bar{\delta} : \mathcal{B} \setminus D \rightarrow (0, \varepsilon]$ , where  $\bar{\delta}(z)$  is the supremum of  $\delta' < \varepsilon$  such that  $\varphi(\mathbb{D}_{\delta'}, z) \subset \mathcal{B} \setminus D$ . Then  $\bar{\delta}$  is a lower semicontinuous function.*

*Proof.* Fix  $z_0$  and let  $\epsilon > 0$ . The set  $\varphi(\overline{\mathbb{D}}_{\bar{\delta}(z_0)-\epsilon}, z_0)$  is compact and is contained in  $\mathcal{B} \setminus D$ . If  $z$  is close enough to  $z_0$  we have that  $\varphi(\overline{\mathbb{D}}_{\bar{\delta}(z_0)-\epsilon}, z)$  is also contained in  $\mathcal{B} \setminus D$ . Then  $\bar{\delta}(z) \geq \bar{\delta}(z_0) - \epsilon$  and it follows that  $\bar{\delta}$  is lower semicontinuous.

Consider  $\tilde{\delta} : h(\mathcal{B} \setminus D) \rightarrow (0, \epsilon]$ , where  $\tilde{\delta}(w)$  is the supremum of  $\delta' < \epsilon$  such that  $\tilde{\varphi}(*, w)$  is injective on  $\mathbb{D}_{\delta'}$ . As in Assertion 1, we can prove that  $\tilde{\delta}$  is a lower semicontinuous function.

*Assertion 3.* Define  $\hat{\delta} : \mathcal{B} \setminus D \rightarrow (0, \epsilon]$ , where  $\hat{\delta}(z)$  is the supremum of  $\delta' < \epsilon$  such that  $h(\varphi(\mathbb{D}_{\delta'}, z))$  is contained in  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z))}, h(z))$ . Then  $\hat{\delta}$  is a lower semicontinuous function.

*Proof.* Fix  $z_0$  and let  $\epsilon > 0$ . Since  $h(\varphi(\mathbb{D}_{\hat{\delta}(z_0)}, z_0))$  is contained in  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z_0))}, h(z_0))$ , there is  $\epsilon' > 0$  such that  $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z_0))$  is contained in  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z_0))-\epsilon'}, h(z_0))$ . Let  $\Sigma$  be a disc passing through  $h(z_0)$  and transverse to the foliation. Since  $\tilde{\delta}$  is lower semicontinuous, we may take  $\Sigma$  small enough such that  $\tilde{\varphi}(*, z)$  is injective on  $\overline{\mathbb{D}}_{\tilde{\delta}(h(z_0))-\epsilon'}$  for all  $z \in \Sigma$ . Moreover, we may take  $\Sigma$  small enough such that  $\tilde{\varphi}$  is injective on  $\mathbb{D}_{\tilde{\delta}(h(z_0))-\epsilon'} \times \Sigma$ . Let  $M$  denote the open set  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z_0))-\epsilon'} \times \Sigma)$  and let  $M' = \tilde{\varphi}(\mathbb{D}_{\epsilon'/2} \times \Sigma)$ . We may take a neighborhood  $\Omega$  of  $z_0$  such that  $h(\Omega) \subset M'$  and  $\tilde{\delta}(h(z)) \geq \tilde{\delta}(h(z_0)) - \epsilon'/2$  for all  $z \in \Omega$ , because  $\tilde{\delta}$  is lower semicontinuous. Since  $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z_0))$  is compact and is contained in  $M$ , we may assume  $\Omega$  small enough such that  $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$  is contained in  $M$  for all  $z \in \Omega$ . Fix  $z \in \Omega$ . Since  $h(z) \in M'$ , there is  $w' \in \Sigma$  and  $t'$ , with  $|t'| < \epsilon'/2$ , such that  $h(z) = \tilde{\varphi}(t', w')$ . Since  $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$  is contained in  $M$ , we deduce that it is contained in  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z_0))-\epsilon'}, w')$ . Then, given  $w$  in  $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$ , we have that  $w = \tilde{\varphi}(t'', w')$  with  $|t''| < \tilde{\delta}(h(z_0)) - \epsilon'$ . Thus

$$w = \tilde{\varphi}(t'', w') = \tilde{\varphi}(t'' - t', \tilde{\varphi}(t', w')) = \tilde{\varphi}(t'' - t', h(z)),$$

where  $|t'' - t'| \leq |t''| + |t'| < \tilde{\delta}(h(z_0)) - \epsilon' + \epsilon'/2 = \tilde{\delta}(h(z_0)) - \epsilon'/2 \leq \tilde{\delta}(h(z))$ . Then  $h(\varphi(\overline{\mathbb{D}}_{\hat{\delta}(z_0)-\epsilon}, z))$  is contained in  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z))}, h(z))$  and it follows that  $\hat{\delta}$  is lower semicontinuous.

It is easy to see that the function  $g = \min\{\delta, \bar{\delta}, \hat{\delta}\}$  is also lower semicontinuous. Then, by Lemma 2.3, there exists a positive continuous function  $\tau$  on  $\mathcal{B} \setminus D$  such that  $\tau < \delta, \bar{\delta}, \hat{\delta}$ . By the definition of  $\bar{\delta}$ , (i) is satisfied. Since  $\varphi(*, z)$  is injective on  $\mathbb{D}_{\bar{\delta}}$  and  $\tau(z) \in \mathbb{D}_{\bar{\delta}}$ , we have that (ii) holds. Now, we shall define  $\tilde{\tau}$ . Let  $w = h(z) \in h(\mathcal{B} \setminus D)$ . Since  $\tau < \hat{\delta}$ , we have that  $h(\varphi(\tau(z), z))$  is contained in  $\tilde{\varphi}(\mathbb{D}_{\tilde{\delta}(h(z))}, h(z))$

and by injectivity there exists a unique  $\tilde{\tau}(h(z))$  in  $\mathbb{D}_{\tilde{\delta}(h(z))}$  such that  $h(\varphi(\tau(z), z)) = \tilde{\varphi}(\tilde{\tau}(h(z)), h(z))$ . Now, it is easy to see that  $\tilde{\tau}$  is continuous and therefore (iii) holds. q.e.d.

*Proof of Lemma 2.3.* Consider  $x \in U$  and  $a_x \in \mathbb{R}$ , such that  $f(x) < a_x < g(x)$ . It follows from the definition of lower and upper semicontinuous function that there exists a neighborhood  $V_x$  of  $x$  in  $U$  such that  $f(y) < a_x < g(y)$  for all  $y \in V_x$ . We may take a subset  $I \subset U$ , such that  $U \subset \bigcup_{i \in I} V_i$  and  $\{V_i\}_{i \in I}$  is locally finite. Thus, we have  $f(x) < a_i < g(x)$  for all  $x \in V_i$ . Let  $\{\psi_i\}_{i \in I}$  be a partition of the unity subordinate to  $\{V_i\}_{i \in I}$ . Then, we define  $h : U \rightarrow \mathbb{R}$  by

$$h(x) = \sum_{i \in I} \psi_i(x) a_i.$$

Clearly,  $h$  is continuous. If  $x \in V_i$ , then  $f(x) < a_i < g(x)$ , hence  $\psi_i(x)f(x) < \psi_i(x)a_i < \psi_i(x)g(x)$  and it follows that  $f < h < g$ . q.e.d.

From Lemma 2.2, we have the maps

$$\begin{aligned} f &: \mathcal{B} \setminus D \rightarrow \mathcal{B} \setminus D, \\ f(z) &= \varphi(\tau(z), z) \end{aligned}$$

and

$$\begin{aligned} \tilde{f} &: \tilde{\mathcal{B}} \setminus \tilde{D} \rightarrow \tilde{\mathcal{B}} \setminus \tilde{D}, \\ \tilde{f}(w) &= \tilde{\varphi}(\tilde{\tau}(w), w) \end{aligned}$$

with

$$h \circ f = \tilde{f} \circ h$$

and such that  $f$  and  $\tilde{f}$  are without fixed points.

There exists  $\psi, \tilde{\psi} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  with the following properties:

- (i)  $\psi(D) = 0$  and  $\tilde{\psi}(\tilde{D}) = 0$ .
- (ii)  $\psi : \mathbb{C}^2 \setminus D \rightarrow \mathbb{C}^2 \setminus \{0\}$  and  $\tilde{\psi} : \mathbb{C}^2 \setminus \tilde{D} \rightarrow \mathbb{C}^2 \setminus \{0\}$  are homeomorphisms.
- (iii)  $\psi$  and  $\tilde{\psi}$  are equal to the identity out of  $B$  and  $\tilde{B}$  respectively.

We define

$$\begin{aligned} f' &= \psi f \psi^{-1} : \mathcal{B} \setminus \{0\} \rightarrow \mathcal{B} \setminus \{0\} \subset \mathbb{C}^2, \\ \tilde{f}' &= \tilde{\psi} \tilde{f} \tilde{\psi}^{-1} : \tilde{\mathcal{B}} \setminus \{0\} \rightarrow \tilde{\mathcal{B}} \setminus \{0\} \subset \mathbb{C}^2, \\ h' &= \tilde{\psi} h \psi^{-1} : V \rightarrow \tilde{V}. \end{aligned}$$

Then we have the following:

- (i)  $f'$  and  $\tilde{f}'$  do not have fixed points.
- (ii) On  $\partial B$ , we have  $f' = f$  and  $\tilde{f}' = \tilde{f}$ .
- (iii)  $h'$  is a homeomorphism with  $h'(0) = 0$  and such that  $h' \circ f' = \tilde{f}' \circ h'$ .

Thus, there are well defined maps:

$$\begin{aligned}(f' - \text{id}) & : \mathcal{B} \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus \{0\}, \\ (\tilde{f}' - \text{id}) & : \tilde{\mathcal{B}} \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus \{0\}.\end{aligned}$$

Observe that  $H_3(\mathcal{B} \setminus \{0\}) \subset H_3(\mathbb{C}^2 \setminus \{0\})$  and this inclusion is an isomorphism between the groups. Then  $(f' - \text{id})$  induces a map

$$(f' - \text{id})_* : H_3(\mathbb{C}^2 \setminus \{0\}) \rightarrow H_3(\mathbb{C}^2 \setminus \{0\})$$

at the homology level.

**Lemma 2.4.**  $(f' - \text{id})_*$  is the multiplication by  $s$ .

*Proof.* We have that  $\partial B \subset \mathcal{B}$  is a generator of  $H_3(\mathbb{C}^2 \setminus \{0\})$ . It is known that, homologically:

$$(f' - \text{id})(\mathbb{S}^3) = (f' - \text{id})(\partial B) = n\mathbb{S}^3,$$

where  $n$  is the degree of the map:

$$\begin{aligned}g : \partial B & \rightarrow \mathbb{S}^3, \\ g(z) & = \frac{(f' - \text{id})}{\|(f' - \text{id})\|}(z).\end{aligned}$$

Thus, it is sufficient to prove that  $\deg(g) = s$ . Observe that  $g = \frac{(f' - \text{id})}{\|(f' - \text{id})\|}$ , since  $f' = f$  on  $\partial B$ . By (ii) of Lemma 2.2 the map

$$\begin{aligned}G : [0, 1] \times \partial B & \rightarrow \mathbb{S}^3, \\ G(t, z) & = \frac{\varphi(t\tau(z), z) - z}{\|\varphi(t\tau(z), z) - z\|}, \quad t \neq 0,\end{aligned}$$

$$G(0, z) = \frac{\tau(z)}{\|\tau(z)\|} \cdot \frac{Z(z)}{\|Z(z)\|}$$

is well defined. Evidently,  $G(1, z) = g(z)$ . On the other hand:

$$\begin{aligned}\lim_{t \rightarrow 0} G(t, z) & = \frac{\tau(z)}{\|\tau(z)\|} \lim_{t \rightarrow 0} \left\| \frac{\varphi(t\tau(z), z) - z}{t\tau(z)} \right\|^{-1} \cdot \lim_{t \rightarrow 0} \frac{\varphi(t\tau(z), z) - z}{t\tau(z)} \\ & = \frac{\tau(z)}{\|\tau(z)\|} \lim_{s \rightarrow 0} \left\| \frac{\varphi(s, z) - z}{s} \right\|^{-1} \cdot \lim_{s \rightarrow 0} \frac{\varphi(s, z) - z}{s} \\ & = \frac{\tau(z)}{\|\tau(z)\|} \cdot \frac{Z(z)}{\|Z(z)\|}.\end{aligned}$$

It follows that  $G$  is continuous and therefore is a homotopy between  $g(z)$  and  $G(0, z) = \frac{\tau(z)}{\|\tau(z)\|} \cdot \frac{Z(z)}{\|Z(z)\|}$ . Now, since  $\pi_3(\mathbb{S}^1) = \{0\}$ , the map  $\tau/|\tau| : \partial B \rightarrow \mathbb{S}^1$  is homotopic to the constant  $1 \in \mathbb{S}^1$  and  $g$  is homotopic to  $Z/\|Z\|$ . Therefore  $\deg(g) = \deg(Z/\|Z\|) = s$ .

In the same way, we have that

$$(\tilde{f}' - \text{id})_* : H_3(\mathbb{C}^2 \setminus \{0\}) \rightarrow H_3(\mathbb{C}^2 \setminus \{0\})$$



is the multiplication by  $\tilde{s}$ .

Let

$$h'_* : H_3(\mathbb{C}^2 \setminus \{0\}) \rightarrow H_3(\mathbb{C}^2 \setminus \{0\})$$

be the isomorphism induced by  $h'$ . Clearly, the following lemma implies Theorem 2.1.

**Lemma 2.5.** *The following diagram commutes:*

$$\begin{array}{ccc} H_3(\mathbb{C}^2 \setminus \{0\}) & \xrightarrow{(f' - \text{id})_*} & H_3(\mathbb{C}^2 \setminus \{0\}) \\ \downarrow h'_* & & \downarrow h'_* \\ H_3(\mathbb{C}^2 \setminus \{0\}) & \xrightarrow{(\tilde{f}' - \text{id})_*} & H_3(\mathbb{C}^2 \setminus \{0\}) \end{array}$$

*Proof.* Recall that  $\mathcal{B}$  was chosen such that

$$\mathcal{B} \subset \{|z_1| < \varepsilon, |z_2| < 2r\} \subset \{|z_1| < 2\varepsilon, |z_2| < 4r\} \subset V.$$

Since  $h' \circ f' = \tilde{f}' \circ h'$  we have  $(\tilde{f}' - \text{id}) \circ h' = \tilde{f}' \circ h' - h' = h' \circ f' - h'$ . It is sufficient to prove that  $h' \circ f' - h'$  and  $h' \circ (f' - \text{id}) : \mathcal{B} \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus \{0\}$  are homotopic. For any  $z \in \mathcal{B} \setminus \{0\}$  and  $t \in [0, 1]$  we have that  $f'(z), (1 - t)z \in \mathbb{D}_\varepsilon \times \mathbb{D}_{2r}$ . Then  $(f'(z) + (1 - t)z)$  is contained in  $\mathbb{D}_{2\varepsilon} \times \mathbb{D}_{4r} \subset V$ . Therefore, the map:

$$F : [0, 1] \times (\mathcal{B} \setminus \{0\}) \rightarrow \mathbb{C}^2 \setminus \{0\},$$

$$F(t, z) = h'(f'(z) - (1 - t)z) - h'(tz)$$

is well defined.  $F$  is continuous and  $F(t, z) \neq 0$  for all  $(t, z) \in [0, 1] \times (\mathcal{B} \setminus \{0\})$  because  $F(t, z) = 0$  implies  $h'(f'(z) - (1 - t)z) = h'(tz)$  and since  $h'$  is a homeomorphism  $f'(z) - (1 - t)z = tz$ , hence  $f'(z) = z$ , which contradicts  $f'(z) \neq z$ . Thus  $F$  is a homotopy between  $h' \circ f' - h'$  and  $h' \circ (f' - \text{id})$ . q.e.d.

### 3. A topological fact.

Let  $M$  be a complex manifold. We say that  $\mathcal{D}$  is a complex disc in  $M$ , if  $\mathcal{D} \subset M$  and there is a map  $f : \overline{\mathbb{D}} \rightarrow M$ , which is a homeomorphism onto  $\mathcal{D}$  and is holomorphic on  $\mathbb{D}$ . Let  $V$  be any subset of  $M$  containing  $\partial\mathcal{D}$ . The map  $f|_{S^1} : S^1 \rightarrow \partial\mathcal{D} \subset M$  defines a 1-cycle in  $V$  and represents an element in  $H_1(V)$  which does not depend on  $f$ . We denote this 1-cycle by  $\partial\mathcal{D}$  independently of the set  $V$ . For simplicity, we write  $\gamma = \gamma'$  in  $H_1(M)$  for means that the 1-cycles  $\gamma$  and  $\gamma'$  represents the same element in the group  $H_1(M)$ . Let  $\pi : \widehat{\mathbb{C}^2} \rightarrow \mathbb{C}^2$  be the blow up at  $0 \in \mathbb{C}^2$  and let  $E = \pi^{-1}(0)$ . Let  $\rho : \widehat{\mathbb{C}^2} \rightarrow E$  be the natural projection. (If  $L$  is the strict transform by  $\pi$  of a complex line passing through  $0 \in \mathbb{C}^2$ , then  $\rho(L) = L \cap E$ .) The following Lemma is a reason for assuming that the topological equivalence  $h$  preserves the orientation of  $\mathbb{C}^2$ .

