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A PROOF OF THE FABER INTERSECTION NUMBER CONJECTURE

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Abstract

We prove the famous Faber intersection number conjecture and other more general results by using a recursion formula of n-point functions for intersection numbers on moduli spaces of curves. We also present some vanishing properties of Gromov-Witten invariants.

1. Introduction

Starting from the work of Mumford, one fundamental problem in algebraic geometry is the study of intersection theory on moduli spaces of stable curves. Through the work of Witten and Kontsevich we learned that the intersection theory of moduli spaces also has striking connection to string theory and two dimensional gravity. Denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g complex algebraic curves. We have the morphism that forgets the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π, and by $D_1, \ldots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf, we have the following tautological classes on moduli spaces of curves.

$$\begin{aligned} \psi_i &= c_1(\sigma_i^*(\omega_\pi)) \\ \kappa_i &= \pi_*\left(c_1\left(\omega_\pi\left(\sum D_i\right)\right)^{i+1}\right) \\ \lambda_k &= c_k(\mathbb{E}), \quad 1 \le k \le g, \end{aligned}$$

where $\mathbb{E} = \pi_*(\omega_\pi)$ is the Hodge bundle.

Intuitively, ψ_i is the first Chern class of the line bundle corresponding to the cotangent space of the universal curve at the i-th marked point and the fiber of $\mathbb{E}$ is the space of holomorphic one forms on the algebraic curve.

The classes κ_i were first introduced by Mumford [**22**] on $\overline{\mathcal{M}}_g$, their generalization to $\overline{\mathcal{M}}_{g,n}$ here is due to Arbarello-Cornalba [**1**].

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We use Witten's notation

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle$$
$$\triangleq \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

These intersection numbers are called the Hodge integrals. They are rational numbers because the moduli space of curves are orbifolds (with quotient singularities) except in genus zero. Their degrees should add up to $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

Intersection numbers of pure ψ classes $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ are often called intersection indices or descendant integrals. Faber's algorithm [**3**] reduces the calculation of general Hodge integrals to intersection indices, based on Mumford's Chern character formula [**22**]

$$\mathrm{ch}_{2g-1}(\mathbb{E}) = \frac{B_{2g}}{(2g)!} \Bigg[\kappa_{2g-1} - \sum_{i=1}^{n} \psi_i^{2g-1} + \frac{1}{2} \sum_{\xi \in \Delta} l_{\xi_*} \left(\sum_{i=0}^{2g-2} \psi_{n+1}^i (-\psi_{n+2})^{2g-2-i} \right) \Bigg],$$

where Δ enumerates all boundary divisors and l_{ξ_*} is the push-forward map under the natural inclusion.

The celebrated Witten-Kontsevich theorem [**13**, **25**] asserts that the generating function of intersection indices

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is governed by the KdV hierarchy, which provides a recursive way to compute all these intersection numbers.

The tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ is defined to be the smallest $\mathbb{Q}$-subalgebra of the Chow ring $\mathcal{A}^*(\mathcal{M}_g)$ generated by the tautological classes κ_i and λ_i. Mumford [**22**] proved that the ring $\mathcal{R}^*(\mathcal{M}_g)$ is in fact generated by the $g-2$ classes $\kappa_1, \ldots, \kappa_{g-2}$.

It is a theorem of Looijenga [**19**] that $\dim \mathcal{R}^k(\mathcal{M}_g) = 0,\ k > g-2$ and $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) \leq 1$. Later Faber proved that actually $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) = 1$.

Faber's conjecture. Around 1993, Faber [**2**] proposed three remarkable conjectures about the structure of the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ which we briefly state as follows:

i) For $0 \leq k \leq g-2$, the natural product

$$R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

is a perfect pairing.

ii) The $[g/3]$ classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the ring $\mathcal{R}^*(\mathcal{M}_g)$, with no relations in degrees $\leq [g/3]$.

iii) Let $\sum_{j=1}^n d_j = g-2$ and $d_j \geq 0$. Then

$$\pi_*(\psi_1^{d_1+1} \ldots \psi_n^{d_n+1}) = \sum_{\sigma \in S_n} \kappa_\sigma = \frac{(2g-3+n)!}{(2g-2)!! \prod_{j=1}^n (2d_j+1)!!} \kappa_{g-2}, \tag{1}$$

where κ_σ is defined as follows: write the permutation σ as a product of $\nu(\sigma)$ disjoint cycles $\sigma = \beta_1 \cdots \beta_{\nu(\sigma)}$, where we think of the symmetric group S_n as acting on the n-tuple $(d_1, \ldots, d_n)$. Denote by $|\beta|$ the sum of the elements of a cycle β. Then $\kappa_\sigma = \kappa_{|\beta_1|}\kappa_{|\beta_2|} \cdots \kappa_{|\beta_{\nu(\sigma)}|}$.

Part (i) is called Faber's perfect pairing conjecture, which is still open. Faber has verified it for $g \leq 23$.

Part (ii) has been proved independently by Morita [**21**] and Ionel [**12**] with very different methods. As pointed out by Faber [**2**], Harer's stability result implies that there is no relation in degrees $\leq [g/3]$.

Part (iii) of Faber's conjectures is the intersection number conjecture, whose importance lies in that it computes all top intersections in the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ and determines its ring structure if we assume Faber's perfect pairing conjecture. Theoretically it gives the dimension of tautological rings by computing the rank of intersection matrices which we will discuss in a subsequent work.

Faber's conjecture is a fundamental question mentioned in monographs such as [**6**, **11**] that many algebraic geometers have worked on. In this paper, we prove the Faber intersection number conjecture completely. First we recall two equivalent formulations.

The Faber intersection number conjecture is equivalent to

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)! \prod_{j=1}^n (2d_j-1)!!}, \tag{2}$$

where B_{2g} denotes the $2g$-th Bernoulli number. By Mumford's formula for the Chern character of the Hodge bundle, the above identity is equivalent to

$$\frac{(2g-3+n)!}{2^{2g-1}(2g-1)! \prod_{j=1}^n (2d_j-1)!!} = \tag{3}$$

$$\langle \tau_{2g} \prod_{j=1}^n \tau_{d_j} \rangle_g - \sum_{j=1}^n \langle \tau_{d_j+2g-1} \prod_{i \neq j} \tau_{d_i} \rangle_g + \frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j} \tau_j \prod_{i=1}^n \tau_{d_i} \rangle_{g-1}$$
$$+ \frac{1}{2} \sum_{\underline{n} = I \coprod J} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},$$

where $d_j \geq 1$, $\sum_{j=1}^{n} d_j = g+n-2$. We refer to [**2**, **17**] for discussions of the above equivalences.