**Lemma 3.1.** *Let  $h : U \rightarrow U'$  be a homeomorphism, where  $U$  and  $U'$  are neighborhoods of  $0 \in \mathbb{C}^2$  homeomorphic to balls. Let  $P$  and  $P'$  be two complex lines passing through  $0 \in \mathbb{C}^2$ . Suppose that  $P \cap U$  is homeomorphic to a disc and  $h(P \cap U) = P' \cap U'$ . Let  $L$  and  $L'$  be the strict transforms by  $\pi$  of  $P$  and  $P'$  respectively. Let  $p$  and  $p'$  be the points of intersection of  $L$  and  $L'$  with  $E$  respectively. Denote by  $W$  and  $W'$  the sets  $\pi^{-1}(U)$  and  $\pi^{-1}(U')$  in  $\widehat{\mathbb{C}^2}$  and let  $h : W \setminus E \rightarrow W' \setminus E$  be the homeomorphism defined by  $h = \pi^{-1} h \pi$ . Let  $V \subset W$  be a neighborhood of  $p$  and let*

$$\varphi : \mathbb{D} \times \mathbb{D} \rightarrow V$$

*be a biholomorphism such that  $\varphi(\{0\} \times \mathbb{D}) = L \cap V$  and  $\varphi(\mathbb{D} \times \{0\}) = E \cap V$ . Let  $r$  with  $0 < r < 1$  and consider the disc  $\mathcal{B}_w = \varphi(w, |z| \leq r)$ , where  $w \in \mathbb{D}$ . Let  $\Omega$  be a neighborhood of  $p'$  in  $E$ , homeomorphic to a disc. Let  $V' \subset \widehat{\mathbb{C}^2}$  be the set  $\rho^{-1}(\Omega)$ . Let  $\mathcal{A}' \subset V' \setminus E$  and  $\mathcal{B}' \subset V' \setminus L'$  be complex discs transverse to  $L'$  and  $E$  respectively. Then, for  $|w|$  small enough we have the following:*

(i) *If  $h$  preserves the orientation of  $\mathbb{C}^2$ , then*

$$h(\partial\mathcal{B}_w) = \xi\partial\mathcal{B}' \quad \text{in } H_1(V' \setminus (L' \cup E)),$$

*where  $\xi = +1$  or  $-1$ .*

(ii) *If  $h$  inverts the orientation of  $\mathbb{C}^2$ , then*

$$h(\partial\mathcal{B}_w) = -2\xi\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in } H_1(V' \setminus (L' \cup E)),$$

*where  $\xi = +1$  or  $-1$ .*

**Remark.** With some hypothesis on the foliation  $\mathcal{F}$ , we have in fact that the topological equivalence  $h$  necessarily preserves the orientation of  $\mathbb{C}^2$ . Precisely, we have the following.

**Proposition 3.2.** *Let  $\mathcal{F}$  be a holomorphic foliation by curves on  $U$  which has  $0 \in \mathbb{C}^2$  as its unique singularity. Suppose that  $\mathcal{F}$  has three smooth and transverse separatrices. Suppose that  $\tilde{\mathcal{F}}$  is another holomorphic foliation of a neighborhood  $\tilde{U}$  of  $0 \in \mathbb{C}^2$  and let*

$$h : U \rightarrow \tilde{U}$$

*be a topological equivalence between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$ . Then  $h$  preserves the orientation of  $\mathbb{C}^2$ .*

Let  $U \subset \mathbb{C}^2$  be an open set homeomorphic to a ball. Let  $P$  be a complex line in  $\mathbb{C}^2$  and suppose that  $U \cap P$  is homeomorphic to a disc. It follows by Alexander's duality theorem that  $H_1(U \setminus P) \simeq \mathbb{Z}$ . Let  $\mathcal{D} \subset \mathbb{C}^2$  be a complex disc transverse to  $P$ . The 1-cycle  $\partial\mathcal{D}$  represents an element in  $H_1(U \setminus P) \simeq \mathbb{Z}$ , which does not depend on the disc  $\mathcal{D}$ . We know that  $\partial\mathcal{D}$  is a generator of the group and we say that it is the *positive generator* of  $H_1(U \setminus P)$ . Given a homeomorphism  $f : M \rightarrow M'$ ,

where  $M$  and  $M'$  are oriented manifolds, we define  $\deg(f)$  to be 1 or  $-1$  depending on whether  $f$  preserves or reverses orientation.

**Lemma 3.3.** *Let  $h : U \rightarrow U'$  be a homeomorphism, where  $U$  and  $U'$  are neighborhoods of  $0 \in \mathbb{C}^2$  homeomorphic to balls. Let  $P$  and  $P'$  be two complex lines passing through  $0 \in \mathbb{C}^2$ . Suppose that  $P \cap U$  is homeomorphic to a disc and  $h(P \cap U) = P' \cap U'$ . Let  $a$  and  $a'$  be 1-cycles in  $U \setminus P$  and  $U' \setminus P'$  representing the positive generators of  $H_1(U \setminus P)$  and  $H_1(U' \setminus P')$  respectively. Then*

$$h(a) = \deg(h) \deg(h|_P) a' \quad \text{in} \quad H_1(U' \setminus P').$$

*Proof of Lemma 3.1.* If  $\mathcal{B}'' \subset V' \setminus L'$  is any complex disc transverse to  $E$ , we have that  $\partial \mathcal{B}''$  is homologous  $\partial \mathcal{B}'$  in  $H_1(V' \setminus (L' \cup E))$ . Thus, we may change the disc  $\mathcal{B}''$  if necessary and assume that it is contained in  $W'$ . Let  $b'$  be the 1-cycle defined by  $b' = \pi(\partial \mathcal{B}'')$ . Then, since  $\pi(\mathcal{B}'') \subset U'$  is a complex disc transverse to  $P'$  and  $\pi(\partial \mathcal{B}'') = \partial \pi(\mathcal{B}'')$ , we have that  $b'$  is a positive generator of  $H_1(U' \setminus P')$ . Analogously, if  $b = \pi(\partial \mathcal{B}_w)$ , we deduce that  $b$  is a positive generator of  $H_1(U \setminus P)$ . It follows from Lemma 3.3 that:

$$h(b) = \psi \xi b' \quad \text{in} \quad H_1(U' \setminus P'),$$

where  $\psi = \deg(h)$  and  $\xi = \deg(h|_P)$ . Then, since  $\pi^{-1} : U' \setminus P' \rightarrow W' \setminus (L' \cup E)$  is well defined, we have that

$$\pi^{-1}(h(b)) = \psi \xi \pi^{-1}(b') \quad \text{in} \quad H_1(W' \setminus (L' \cup E))$$

and thus

$$(1) \quad h(\partial \mathcal{B}_w) = \psi \xi \partial \mathcal{B}' \quad \text{in} \quad H_1(W' \setminus (L' \cup E)).$$

Observe that  $\pi(\mathcal{A}')$  is a complex disc transverse to  $P'$ . Then the cycle  $\partial \pi(\mathcal{A}') = \pi(\partial \mathcal{A}')$  represents the positive generator of  $H_1(U' \setminus P')$ . Thus, we deduce that  $\pi(\partial \mathcal{A}') = \pi(\partial \mathcal{B}')$  in  $H_1(U' \setminus P')$  and therefore

$$(2) \quad \partial \mathcal{A}' = \partial \mathcal{B}' \quad \text{in} \quad H_1(W' \setminus (L' \cup E)).$$

Let  $\mathcal{C}$  be the disc  $\varphi(0, |z| \leq r)$  in  $L$ . Let  $\mathcal{C}'$  be a disc in  $L'$  containing  $p'$ . Since  $h$  maps  $\mathcal{C}$  homeomorphically into  $L'$  with  $h(p) = p'$ , the cycle  $h(\partial \mathcal{C})$  is a generator of the group  $H_1(L' \setminus \{p'\})$  and we have  $h(\partial \mathcal{C}) = \deg(h|_L) \partial \mathcal{C}'$ . Thus, since  $h|_L$  preserves orientation if and only if  $h|_P$  does, we have that  $h(\partial \mathcal{C}) = \xi \partial \mathcal{C}'$  in  $H_1(L' \setminus \{p'\})$ . Since  $L' \setminus \{p'\}$  is contained in  $V' \setminus E$ , we conclude that

$$(3) \quad h(\partial \mathcal{C}) = \xi \partial \mathcal{C}' \quad \text{in} \quad H_1(V' \setminus E).$$

Observe that  $\partial \mathcal{C}' = \partial \mathcal{B}'$  in  $H_1(V' \setminus E)$ . Moreover,  $\partial \mathcal{C} = \varphi(0, |z| = r)$  is homologous to  $\partial \mathcal{B}_w = \varphi(w, |z| = r)$  in the set  $T = \varphi(|z| \leq |w|, |z| = r)$ . It is easy to see that for  $|w|$  small enough, the set  $h(T)$  is contained

in  $V' \setminus E$ . Then  $h(\partial\mathcal{C})$  and  $h(\partial\mathcal{B}_w)$  are homologous in  $V' \setminus E$ . It follows from (3) and the observations above that for  $|w|$  small enough:

$$(4) \quad h(\partial\mathcal{B}_w) = \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(V' \setminus E).$$

We know that there exists integers  $n$  and  $m$  such that

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + m\partial\mathcal{B}' \quad \text{in} \quad H_1(V' \setminus (L' \cup E)).$$

Then, since  $V' \setminus (L' \cup E) \subset V' \setminus E$ :

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + m\partial\mathcal{B}' \quad \text{in} \quad H_1(V' \setminus E),$$

hence

$$h(\partial\mathcal{B}_w) = m\partial\mathcal{B}' \quad \text{in} \quad H_1(V' \setminus E),$$

because  $\partial\mathcal{A}' = 0$  in  $H_1(V' \setminus E)$ . From this and (4) we conclude that  $m = \xi$ . Then

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(V' \setminus (L' \cup E))$$

and, since  $V' \setminus (E \cup L')$  is contained in  $W' \setminus (E \cup L')$ , we have that

$$(5) \quad h(\partial\mathcal{B}_w) = n\partial\mathcal{A}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W' \setminus (L' \cup E)).$$

From (2) we have  $\partial\mathcal{A}' = \partial\mathcal{B}'$  in  $H_1(W' \setminus (L' \cup E))$ . Replacing in (5) we obtain:

$$h(\partial\mathcal{B}_w) = n\partial\mathcal{B}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W' \setminus (L' \cup E)).$$

Thus, from (1) we have:

$$\psi\xi\partial\mathcal{B}' = n\partial\mathcal{B}' + \xi\partial\mathcal{B}' \quad \text{in} \quad H_1(W' \setminus (L' \cup E))$$

and therefore  $n = (\psi - 1)\xi$ . This proves the Lemma. q.e.d.

*Proof of Proposition 3.2.* It is known that the germ of three smooth and transverse curves is equivalent to the germ given by its tangents lines. Therefore we may assume that  $\mathcal{F}$  has three transverse complex lines  $P_1, P_2$  and  $P_3$  as separatrices. Then  $h(P_1), h(P_2)$  and  $h(P_3)$  are smooth and transverse separatrices of  $\tilde{\mathcal{F}}$  and we can also assume that they are contained in complex lines  $\tilde{P}_1, \tilde{P}_2$  and  $\tilde{P}_3$ . By reducing  $U$  we may assume that  $U \cap P_1, U \cap P_2$  and  $U \cap P_3$  are homeomorphic to discs. We may take a neighborhood  $\tilde{U}' \subset h(U)$  of  $0 \in \mathbb{C}^2$  such that  $\tilde{U}' \cap \tilde{P}_1, \tilde{U}' \cap \tilde{P}_2$  and  $\tilde{U}' \cap \tilde{P}_3$  are homeomorphic to discs and are contained in  $h(U \cap P_1), h(U \cap P_2)$  and  $h(U \cap P_3)$  respectively. Then if we make  $U' = h^{-1}(\tilde{U}')$ , it is easy to see that  $U' \cap P_1, U' \cap P_2$  and  $U' \cap P_3$  are homeomorphic to discs and  $h(U' \cap P_1) = \tilde{U}' \cap \tilde{P}_1, h(U' \cap P_2) = \tilde{U}' \cap \tilde{P}_2, h(U' \cap P_3) = \tilde{U}' \cap \tilde{P}_3$ . We may choose two of the complex lines  $P_1, P_2$  and  $P_3$ , say us  $P_1$  and  $P_2$ , such that  $\deg(h|_{P_1}) = \deg(h|_{P_2})$ . Let  $\mathcal{D} \subset P_1$  be a disc containing  $0 \in \mathbb{C}^2$ . Then  $h(\partial\mathcal{D}) = \deg(h|_{P_1})\partial h(\mathcal{D})$  in  $H_1(\tilde{P}_1 \cap \tilde{U}' \setminus \{0\})$  and, since  $\tilde{P}_1 \cap \tilde{U}' \setminus \{0\} \subset \tilde{U}' \setminus \tilde{P}_2$ , we have that

$$h(\partial\mathcal{D}) = \deg(h|_{P_1})\partial h(\mathcal{D}) \quad \text{in} \quad H_1(\tilde{U}' \setminus \tilde{P}_2).$$

On the other hand, since  $\partial\mathcal{D}$  and  $\partial h(\mathcal{D})$  are positive generators of  $H_1(U' \setminus P_2)$  and  $H_1(\tilde{U}' \setminus \tilde{P}_2)$  respectively, we have by Lemma 3.3 that

$$h(\partial\mathcal{D}) = \deg(h) \deg(h|_{P_2}) \partial h(\mathcal{D}) \quad \text{in } H_1(\tilde{U}' \setminus \tilde{P}_2).$$

Finally, since  $\deg(h|_{P_1}) = \deg(h|_{P_2})$ , it follows from the equations above that  $\deg(h) = 1$  and therefore  $h$  preserves orientation. q.e.d.

*Proof of Lemma 3.3.* We only sketch the proof. Let  $\mathcal{D}$  and  $\mathcal{D}'$  be complex discs transverse to  $P$  and  $P'$  respectively. Thus  $\partial\mathcal{D}$  and  $\partial\mathcal{D}'$  are homologous to  $a$  and  $a'$  respectively. Clearly  $h(\partial\mathcal{D}) = \xi \partial\mathcal{D}'$ , where  $\xi = 1$  or  $-1$ . Let  $L = P \cap U$  and  $L' = P' \cap U'$ . It follows from the topological invariance of the intersection number (see [6], p.200) that

$$h(L) \cdot h(\mathcal{D}) = \deg(h) L' \cdot \mathcal{D}'.$$

On the other hand it is easy to see that

$$h(L) \cdot h(\mathcal{D}) = (\deg(h|_P) L') \cdot (\xi \mathcal{D}') = \deg(h|_P) \xi L' \cdot \mathcal{D}'.$$

Then  $\deg(h|_P) \xi = \deg(h)$  and therefore  $\xi = \deg(h|_P) \deg(h)$ , which proves the lemma. q.e.d.

#### 4. Proof of theorem 1.2

Let  $\rho : \widehat{\mathbb{C}^2} \rightarrow \pi^{-1}(0)$  be the projection associated to the natural fibration on a neighborhood of the divisor  $\pi^{-1}(0)$ . Let  $h : U \rightarrow \tilde{U}$ ,  $\mathcal{F}$ ,  $\tilde{\mathcal{F}}$ ,  $P$ , and  $\tilde{P}$  be as in Theorem 1.2. We know that the strict transforms of  $P$  and  $\tilde{P}$  are fibers of  $\rho$ . Let  $L_p$  and  $L_{\tilde{p}}$ , the fibers passing through  $p$  and  $\tilde{p}$ , be the strict transforms of  $P$  and  $\tilde{P}$  respectively. By the hypothesis on  $P$  and  $\tilde{P}$  we have that  $p$  and  $\tilde{p}$  are regular points of  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  respectively. Let  $W$  and  $\tilde{W}$  denote the sets  $\pi^{-1}(U)$  and  $\pi^{-1}(\tilde{U})$  and let  $E$  be the divisor  $\pi^{-1}(0)$ . Since  $h(P \cap U) = \tilde{P} \cap \tilde{U}$ , if

$$h : W \setminus E \rightarrow \tilde{W} \setminus E$$

is the homeomorphism given by  $h = \pi^{-1} h \pi$ , we have that

$$h(L_p \cap W \setminus \{p\}) = L_{\tilde{p}} \cap \tilde{W} \setminus \{\tilde{p}\}.$$

Now, it is easy to see that Theorem 1.2 is a direct consequence of the following proposition.

**Proposition 4.1.** *Let  $p$  and  $\tilde{p}$  be points in the divisor which are nonsingular for  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  respectively. Let  $L_p$  and  $L_{\tilde{p}}$  be the fibers through  $p$  and  $\tilde{p}$  respectively and suppose that*

$$h(L_p \cap W \setminus \{p\}) = L_{\tilde{p}} \cap \tilde{W} \setminus \{\tilde{p}\}.$$

Then there exists neighborhoods  $U \subset U$  and  $\tilde{U} \subset \tilde{U}$  of  $0 \in \mathbb{C}^2$ , and another topological equivalence

$$\hat{h} : U \rightarrow \tilde{U}$$

between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$ , for which the hypothesis of Theorem 2.1 holds.

We need some lemmas. Let  $U \subset \mathbb{C}$  be the domain bounded by the Jordan curve  $J$ . Let  $p \in U$  and  $\zeta \in J$ . We know that any biholomorphism between  $\mathbb{D}$  and  $U$  extends as a homeomorphism between  $\overline{\mathbb{D}}$  and  $\overline{U} = U \cup J$  and there exists a unique biholomorphism  $f : \mathbb{D} \rightarrow U$  with  $f(0) = p$  and such that its extension to  $\overline{\mathbb{D}}$  satisfies  $f(1) = \zeta$ . In other words,  $f : \overline{\mathbb{D}} \rightarrow \overline{U}$  is the unique orientation preserving homeomorphism, which is conformal on  $\mathbb{D}$  and maps 0 to  $p$  and 1 to  $\zeta$ . It is easy to prove that  $g : \overline{\mathbb{D}} \rightarrow \overline{U}$  defined by  $g(z) = f(\bar{z})$  is the unique orientation reversing homeomorphism, which is conformal on  $\mathbb{D}$  and maps 0 to  $p$  and 1 to  $\zeta$ . Therefore we have the following.

**Lemma 4.2.** *Let  $U, U' \subset \mathbb{C}$  be the domains bounded by the Jordan curves  $J$  and  $J'$  respectively. Let  $p \in U$ ,  $\zeta \in J$  and  $p' \in U'$ ,  $\zeta' \in J'$ . Then there exists exactly two homeomorphisms between  $\overline{U}$  and  $\overline{U}'$  which are conformal and maps  $p$  to  $p'$  and  $\zeta$  to  $\zeta'$ . The first one preserves orientation and the other one reverses orientation.*

**Lemma 4.3.** *Let  $J_k : S^1 \rightarrow \mathbb{C}$  be a Jordan curve for all  $k \geq 1$ . Suppose that  $J_k$  converges uniformly on  $S^1$  to the Jordan curve  $J : S^1 \rightarrow \mathbb{C}$ . Let  $U$  and  $U_k$ ,  $k \geq 1$  be the domains bounded by  $J$  and  $J_k$ ,  $k \geq 1$  respectively. Let  $p_k \in U_k$  and  $\zeta_k \in J_k$  be such that  $p_k \rightarrow p \in U$  and  $\zeta_k \rightarrow \zeta \in J$  as  $k \rightarrow \infty$ . Let  $f : \overline{\mathbb{D}} \rightarrow \overline{U}$  and  $f_k : \overline{\mathbb{D}} \rightarrow \overline{U}_k$  be the orientation preserving homeomorphisms which are conformal on  $\mathbb{D}$  and such that  $f(0) = p$ ,  $f(1) = \zeta$ ,  $f_k(0) = p_k$  and  $f_k(1) = \zeta_k$ . Then  $f_k$  converges to  $f$  uniformly on  $\overline{\mathbb{D}}$ . If under the same hypothesis, we change “orientation preserving homeomorphisms” by “orientation reversing homeomorphisms”, the conclusion is also true.*

**Lemma 4.4.** *Let  $\phi : X \rightarrow \mathbb{C} \setminus \{0\}$  be a continuous function. Suppose that  $\phi_* : \pi_1(X) \rightarrow \pi_1(\mathbb{C} \setminus \{0\})$  is trivial. Then there exists a continuous function  $\log_\phi : X \rightarrow \mathbb{C}$  such that  $e^{\log_\phi} = \phi$ .*

**Lemma 4.5.** *Let  $\phi : S^1 \rightarrow S^1$  be an orientation preserving homeomorphism. Consider  $S^1$  as a subset of  $\mathbb{C}$  and define the closed curve  $\alpha : S^1 \rightarrow \mathbb{C} \setminus \{0\}$  by  $\alpha(\zeta) = \phi(\zeta)/\zeta$ . Then  $\alpha$  is homotopically trivial in  $\mathbb{C} \setminus \{0\}$ .*

**Lemma 4.6.** *Let  $\phi : S^1 \rightarrow S^1$  be an orientation preserving homeomorphism and let  $\tau : S^1 \rightarrow \mathbb{C}$  be such that  $e^{\tau(\zeta)} = \phi(\zeta)/\zeta$ . Let  $A \subset \mathbb{C}$  be the annulus  $\{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$ . Then the map*

$$g : A \rightarrow A$$

$$g(z) = ze^{(2|z|-1)\tau(z/|z|)}$$

is a homeomorphism. Moreover,  $g = \phi$  on  $\{|z| = 1\}$  and  $g = \text{id}$  on  $\{|z| = 1/2\}$ .

**Lemma 4.7.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a conformal map. Then there exists  $\delta_0 > 0$  such that for all  $\delta \leq \delta_0$  the set  $f(|z| \leq \delta)$  is convex. (For convenience, we define a set  $\bar{U} \subset \mathbb{C}$  to be convex if  $U$  is the domain bounded by a smooth Jordan curve with positive curvature.)*

**Lemma 4.8.** *Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a conformal map. Let  $U$  be an open set in  $\mathbb{C}$  and let  $\delta_0 > 0$ . Suppose for all  $\delta \leq \delta_0$  the set  $f(|z| \leq \delta)$  is convex and contained in  $U$ . Then there exists  $\epsilon > 0$  with the following property: if  $g : \mathbb{D} \rightarrow \mathbb{C}$  is a conformal map with  $\|f - g\|_{\{|z| \leq \delta_0\}} < \epsilon$ , then for all  $\delta \leq \delta_0$  the set  $g(|z| \leq \delta)$  is convex and contained in  $U$ . (If  $K$  is compact and  $f$  is continuous,  $\|f\|_K$  is defined as the supremum of  $|f(x)|$  for  $x \in K$ .)*

Any leaf of  $\mathcal{F}_0$  or  $\tilde{\mathcal{F}}_0$  has a natural orientation induced by the complex structure. Thus, given a leaf  $L$  of  $\mathcal{F}_0$  out of the divisor, we may state if  $h|_L : L \rightarrow \tilde{L}$  preserves or reverses orientation. Suppose that  $h|_L$  preserves orientation. Then it is not difficult to prove that  $h|_{L'}$  preserves orientation of any leaf  $L'$  close enough to  $L$ . On the other hand, if  $h|_L$  reverses orientation, the same holds for  $h|_{L'}$  provided the leaf  $L'$  is close enough to  $L$ . By connectedness we have in fact that: either  $h$  preserves orientation for every leaf, or  $h$  reverses orientation for every leaf.

*Proof of Proposition 4.1.* Let  $V$  and  $\tilde{V}$  be neighborhoods of  $p$  and  $\tilde{p}$  and let  $\varphi : \bar{\mathbb{D}} \times \bar{\mathbb{D}} \rightarrow V$  and  $\tilde{\varphi} : \bar{\mathbb{D}} \times \bar{\mathbb{D}} \rightarrow \tilde{V}$  be diffeomorphisms with the following properties:

- (i) If restricted to  $\mathbb{D} \times \mathbb{D}$ , the maps  $\varphi$  and  $\tilde{\varphi}$  are biholomorphisms.
- (ii) The leaves of  $\mathcal{F}_0|_V$  and the leaves of  $\tilde{\mathcal{F}}_0|_{\tilde{V}}$  are given by the sets  $\varphi(\bar{\mathbb{D}} \times \{*\})$  and  $\tilde{\varphi}(\bar{\mathbb{D}} \times \{*\})$  respectively.
- (iii) We have  $L_p \cap V = \varphi(\{0\} \times \bar{\mathbb{D}})$ ,  $E \cap V = \varphi(\bar{\mathbb{D}} \times \{0\})$ ,  $L_{\tilde{p}} \cap \tilde{V} = \tilde{\varphi}(\{0\} \times \bar{\mathbb{D}})$  and  $E \cap \tilde{V} = \tilde{\varphi}(\bar{\mathbb{D}} \times \{0\})$ .

Let  $\varrho : V \rightarrow \bar{\mathbb{D}}$  be the projection  $\varrho(\varphi(z_1, z_2)) = z_1$  and we also denote by  $\varrho$  the projection  $\varrho : \tilde{V} \rightarrow \bar{\mathbb{D}}$ ,  $\varrho(\tilde{\varphi}(z_1, z_2)) = z_1$ . Let  $\Sigma$  be the set  $L_p \cap V = \varphi(\{0\} \times \bar{\mathbb{D}})$ . We have that  $h(\Sigma) \subset L_{\tilde{p}}$  and we may assume  $V$  small enough such that  $h(\Sigma) \subset \tilde{V}$ . Given  $x = \varphi(0, z_2) \in \Sigma$ , we denote by  $D_x$  the plaque  $\varphi(\bar{\mathbb{D}} \times \{z_2\})$  passing through  $x$ . We have that  $D_x$  is a closed disc in the leaf of  $\mathcal{F}_0$  passing through  $x$ .

**Step 1.** Fix a point  $q$  in  $\partial\mathbb{D} = S^1$  and denote by  $q_x$  the unique point in  $\partial D_x$  such that  $\varrho(q_x) = q$ . If  $h$  preserves the orientation of the leaves, by Lemma 4.2 we may define  $f_x : D_x \rightarrow h(D_x)$  as the unique orientation-preserving-homeomorphism which is conformal on the interior of  $D_x$

and such that  $f_x(x) = h(x)$  and  $f_x(q_x) = h(q_x)$ . Otherwise, we define  $f_x : D_x \rightarrow h(D_x)$  as the unique orientation reversing homeomorphism which is conformal on the interior of  $D_x$  and such that  $f_x(x) = h(x)$  and  $f_x(q_x) = h(q_x)$ . Let  $\varrho_x^{-1} : \overline{\mathbb{D}} \rightarrow D_x$  be the inverse of  $\varrho|_{D_x} : D_x \rightarrow \overline{\mathbb{D}}$ .

Assertion 1. *Let  $f : V \setminus E \rightarrow \widehat{\mathbb{C}^2}$  be defined by  $f|_{D_x} = f_x$  for all  $x \in \Sigma \setminus \{p\}$ . Then  $f$  is continuous.*

*Proof.* Let  $g_x : \overline{\mathbb{D}} \rightarrow h(D_x)$  be defined by  $g_x = f_x \circ \varrho_x^{-1}$ . It is sufficient to prove that  $g_x$  varies continuously with  $x$ , precisely: fix  $x_0 \in \Sigma \setminus \{p\}$  and let  $x_k (k \geq 1)$  be such that  $x_k \rightarrow x_0$  as  $k \rightarrow \infty$ ; then we shall prove that  $g_{x_k} \rightarrow g_{x_0}$  uniformly on  $\overline{\mathbb{D}}$ . Since  $h(D_{x_0})$  is a compact and simply connected subset of a leaf of  $\widetilde{\mathcal{F}}_0$ , there exists a neighborhood  $U$  of  $h(D_{x_0})$  and a biholomorphism  $\phi = (Z, W) : U \rightarrow \mathbb{D} \times \mathbb{D}$  such that the leaves of  $\widetilde{\mathcal{F}}_0$  are mapped to the sets  $\mathbb{D} \times \{z\}$ . We may assume that  $h(D_{x_k})$  is contained in  $U$  for all  $k \geq 0$ . Thus, we define  $G_k : \overline{\mathbb{D}} \rightarrow \mathbb{D} \times \mathbb{D}$  by  $G_k = \phi \circ g_{x_k} = (Z \circ g_{x_k}, W \circ g_{x_k})$ . Since  $g_{x_k}(\overline{\mathbb{D}}) = h(D_{x_k}) \subset U$  is contained in a leaf, there is  $z_k \in \mathbb{D}$  such that  $G_k(\overline{\mathbb{D}})$  is contained in  $\mathbb{D} \times \{z_k\}$ . Thus  $W \circ g_{x_k} \equiv z_k$  and it is sufficient to show that  $F_k = Z \circ g_{x_k} : \overline{\mathbb{D}} \rightarrow \mathbb{D}$  converges to  $F_0 = W \circ g_{x_0}$  uniformly on  $\overline{\mathbb{D}}$ . Observe that  $F_k$  is a homeomorphism onto its image and is conformal on  $\mathbb{D}$ . Moreover, we have that

$$F_k(0) = Z \circ g_{x_k}(0) = Z(h(x_k)) \rightarrow Z(h(x_0)) = Z \circ g_{x_0}(0) = F_0(0)$$

and

$$F_k(q) = Z \circ g_{x_k}(q) = h(q_{x_k}) \rightarrow h(q_{x_0}) = g_{x_0}(q) = F_0(q).$$

Then Assertion 1 follows from Lemma 4.3

Let

$$\theta_x : S^1 \rightarrow S^1$$

be the homeomorphism defined by  $\theta_x = \varrho|_{S^1}^{-1} \circ h \circ \varrho_x^{-1}|_{S^1}$ . It is easy to see that  $\theta_x$  preserves the orientation of  $S^1$ .

Define the function

$$\begin{aligned} \phi : S^1 \times (\Sigma \setminus \{p\}) &\rightarrow \mathbb{C} \setminus \{0\} \\ \phi(\zeta, x) &= \frac{\theta_x(\zeta)}{\zeta}. \end{aligned}$$

Assertion 2. *At homotopy level,  $\phi_* : \pi_1(S^1 \times (\Sigma \setminus \{p\})) \rightarrow \pi_1(\mathbb{C} \setminus \{0\})$  is trivial.*

*Proof.* The generators of  $\pi_1(S^1 \times (\Sigma \setminus \{p\}))$  are represented by the paths

$$\alpha, \beta : S^1 \rightarrow S^1 \times (\Sigma \setminus \{p\}),$$



defined by  $\alpha(\zeta) = (\zeta, x_0)$  and  $\beta(\zeta) = (q, \gamma(\zeta))$ , where  $x_0 \in \Sigma \setminus \{p\}$  and  $\gamma$  is a simple closed curve around  $p$  in  $\Sigma$ . Recall that  $q \in S^1$ , then  $|q| = 1$  and we have

$$\begin{aligned} \phi(\beta(\zeta)) &= \phi(q, \gamma(\zeta)) = \frac{\theta_{\gamma(\zeta)}(q)}{q} = \frac{\varrho f_{\gamma(\zeta)}^{-1} h \varrho_{\gamma(\zeta)}^{-1}(q)}{q} \\ &= \frac{\varrho f_{\gamma(\zeta)}^{-1} h(q_{\gamma(\zeta)})}{q} = \frac{\varrho f_{\gamma(\zeta)}^{-1} f_{\gamma(\zeta)}(q_{\gamma(\zeta)})}{q} \\ &= \frac{\varrho(q_{\gamma(\zeta)})}{q} \\ &= 1. \end{aligned}$$

Then  $\phi_*(\beta) = 0$ . On the other hand, since  $\theta_{x_0} : S^1 \rightarrow S^1$  is an orientation preserving homeomorphism, we have by Lemma 4.5 that

$$\begin{aligned} \phi \circ \alpha : S^1 &\rightarrow \mathbb{C} \setminus \{0\}, \\ \phi \circ \alpha(\zeta) &= \frac{\theta_{x_0}(\zeta)}{\zeta} \end{aligned}$$

is homotopically trivial and thus  $\phi_*(\alpha) = 0$ .