The following interesting relation is observed by Faber and proved by Zagier using the Faber intersection number conjecture (see [**2**])

$$\kappa_1^{g-2} = \frac{1}{g-1} 2^{2g-5}((g-2)!)^2 \kappa_{g-2}.$$

In fact, from (1), the above relation is equivalent to a combinatorial identity

$$\sum_{k=1}^{g} \left(\frac{(-1)^k}{k!}(2g+1+k) \sum_{\substack{g=m_1+\cdots+m_k \\ m_i>0}} \binom{2g+k}{2m_1+1,\ldots,2m_k+1} \right) = (-1)^g 2^{2g}(g!)^2.$$

We learned of an elegant proof from Jian Zhou using the residue theorem.

Faber [**2**] proved identity (3) when $n=1$ using explicit formulae of up to three-point functions. The identity (2) was shown to follow from the degree 0 Virasoro conjecture for $\mathbb{P}^2$ by Getzler and Pandharipande [**8**]. In 2001 Givental [**9**] has announced a proof of Virasoro conjecture for $\mathbb{P}^n$. Y.-P. Lee and R. Pandharipande are writing a book [**16**] giving details. Recently Teleman [**23**] announced a proof of the Virasoro conjecture for manifolds with semi-simple quantum cohomology. His argument depends crucially on the Mumford conjecture about the stable rational cohomology rings of the moduli spaces proved by Madsen and Weiss [**20**].

Goulden, Jackson and Vakil [**10**] recently give an enlightening proof of identity (1) for up to three points. Their remarkable proof uses relative virtual localization and a combinatorialization of the Hodge integrals, establishing connections to double Hurwitz numbers.

Our alternative approach is quite direct, we prove identity (3) for all g and n by using a recursive formula of n-point functions. Actually, the n-point function formula has far-reaching applications. Recently Zhou [**28**] used our results on n-point functions in his computation of Hurwitz-Hodge integrals, which leads to a proof of the crepant resolution conjecture of type A surface singularities for all genera.

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2. The n-point functions

Definition 2.1. We call the following generating function

$$\begin{aligned} F(x_1,\dots,x_n) &= \sum_{g=0}^{\infty} F_g(x_1,\dots,x_n) \\ &= \sum_{g=0}^{\infty} \sum_{\sum d_j = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^{n} x_j^{d_j} \end{aligned}$$

the n-point functions.

Consider the following "normalized" n-point function

$$G(x_1,\dots,x_n) = \exp\left(\frac{-\sum_{j=1}^n x_j^3}{24} \right) F(x_1,\dots,x_n).$$

We will let $G_g(x_1,\dots,x_n)$ denote the degree $3g-3+n$ homogenous component of $G(x_1,\dots,x_n)$.

In contrast with the original n-point function, its normalization has some distinct properties (see [**18**]). For example, the coefficient of $z^k \prod_{j=1}^n x_j^{d_j}$ in $G_g(z,x_1,\dots,x_n)$ is zero whenever $k > 2g-2+n$.

It's well-known that

$$F_0(x_1,\dots,x_n) = G_0(x_1,\dots,x_n) = (x_1+\cdots+x_n)^{n-3}.$$

There are explicit formulae for one and two-point functions due to Witten [**25**] and Dijkgraaf (see [**2**]) respectively

$$G(x) = \frac{1}{x^2}, \qquad G(x,y) = \frac{1}{x+y} \sum_{k\geq 0} \frac{k!}{(2k+1)!} \left(\frac{1}{2} xy(x+y) \right)^k .$$

In an unpublished note [**27**] (kindly sent to us by Faber), Zagier obtained a marvelous formula of the three-point function (see [**18**]).

We proved in [**18**] the following recursion formula for general normalized n-point function.

Proposition 2.2. [**18**] *For $n \geq 2$,*

$$G(x_1,\dots,x_n) = \sum_{r,s\geq 0} \frac{(2r+n-3)!!}{4^s(2r+2s+n-1)!!} P_r(x_1,\dots,x_n) \Delta(x_1,\dots,x_n)^s,$$

where P_r and Δ are homogeneous symmetric polynomials defined by

$$\Delta(x_1,\dots,x_n) = \frac{\left(\sum_{j=1}^n x_j \right)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$P_r(x_1,\dots,x_n) =$$

$$\left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}\Big(\sum_{i\in I}x_i\Big)^2\Big(\sum_{i\in J}x_i\Big)^2 G(x_I)G(x_J)\right)_{3r+n-3}$$
$$=\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}\Big(\sum_{i\in I}x_i\Big)^2\Big(\sum_{i\in J}x_i\Big)^2\sum_{r'=0}^{r}G_{r'}(x_I)G_{r-r'}(x_J),$$

where $I, J\neq\emptyset$, $\underline{n}=\{1,2,\ldots,n\}$ and $G_g(x_I)$ denotes the degree $3g+|I|-3$ homogeneous component of the normalized $|I|$-point function $G(x_{k_1},\ldots,x_{k_{|I|}})$, where $k_j\in I$.

The proof amounts to check that $G(x_1,\ldots,x_n)$, as recursively defined in Proposition 2.2, satisfies the following Witten-Kontsevich differential equation (see [**18**]),

$$y\frac{\partial}{\partial y}\left(\Big(y+\sum_{j=1}^n x_j\Big)^2 G_g(y,x_1,\ldots,x_n)\right)=$$
$$\frac{y}{8}\Big(y+\sum_{j=1}^n x_j\Big)^4 G_{g-1}(y,x_1,\ldots,x_n)-\frac{y^3}{8}\Big(y+\sum_{j=1}^n x_j\Big)^2 G_{g-1}(y,x_1,\ldots,x_n)$$
$$+\frac{y}{2}\sum_{\underline{n}=I\coprod J}\left(\left(y+\sum_{i\in I}x_i\right)\left(\sum_{i\in J}x_i\right)^3+2\left(y+\sum_{i\in I}x_i\right)^2\left(\sum_{i\in J}x_i\right)^2\right)$$
$$\times G_{g'}(y,x_I)G_{g-g'}(x_J)$$
$$-\frac{1}{2}\left(y+\sum_{j=1}^n x_j\right)\left(\sum_{j=1}^n x_j\right)G_g(y,x_1,\ldots,x_n).$$

The verification is tedious but straightforward. It will be included in a updated version of the paper [**18**].