It follows from Assertion 2 and Lemma 4.4 that there exists a continuous function

$$\tau : S^1 \times (\Sigma \setminus \{p\}) \rightarrow \mathbb{C}$$

such that  $e^\tau = \phi$ , that is,  $e^{\tau(\zeta, x)} = \theta_x(\zeta)/\zeta$ . Consider the annulus  $A = \{1/2 \leq ||z|| \leq 1\} \subset \overline{\mathbb{D}}$  and define the map

$$\begin{aligned} g : A \times (\Sigma \setminus \{p\}) &\rightarrow A, \\ g(z, x) &= z e^{(2|z|-1)\tau(z/|z|, x)}. \end{aligned}$$

It follows from Lemma 4.6 that for all  $x$  the map

$$\begin{aligned} g_x : A &\rightarrow A, \\ g_x(z) &= g(z, x) \end{aligned}$$

is a homeomorphism such that  $g_x = \text{id}$  on  $\{|z| = 1/2\}$  and  $g_x = \theta_x$  on  $S^1$ . Let  $A_x$  be the annulus  $\varrho_x^{-1}(A)$  in  $D_x$  and let  $\partial A'_x = \varrho_x^{-1}(|z| = 1/2)$  and  $\partial A''_x = \varrho_x^{-1}(|z| = 1)$  be the interior and the exterior boundary of  $A_x$  respectively. Then the map

$$\bar{g} : A_x \rightarrow f_x(A_x)$$

defined by  $\bar{g}_x = f_x \varrho_x^{-1} g_x \varrho : A_x \rightarrow f_x(A_x)$  is a homeomorphism and it is easy to see that  $\bar{g}_x$  coincides with  $f_x$  on  $\partial A'_x$  and with  $h$  on  $\partial A''_x$ . Then we may define the homeomorphism

$$h_x : D_x \rightarrow h(D_x)$$

by

$$\begin{aligned} h_x &= f_x \quad \text{on} \quad \varrho_x^{-1}(|z| \leq 1/2), \\ h_x &= g_x \quad \text{on} \quad A_x. \end{aligned}$$

Clearly,  $h_x$  coincides with  $h$  on  $\partial D_x$  and it is easy to see that  $h_x$  depends continuously on  $x$ . Finally, we define the function  $h'$  by

$$\begin{aligned} h'|_{D_x} &= h_x \quad \text{for all} \quad x \in \Sigma \setminus \{p\}, \\ h' &= h, \quad \text{otherwise.} \end{aligned}$$

It is easy to see that  $h'$  is injective and take leaves to leaves. Moreover, if we restrict  $h'$  to a small enough neighborhood of the divisor,  $h'$  is continuous. Hence,  $h'$  restricted to a neighborhood of the divisor is a homeomorphism onto its image and is therefore a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$ . By definition  $h'$  is conformal on every plaque  $\varrho_x^{-1}(|z| \leq 1/2)$ , because coincides with  $f_x$ . In other words, there is  $\epsilon > 0$  such that  $h'$  restricted to  $\varphi(|z_1| \leq 1/2, |z_2| \leq \epsilon)$  is conformal along the leaves.

**Step 2.** From step 1 and by reducing  $V$ , we may assume that  $h$  restricted to  $V$  is conformal along the leaves. Then for all  $x \in \Sigma \setminus \{p\}$  the map

$$h\varrho_x^{-1} : \overline{\mathbb{D}} \rightarrow h(D_x)$$

is conformal and maps 0 to  $h(x)$ . Given  $x \in \Sigma \setminus \{p\}$ , since  $h\varrho_x^{-1}(0) = h(x)$  is contained in  $L_{\tilde{p}} \cap \tilde{V}$ , there is  $\delta > 0$  such that the disc  $\{|z| \leq \delta\}$  in  $\overline{\mathbb{D}}$  is mapped by  $h\varrho_x^{-1}$  into the interior of  $\tilde{V}$ . Then the map

$$\varrho h\varrho_x^{-1} : \{|z| \leq \delta\} \rightarrow \mathbb{D}$$

is well defined and assuming  $\delta$  be small, by Lemma 4.7 we have that for all  $\delta' \leq \delta$  the disc  $\{|z| \leq \delta'\}$  is mapped by  $\varrho h\varrho_x^{-1}$  onto a convex subset of  $\mathbb{D}$ . Define  $\delta(x) > 0$  as the supremum of  $0 < \delta < 1$  such that for all  $\delta' \leq \delta$ , the disc  $\{|z| \leq \delta'\}$  in  $\overline{\mathbb{D}}$  is mapped by  $\varrho h\varrho_x^{-1}$  onto a convex subset of  $\mathbb{D}$ .

**Assertion 3.** *The function  $\delta : \Sigma \setminus p \rightarrow \mathbb{R}^+$  is lower semi-continuous.*

*Proof.* Fix  $x_0 \in \Sigma \setminus p$  and let  $\epsilon > 0$ . Take  $\delta_0$  be such that  $\delta(x_0) - \epsilon < \delta_0 < \delta(x_0)$ . Then the disc  $\{|z| \leq \delta_0\}$  is mapped by  $\varrho h\varrho_{x_0}^{-1}$  onto a compact subset of  $\mathbb{D}$ . Then, if  $\Omega$  is a small enough neighborhood of  $x_0$  in  $\Sigma \setminus p$ , we have that

$$\varrho h\varrho_x^{-1} : \{|z| \leq \delta_0\} \rightarrow \overline{\mathbb{D}}$$

is well defined for all  $x \in \Omega$ . If we write  $f = \varrho h\varrho_{x_0}^{-1}$ , it follows from the definition of  $\delta(x_0)$  that for all  $\delta' \leq \delta(x_0) - \epsilon$ , the set  $f(|z| \leq \delta')$  is a

convex subset of  $\mathbb{D}$ . Let  $\epsilon_0 > 0$  be given by Lemma 4.8 for  $f = \varrho h \varrho_{x_0}^{-1}$  and  $U = \mathbb{D}$ . Then if

$$g : \{|z| \leq \delta_0\} \rightarrow \overline{\mathbb{D}}$$

is a conformal map with  $\|f - g\|_{\{|z| \leq \delta(x_0) - \epsilon\}} < \epsilon_0$ , we have that for all  $\delta' \leq \delta(x_0) - \epsilon$ , the set  $g(|z| \leq \delta')$  is also convex and contained in  $\mathbb{D}$ . By reducing the neighborhood  $\Omega$  of  $x_0$  we may assume that

$$\|\varrho h \varrho_{x_0}^{-1} - \varrho h \varrho_x^{-1}\|_{\{|z| \leq \delta(x_0) - \epsilon\}} < \epsilon_0$$

for all  $x \in \Omega$ . Then, for all  $\delta' \leq \delta(x_0) - \epsilon$  the set  $\varrho h \varrho_x^{-1}(|z| \leq \delta')$  is convex and contained in  $\mathbb{D}$ . Thus by the definition of  $\delta(x)$  we conclude that

$$\delta(x) \geq \delta(x_0) - \epsilon.$$

It follows that  $\delta$  is a lower semi-continuous function.

Assertion 4. *There exists a positive continuous function*

$$r : \Sigma \setminus \{p\} \rightarrow (0, 1)$$

such that for all  $x$  the map

$$\varrho h \varrho_x^{-1} : \{|z| \leq r(x)\} \rightarrow \overline{\mathbb{D}}$$

is well defined and its image  $U_x := \varrho h \varrho_x^{-1}(|z| \leq r(x))$  is a convex subset of  $\mathbb{D}$ .

*Proof.* We take any continuous function  $r < \delta$  given by Lemma 2.3. Then Assertion 4 is a direct consequence of the definition of  $\delta$ .

For all  $0 < r < 1$  let  $\beta_r : [0, 1] \rightarrow [0, 1]$  be the homeomorphism defined by

$$\beta_r(t) = t^{\frac{\ln(1/r)}{\ln 2}}.$$

We have that  $\beta_r(0) = 0$ ,  $\beta_r(1) = 1$  and it is easy to see that  $\beta_r(1/2) = r$ . In fact

$$\begin{aligned} \beta_r(1/2) &= (1/2)^{\frac{\ln(1/r)}{\ln 2}} = \left(2^{\frac{1}{2}}\right)^{-\ln(1/r)} \\ &= \left((e^{\ln 2})^{\frac{1}{2}}\right)^{\ln(r)} = e^{\ln(r)} = r. \end{aligned}$$

For each  $x \in \Sigma \setminus \{p\}$  we define the homeomorphism:

$$\begin{aligned} f_x : \overline{\mathbb{D}} &\rightarrow \overline{\mathbb{D}}, \\ f_x(z) &= \beta_{r(x)}(|z|)z. \end{aligned}$$

Observe that  $f_x$  maps each ratio of  $\overline{\mathbb{D}}$  homeomorphically onto itself and this homeomorphism is “given” by  $\beta_{r(x)}$ . We have that  $f_x(0) = 0$ ,  $f_x = \text{id}$  on  $\partial\overline{\mathbb{D}}$  and that  $f_x$  maps the disc  $\{|z| \leq 1/2\}$  onto the disc  $\{|z| \leq r(x)\}$ . For all  $y \in L_{\tilde{p}} \cap \tilde{V}$ , let  $\varrho_y^{-1} : \overline{\mathbb{D}} \rightarrow D_y$  be the inverse of

$$\varrho|_{D_y} : D_y \rightarrow \overline{\mathbb{D}}.$$

Assertion 5. For each  $x \in \Sigma \setminus \{p\}$ , define the homeomorphism

$$h_x = h\varrho_x^{-1}f_x\varrho : D_x \rightarrow h(D_x).$$

Then  $h_x$  coincides with  $h$  on  $\partial D_x$  and maps the disc  $\varrho_x^{-1}(|z| \leq 1/2)$  onto  $\varrho_{h(x)}^{-1}(U_x)$ . Moreover,  $h_x$  depends continuously on  $x$ .

*Proof.* If  $\zeta \in \partial D_x$ , then  $\varrho(\zeta) \in S^1$  and since  $f_x = \text{id}$  on  $S^1$  we have that  $f_x(\varrho(\zeta)) = \varrho(\zeta)$ . Then

$$h_x(\zeta) = h\varrho_x^{-1}f_x\varrho(\zeta) = h\varrho_x^{-1}\varrho(\zeta) = h(\zeta).$$

On the other hand,

$$h_x(\varrho_x^{-1}(|z| \leq 1/2)) = h\varrho_x^{-1}f_x\varrho(\varrho_x^{-1}(|z| \leq 1/2)) = h\varrho_x^{-1}f_x(|z| \leq 1/2)$$

and, since  $f_x(|z| \leq 1/2) = \{|z| \leq r(x)\}$ , we obtain:

$$h_x(\varrho_x^{-1}(|z| \leq 1/2)) = h\varrho_x^{-1}(|z| \leq r(x)).$$

Recall that  $U_x = \varrho h\varrho_x^{-1}(|z| \leq r(x))$  and so

$$\varrho_{h(x)}^{-1}(U_x) = h\varrho_x^{-1}(|z| \leq r(x)).$$

therefore

$$h_x(\varrho_x^{-1}(|z| \leq 1/2)) = \varrho_{h(x)}^{-1}(U_x).$$

Finally,  $h$  depends continuously on  $x$  because  $\beta_r$  depends continuously on  $r$ .

We now define the function  $h'$  by

$$\begin{aligned} h'|_{D_x} &= h_x \quad \text{for all } x, \\ h' &= h, \quad \text{otherwise.} \end{aligned}$$

It is easy to see that  $h'$  is injective and take leaves to leaves. Moreover, if we restrict  $h'$  to a small enough neighborhood of the divisor, it is continuous. Hence,  $h'$  restricted to a neighborhood of the divisor is a homeomorphism onto its image and is therefore a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$ . By definition,  $h'$  maps each plaque  $\varrho_x^{-1}(|z| \leq 1/2)$  onto  $\varrho_{h(x)}^{-1}(U_x)$ . In other words, any plaque  $\varrho_x^{-1}(|z| \leq 1/2)$  is mapped by  $h'$  onto a set which projection by  $\varrho$  is a convex set  $U_x$  in  $\mathbb{D}$ .

**Step 3.** From step 2 and by reducing  $V$  we may assume that  $h$  maps each plaque  $D_x$  onto  $\varrho_{h(x)}^{-1}(U_x)$ . Since  $U_x \subset \mathbb{D}$  is convex and contains 0, given  $w \in \overrightarrow{\mathbb{D}}$  there exists a unique point in the intersection of  $\partial U_x$  with the ray  $0\overrightarrow{w}$ . Let  $r_x(w)$  be the norm of this point. It is easy to see that  $r_x(w)$  depends continuously on  $x$  and  $w$ . We define the homeomorphism:

$$f_x : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}},$$

$$f_x(w) = \beta_{r_x(w)}(|w|)w.$$

Observe that  $f_x$  maps the ratio of  $\mathbb{D}$  passing through  $w$  homeomorphically onto itself and this homeomorphism is “given” by  $\beta_{r_x(w)}$ . We have that  $f_x$  maps the disc  $\{|z| \leq 1/2\}$  onto  $U_x$ .

Assertion 6. *For each  $x \in \Sigma \setminus \{p\}$  define the homeomorphism*

$$g_x = \varrho_{h(x)}^{-1} f_x^{-1} \varrho : D_{h(x)} \rightarrow D_{h(x)}.$$

*Then  $g_x = \text{id}$  on  $\partial D_{h(x)}$  and maps  $\varrho_{h(x)}^{-1}(U_x)$  onto  $\varrho_{h(x)}^{-1}(|z| \leq 1/2)$ . Moreover,  $g_x$  depends continuously on  $x$ .*

*Proof.* If  $\zeta \in \partial D_{h(x)}$ , then  $\varrho(\zeta) \in S^1$  and since  $f_x = \text{id}$  on  $S^1$  we have that  $f_x^{-1}(\varrho(\zeta)) = \varrho(\zeta)$ . Then

$$g_x(\zeta) = \varrho_{h(x)}^{-1} f_x^{-1} \varrho(\zeta) = \varrho_{h(x)}^{-1} \varrho(\zeta) = \zeta.$$

On the other hand:

$$g_x(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1} f_x^{-1} \varrho(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1} f_x^{-1}(U_x).$$

From the definition of  $f_x$ , we have that  $f_x^{-1}(U_x) = \{|z| \leq 1/2\}$ . Then

$$g_x(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1} f_x^{-1}(U_x) = \varrho_{h(x)}^{-1}(|z| \leq 1/2).$$

Finally,  $g_x$  depends continuously on  $x$  because  $r_x$  depends continuously on  $x$ .

Now, define the function  $g$  by

$$\begin{aligned} g|_{D_{h(x)}} &= g_x \quad \text{for all } x, \\ g &= \text{id}, \quad \text{otherwise.} \end{aligned}$$

It is easy to see that  $g$  is injective and maps leaves of  $\tilde{\mathcal{F}}_0$  to leaves of  $\tilde{\mathcal{F}}_0$ . Moreover, if we restrict  $g$  to a small enough neighborhood of the divisor,  $g$  is continuous. Hence,  $g$  restricted to a neighborhood of the divisor is a homeomorphism onto its image and is therefore a topological equivalence of  $\tilde{\mathcal{F}}_0$  with itself. Finally we define  $h' = g \circ h$ . Then  $h'$  is a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$  and from the definition of  $g$  we have

$$h'(D_x) = g(h(D_x)) = g(\varrho_{h(x)}^{-1}(U_x)) = \varrho_{h(x)}^{-1}(|z| \leq 1/2).$$

Thus  $h'$  maps each plaque  $D_x$  onto the plaque  $\varrho_{h(x)}^{-1}(|z| \leq 1/2)$ .

**Step 4.** From step 3 and by redefining  $\tilde{V}$  we may assume that for all  $y \in \overline{\mathbb{D}} \setminus \{0\}$  the equivalence  $h$  maps the plaque  $\varphi(\overline{\mathbb{D}} \times \{y\})$  onto the plaque  $\tilde{\varphi}(\overline{\mathbb{D}} \times \{f(y)\})$ , where  $f : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  is a homeomorphism onto its image. Therefore  $h|_{V \setminus E} : V \setminus E \rightarrow \tilde{V} \setminus E$  is expressed as

$$h(\varphi(x, y)) = \tilde{\varphi}(h_y(x), f(y)),$$

where  $h_y : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  is a homeomorphism such that  $h_y(0) = 0$  (because  $h(\Sigma) \subset L_{\tilde{p}}$ ). As a first case we assume that the homeomorphisms  $h_y$  preserve orientation. Define the function

$$\begin{aligned} \phi : S^1 \times (\overline{\mathbb{D}} \setminus \{0\}) &\rightarrow \mathbb{C} \setminus \{0\} \\ \phi(\zeta, y) &= \frac{h_y(\zeta)}{\zeta}. \end{aligned}$$

Assertion 7. *At homotopy level,  $\phi_* : \pi_1(S^1 \times (\overline{\mathbb{D}} \setminus \{0\})) \rightarrow \pi_1(\mathbb{C} \setminus \{0\})$  is trivial.*

*Proof.* The generators of  $\pi_1(S^1 \times (\overline{\mathbb{D}} \setminus \{0\}))$  are represented by the paths

$$\alpha, \beta : S^1 \rightarrow S^1 \times (\overline{\mathbb{D}} \setminus \{0\}),$$

defined as  $\alpha(\zeta) = (\zeta, 1)$  and  $\beta(\zeta) = (1, \zeta)$ . Then we have that

$$\phi \circ \alpha(\zeta) = \phi(\zeta, 1) = \frac{h_1(\zeta)}{\zeta}$$

and, since  $h_1|_{S^1} : S^1 \rightarrow S^1$  preserves the orientation, we have by Lemma 4.5 that  $\phi \circ \alpha$  is homotopically trivial in  $\mathbb{C} \setminus \{0\}$ . Observe that  $\beta$  is the boundary of the disc  $\{(1, y) : |y| \leq 1\}$ . Thus,  $\varphi(\beta)$  is the boundary of the complex disc  $\mathcal{B} = \varphi(1, |y| \leq 1)$ . Consider the disc  $\mathcal{B}_w = \varphi(w, |y| \leq 1)$ , where  $w \in \mathbb{D} \setminus \{0\}$ . By Lemma 3.1 we may chose  $w$  such that the path  $h(\partial\mathcal{B}_w)$  in  $\tilde{V}$  does not link the fiber  $L_{\tilde{p}}$ . Thus, since  $\partial\mathcal{B} = \partial\mathcal{B}_w$  in  $H_1(V \setminus (L_p \cup E))$  and  $h(V \setminus (L_p \cup E)) \subset \tilde{V} \setminus (L_{\tilde{p}} \cup E)$ , we have that  $h(\partial\mathcal{B})$  does not link the fiber  $L_{\tilde{p}}$ . Therefore the path  $\tilde{\varphi}^{-1}h(\partial\mathcal{B})$  in  $(\overline{\mathbb{D}} \setminus \{0\}) \times \overline{\mathbb{D}}$  does not link  $\{0\} \times \overline{\mathbb{D}}$  and, since

$$\begin{aligned} \tilde{\varphi}^{-1}h(\partial\mathcal{B}) &= \tilde{\varphi}^{-1}h(\varphi(\beta)) = \tilde{\varphi}^{-1}h(\varphi(1, \zeta)) \\ &= \tilde{\varphi}^{-1}\tilde{\varphi}(h_\zeta(1), f(\zeta)) = (h_\zeta(1), f(\zeta)), \end{aligned}$$

we conclude that the path  $\zeta \rightarrow h_\zeta(1) = \phi(\beta(\zeta))$  is homotopically trivial in  $\mathbb{C} \setminus \{0\}$ .

Assertion 7 and Lemma 4.4 imply that there exists a continuous function

$$\tau : S^1 \times (\overline{\mathbb{D}} \setminus \{0\}) \rightarrow \mathbb{C}$$

such that  $e^{\tau(\zeta, y)} = h_y(\zeta)/\zeta$ . We define the map:

$$h' : V \setminus E \rightarrow \tilde{V} \setminus E$$

by:

$$\begin{aligned} h'(\varphi(x, y)) &= \tilde{\varphi}(x, f(y)), \quad \text{for } |x| < 1/2, \quad \text{and} \\ h'(\varphi(x, y)) &= \tilde{\varphi}\left(xe^{(2|x|-1)\tau(x/|x|, y)}, f(y)\right), \quad \text{for } |x| \geq 1/2. \end{aligned}$$

By Lemma 4.6 we have that  $h'$  maps the plaque  $\varphi(\overline{\mathbb{D}} \times \{y\})$  homeomorphically onto the plaque  $\tilde{\varphi}(\overline{\mathbb{D}} \times \{f(y)\})$ . Thus  $h'$  is a homeomorphism

which preserves the plaques and it is easy to see that  $h'$  coincides with  $h$  on  $\varphi(\partial\mathbb{D} \times (\overline{\mathbb{D}} \setminus \{0\}))$ . Moreover  $h'$  extends to  $\varphi(|x| < 1/2, y = 0) \subset E$  as  $h'(\varphi(x, 0)) = \tilde{\varphi}(x, 0)$ . It is easy to see that this extension is a homeomorphism onto its image. We now define:

$$\begin{aligned} \hat{h} &= h' \quad \text{on } V \setminus E, \\ \hat{h} &= h \quad \text{otherwise.} \end{aligned}$$

As before, on a neighborhood of the divisor,  $\hat{h}$  is also a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$ . Moreover, from above  $\hat{h}$  extends to the open set  $\varphi(|x| < 1/2, |y| < 1)$  and Proposition 4.1 is therefore proved in this case. We now suppose that the homeomorphisms  $h_x$  inverts orientation. Then we define

$$h' : V \setminus E \rightarrow \tilde{V} \setminus E$$

by:

$$\begin{aligned} h'(\varphi(x, y)) &= \tilde{\varphi}(\bar{x}, f(y)), \quad \text{for } |x| < 1/2, \quad \text{and} \\ h'(\varphi(x, y)) &= \tilde{\varphi}\left(\bar{x}e^{(2|x|-1)\tau(\bar{x}/|x|, y)}, f(y)\right), \quad \text{for } |x| \geq 1/2. \end{aligned}$$

and the proof follows in the same way. q.e.d.

*Proof of Lemma 4.3.* This lemma is a direct consequence of a theorem of Rado (see [11], p.26). q.e.d.

*Proof of Lemma 4.4.* Fix  $x_0 \in X$ . There is a neighborhood  $\Omega$  of  $z_0 = \phi(x_0)$  in  $\mathbb{C} \setminus \{0\}$  where a branch of logarithm function is well defined. Then there exist a holomorphic function

$$f : \Omega \rightarrow \mathbb{C}$$

such that  $e^{f(z)} = z$  for all  $z \in \Omega$ . We know that  $f$  can be analytically continued along any path  $\gamma$  in  $\mathbb{C} \setminus \{0\}$  with  $\gamma(0) = z_0$  and  $\gamma(1) = z \in \mathbb{C} \setminus \{0\}$ . This analytic continuation has a value at  $\gamma(1) = z$ , which we denote by  $f_\gamma(z)$ . Let  $x \in X$ . Take a path  $\alpha$  in  $X$  connecting  $x_0$  to  $x$ . Then we define  $F_\alpha(x) = f_{\phi \circ \alpha}(\phi(x))$ . Let  $\alpha'$  be other path in  $X$  connecting  $x_0$  to  $x$ . Then, since

$$\phi_* : \pi_1(X) \rightarrow \pi_1(\mathbb{C} \setminus \{0\})$$

is trivial, it follows that  $\phi \circ \alpha$  and  $\phi \circ \alpha'$  are homotopic in  $\mathbb{C} \setminus \{0\}$ . Then

$$f_{\phi \circ \alpha}(\phi(x)) = f_{\phi \circ \alpha'}(\phi(x))$$

and so  $F_\alpha(x) = F_{\alpha'}(x)$ . Therefore we define  $\log_\phi(x) = F_\alpha(x)$  for any  $\alpha$ . q.e.d.

*Proof of Lemma 4.5.* It is known that a map  $\phi : S^n \rightarrow S^n$  is homotopically determined by its degree (Brouwer). Thus, a preserving-orientation homeomorphism of  $S^1$  is homotopic to the identity map

$\text{id} : S^1 \rightarrow S^1$ , that is, there exists a map

$$F : S^1 \times [0, 1] \rightarrow S^1$$

such that  $F(\zeta, 0) = \phi(\zeta)$  and  $F(\zeta, 1) = \zeta$  for all  $\zeta \in S^1$ . Then the map

$$G : S^1 \times [0, 1] \rightarrow S^1 \subset \mathbb{C} \setminus \{0\}$$

defined by

$$G(\zeta, t) = \frac{F(\zeta, t)}{\zeta}$$

is a homotopy between  $\alpha$  and the constant 1.

q.e.d.

*Proof of Lemma 4.6.* We first observe that each circle  $\{|z| = r\}$  in  $A$  is mapped into itself. Let  $z \in A$  with  $|z| = r$ . Since

$$e^{\tau(\zeta)} = \phi(\zeta)/\zeta \in S^1$$

for all  $\zeta \in S^1$ , it follows that  $\tau(z/|z|) = 2\pi it$  with  $t \in \mathbb{R}$ . Then

$$|g(z)| = \left| z e^{(2|z|-1)\tau(z/|z|)} \right| = |z| \left| e^{(2|z|-1)(2\pi it)} \right| = |z| = r.$$

Now, it is sufficient to prove that  $g$  maps each  $\{|z| = r\}$  homeomorphically onto itself, which is equivalent to prove that the map  $h : S^1 \rightarrow S^1$  defined by  $h(\zeta) = g(r\zeta)/r$  is a homeomorphism. We have that

$$h(\zeta) = g(r\zeta)/r = (r\zeta) e^{(2|r\zeta|-1)\tau(r\zeta/|r\zeta|)}/r = \zeta e^{(2r-1)\tau(\zeta)},$$

where  $1/2 \leq r \leq 1$ . Since  $\phi$  is a homeomorphism and preserves the orientation, there exists a homeomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\phi(e^{2\pi it}) = e^{2\pi i f(t)}$  and  $f(t+1) = f(t) + 1$  for all  $t \in \mathbb{R}$ . Then, since  $e^{\tau(\zeta)} = \phi(\zeta)/\zeta$ , we obtain

$$e^{\tau(e^{2\pi it})} = \phi(e^{2\pi it})/e^{2\pi it} = e^{2\pi i f(t)}/e^{2\pi it} = e^{2\pi i(f(t)-t)}.$$

Hence  $\tau(e^{2\pi it}) = 2\pi i(f(t) - t + N)$ , where  $N \in \mathbb{Z}$ . Then

$$\begin{aligned} h(e^{2\pi it}) &= e^{2\pi it} e^{(2r-1)\tau(e^{2\pi it})} = e^{2\pi it} e^{(2r-1)(2\pi i)(f(t)-t+N)} \\ &= e^{(2\pi i)(t+(2r-1)f(t)-(2r-1)t+(2r-1)N)} \\ (6) \quad &= e^{(2\pi i)((2r-1)f(t)+(2-2r)t+(2r-1)N)} \end{aligned}$$

and we have therefore

$$h(e^{2\pi it}) = e^{2\pi i \bar{f}(t)},$$

where  $\bar{f}(t) = (2r-1)f(t) + (2-2r)t + (2r-1)N$ . An easy computation shows that  $\bar{f}(t+1) = \bar{f}(t) + 1$ . Moreover, since  $f$  is increasing, it is easy to see that  $\bar{f}$  also is. Then  $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism and the lemma follows. q.e.d.

*Proof of Lemma 4.7.* Since the conjugation  $z \rightarrow \bar{z}$  preserves the convex sets, by replacing  $f$  with  $\bar{f}$  we may assume that  $f$  is holomorphic. For  $r > 0$  small enough, define  $g_r : \mathbb{D} \rightarrow \mathbb{C}$ ,  $g_r(z) = f(rz/a)/r$ , where



$a = f'(0)$ . It is easy to see that  $g_r(z) \rightarrow z$  as  $r \rightarrow 0$  for all  $z$ . Then  $g_r$  converges uniformly on compact sets to the identity  $\text{id} : \mathbb{D} \rightarrow \mathbb{D}$  as  $r \rightarrow 0$ . Hence there is  $r_0$  such that for all  $r \leq r_0$  we have

$$\|\text{id} - g_r\|_{\{|z| \leq 1/2\}} < \epsilon,$$

where  $\epsilon$  is given by Lemma 4.8 for  $\delta_0 = 1/2$ . Therefore  $g_r(|z| \leq 1/2)$  is convex for all  $r \leq r_0$ . But

$$g_r(|z| \leq 1/2) = g\left(\frac{r\{|z| \leq 1/2\}}{a}\right) / r = g\left(|z| \leq \frac{r}{|2a|}\right) / r,$$

which is convex in and only if the set  $g(|z| \leq r/|2a|)$  is convex. Then, if we take  $\delta_0 = r_0/(2|a|)$ , we have that the set  $g(|z| \leq \delta)$  is convex for all  $\delta \leq \delta_0$ . q.e.d.

*Proof of Lemma 4.8.* We may assume that  $f$  is holomorphic. Thus, if the conformal map  $g$  is close enough to  $f$ , it will be holomorphic too. If  $\alpha : (a, b) \rightarrow \mathbb{C}$  is a smooth curve, the curvature of  $\alpha$  at the point  $\alpha(t)$  is given by

$$\begin{aligned} k_\alpha(t) &= \left| \frac{d}{dt} \left( \frac{\alpha'(t)}{|\alpha'(t)|} \right) \right| = \frac{||\alpha''(t)|\alpha'(t)| - \alpha'(t)|\alpha'(t)'||}{|\alpha'(t)|^2} \\ &= \frac{\left| |\alpha''(t)|\alpha'(t)| - \alpha'(t) \left( \frac{\alpha''(t)\overline{\alpha'(t)} + \alpha'(t)\overline{\alpha''(t)}}{2|\alpha'(t)|} \right) \right|}{|\alpha'(t)|^2} \\ (7) \quad &= \frac{||\alpha''(t)|\alpha'(t)|^2 - \overline{\alpha''(t)}(\alpha'(t))^2||}{2|\alpha'(t)|^3}. \end{aligned}$$

Let  $r > 0$  and parametrizes the boundary of the disc  $\{|z| \leq r\}$  by  $\gamma_r(t) = re^{it/r}$ ,  $t \in \mathbb{R}$ . Let  $g : \mathbb{D} \rightarrow \mathbb{C}$  be any holomorphic conformal map and let  $\alpha_{rg}$  be the curve  $\alpha_{rg} = g \circ \gamma_r$ . We have  $\alpha'_{rg}(t) = g'(\gamma_r(t))\gamma'_r(t)$ ,  $\alpha''_{rg}(t) = g''(\gamma_r(t))(\gamma'_r(t))^2 + g'(\gamma_r(t))\gamma''_r(t)$ ,  $|\gamma'_r(t)| = 1$  and  $|\gamma''_r(t)| = 1/r$ . Then from (7):

$$\begin{aligned} k_{\alpha_{rg}}(t) &= \frac{|(g''(\gamma_r)(\gamma'_r)^2 + g'(\gamma_r)\gamma''_r)|g'(\gamma_r)| - g'(\gamma_r)\gamma'_r|g'(\gamma_r)'|}{|g'(\gamma_r)|^2} \\ &= \frac{|g'(\gamma_r)\gamma''_r|g'(\gamma_r)| + g''(\gamma_r)(\gamma'_r)^2|g'(\gamma_r)| - g'(\gamma_r)\gamma'_r|g'(\gamma_r)'|}{|g'(\gamma_r)|^2}, \end{aligned}$$

Hence

$$\begin{aligned} k_{\alpha_{rg}}(t) &\geq \frac{|g'(\gamma_r)|^2/r - |g''(\gamma_r)||g'(\gamma_r)| - |g'(\gamma_r)||g'(\gamma_r)'|}{|g'(\gamma_r)|^2} \\ (8) \quad &= \frac{|g'(\gamma_r)|/r - |g''(\gamma_r)| - |g'(\gamma_r)'|}{|g'(\gamma_r)|}. \end{aligned}$$

Observe that

$$|g'(\gamma_r)|' = \frac{g''(\gamma_r)\overline{\gamma_r'g'(\gamma_r)} + g'(\gamma_r)\overline{g''(\gamma_r)\gamma_r'}}{|g'(\gamma_r)|}$$

and thus

$$|g'(\gamma_r)|' \leq \frac{|g''(\gamma_r)||g'(\gamma_r)| + |g'(\gamma_r)||g''(\gamma_r)|}{|g'(\gamma_r)|} \leq 2|g''(\gamma_r)|.$$

Replacing in equation (8) we obtain

$$\begin{aligned} k_{\alpha_{rg}}(t) &\geq \frac{|g'(\gamma_r)|/r - |g''(\gamma_r)| - 2|g''(\gamma_r)|}{|g'(\gamma_r)|} \\ &= 1/r - 3|g''(\gamma_r)|/|g'(\gamma_r)|. \end{aligned}$$

We know that if  $g \rightarrow f$ , then  $g''/g' \rightarrow f''/f'$  (uniformly on the compact sets). Then there is  $\epsilon > 0$  such that  $\|f - g\|_{\{|z| \leq r_0\}} < \epsilon_1$  implies  $k_{\alpha_{rg}}(t) \geq 1/r - 3\|f''/f'\|_{\{|z| \leq \delta_0\}} - 1$  for all  $r \in (0, \delta_0)$ . Thus we make take  $r_0 \in (0, \delta_0)$  such that  $k_{\alpha_{rg}}(t) > 0$  whenever  $\|f - g\|_{\{|z| \leq r_0\}} < \epsilon$  and  $r < r_0$ . On the other hand, clearly if  $g \rightarrow f$  then  $g'(\gamma_r) \rightarrow f'(\gamma_r)$  and  $g''(\gamma_r) \rightarrow f''(\gamma_r)$  uniformly on  $\{r_0 \leq |r| \leq \delta_0\}$ ,  $t \in \mathbb{R}$ . Consequently, from (8), we have  $k_{\alpha_{rg}}(t) \rightarrow k_{\alpha_{rf}}(t)$  uniformly on  $t \in \mathbb{R}$ ,  $r \in [r_0, \delta_0]$ . Then, since  $k(\alpha_{rf}(t)) > 0$  for all  $t \in \mathbb{R}$ ,  $r \in [r_0, \delta_0]$  (from convexity), we may reduce  $\epsilon$  in order to have  $k_{\alpha_{rg}}(t) > 0$  for all  $t \in \mathbb{R}$ ,  $r \in [r_0, \delta_0]$ . Thus  $g(|z| \leq \delta)$  is convex for all  $\delta \leq \delta_0$ . Clearly we may assume  $\epsilon$  small enough such that  $g(|z| \leq \delta_0)$  is contained in  $U$ , which finishes the proof. q.e.d.