Recall the well-known string equation

$$\langle\tau_0\prod_{i=1}^n\tau_{k_i}\rangle_g=\sum_{j=1}^n\langle\tau_{k_j-1}\prod_{i\neq j}\tau_{k_i}\rangle_g$$

and the dilaton equation

$$\langle\tau_1\prod_{i=1}^n\tau_{k_i}\rangle_g=(2g-2+n)\langle\prod_{i=1}^n\tau_{k_i}\rangle_g.$$

Note that the string equation can be equivalently written as

$$F(x_1,\ldots,x_n,0)=\Big(\sum_{j=1}^n x_j\Big)F(x_1,\ldots,x_n).$$

Proposition 2.3. *Let $n \geq 2$. We have the following recursive formula of n-point functions.*

$$(2g+n-1)F_g(x_1,\dots,x_n) = \frac{\left(\sum_{j=1}^n x_j\right)^3}{12} F_{g-1}(x_1,\dots,x_n)$$
$$+ \frac{1}{2\left(\sum_{j=1}^n x_j\right)} \sum_{g'=0}^{g} \sum_{\underline{n}=I\coprod J} \left(\sum_{i\in I} x_i\right)^2 \left(\sum_{i\in J} x_i\right)^2 F_{g'}(x_I)F_{g-g'}(x_J).$$

Proof. From Proposition 2.2, we have

$$\begin{aligned}
&G_g(x_1,\dots,x_n)\\
=&\sum_{r+s=g} \frac{(2r+n-3)!!}{4^s(2g+n-1)!!} P_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^s\\
=&\frac{1}{2g+n-1}P_g(x_1,\dots,x_n)\\
&+\sum_{r+s=g-1} \frac{(2r+n-3)!!}{4^{s+1}(2g+n-1)!!} P_r(x_1,\dots,x_n)\Delta(x_1,\dots,x_n)^{s+1}\\
=&\frac{1}{(2g+n-1)}P_g(x_1,\dots,x_n) + \frac{\Delta(x_1,\dots,x_n)}{4(2g+n-1)} G_{g-1}(x_1,\dots,x_n).
\end{aligned}$$

We define

$$H = \exp\left(\frac{\sum_{i=1}^n x_i^3}{24}\right), \qquad H^{-1} = \exp\left(\frac{-\sum_{i=1}^n x_i^3}{24}\right),$$

$$H_d = \frac{1}{d!}\left(\frac{\sum_{i=1}^n x_i^3}{24}\right)^d, \qquad H_d^{-1} = \frac{1}{d!}\left(\frac{-\sum_{i=1}^n x_i^3}{24}\right)^d.$$

Note that $\sum_{i=0}^d H_i H_{d-i}^{-1} = 0$ if $d>0$.

Let LHS and RHS denote the left and right hand side of the recursion in the lemma. We have

$$H^{-1}\cdot RHS = \sum_{g=0}^{\infty}\left(\frac{1}{12}\left(\sum_{i=1}^n x_i\right)^3 G_{g-1}(x_1,\dots,x_n) + P_g(x_1,\dots,x_n)\right)$$
$$= \sum_{g=0}^{\infty}\left((2g+n-1)G_g(x_1,\dots,x_n) + \frac{1}{12}\left(\sum_{i=1}^n x_i^3\right) G_{g-1}(x_1,\dots,x_n)\right)$$

$$
\begin{aligned}
H^{-1}\cdot LHS &= \sum_{g=0}^{\infty}\sum_{a+b+c=g}(2a+2b+n-1)G_a(x_1,\ldots,x_n)H_bH_c^{-1}\\
&= \sum_{g=0}^{\infty}\sum_{a=0}^{g}(2a+n-1)G_a(x_1,\ldots,x_n)\sum_{b+c=g-a}H_bH_c^{-1}\\
&\quad+\sum_{g=0}^{\infty}\sum_{a+b+c=g}G_a(x_1,\ldots,x_n)2bH_bH_c^{-1}\\
&= \sum_{g=0}^{\infty}(2g+n-1)G_g(x_1,\ldots,x_n)+\sum_{g=0}^{\infty}\frac{1}{12}\left(\sum_{i=1}^{n}x_i^3\right)G_{g-1}(x_1,\ldots,x_n).
\end{aligned}
$$

q.e.d.

Only very recently, we realize that Proposition 2.3 has already been embodied in the first KdV equation of the Witten-Kontsevich theorem.

The KdV hierarchy is the following hierarchy of differential equations for $n \geq 1$,

$$\frac{\partial U}{\partial t_n}=\frac{\partial}{\partial t_0}R_{n+1},$$

where R_n are Gelfand-Dikii differential polynomials in $U, \partial U/\partial t_0$, $\partial^2 U/\partial t_0^2, \ldots$, defined recursively by

$$R_1=U,\qquad \frac{\partial R_{n+1}}{\partial t_0}=\frac{1}{2n+1}\left(\frac{\partial U}{\partial t_0}R_n+2U\frac{\partial R_n}{\partial t_0}+\frac{1}{4}\frac{\partial^3}{\partial t_0^3}R_n\right).$$

It is easy to see that

$$
\begin{aligned}
R_2 &= \frac{1}{2}U^2+\frac{1}{12}\frac{\partial^2 U}{\partial t_0^2},\\
R_3 &= \frac{1}{6}U^3+\frac{U}{12}\frac{\partial^3 U}{\partial t_0^3}+\frac{1}{24}\Big(\frac{\partial U}{\partial t_0}\Big)^2+\frac{1}{240}\frac{\partial^4 U}{\partial t_0^4},\\
&\ \ \vdots
\end{aligned}
$$

The Witten-Kontsevich theorem states that the generating function

$$F(t_0,t_1,\ldots)=\sum_g\sum_{\mathbf{n}}\langle\prod_{i=0}^{\infty}\tau_i^{n_i}\rangle_g\prod_{i=0}^{\infty}\frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy, i.e. $\partial^2 F/\partial t_0^2$ obeys all equations in the KdV hierarchy. The first equation in the KdV hierarchy is the classical KdV equation

$$\frac{\partial U}{\partial t_1}=U\frac{\partial U}{\partial t_0}+\frac{1}{12}\frac{\partial^3 U}{\partial t_0^3}.$$

By the Witten-Kontsevich theorem, we have

$$\frac{\partial^3 F}{\partial t_1 \partial t_0^2} = \frac{\partial F}{\partial t_0^2}\frac{\partial F}{\partial t_0^3} + \frac{1}{12}\frac{\partial^5 F}{\partial t_0^5}.$$

Integrating each side with respect to t_0 and putting $\langle\langle \tau_{k_1} \cdots \tau_{k_n} \rangle\rangle := \partial^n F / \partial t_{k_1} \cdots \partial t_{k_n}$, we get

$$\langle\langle \tau_0 \tau_1 \rangle\rangle = \frac{1}{12}\langle\langle \tau_0^4 \rangle\rangle + \frac{1}{2}\langle\langle \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle.$$

Then Proposition 2.3 follows by applying the dilaton equation.