### 5. The differentiable case.

In this section we prove Theorem 1.1. As before, let  $\pi : \widehat{\mathbb{C}^2} \rightarrow \mathbb{C}^2$  be the blow up at  $0 \in \mathbb{C}^2$  and let  $E$  be denote the divisor  $\pi^{-1}(0)$ . Let  $\rho : \widehat{\mathbb{C}^2} \rightarrow E$  be the natural projection associated to the fibration on  $\widehat{\mathbb{C}^2}$  which fibers are given by the strict transforms of the complex lines passing through  $0 \in \mathbb{C}^2$ .

**Definition 5.1.** Let  $\{z_k\}$  be a sequence of points in  $\mathbb{C}^2 \setminus \{0\}$ . Let  $L$  be a complex line passing through  $0 \in \mathbb{C}^2$ . We say that  $\{z_k\}$  is tangent to  $L$  at 0 if  $z_k \rightarrow 0$  and every accumulation point of  $\{z_k/||z_k||\}$  is contained in  $L$ .

**Lemma 5.2.** *Let  $\{x_k\}$  be a sequence of points in  $\widehat{\mathbb{C}^2} \setminus E$ . Let  $x \in E$  and let  $P_x = \pi(L_x)$ , where  $L_x$  is the fiber of  $\rho$  through  $x$ . Then  $x_k \rightarrow x \in E$  if and only if  $\{\pi(x_k)\}$  is tangent to  $P_x$  at 0.*

Let  $C$  be an irreducible separatrix (That is: an irreducible holomorphic curve invariant by  $\mathcal{F}$ ) of  $\mathcal{F}$  (It exists by Separatrix Theorem, see [4]). Then  $\tilde{C} = h(C)$  is an irreducible separatrix of  $\tilde{\mathcal{F}}$ . Let  $P$  and  $\tilde{P}$  be the tangents lines at  $0 \in \mathbb{C}^2$  of  $C$  and  $\tilde{C}$  respectively.

**Proposition 5.3.** *Denote by  $A$  the derivative  $dh(0) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ . Then  $A(P) = \tilde{P}$ .*

*Proof.* Given  $v \in P \setminus \{0\}$ , there exists a path  $\gamma : [0, 1] \rightarrow C$ , with  $\gamma(0) = 0$  and such that  $\gamma'(0) = v$ . Then the path  $h \circ \gamma$  is contained in  $\tilde{C}$  and therefore

$$(h \circ \gamma)'(0) = dh(0)(\gamma'(0)) = A(v)$$

is contained in  $\tilde{P}$ . It follows that  $A(P) \subset \tilde{P}$ , and so  $A(P) = \tilde{P}$ , since  $A$  is an isomorphism. q.e.d.

Let  $L$  and  $\tilde{L}$  denote the strict transforms by  $\pi$ , of  $P$  and  $\tilde{P}$  respectively. Let  $q$  and  $\tilde{q}$  be the points of intersection of  $L$  and  $\tilde{L}$  with  $E$ . We may assume without loss of generality that

$$P = \tilde{P} = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}.$$

Let  $\mathcal{U} = \pi^{-1}(z_1 \neq 0)$  and consider holomorphic coordinates  $(t, x)$  in  $\mathcal{U}$  such that  $\pi$  is given by  $\pi(t, x) = (x, tx)$ . Then the fibers of  $\rho$  are given by the sets  $\{t = cte\}$  and, the fibers  $L$  and  $\tilde{L}$  are represented by  $\{t = 0\}$ , that is,  $q = \tilde{q} = (0, 0)$ . Since  $\tilde{\mathcal{F}}_0$  has a finite number of singularities on  $E$ , we may take  $\epsilon > 0$  such that the set  $\{(t, 0) : 0 < |t| < 2\epsilon\} \subset E$  does not contain singularities of  $\tilde{\mathcal{F}}_0$ . let

$$A : \widehat{\mathbb{C}^2} \setminus E \rightarrow \widehat{\mathbb{C}^2} \setminus E$$

be the homeomorphism defined by  $A = \pi^{-1}A\pi$ .

**Proposition 5.4.** *There exists  $\delta > 0$  such that the set*

$$\{(t, x) : |t| < 2\delta\} \setminus E$$

*is mapped by  $A$  into  $\{(t, x) : |t| < 2\epsilon\}$ . Clearly, we may take  $\delta$  such that the set  $\{(t, 0) : 0 < |t| < 2\delta\} \subset E$  does not contain singularities of  $\mathcal{F}_0$ .*

*Proof.* Let  $A(z) = (A_1(z), A_2(z))$  for all  $z = (z_1, z_2) \in \mathbb{C}^2$ . Since  $A(P) = P'$ , it follows that  $A_2(z_1, 0) = 0$  for all  $z_1 \in \mathbb{C}$ . Hence:

$$\frac{A_2(\zeta, 0)}{A_1(\zeta, 0)} = 0$$

for all  $\zeta \in S^1$ . Then there exists  $\delta > 0$  such that

$$(9) \quad \frac{A_2(\zeta, z_2)}{A_1(\zeta, z_2)} < 2\epsilon$$

for all  $\zeta \in S^1$  and all  $z_2 \in \mathbb{C}$  with  $|z_2| \leq 2\delta$ . Since  $A$  is real linear:

$$\frac{A_2(z_1, z_2)}{A_1(z_1, z_2)} = \frac{|z_1|A_2(z_1/|z_1|, z_2/|z_1|)}{|z_1|A_1(z_1/|z_1|, z_2/|z_1|)} = \frac{A_2(z_1/|z_1|, z_2/|z_1|)}{A_1(z_1/|z_1|, z_2/|z_1|)} < 2\epsilon$$

and, since  $z_1/|z_1| \in S^1$ , it follows from (9) that

$$(10) \quad \frac{A_2(z_1, z_2)}{A_1(z_1, z_2)} < 2\epsilon \quad \text{whenever} \quad |z_2/z_1| \leq 2\delta.$$

If  $w \in \{(t, x) : |t| < 2\delta\} \setminus E$ , then  $\pi(w) = (z_1, z_2)$  with  $z_1 \neq 0$  and  $|z_2/z_1| < 2\delta$ . Therefore

$$\begin{aligned} A(w) &= \pi^{-1}A\pi(w) = \pi^{-1}A(z_1, z_2) = \pi^{-1}(A_1(z_1, z_2), A_2(z_1, z_2)) \\ &= \left( \frac{A_2(z_1, z_2)}{A_1(z_1, z_2)}, A_1(z_1, z_2) \right), \end{aligned}$$

and it follows from (10) that  $A(w)$  is contained in  $\{(t, x) : |t| < 2\epsilon\}$ .

q.e.d.

Let  $p = (\delta, 0) \in E$  and let  $L_p = \{t = \delta\}$  (its fiber). Consider the path

$$\beta : S^1 \rightarrow L_p,$$

$$\beta(\zeta) = (\delta, \zeta),$$

and let  $\beta_A : S^1 \rightarrow \{(t, x) : |t| < 2\epsilon\}$  given by  $\beta_A = A \circ \beta$ .

**Proposition 5.5.** *The set  $\rho(A(L_p \setminus \{p\}))$  is equal to  $\rho(\beta_A(S^1))$ .*

*Proof.* Evidently  $\rho\beta_A(S^1) \subset \rho(A(L_p \setminus \{p\}))$ . On the other hand, let  $(\delta, x) \in L_p \setminus \{p\}$ , then

$$\begin{aligned} \rho A(\delta, x) &= \rho\pi^{-1}A\pi(\delta, x) = \rho\pi^{-1}A(x, \delta x) \\ &= \rho\pi^{-1}(A_1(x, \delta x), A_2(x, \delta x)) = \rho\left(\frac{A_2(x, \delta x)}{A_1(x, \delta x)}, A_1(x, \delta x)\right) \\ &= \left(\frac{A_2(x, \delta x)}{A_1(x, \delta x)}, 0\right) = \left(\frac{A_2(x/|x|, \delta x/|x|)}{A_1(x/|x|, \delta x/|x|)}, 0\right) \\ &= \rho\left(\frac{A_2(x/|x|, \delta x/|x|)}{A_1(x/|x|, \delta x/|x|)}, A_1(x/|x|, \delta x/|x|)\right) \\ &= \rho\pi^{-1}(A_1(x/|x|, \delta x/|x|), A_2(x/|x|, \delta x/|x|)) \\ &= \rho\pi^{-1}A(x/|x|, \delta x/|x|) = \rho\pi^{-1}A\pi(\delta, x/|x|) \\ &= \rho A(\beta(x/|x|)) = \rho(\beta_A(x/|x|)). \end{aligned}$$

Therefore  $\rho(A(L_p \setminus \{p\})) \subset \rho\beta_A(S^1)$ .

q.e.d.

Define  $K$  as the set of points  $y \in E$  such that there exists a sequence  $\{x_k\}$  in  $L_p \setminus \{p\}$  with  $h(x_k) \rightarrow y$  as  $k \rightarrow \infty$ .

**Proposition 5.6.** *Given a neighborhood  $\Omega$  of  $K$  in  $\widehat{\mathbb{C}^2}$ , there exist a disc  $\Sigma$  in  $L_p$  containing  $p$ , such that the set  $h(\Sigma \setminus \{p\})$  is contained in  $\Omega$ .*

*Proof.* Is a direct consequence of the definition of  $K$ . q.e.d.

**Proposition 5.7.** *The set  $K$  is equal to  $\rho\beta_A(S^1)$ . Thus, since  $\beta_A(S^1) \subset A(L_p \setminus \{p\})$  does not intersect  $\tilde{L}$ , the set  $K$  is contained in  $\{(t, 0) : 0 < |t| < 2\epsilon\}$ .*

*Proof.* Let  $y \in K$ . Then there exist a sequence  $\{x_k\}$  in  $L_p \setminus \{p\}$  with  $h(x_k) \rightarrow y$  as  $k \rightarrow \infty$ . Let  $P_y = \pi(L_y)$ , where  $L_y$  is the fiber of  $\rho$  through  $y$ . It follows from Lemma 5.2 that the sequence  $\{\pi(h(x_k))\}$  is tangent to  $P_y$  at 0. Since  $\pi(x_k) \rightarrow 0$  as  $k \rightarrow \infty$  and  $A$  is the derivate of  $h$  at 0, we have that

$$h(\pi(x_k)) = A(\pi(x_k)) + R(\pi(x_k)),$$

where  $R(\pi(x_k))/\|\pi(x_k)\| \rightarrow 0$  as  $k \rightarrow \infty$ . Therefore

$$(11) \quad \frac{h(\pi(x_k))}{\|\pi(x_k)\|} = \frac{A(\pi(x_k))}{\|\pi(x_k)\|} + \frac{R(\pi(x_k))}{\|\pi(x_k)\|},$$

with  $R(\pi(x_k))/\|\pi(x_k)\| \rightarrow 0$  as  $k \rightarrow \infty$ . Since the sequence  $\{h(\pi(x_k))\} = \{\pi h(x_k)\}$  is tangent to  $P_y$  at 0, we have by definition that any accumulation point of

$$\frac{h(\pi(x_k))}{\|h(\pi(x_k))\|}$$

is contained in  $P_y$  and the same holds for the sequence

$$\frac{h(\pi(x_k))}{\|\pi(x_k)\|} = \frac{h(\pi(x_k))}{\|h(\pi(x_k))\|} \cdot \frac{\|h(\pi(x_k))\|}{\|\pi(x_k)\|}.$$

Then, it follows from (11) that any accumulation point of the sequence

$$\frac{A(\pi(x_k))}{\|\pi(x_k)\|}$$

is contained in  $P_y$  and the same property is satisfied by

$$\frac{A(\pi(x_k))}{\|A(\pi(x_k))\|} = \frac{A(\pi(x_k))}{\|\pi(x_k)\|} \cdot \frac{\|\pi(x_k)\|}{\|A(\pi(x_k))\|}.$$

Then the sequence

$$\frac{A(\pi(x_k))}{\|A(\pi(x_k))\|} = \frac{\pi(A(x_k))}{\|\pi(A(x_k))\|}$$

is tangent to  $P_y$  at 0. By Lemma 5.2 we have that  $A(x_k) \rightarrow y$  as  $k \rightarrow \infty$ , hence  $\rho(A(x_k)) \rightarrow y$  as  $k \rightarrow \infty$ . Then  $y$  is a limit point of  $\rho(A(L_p \setminus \{p\}))$ . But  $\rho(A(L_p \setminus \{p\}))$  is equal to  $\rho\beta_A(S^1)$  by Proposition 5.5. Then, since  $\rho\beta_A(S^1)$  is compact, we have that  $y \in \rho\beta_A(S^1)$  and therefore  $K \subset \rho\beta_A(S^1)$ . On the other hand, let  $y \in \rho\beta_A(S^1)$ . Then  $y = \rho(A(\delta, \zeta))$ . For all  $k \in \mathbb{N}$  let  $x_k = (\delta, s_k\zeta) \in L_p$ , where  $s_k > 0$

and  $s_k \rightarrow 0$  as  $k \rightarrow \infty$ . Clearly  $x_k \rightarrow p = (\delta, 0)$  as  $k \rightarrow \infty$ . Then  $\pi(x_k) \rightarrow 0 \in \mathbb{C}^2$  as  $k \rightarrow \infty$  and we have that

$$h(\pi(x_k)) = A(\pi(x_k)) + R(\pi(x_k))$$

with  $\|R(\pi(x_k))\|/\|\pi(x_k)\| \rightarrow 0$  as  $k \rightarrow \infty$ . Therefore

$$\frac{h(\pi(x_k))}{\|\pi(x_k)\|} = \frac{A(\pi(x_k))}{\|\pi(x_k)\|} + \frac{R(\pi(x_k))}{\|\pi(x_k)\|}.$$

Hence, since

$$\frac{A(\pi(x_k))}{\|\pi(x_k)\|} = \frac{A(s_k \zeta, s_x \zeta \delta)}{\|(s_k \zeta, s_x \zeta \delta)\|} = \frac{s_k A(\zeta, \zeta \delta)}{|s_k| \|(\zeta, \zeta \delta)\|} = \frac{A(\zeta, \zeta \delta)}{\|(\zeta, \zeta \delta)\|}$$

and  $\|R(\pi(x_k))\|/\|\pi(x_k)\| \rightarrow 0$  as  $k \rightarrow \infty$ , we have that

$$(12) \quad \frac{h(\pi(x_k))}{\|\pi(x_k)\|} \rightarrow \frac{A(\zeta, \zeta \delta)}{\|(\zeta, \zeta \delta)\|}$$

as  $k \rightarrow \infty$ . Let  $L_y$  be the fiber of  $\rho$  through  $y$  and let  $P_y = \pi(L_y)$ . Since  $\rho(A(\delta, \zeta)) = y$  we have  $A(\delta, \zeta) \in L_y$ , hence  $\pi A(\delta, \zeta) \in P_y$ . Then

$$\frac{A(\zeta, \zeta \delta)}{\|(\zeta, \zeta \delta)\|} = \frac{A(\pi(\delta, \zeta))}{\|(\pi(\delta, \zeta))\|} = \frac{\pi A(\delta, \zeta)}{\|(\pi(\delta, \zeta))\|}$$

is contained in  $P_y$  and it follows from (12) that any accumulation point of the sequence

$$\frac{\pi(h(x_k))}{\|\pi(h(x_k))\|} = \frac{h(\pi(x_k))}{\|\pi(x_k)\|} \cdot \frac{\|\pi(x_k)\|}{\|\pi(x_k)\|}$$

is contained in  $P_y$ . Then, by Lemma 5.2 we have that  $\pi(h(x_k)) \rightarrow y$  as  $k \rightarrow \infty$ . Thus  $y \in K$  and therefore  $\rho\beta_A(S^1) \subset K$ . q.e.d.