3. The Faber intersection number conjecture

Now we explain our approach to prove identity (3), hence the Faber intersection number conjecture. We establish its relationship with n-point functions.

For the sake of brevity, we introduce the following notations

$$\begin{aligned} &L_g^{a,b}(y, x_1 \ldots, x_n) \\ &= \sum_{g'=0}^{g} \sum_{\underline{n}=I \coprod J} \Big(y + \sum_{i \in I} x_i\Big)^a \Big(-y + \sum_{i \in J} x_i\Big)^b F_{g'}(y, x_I) F_{g-g'}(-y, x_J), \end{aligned}$$

where $a, b \in \mathbb{Z}$. We regard $L_g^{a,b}(y, x_1 \ldots, x_n)$ as a formal series in $\mathbb{Q}[x_1, \ldots, x_n][[y, y^{-1}]]$ with $\deg y < \infty$.

We now prove that the Faber intersection number conjecture can be reduced to three statements about the coefficients of the above functions.

Proposition 3.1. *We have*

i)
$$\big[L_g^{0,0}(y, x_1 \ldots, x_n)\big]_{y^{2g-2}} = 0;$$

ii) *For* $k > 2g$,
$$\big[L_g^{2,2}(y, x_1 \ldots, x_n)\big]_{y^k} = 0;$$

iii) *For* $d_j \geq 1$ *and* $\sum_{j=1}^n d_j = g + n$,
$$\big[L_g^{2,2}(y, x_1 \ldots, x_n)\big]_{y^{2g} \prod_{j=1}^n x_j^{d_j}} = \frac{(2g+n+1)!}{4^g (2g+1)! \prod_{j=1}^n (2d_j - 1)!!}.$$

In fact, Proposition 3.1 is a special case of more general results proved in the next section. Clearly identities (i) and (ii) of the following corollary add up to the desired identity (3).

Corollary 3.2. *We have*

i) *Let* $d_j \geq 0$ *and* $\sum_{j=1}^n d_j = g+n-2$. *Then*

$$\begin{aligned}\langle\prod_{j=1}^n \tau_{d_j}\tau_{2g}\rangle_g &= \sum_{j=1}^n \langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g \\ &\quad -\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'};\end{aligned}$$

ii) *Let* $d_j \geq 1$ *and* $\sum_{j=1}^n (d_j-1) = g-1$. *Then*

$$\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_g = \frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^n(2d_j-1)!!};$$

iii) *Let* $k>g$, $d_j\geq 0$ *and* $\sum_{j=1}^n d_j = 3g+n-2k-2$. *Then*

$$\sum_{j=0}^{2k}(-1)^j\langle\tau_{2k-j}\tau_j\prod_{i=1}^n\tau_{d_i}\rangle_g = 0.$$

Proof. Since one and two-point functions in genus 0 are

$$F_0(x)=\frac{1}{x^2},\qquad F_0(x,y)=\frac{1}{x+y}=\sum_{k=0}^\infty(-1)^k\frac{x^k}{y^{k+1}},$$

it is consistent to define

$$\langle\tau_{-2}\rangle_0 = 1,\qquad \langle\tau_k\tau_{-1-k}\rangle_0=(-1)^k,\ k\geq 0.$$

By allowing the index to run over all integers, we have

$$\begin{aligned}&\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\ &\qquad\qquad + \langle\prod_{j=1}^n\tau_{d_j}\tau_{2g}\rangle_g - \sum_{j=1}^n\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g \\ &\quad = \frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j\in\mathbb{Z}}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\ &\quad = \left[\sum_{g'=0}^g\sum_{\underline{n}=I\coprod J}F_{g'}(y,x_I)F_{g-g'}(-y,x_J)\right]_{y^{2g-2}\prod_{i=1}^n x_i^{d_i}} \\ &\quad = \left[L_g^{0,0}(y,x_1,\ldots,x_n)\right]_{y^{2g-2}\prod_{i=1}^n x_i^{d_i}} = 0.\end{aligned}$$

From Proposition 2.3, we have

$$\frac{1}{2}\left(\sum_{j=1}^{n} x_j\right) F_g(x_1,\dots,x_n) = \frac{\left(\sum_{j=1}^{n} x_j\right)^4}{24(2g+n-1)} F_{g-1}(x_1,\dots,x_n)$$
$$+ \frac{1}{2(2g+n-1)}\left(L_g^{2,2}(y,x_{\underline{n}}) + \sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I} x_i\right)^2\left(\sum_{i\in J} x_i\right)^2 \right.$$
$$\left. \times F_{g'}(y,-y,x_I)F_{g-g'}(x_J)\right).$$

By Proposition 3.1(ii)-(iii), we can use Proposition 2.3 to inductively prove

$$\sum_{j=0}^{2k}(-1)^j\langle\tau_{2k-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_g = [F_g(y,-y,x_1,\dots,x_n)]_{y^{2k}} = 0, \quad \text{for } k>g$$

and we have

$$\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\tau_0\prod_{i=1}^{n}\tau_{d_i}\rangle_g = \frac{(2g+n)!}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!},$$

which, from the string equation and induction on the maximum index (say d_1) among $\{d_i\}$, implies (by the dilaton equation, we may assume $d_i \geq 2$)

$$\begin{aligned}
&\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_g \\
= \; &\sum_{j=0}^{2g}(-1)^j\langle\tau_0\tau_{2g-j}\tau_j\tau_{d_1+1}\prod_{i=2}^{n}\tau_{d_i}\rangle_g \\
&-\sum_{k=2}^{n}\sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\tau_{d_1+1}\tau_{d_k-1}\prod_{i\neq 1,k}\tau_{d_i}\rangle_g \\
= \; &\frac{(2g+n)!}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!(2d_1+1)} \\
&-\sum_{k=2}^{n}\frac{(2g+n-1)!(2d_k-1)}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!(2d_1+1)} \\
= \; &\frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!}.
\end{aligned}$$

q.e.d.

So in order to prove the Faber intersection number conjecture, we only need to prove the three statements (i)-(iii) in Proposition 3.1 about n-point functions. Actually we will prove more general results which are stated as main theorems, Theorems 4.4 and 4.5 in the next section. Proposition 3.1, therefore the Faber intersection number conjecture, is a special case of these theorems.

4. Proof of main theorems

The binomial coefficients $\binom{p}{k}$, for $k \geq 0, p \in \mathbb{Z}$ are given by

$$\binom{p}{k} = \begin{cases} 0, & k < 0, \\ 1, & k = 0, \\ \frac{p(p-1)\cdots(p-k+1)}{k!}, & k \geq 1. \end{cases}$$

Lemma 4.1. *Let $a, b \in \mathbb{Z}$ and $n \geq 0$. Then*

$$\sum_{i=0}^{n} \binom{i+a}{i} \binom{n-i+b}{n-i} = \binom{n+a+b+1}{n}.$$

Proof. Note that

$$\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}.$$

By denoting the left-hand side of the above equation by $A_n(a, b)$, we have

$$A_n(a, b) = A_n(a-1, b) + A_{n-1}(a, b).$$

First we argue by induction on n and $|b|$ to prove

$$A_n(0, b) = \binom{n+b+1}{n}.$$

Then we argue by induction on n and $|a|$ to prove

$$A_n(a, b) = \binom{n+a+b+1}{n}.$$

q.e.d.

We now prove two lemmas that will serve as base cases for our inductive arguments.

Lemma 4.2. *Let $a, b \in \mathbb{Z}$ and $k \geq 2g - 3 + a + b$. Then*

i)

$$\left[L_g^{a,b}(y, x)\right]_{y^k} = 0,$$

ii)

$$\Big[L_g^{a,b}(y,x)\Big]_{y^{2g-4+a+b}x^{g+1}} = \frac{(-1)^b(2g-2+a+b)}{4^g(2g+1)!!}.$$

Proof. Here we recall the definition of normalized n-point functions

$$G(x_1,\dots,x_n) = \exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\dots,x_n).$$

In particular, we have

$$G(x) = \frac{1}{x^2}, \qquad G(x,y) = \frac{1}{x+y}\sum_{k\ge 0}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k.$$

By definition

$$\sum_{g\ge 0} L_g^{a,b}(y,x_1\dots,x_n) =$$

$$\exp\left(\frac{\sum_{j=1}^n x_j^3}{24}\right)\sum_{\underline{n}=I\coprod J}\Big(y+\sum_{i\in I}x_i\Big)^a\Big(-y+\sum_{i\in J}x_i\Big)^b G(y,x_I)G(-y,x_J),$$

So for statements (i) and (ii), it is not difficult to see that we only need to prove that for $k\ge 2g-3+a+b$,

$$\Big[y^{a-2}(-y+x)^bG_g(-y,x)+(-y)^{b-2}(y+x)^aG_g(y,x)\Big]_{y^k}=0$$

and

$$\Big[y^{a-2}(-y+x)^bG_g(-y,x)+(-y)^{b-2}(y+x)^aG_g(y,x)\Big]_{y^{2g-4+a+b}x^{g+1}}$$
$$= \frac{(-1)^b(2g-2+a+b)}{4^g(2g+1)!!}.$$

Both follow easily from the explicit formula of $G(y,x)$. q.e.d.

Lemma 4.3. *Let $a,b\in\mathbb{Z}$ and $k\ge a+b-3$. Then*

i)

$$\Big[L_0^{a,b}(y,x_1,\dots,x_n)\Big]_{y^k}=0,$$

ii)

$$\Big[L_0^{a,b}(y,x_1,\dots,x_n)\Big]_{y^{a+b-4}\prod_{j=1}^n x_j} = \frac{(-1)^b(a+b+n-3)!}{(a+b-3)!}.$$

Proof. Since

$$F_0(x_1,\dots,x_n) = (x_1+\cdots+x_n)^{n-3},$$

we have by definition

$$L_0^{a,b}(y,x_1\dots,x_n) = \sum_{\underline{n}=I\coprod J}\Big(y+\sum_{i\in I}x_i\Big)^{|I|-2+a}\Big(-y+\sum_{i\in J}x_i\Big)^{|J|-2+b}.$$

For any monomial $y^k \prod_{j=1}^n x_j^{d_j}$ in $L_0^{a,b}(y, x_1 \ldots, x_n)$, if $k \geq a+b-3$, then there must be some $d_j = 0$. We may assume $d_n = 0$, then

$$\begin{aligned}
& L_0^{a,b}(y, x_1 \ldots, x_{n-1}, 0) \\
= & \sum_{\{1,\ldots,n-1\}=I \coprod J} \Bigg(\Big(y + \sum_{i\in I} x_i\Big)^{|I|-1+a} \Big(-y + \sum_{i\in J} x_i\Big)^{|J|-2+b} \\
& \qquad + \Big(y + \sum_{i\in I} x_i\Big)^{|I|-2+a} \Big(-y + \sum_{i\in J} x_i\Big)^{|J|-1+b} \Bigg) \\
= & \left(\sum_{j=1}^{n-1} x_j\right) \sum_{\{1,\ldots,n-1\}=I \coprod J} \Big(x_1 + \sum_{i\in I} x_i\Big)^{|I|-2+a} \Big(-x_1 + \sum_{i\in J} x_i\Big)^{|J|-2+b} \\
= & \left(\sum_{j=1}^{n-1} x_j\right) L_0^{a,b}(y, x_1 \ldots, x_{n-1}).
\end{aligned}$$

So (i) follows by induction on n. By applying Lemma 4.1 we have

$$\begin{aligned}
& \Big[L_0^{a,b}(y, x_1, \ldots, x_n)\Big]_{y^{a+b-4}\prod_{j=1}^n x_j} \\
& = (-1)^b \sum_{|I|=0}^{n} \binom{|I|-2+a}{|I|} |I|! \binom{|j|-2+b}{|J|} |J|! \binom{n}{|I|} \\
& = (-1)^b n! \sum_{i=0}^{n} \binom{i-2+a}{i} \binom{n-i-2+b}{n-i} \\
& = (-1)^b n! \binom{a+b+n-3}{n} \\
& = \frac{(-1)^b (a+b+n-3)!}{(a+b-3)!}.
\end{aligned}$$

So we have proved (ii). q.e.d.