**Proposition 5.8.** *Define  $\theta : [0, 1] \rightarrow E$  by  $\theta(s) = \rho\beta_A(e^{\pi i s})$  for all  $s \in [0, 1]$ . Then*

$$\begin{aligned} \rho \circ \beta_A(e^{2\pi i s}) &= \theta(2s), & \text{if } 0 \leq s \leq 1/2, \\ \rho \circ \beta_A(e^{2\pi i s}) &= \theta(2s - 1), & \text{if } 1/2 \leq s \leq 1. \end{aligned}$$

*In particular,  $\rho\beta(S^1) = \theta([0, 1])$  and, by Proposition 5.7, we have that  $K = \theta([0, 1])$ .*

*Proof.* If  $s \in [0, 1/2]$ , then  $\rho\beta_A(e^{2\pi is}) = \rho\beta_A(e^{\pi i(2s)}) = \theta(2s)$ . Suppose now that  $s \in [1/2, 1]$ . Then, since  $A$  is real linear:

$$\begin{aligned}
 w &= \rho A\beta(e^{2\pi is}) = \rho\pi^{-1}A\pi(\delta, e^{2\pi is}) = \rho\pi^{-1}A(e^{2\pi is}, \delta e^{2\pi is}) \\
 &= \rho\pi^{-1}(-1)A((-1)e^{2\pi is}, (-1)\delta e^{2\pi is}) \\
 &= \rho\pi^{-1}(-1)(A_1(e^{-\pi i}e^{2\pi is}, e^{-\pi i}\delta e^{2\pi is}), A_2(e^{-\pi i}e^{2\pi is}, e^{-\pi i}\delta e^{2\pi is})) \\
 &= \rho\pi^{-1}(-A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}), -A_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})) \\
 &= \rho \left( \frac{A_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}{A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}, -A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}) \right) \\
 &= \left( \frac{A_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}{A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}, 0 \right) \\
 &= \rho \left( \frac{A_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}{A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})}, A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}) \right) \\
 &= \rho\pi^{-1}(A_1(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}), A_2(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)})) \\
 &= \rho\pi^{-1}A(e^{\pi i(2s-1)}, \delta e^{\pi i(2s-1)}) = \rho\pi^{-1}A\pi(\delta, e^{\pi i(2s-1)}) \\
 &= \rho A(\delta, e^{\pi i(2s-1)}) = \rho A\beta(e^{\pi i(2s-1)}) = \rho\beta_A(e^{\pi i(2s-1)}) \\
 &= \theta(2s-1),
 \end{aligned}$$

since  $(2s-1) \in [0, 1]$ .

**Proposition 5.9.** *We have that: either  $K$  is a point, or  $K$  is equal to a Jordan curve.*

*Proof.* By Proposition 5.7 and Proposition 5.8, it is sufficient to prove that: either  $\theta$  is constant or it is a simple closed curve. By Proposition 5.8, we have that  $\theta(0) = \theta(2(1/2) - 1) = \rho\beta_A(e^{2\pi i(1/2)}) = \theta(2(1/2)) = \theta(1)$ . Thus  $\theta$  defines a closed curve in  $E$ . Suppose that  $\theta$  is not a simple curve, that is,  $\theta(s') = \theta(s'')$  for  $0 \leq s' < s'' < 1$ . Observe that

$$\theta(s') = \rho\pi^{-1}A\pi(\delta, e^{\pi is'}) = \rho\pi^{-1}A(e^{\pi is'}, \delta e^{\pi is'}).$$

Writing  $A(e^{\pi is'}, \delta e^{\pi is'}) = (A'_1, A'_2)$  we have that

$$\theta(s') = \rho\pi^{-1}(A'_1, A'_2) = \rho \left( \frac{A'_2}{A'_1}, A'_1 \right) = \left( \frac{A'_2}{A'_1}, 0 \right).$$

Analogously, making  $A(e^{\pi is''}, \delta e^{\pi is''}) = (A''_1, A''_2)$  we obtain

$$\theta(s'') = \left( \frac{A''_2}{A''_1}, 0 \right).$$

Then  $\frac{A'_2}{A'_1} = \frac{A''_2}{A''_1}$  and we have therefore that

$$\frac{aA'_2 + bA''_2}{aA'_1 + bA''_1} = \frac{A'_2}{A'_1} = \frac{A''_2}{A''_1}$$

for all  $a, b \in \mathbb{R}$  such that  $aA'_1 + bA''_1 \neq 0$ . Computing as above

$$\rho\pi^{-1}(aA'_1 + bA''_1, aA'_2 + bA''_2) = \left( \frac{aA'_2 + bA''_2}{aA'_1 + bA''_1}, 0 \right) = \left( \frac{A'_2}{A'_1}, 0 \right) = \theta(s'),$$

that is,

$$(13) \quad \rho\pi^{-1}(a(A'_1, A'_2) + b(A''_1, A''_2)) = \theta(s').$$

Since  $0 \leq s' < s'' < 1$ , the vectors  $e^{\pi is'}$  and  $e^{\pi is''}$  are real-linearly independent. Thus, for all  $s \in [0, 1]$  we have that  $e^{\pi is} = ae^{\pi is'} + be^{\pi is''}$  with  $a, b \in \mathbb{R}$ . Therefore:

$$\begin{aligned} \theta(s) &= \rho A\beta(e^{\pi is}) = \rho\pi^{-1}A\pi(\delta, e^{2\pi is}) = \rho\pi^{-1}A(e^{2\pi is}, \delta e^{2\pi is}) \\ &= \rho\pi^{-1}A(ae^{\pi is'} + be^{\pi is''}, \delta(ae^{\pi is'} + be^{\pi is''})) \\ &= \rho\pi^{-1}A(a(e^{\pi is'}, \delta e^{\pi is'}) + b(e^{\pi is''}, \delta e^{\pi is''})) \\ &= \rho\pi^{-1}(aA(e^{\pi is'}, \delta e^{\pi is'}) + bA(e^{\pi is''}, \delta e^{\pi is''})) \\ &= \rho\pi^{-1}(a(A'_1, A'_2) + b(A''_1, A''_2)), \end{aligned}$$

and by using (13):

$$\theta(s) = \theta(s').$$

It follows that  $\theta$  is constant and the assertion is therefore proved.

We denote by  $V$  and  $\tilde{V}$  the sets  $\{(t, x) : |t| \leq 2\delta\}$  and  $\{(t, x) : |t| \leq 2\epsilon\}$  respectively. Let

$$\tilde{\beta} : S^1 \rightarrow \tilde{V}$$

be the path defined by  $\tilde{\beta}(\zeta) = (\epsilon, \zeta)$ .

**Proposition 5.10.** *The path  $\beta_A$  is homologous to  $\xi\tilde{\beta}$  in  $\tilde{V} \setminus (\tilde{L} \cup E)$ , where  $\xi = 1$  or  $-1$ .*

*Proof.* Let  $\mathcal{B}_w$  be the disc  $\{(t, x) : t = w, |x| \leq 1\}$  in  $V$ . Observe that  $\tilde{\beta}$  is equal to  $\partial\tilde{\mathcal{B}}$ , where  $\tilde{\mathcal{B}}$  is the disc  $\{(\epsilon, x) : |x| \leq 1\}$  in  $\tilde{V}$ . Then, since  $A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  preserves orientation, it follows from Lemma 3.1 that for some  $w \neq 0$ :

$$(14) \quad A(\partial\mathcal{B}_w) = \xi\partial\tilde{\mathcal{B}} = \xi\tilde{\beta} \quad \text{in} \quad H_1(\tilde{V} \setminus (\tilde{L} \cup E)).$$

Observe that  $\partial\mathcal{B}_w$  is homologous to  $\beta$  in  $V \setminus (L \cup E)$ . Then, since  $A(V \setminus (L \cup E))$  is contained in  $\tilde{V} \setminus (\tilde{L} \cup E)$ , it follows that

$$(15) \quad A(\partial\mathcal{B}_w) = A(\beta) = \beta_A \quad \text{in} \quad H_1(\tilde{V} \setminus (\tilde{L} \cup E)).$$

Thus the proposition follows from (15) and (14).

**Proposition 5.11.** *Suppose that  $K$  is a Jordan curve and let  $U \subset \{(t, 0) : |t| < 2\epsilon\}$  be the domain bounded by  $K$ . Then  $q = (0, 0) \notin U$ .*



*Proof.* Making  $C = \{(t, 0) : |t| < \epsilon\}$  and since  $\rho : \tilde{V} \setminus (\tilde{L} \cup E) \rightarrow C \setminus \{p'\}$  is well defined, it follows from Proposition 5.10 that

$$\rho(\beta_A) = \xi \rho(\tilde{\beta}) \quad \text{in} \quad H_1(C \setminus \{p'\}).$$

Then, since  $\rho(\tilde{\beta}) = 0$  in  $H_1(C \setminus \{p'\})$ , we have that

$$(16) \quad \rho \circ \beta_A = 0 \quad \text{in} \quad H_1(C \setminus \{p'\}).$$

If we consider  $\rho \circ \beta_A$  as defined on  $[0, 1]$  by  $s \rightarrow \rho \beta_A(e^{2\pi i s})$ , it follows from Proposition 5.8 that  $\rho \circ \beta_A = \theta * \theta$ . Then

$$\rho \circ \beta_A = 2\theta \quad \text{in} \quad H_1(C \setminus \{p'\})$$

and it follows from (16) that

$$\theta = 0 \quad \text{in} \quad H_1(C \setminus \{p'\}),$$

since  $H_1(C \setminus \{p'\})$  does not have torsion. Therefore  $p' \notin U$ .

**Proposition 5.12.** *Let  $\Sigma$  be a disc in  $L_p$  containing  $p$  and such that  $\mathcal{A} = h(\Sigma \setminus \{p\})$  is contained in  $\tilde{V} \setminus E$ . Let  $\gamma$  be a path in  $\mathcal{A}$ , which represents a generator of  $H_1(\mathcal{A})$ . Then  $\gamma$  is homologous to  $\xi \tilde{\beta}$  in  $\tilde{V} \setminus E$  with  $\xi = 1$  or  $-1$ .*

*Proof.* Since  $\tilde{V} \setminus (\tilde{L} \cup E)$  is contained in  $\tilde{V} \setminus E$ , it follows from Proposition 5.10 that  $\beta_A$  is homologous to  $\xi \tilde{\beta}$  in  $\tilde{V} \setminus E$  where  $\xi = 1$  or  $-1$ . Therefore it is sufficient to show that  $\gamma$  is homologous to  $\xi \beta_A$  with  $\xi = 1$  or  $-1$ . Let

$$\vartheta_r : S^1 \rightarrow L_p = \{t = \delta\}$$

be the path defined by  $\vartheta_r(\zeta) = (\delta, r\zeta)$  with  $0 < r < 1$  small enough such that  $\{(\delta, x) : |x| \leq r\}$  is contained in  $\Sigma$ . Then  $\vartheta_r$  is a generator of  $H_1(\Sigma \setminus \{p\})$  and consequently  $h \circ \vartheta_r$  is a generator of  $H_1(\mathcal{A})$ . Thus  $\gamma$  is homologous to  $\xi h \circ \vartheta_r$  in  $\tilde{V} \setminus E$ , where  $\xi = 1$  or  $-1$ . Therefore it is sufficient to prove that  $h \circ \vartheta_r$  is homologous to  $\beta_A$  in  $\tilde{V} \setminus E$ . Recall that  $\beta(\zeta) = (\delta, \zeta)$ . Then  $\beta$  and  $\vartheta_r$  are homologous in  $C = \{(\delta, x) : 0 < |x| \leq 1\} \subset L_p$  and, since  $A(C) \subset \tilde{V} \setminus E$ , it follows that the paths  $A \circ \beta = \beta_A$  and  $A \circ \vartheta_r$  are homologous in  $\tilde{V} \setminus E$ . Then, it suffices to show that  $h \circ \vartheta_r$  and  $A \circ \vartheta_r$  are homologous in  $\tilde{V} \setminus E$  for some  $r > 0$ .

Let  $P' = \pi(L_p)$  and consider the path  $\theta_r : S^1 \rightarrow P'$  defined by  $\theta_r = \pi \circ \vartheta_r$ , that is  $\theta_r(\zeta) = (r\zeta, \delta r\zeta)$ . Recall that  $A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is an isomorphism, then there exist a constant  $c > 0$  such that

$$(17) \quad \|A(z)\| > c\|z\| \quad \text{for all} \quad z \in \mathbb{C}^2.$$

Since  $A$  is the derivate of  $h$  at  $0$ , there exists  $\epsilon > 0$  such that

$$(18) \quad h(z) = A(z) + R(z),$$

with  $|R(z)| < c|z|$  whenever  $|z| < \varepsilon$ . Now, assume that

$$r < \min \left\{ \frac{\varepsilon}{\sqrt{1+\delta^2}}, c, c/(2\varepsilon+1), \frac{\varepsilon_0}{\sqrt{1+\delta^2}} \right\},$$

where the constant  $\varepsilon_0 > 0$  will be defined later. Then, since  $\theta_r(\zeta) = (r\zeta, \delta r\zeta)$  satisfies

$$(19) \quad \|\theta_r(\zeta)\| = r\sqrt{1+\delta^2} < \varepsilon,$$

we have that

$$(20) \quad \|R(\theta_r(\zeta))\| < c\|\theta_r(\zeta)\|.$$

Therefore the map

$$F : S^1 \times [0, 1] \rightarrow \mathbb{C}^2, \\ F(\zeta, s) = A(\theta_r(\zeta)) + sR(\theta_r(\zeta))$$

is such that

$$\begin{aligned} \|F(\zeta, s)\| &= \|A(\theta_r(\zeta)) + sR(\theta_r(\zeta))\| \\ &\geq \|A(\theta_r(\zeta))\| - \|sR(\theta_r(\zeta))\| \geq c\|\theta_r(\zeta)\| - \|R(\theta_r(\zeta))\| > 0. \end{aligned}$$

Observe that  $F(\zeta, 0) = A(\theta_r(\zeta))$  and  $F(\zeta, 1) = A(\theta_r(\zeta)) + R(\theta_r(\zeta)) = h(\theta_r(\zeta))$ . Then  $F$  defines a homotopy between  $A(\theta_r)$  and  $h(\theta_r)$  in  $\mathbb{C}^2 \setminus \{0\}$ . Thus, since  $\pi^{-1}A(\theta_r) = A(\vartheta_r)$  and  $\pi^{-1}h(\theta_r) = h(\vartheta_r)$ , it follows that  $\pi^{-1} \circ F$  defines a homotopy between  $A \circ \vartheta_r$  and  $h \circ \vartheta_r$  in  $\widehat{\mathbb{C}^2} \setminus E$ . Therefore, in order to prove that  $A \circ \vartheta_r = h \circ \vartheta_r$  in  $H_1(\widetilde{V} \setminus E)$ , it suffices to show that  $\pi^{-1} \circ F(\zeta, s)$  belongs to  $\widetilde{V}$  for all  $s \in [0, 1]$ ,  $\zeta \in S^1$ . We write  $F(\zeta, s) = (x_F, y_F)$ ,  $A(\theta_r(\zeta)) = (x_A, y_A)$  and  $R(\theta_r(\zeta)) = (x_R, y_R)$ , then

$$(21) \quad (x_F, y_F) = (x_A, y_A) + s(x_R, y_R).$$

Observe that

$$\left( \frac{y_A}{x_A}, x_A \right) = \pi^{-1}(x_A, y_A) = \pi^{-1}A(\theta_r(\zeta)) = A\pi^{-1}\theta_r(\zeta) = A \circ \vartheta_r(\zeta),$$

hence  $(y_A/x_A, 0) = \rho A \vartheta_r(\zeta)$ . Then, since  $A\vartheta_r(\zeta)$  is contained in  $A(L_p \setminus \{p\})$ , it follows from Proposition 5.5 and Proposition 5.7 that  $(y_A/x_A, 0)$  is contained in  $K$ . Thus, since  $K$  a compact subset of  $\{(t, 0) : |t| < 2\varepsilon\}$ , we have that

$$(22) \quad \frac{|y_A|}{|x_A|} + \varepsilon_1 < 2\varepsilon$$

for some  $\varepsilon_1 > 0$  small enough. Take  $\varepsilon_2 > 0$  be such that

$$(23) \quad \frac{\varepsilon_2(1+2\varepsilon)}{c/(1+2\varepsilon) - \varepsilon_2} < \varepsilon_1.$$

Now, we chose  $\varepsilon_0$  be such that

$$(24) \quad \|R(z)\| < \varepsilon_2\|z\|$$

whenever  $\|z\| < \varepsilon_0$ . Observe that  $\pi^{-1} \circ (x_F, y_F)$  belongs to

$$\tilde{V} = \{(t, x) : |x| < 2\epsilon\}$$

if and only if  $\frac{y_F}{x_F} < 2\epsilon$ , and by (21), if and only if

$$(25) \quad \frac{y_A + sy_R}{x_A + sy_R} < 2\epsilon.$$

An easy computation shows that

$$\frac{y_A + sy_R}{x_A + sy_R} = \frac{y_A}{x_A} + \frac{sy_R - sy_R(y_A/x_A)}{x_A + sy_R}.$$

Thus, in view of (22), it is sufficient to prove that

$$(26) \quad \frac{|sy_R - sy_R(y_A/x_A)|}{|x_A + sy_R|} \leq \epsilon_1.$$

Since that  $\|\theta_r(\zeta)\| = r\sqrt{1 + \delta^2} < \varepsilon_0$ , it follows from (24) that  $\|(y_R, y_R)\| = \|R(\theta_r(\zeta))\| < \varepsilon_2\|\theta_r(\zeta)\|$ , hence  $|y_R| < \varepsilon_2\|\theta_r(\zeta)\|$ . Then

$$\begin{aligned} |sy_R - sy_R(y_A/x_A)| &= |sy_R| \cdot |1 - y_A/x_A| \\ &< \varepsilon_2\|\theta_r(\zeta)\|(1 + |y_A|/|x_A|) \end{aligned}$$

and, by using (22), we obtain

$$(27) \quad |sy_R - s(y_A/x_A)y_R| < \varepsilon_2(1 + 2\epsilon)\|\theta_r(\zeta)\|.$$

On the other hand, also from (22) we have that  $|y_A| < 2\epsilon|x_A|$ , hence

$$(1 + 2\epsilon)|x_A| \geq |x_A| + |y_A| \geq \|(x_A, y_A)\| = \|A(\theta_r(\zeta))\| \geq c\|\theta_r(\zeta)\|$$

and therefore

$$|x_A| \geq \frac{c}{1 + 2\epsilon} \cdot \|\theta_r(\zeta)\|.$$

Then

$$|x_A + sy_R| \geq |x_A| - |sy_R| \geq |x_A| - |y_R| \geq \frac{c}{1 + 2\epsilon}\|\theta_r(\zeta)\| - \varepsilon_2\|\theta_r(\zeta)\|$$

and so

$$|x_A + sy_R| \geq (c/(1 + 2\epsilon) - \varepsilon_2)\|\theta_r(\zeta)\|.$$

From this and (27) we obtain

$$\frac{|sy_R - s(y_A/x_A)y_R|}{|x_A + sy_R|} \leq \frac{\varepsilon_2(1 + 2\epsilon)\|\theta_r(\zeta)\|}{(c/(1 + 2\epsilon) - \varepsilon_2)\|\theta_r(\zeta)\|} = \frac{\varepsilon_2(1 + 2\epsilon)}{(c/(1 + 2\epsilon) - \varepsilon_2)}$$

and from (23):

$$\frac{|sy_R - sy_A/x_A y_R|}{|x_A + sy_R|} \leq \varepsilon_1,$$

which finishes the proof.

q.e.d.