Theorem 4.4. *Let $a, b \in \mathbb{Z}$ and $k \geq 2g-3+a+b$. Then*

$$\Big[L_g^{a,b}(y, x_1 \ldots, x_n)\Big]_{y^k} = 0.$$

Proof. We will argue by induction on g and n, since the theorem holds for $g=0$ or $n=1$ as proved in the above lemmas. We have

$$\begin{aligned}(2g+n)&L_g^{a,b}(y,x_1\dots,x_n)\\ &=\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}\Big(y+\sum_{i\in I}x_i\Big)^a\Big(-y+\sum_{i\in J}x_i\Big)^b\\ &\qquad\qquad (2g'+|I|)F_{g'}(y,x_I)F_{g-g'}(-y,x_J)\\ &+\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}\Big(y+\sum_{i\in I}x_i\Big)^a\Big(-y+\sum_{i\in J}x_i\Big)^b\\ &\qquad\qquad F_{g'}(y,x_I)(2g-2g'+|J|)F_{g-g'}(-y,x_J).\end{aligned}$$

Substituting $F_{g'}(y,x_I)$ by Propostion 2.3,

$$\begin{aligned}&\Bigg[\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}\Big(y+\sum_{i\in I}x_i\Big)^a\Big(-y+\sum_{i\in J}x_i\Big)^b(2g'+|I|)\\ &\qquad\qquad\times F_{g'}(y,x_I)F_{g-g'}(-y,x_J)\Bigg]_{y^k}\\ &\qquad\qquad=\frac{1}{12}\Big[L_{g-1}^{a+3,b}(y,x_1,\dots,x_n)\Big]_{y^k}\\ &+\Bigg[\sum_{g'=0}^{g}\sum_{s\geq 0}\binom{a-1}{s}\sum_{\underline{n}=I\coprod J}F_{g'}(x_I)\Big(\sum_{i\in I}x_i\Big)^{s+2}L_{g-g'}^{a+1-s,b}(y,x_J)\Bigg]_{y^k}.\end{aligned}$$

Note that in the last term of the above equation, $|J|<n$. So by induction, for $k\geq 2g-3+a+b$, the sums vanish except for $g'=0$ and $s=0$, namely the term

$$\Bigg[\sum_{\underline{n}=I\coprod J}\Big(\sum_{i\in I}x_i\Big)^{|I|-1}L_g^{a+1,b}(y,x_J)\Bigg]_{y^k}.$$

Let $d_j\geq 1$ for $1\leq j\leq n$. By induction, it is not difficult to see from the above that

$$\begin{aligned}(2g+n)&\Big[L_g^{a,b}(y,x_1\dots,x_n)\Big]_{y^k\prod_{j=1}^n x_j^{d_j}}\\ &=\frac{1}{12}\Big[L_{g-1}^{a+3,b}(y,x_1,\dots,x_n)+L_{g-1}^{a,b+3}(y,x_1,\dots,x_n)\Big]_{y^k\prod_{j=1}^n x_j^{d_j}}.\end{aligned}$$

By induction, we have

$$
\begin{aligned}
0 &= \left(\sum_{j=1}^{n} x_j\right) \left[L_{g-1}^{a+1,b+1}(y, x_1, \ldots, x_n)\right]_{y^k} \quad \text{for } k \geq 2g-3+a+b \\
&= \left[L_{g-1}^{a+2,b+1}(y, x_1, \ldots, x_n) + L_{g-1}^{a+1,b+2}(y, x_1, \ldots, x_n)\right]_{y^k}
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \left(\sum_{j=1}^{n} x_j\right)^3 \left[L_{g-1}^{a,b}(y, x_1, \ldots, x_n)\right]_{y^k} \quad \text{for } k \geq 2g-5+a+b \\
&= \left[L_{g-1}^{a+3,b}(y, x_1, \ldots, x_n) + L_{g-1}^{a,b+3}(y, x_1, \ldots, x_n)\right]_{y^k} \\
&\quad + 3\left[L_{g-1}^{a+2,b+1}(y, x_1, \ldots, x_n) + L_{g-1}^{a+1,b+2}(y, x_1, \ldots, x_n)\right]_{y^k} \\
&= \left[L_{g-1}^{a+3,b}(y, x_1, \ldots, x_n) + L_{g-1}^{a,b+3}(y, x_1, \ldots, x_n)\right]_{y^k}.
\end{aligned}
$$

So we have proved that

$$
\left[L_g^{a,b}(y, x_1 \ldots, x_n)\right]_{y^k \prod_{j=1}^n x_j^{d_j}} = 0, \quad \text{for } d_j \geq 1.
$$

If some d_j is zero, the above identity still holds by applying the string equation

$$
L_g^{a,b}(y, x_1 \ldots, x_n, 0) = \left(\sum_{j=1}^{n} x_j\right) L_g^{a,b}(y, x_1 \ldots, x_n).
$$

So we proved the theorem. q.e.d.

Theorem 4.5. *Let $a, b \in \mathbb{Z}$, $d_j \geq 1$ and $\sum_j d_j = g + n$. Then*

$$
\left[L_g^{a,b}(y, x_1 \ldots, x_n)\right]_{y^{2g-4+a+b} \prod_{j=1}^n x_j^{d_j}}
$$

$$
= \frac{(-1)^b (2g-3+n+a+b)!}{4^g (2g-3+a+b)! \prod_{j=1}^n (2d_j - 1)!!}.
$$

Proof. By the dilaton equation, we may assume $d_j \geq 2$. As in the proof of the above theorem, we have

$$
\begin{aligned}
&(2g+n)\left[L_g^{a,b}(y,x_1\dots,x_n)\right]_{y^{2g-4+a+b}\prod_{j=1}^n x_j^{d_j}}\\
&=\frac{1}{12}\left[L_{g-1}^{a+3,b}(y,x_{\underline{n}})+L_{g-1}^{a,b+3}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^n x_j^{d_j}}\\
&=-\frac{1}{4}\left[L_{g-1}^{a+2,b+1}(y,x_{\underline{n}})+L_{g-1}^{a+1,b+2}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^n x_j^{d_j}}\\
&=-\frac{1}{4}\left[\left(\sum_{i=1}^n x_i\right)L_{g-1}^{a+1,b+1}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^n x_j^{d_j}}\\
&=-\frac{1}{4}\sum_{j=1}^n\left[L_{g-1}^{a+1,b+1}(y,x_{\underline{n}})\right]_{y^{2g-4+a+b}x_j^{d_j-1}\prod_{i\neq j}x_i^{d_i}}\\
&=\frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^n(2d_j-1)!!}\sum_{j=1}^n(2d_j-1)\\
&=(2g+n)\frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^n(2d_j-1)!!}.
\end{aligned}
$$

So we have proved the theorem. q.e.d.

All the three statements in Proposition 3.1 are particular cases of Theorems 2.4 and 2.5. We thus conclude the proof of the Faber intersection number conjecture.

The following corollaries ware stated as conjectures in our previous paper [**17**].

Corollary 4.6. *Let $d_j \geq 1$ and $\sum_{j=1}^n (d_j-1)=g$. Then*

$$
\begin{aligned}
&\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!\prod_{j=1}^n(2d_j-1)!!}\\
&\qquad=\langle\tau_{2g-2}\prod_{j=1}^n\tau_{d_j}\rangle_g-\sum_{j=1}^n\langle\tau_{d_j+2g-3}\prod_{i\neq j}\tau_{d_i}\rangle_g\\
&\qquad+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

Proof. Since the right hand side is just

$$\frac{1}{2}\left[L_g^{0,0}(y,x_1\dots,x_n)\right]_{y^{2g-4}\prod_{j=1}^n x_j^{d_j}},$$

the result follows from Theorem 4.5. q.e.d.

Corollary 4.7. *Let $g \geq 2$, $d_j \geq 1$ and $\sum_{j=1}^{n}(d_j - 1) = g$. Then*

$$
\begin{aligned}
&-\frac{(2g-2)!}{|B_{2g-2}|}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\mathrm{ch}_{2g-3}(\mathbb{E})\\
&= \frac{2g-2}{|B_{2g-2}|}\left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-1}\lambda_{g-2} - 3\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-3}\lambda_g\right)\\
&= \frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_{2g-4-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}\\
&\qquad\qquad + \frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\cdot\frac{1}{\prod_{j=1}^{n}(2d_j-1)!!}.
\end{aligned}
$$

Proof. We apply Mumford's formulae [**22**]

$$(2g-3)!\cdot\mathrm{ch}_{2g-3}(\mathbb{E}) = (-1)^{g-1}(3\lambda_{g-3}\lambda_g - \lambda_{g-1}\lambda_{g-2}),$$

$$
\begin{aligned}
\mathrm{ch}_{2g-3}(\mathbb{E}) = \frac{B_{2g-2}}{(2g-2)!}\Bigg[&\kappa_{2g-3} - \sum_{i=1}^{n}\psi_i^{2g-3}\\
&+ \frac{1}{2}\sum_{\xi\in\Delta} l_{\xi *}\left(\sum_{i=0}^{2g-4}\psi_{n+1}^{i}(-\psi_{n+2})^{2g-4-i}\right)\Bigg].
\end{aligned}
$$

So the identity follows from Corollary 4.6. q.e.d.

Both Theorems 4.4 and 4.5 can be extended without difficulty. Let us use the notation

$$
\begin{aligned}
&L_g(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}})\\
&= \sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J} F_{g'}(y, z_1, \ldots, z_a, x_I)F_{g-g'}(-y, w_1, \ldots, w_b, x_J).
\end{aligned}
$$

Theorem 4.8. *Let $a \geq 0$, $b \geq 0$, $n \geq 1$. We have*

i) *For $k \geq 2g-3+a+b$,*

$$\left[L_g(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}})\right]_{y^k} = 0.$$

ii) *For $r_j \geq 0$, $s_j \geq 0$, $d_j \geq 1$ and $\sum r_j + \sum s_j + \sum d_j = g+n$,*

$$
\begin{aligned}
&\left[L_g(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}})\right]_{y^{2g-4+a+b}\prod_{j=1}^{a} z_j^{r_j}\prod_{j=1}^{b} w_j^{s_j}\prod_{j=1}^{n} x_j^{d_j}}\\
&\qquad = \frac{1}{\prod_{j=1}^{a}(2r_j+1)!!\prod_{j=1}^{b}(2s_j+1)!!}\\
&\qquad\qquad\cdot\frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-3+a+b)!\prod_{j=1}^{n}(2d_j-1)!!}.
\end{aligned}
$$

iii) *For* $r_j \geq 0$, $s_j \geq 0$, $d_j \geq 1$, $\sum r_j + \sum s_j + \sum d_j = g + n + 1$ *and* $u \triangleq \#\{r_j = 0\}$, $v \triangleq \#\{s_j = 0\}$, $w \triangleq \#\{d_j = 1\}$,

$$\begin{aligned}&\left[L_g(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}})\right]_{y^{2g-5+a+b}\prod_{j=1}^{a} z_j^{r_j} \prod_{j=1}^{b} w_j^{s_j} \prod_{j=1}^{n} x_j^{d_j}} = \\ &\frac{C}{\prod_{j=1}^{a}(2r_j+1)!!\prod_{j=1}^{b}(2s_j+1)!!} \cdot \frac{(-1)^b(2g-3+n+a+b)!}{4^g(2g-4+a+b)!\prod_{j=1}^{n}(2d_j-1)!!},\end{aligned}$$

where the constant C *is given by*

$$C \triangleq \sum_{j=1}^{a} r_j - \sum_{j=1}^{b} s_j + \frac{a-b}{2} + \frac{(5-u)u-(5-v)v}{2(2g+n+a+b-3-w)}.$$

Proof. When $g = 0$, the proof is an easy verification. Let $p, q \in \mathbb{Z}$.

$$\begin{aligned}&L_g^{p,q}(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}}) \\ &\quad = \sum_{g'=0}^{g} \sum_{\underline{n}=I\coprod J} (y + \sum_{i=1}^{a} z_i + \sum_{i\in I} x_i)^p (-y + \sum_{i=1}^{b} w_i + \sum_{i\in J} x_i)^q \\ &\qquad\qquad \times F_{g'}(y, z_1, \ldots, z_a, x_I) F_{g-g'}(-y, w_1, \ldots, w_b, x_J).\end{aligned}$$

Exactly the same argument of Theorem 4.4 will prove that for $k \geq 2g - 3 + p + q + a + b$,

$$[L_g^{p,q}(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}})]_{y^k} = 0.$$

Statements (ii) and (iii) can also be proved similarly as Theorem 4.5. q.e.d.