It follows from Proposition 5.7 and Proposition 5.9 that there exists a subset  $D$  of the divisor  $E$  with the following properties:

- (i)  $D$  is diffeomorphic to a closed disc.

- (ii)  $D$  is contained in  $\{(t, 0) : 0 < |t| < 2\epsilon\}$
- (iii)  $K$  is contained in the interior of  $D$ .

Let  $\tilde{p}$  be a point in the interior of  $D$  and let  $L_{\tilde{p}}$  be the fiber of  $\rho$  through  $\tilde{p}$ . Since  $D$  is contained in a leaf of  $\tilde{\mathcal{F}}_0$ , there is a disc  $\Sigma'$  in  $L_{\tilde{p}}$  containing  $\tilde{p}$  with the following property: if  $z \in \Sigma'$ , then there exists a closed disc  $D_z$  in the leaf of  $\tilde{\mathcal{F}}_0$  passing through  $z$ , such that  $\rho$  maps  $D_z$  diffeomorphically onto  $D$ . Let  $W$  denote the set  $\bigcup_{z \in \Sigma'} D_z$ . By Proposition 5.6, there exists a disc  $\Sigma$  in  $L_p$  containing  $p$ , such that the set  $\mathcal{A} = h(\Sigma \setminus \{p\})$  is contained in the interior of  $W$ . We assume  $\Sigma$  be small enough such that  $\mathcal{F}_0$  is transverse to  $\Sigma$ .

**Proposition 5.13.** *There exists a disc  $\tilde{\Sigma} \subset \Sigma'$  containing  $\tilde{p}$ , with the following property. Given  $x \in \tilde{\Sigma} \setminus \{\tilde{p}\}$ , the disc  $D_x$  intersects  $\mathcal{A}$  in a unique point  $f(x)$ . Moreover, the map  $f : \tilde{\Sigma} \setminus \{\tilde{p}\} \rightarrow \mathcal{A}$  is continuous.*

*Proof.* The foliation  $\tilde{\mathcal{F}}_0$  induces a complex structure in  $\mathcal{A}$  as follows. Let  $y \in \mathcal{A}$  and  $x \in \Sigma \setminus \{p\}$  with  $h(x) = y$ . Since  $\Sigma$  is transverse to  $\mathcal{F}_0$ , there exists a neighborhood  $W_x$  of  $x$  in  $\widehat{\mathbb{C}^2} \setminus E$  such that each leaf of  $\mathcal{F}_0|_{W_x}$  intersects  $\Sigma$  only one time. Let  $W_y$  be a neighborhood of  $y$  where  $\tilde{\mathcal{F}}_0$  is trivial. Thus, there exists a disc  $\tilde{\Sigma}_y$  (complex sub-manifold of  $W_y$ ) such that each leaf of  $\tilde{\mathcal{F}}_0|_{W_y}$  intersects  $\tilde{\Sigma}_y$  at a unique point. We may assume that  $h^{-1}(W_y)$  is contained in  $W_x$ . Let  $\Sigma_x \subset \Sigma \cap W_x$  be a disc with  $x \in \Sigma_x$  and such that the closure of  $\Sigma_y = h(\Sigma_x) \subset \mathcal{A}$  is contained in  $W_y$ . If  $w$  is a point contained in  $\Sigma_y$ , the leaf of  $\tilde{\mathcal{F}}_0|_{W_y}$  passing through it intersects  $\tilde{\Sigma}_y$  in a unique point  $\psi_y(w)$ . Clearly,  $\psi_y$  is continuous and we claim that  $\psi_y$  is a homeomorphism of  $\Sigma_y$  onto its image. Since  $\tilde{\Sigma}_y$  is compact, it suffices to prove that  $\psi_y$  is injective on  $\tilde{\Sigma}_y$ . Suppose that  $w_1$  and  $w_2$  are two different points in  $\tilde{\Sigma}_y$  contained in the same leaf  $L$  of  $\tilde{\mathcal{F}}_0|_{W_y}$ . Then, since  $\pi_y^{-1}(W_y) \subset W_x$ , we have that  $\pi_y^{-1}(L)$  is contained in a leaf  $L'$  of  $\mathcal{F}_0|_{W_x}$ . Then  $h^{-1}(w_1)$  and  $h^{-1}(w_2)$  are two different points in the intersection of  $L'$  with  $\Sigma_0$ , which is a contradiction. Then we consider  $\psi_y : \Sigma_y \rightarrow \tilde{\Sigma}_y$  as a local chart of  $\mathcal{A}$ . We may assume the sets  $\Sigma_y$  be small enough such that, if  $\Sigma_y \cap \Sigma_{y'} \neq \emptyset$ , then  $\Sigma_y \cup \Sigma_{y'}$  is contained in an open set where  $\tilde{\mathcal{F}}_0$  is trivial. Then it is easy to see that the map  $\psi_{y'} \circ \psi_y^{-1}$ , which preserves the leaves, is a holonomy map and therefore holomorphic.

Given  $y \in \mathcal{A}$ , denote by  $g(y)$  the point in  $\Sigma' \setminus \{\tilde{p}\}$  such that  $y \in D_{g(y)}$ . It is not difficult to see that the map  $g \circ \psi_y^{-1} : \tilde{\Sigma}_y \rightarrow \Sigma'$  is a holonomy map. Therefore  $g : \mathcal{A} \rightarrow \Sigma'$  is holomorphic and regular. It is known (see [1]) that there exists a biholomorphism

$$\varphi : A_r = \{z \in \mathbb{C} : 0 \leq r < |z| < 1\} \rightarrow \mathcal{A}$$

and we may take  $\varphi$  such that  $\varphi(z) \rightarrow E$  as  $|z| \rightarrow r$ . Hence  $g \circ \varphi(z) \rightarrow \tilde{p}$  as  $|z| \rightarrow r$ . Then the map  $g \circ \varphi : A_r \rightarrow \Sigma'$  extends as  $g \circ \varphi \equiv \tilde{p}$  on  $|z| = r$ . This implies that  $r = 0$ . Then  $g \circ \varphi$  extends holomorphically to  $\mathbb{D}$  with  $g \circ \varphi(0) = \tilde{p}$ .

*Assertion.* The map  $g \circ \varphi$  is regular at 0.

*Proof.* Let  $\gamma$  be a path in  $\mathbb{D} \setminus \{0\}$  which winds once around 0. It is sufficient to prove that the path  $g \circ \varphi(\gamma)$  in  $\Sigma'$  winds once around  $\tilde{p}$ . Let  $\beta'$  be a path in  $\Sigma' \setminus \{\tilde{p}\}$  such that

$$(28) \quad \beta' = \tilde{\beta} \quad \text{in} \quad H_1(\tilde{V} \setminus E).$$

Clearly  $\beta'$  represents generators in  $H_1(\Sigma' \setminus \{\tilde{p}\})$  and  $H_1(W \setminus E)$ . Let  $N$  and  $N'$  be integers such that

$$(29) \quad g \circ \varphi(\gamma) = N\beta' \quad \text{in} \quad H_1(\Sigma' \setminus \{\tilde{p}\})$$

and

$$(30) \quad \varphi(\gamma) = N'\beta' \quad \text{in} \quad H_1(W \setminus E).$$

We shall prove that  $N = 1$  or  $-1$ . Observe that  $g$  is the restriction of the map

$$G : W \setminus E \rightarrow \Sigma' \setminus \{\tilde{p}\}$$

defined by  $G(D_x) = \{x\}$  for all  $x \in \Sigma' \setminus \{\tilde{p}\}$ . Then, since  $g(\beta') = \beta'$ , it follows from (29) that

$$g \circ \varphi(\gamma) = N'\beta' \quad \text{in} \quad H_1(\Sigma' \setminus \{\tilde{p}\})$$

and, in view of (29), we conclude that  $N' = N$ . Thus, since  $W \setminus E \subset \tilde{V} \setminus E$ , equation (30) gives:

$$\varphi(\gamma) = N\beta' \quad \text{in} \quad H_1(\tilde{V} \setminus E).$$

Then, by (28), we have that

$$\varphi(\gamma) = N\tilde{\beta} \quad \text{in} \quad H_1(\tilde{V} \setminus E).$$

Thus, since  $\varphi(\gamma)$  is a generator of  $H_1(\mathcal{A})$ , Proposition 5.12 implies that  $N = 1$  or  $-1$ .

Now, since  $g \circ \varphi$  is regular at 0, there exists a disc  $\Omega$  in  $\mathbb{D}$  containing 0, such that  $g \circ \varphi|_{\Omega}$  is a homeomorphism onto its image. Then, since  $\varphi$  is a diffeomorphism, it follows that  $\bar{g} = g|_{\varphi(\Omega \setminus \{0\})}$  is a homeomorphism onto its image. Thus we take a disc  $\tilde{\Sigma} \subset g\varphi(\Omega) \subset \Sigma'$  containing  $\tilde{p}$  and define  $f = \bar{g}^{-1}$  on  $\tilde{\Sigma} \setminus \{\tilde{p}\}$ . Let  $x \in \tilde{\Sigma} \setminus \{\tilde{p}\}$ . Clearly  $f(x) \in \mathcal{A}$  and since  $g(f(x)) = x$ , we have that  $f(x) \in D_x$  and so  $f(x) \in D_x \cap \mathcal{A}$ . If  $y \in D_x \cap \mathcal{A}$ , then  $g(y) = x$  and therefore  $y = f(x)$ . Then  $f(x)$  is the unique point in the intersection of  $D_x$  and  $\mathcal{A}$ . This proves the proposition. q.e.d.

We need the following lemma.

**Lemma 5.14.** *For each  $x \in \mathbb{D}$ , we may take a homeomorphism  $h_x : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  such that:*

- (i)  $h_x(x) = 0$  for all  $x \in \mathbb{D}$ .
- (ii)  $h_x = \text{id}$  on  $S^1$ .
- (iii)  $h_x$  depends continuously on  $x$ .

*Proof of Theorem 1.1.* From Lemma 5.14, for each  $x \in \tilde{\Sigma}$  we may take a homeomorphism  $h_x : D \rightarrow D$  such that:

- (i)  $h_x(\rho(f(x))) = \tilde{p}$
- (ii)  $h_x = \text{id}$  on  $\partial D$
- (iii)  $h_x$  depends continuously on  $x$ .

Then the homeomorphism  $g_x : D_x \rightarrow D_x$  defined by

$$(31) \quad \rho \circ g_x = h_x \circ \rho$$

depends continuously on  $x \in \tilde{\Sigma} \subset L_{\tilde{p}}$ . Consider the map  $g$  defined ( $g$  is not the same function that one in previous pages) as

$$\begin{aligned} g &= g_x \quad \text{on } D_x, \\ g &= \text{id} \quad \text{otherwise.} \end{aligned}$$

We have that  $g$  is univalent and preserves the leaves of  $\tilde{\mathcal{F}}_0$ . Moreover, in a small enough neighborhood of the divisor,  $g$  is continuous. Thus, if restricted to a small enough neighborhood of the divisor,  $g$  is a topological equivalence between  $\tilde{\mathcal{F}}_0$  and itself. Then, in a neighborhood of the divisor,  $g \circ h$  gives a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$ . Therefore for some neighborhoods  $U$  and  $\tilde{U}$  of  $0 \in \mathbb{C}^2$ , the map

$$\hat{h} = \pi g h \pi^{-1} : U \rightarrow \tilde{U}$$

is a topological equivalence between  $\mathcal{F}$  and  $\tilde{\mathcal{F}}$ . Let  $P = \pi(L_p)$  and  $\tilde{P} = \pi(L_{\tilde{p}})$ .

*Assertion.* *There exists a disc  $\mathcal{D}$  in  $P$  containing  $0 \in \mathbb{C}^2$ , such that  $\hat{h}(\mathcal{D})$  is contained in  $\tilde{P}$ .*

*Proof.* If  $y \in \mathcal{A}$  is close enough to  $E$ , we have that  $y \in D_x$  for some  $x \in \tilde{\Sigma}$ . Thus, there is a disc  $\Sigma_0 \subset \Sigma$  containing  $p$ , such that for all  $y$  in  $h(\Sigma_0 \setminus \{p\}) \subset \mathcal{A}$  we have  $y = f(x)$  for some  $x \in \tilde{\Sigma}$ . Then, from (31) and (i) we have that

$$\rho \circ g(y) = \rho \circ g(f(x)) = h_x \circ \rho(f(x)) = \tilde{p}.$$

Thus  $g(y) \in L_{\tilde{p}}$  for all  $y \in h(\Sigma_0 \setminus \{p\})$  and therefore

$$g \circ h(\Sigma_0 \setminus \{p\}) \subset L_{\tilde{p}}.$$

Then, if  $\mathcal{D} \subset \pi(\Sigma_0) \subset P$ , we have that  $\hat{h}(\mathcal{D}) \subset \tilde{P}$ .

Consider a neighborhood  $U' \subset U$  of  $0 \in \mathbb{C}^2$  homeomorphic to a ball and such that  $U' \cap P \subset \mathcal{D}$ . We take  $U'$  small enough such that  $\hat{h}(U') \cap \tilde{P}$  is contained in  $h(\mathcal{D})$ . Thus, making  $\tilde{U}' = \hat{h}(U')$ , it is easy to see that

$$h(U' \cap P) = \tilde{U}' \cap \tilde{P}.$$

Then,

$$\hat{h}|_{U'} : U' \rightarrow \tilde{U}'$$

is a topological equivalence between  $\mathcal{F}_0$  and  $\tilde{\mathcal{F}}_0$ , which satisfies the hypothesis of Theorem 1.2. Therefore Theorem 1.1 is proved. q.e.d.

*Proof of Lemma 5.14.* Let  $\psi : \overline{\mathbb{D}} \rightarrow [0, 1]$  be such that  $\psi = 1$  on  $\{|z| \leq 1/2\}$  and  $\psi = 0$  on  $S^1$ . Let

$$\beta_r(t) : [0, 1] \rightarrow [0, 1]$$

be a diffeomorphism with  $\beta_r(0) = 0$ ,  $\beta_r(1) = 1$ ,  $\beta_r(r) = 1/2$  and such that  $\beta_r$  depends continuously on  $r \geq 0$ . Given  $x \in \mathbb{D}$ , define the vector field

$$V_x : \overline{\mathbb{D}} \rightarrow \mathbb{C}$$

$$V_x(z) = -\psi(\beta_{|x|}(|z|))x,$$

and let  $\varphi_x$  the flow associated to  $V_x$ . Then define  $h_x : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  by  $h_x(z) = \varphi_x(1, z)$ . It is easy to see that  $h_x$  satisfy the conditions of Lemma 5.14. q.e.d.

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