Theorem 4.8 proves all conjectures in Section 3 of [**17**]. We may write down the coefficients of $L_g(y, z_{\underline{a}}, w_{\underline{b}}, x_{\underline{n}})$ explicitly to get a lot of interesting identities of intersection numbers. For example, when $a = 1, b = 0$,

$$\begin{aligned}&\left[L_g(y, z, x_{\underline{n}})\right]_{y^k z^r \prod_{j=1}^{n} x_j^{d_j}} \\ &\quad = \sum_{\underline{n}=I\coprod J} \sum_{j=0}^{k} (-1)^j \langle \tau_j \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_{k-j} \tau_r \prod_{i\in J} \tau_{d_i} \rangle_{g-g'} \\ &+ \langle \tau_{k+2}\tau_r \prod_{j=1}^{n} \tau_{d_j} \rangle_g - (-1)^k \langle \tau_{k+r+1} \prod_{j=1}^{n} \tau_{d_j} \rangle_g - \sum_{j=1}^{n} \langle \tau_r \tau_{d_j+k+1} \prod_{i\neq j} \tau_{d_i} \rangle_g.\end{aligned}$$

When $a=b=1$,

$$
\begin{aligned}
&\left[L_g(y,z,w,x_{\underline{n}})\right]_{y^k z^r w^s \prod_{j=1}^n x_j^{d_j}}\\
&\qquad = \sum_{\underline{n}=I\coprod J}\sum_{j=0}^{k}(-1)^j\langle\tau_j\tau_s\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{k-j}\tau_r\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\\
&\qquad\qquad - \langle\tau_{k+s+1}\tau_r\prod_{j=1}^{n}\tau_{d_j}\rangle_g-(-1)^k\langle\tau_{k+r+1}\tau_s\prod_{j=1}^{n}\tau_{d_j}\rangle_g.
\end{aligned}
$$

5. Gromov-Witten invariants

We will generalize vanishing identities in previous sections to Gromov-Witten invariants.

Let X be a smooth projective variety and $\overline{\mathcal{M}}_{g,n}(X,\beta)$ denote the moduli stack of stable maps of genus g and degree $\beta\in H_2(X,\mathbb{Z})$ with n marked points. There are several canonical morphisms:

i) Let $\mathrm{ev}:\overline{\mathcal{M}}_{g,n}(X,\beta)\to X^n$ be the evaluation maps at the marked points:
$$\mathrm{ev}:(f:C\to X,x_1,\dots,x_n)\mapsto \big(f(x_1),\dots,f(x_n)\big)\in X^n.$$

ii) Let $\pi:\overline{\mathcal{M}}_{g,n+1}(X,\beta)\to\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the map of forgetting the last marked point x_{n+1} and stabilizing the resulting curve.

The forgetful morphism π has n canonical sections
$$\sigma_i:\overline{\mathcal{M}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n+1}(X,\beta),$$
corresponding to the n marked points. Let
$$\omega=\omega_{\overline{\mathcal{M}}_{g,n+1}(V,\beta)/\overline{\mathcal{M}}_{g,n}(X,\beta)}$$
be the relative dualizing sheaf and Ψ_i the cohomology class $c_1(\sigma_i^*\omega)$.

If $\gamma_1,\dots,\gamma_n\in H^*(X,\mathbb{Q})$, the Gromov-Witten invariants are defined by
$$\langle\tau_{d_1}(\gamma_1)\dots\tau_{d_n}(\gamma_n)\rangle_{g,\beta}^V=\int_{[\overline{\mathcal{M}}_{g,n}(V,\beta)]^{\mathrm{virt}}}\Psi_1^{d_1}\cdots\Psi_n^{d_n}\cup\mathrm{ev}^*(\gamma_1\boxtimes\cdots\boxtimes\gamma_n).$$

Given a basis $\{T_a\}$ for $H^*(X,\mathbb{Q})$, we may use $g_{ab}=\int_X T_a\cup T_b$ and its inverse g^{ab} to lower and raise indices. We denote by $T^a=g^{ab}T_b$ and apply the Einstein summation convention.

The genus g Gromov-Witten potential of X is defined by

$$
\begin{aligned}
&\langle\langle\tau_{d_1}(\gamma_1)\cdots\tau_{d_n}(\gamma_n)\tau\rangle\rangle_g\\
&\qquad=\sum_\beta\left\langle\tau_{d_1}(\gamma_1)\cdots\tau_{d_n}(\gamma_n)\exp\left(\sum_{m,a}t_m^a\tau_m(T_a)\right)\right\rangle_{g,\beta}^X q^\beta.
\end{aligned}
$$

Very readable expositions of Gromov-Witten invariants can be found in [**7**, **24**].

We adopt Gathmann's convention [**5**] in the following which will simplify the notation, namely we define

$$\langle\tau_{-2}(pt)\rangle_{0,0}^{X}=1,$$
$$\langle\tau_m(\gamma_1)\tau_{-1-m}(\gamma_2)\rangle_{0,0}^{X}=(-1)^{\max(m,-1-m)}\int_X\gamma_1\cdot\gamma_2,\quad m\in\mathbb{Z}.$$

All other Gromov-Witten invariants that contain a negative power of a cotangent line are defined to be zero.

Motivated by our previous results, we conjecture the following relations for Gromov-Witten invariants, which we have checked in various cases. We deem they are interesting constraints on Gromov-Witten invariants.

Conjecture 5.1. *Let $x_i, y_i\in H^*(X)$ and $k\geq 2g-3+r+s$. Then*

$$\sum_{g'=0}^{g}\sum_{j\in\mathbb{Z}}(-1)^j\langle\langle\tau_j(T_a)\prod_{i=1}^{r}\tau_{p_i}(x_i)\rangle\rangle_{g'}\langle\langle\tau_{k-j}(T^a)\prod_{i=1}^{s}\tau_{q_i}(y_i)\rangle\rangle_{g-g'}=0.$$

Note that j runs over all integers.

Conjecture 5.1 is a direct generalization of Theorem 4.8(i) in the point case. For example, when $r=s=0$, Conjecture 5.1 becomes

$$\langle\langle\tau_{2k}(1)\rangle\rangle_g-\sum_{m,a}t_m^a\langle\langle\tau_{m+2k-1}(T_a)\rangle\rangle_g$$
$$+\frac{1}{2}\sum_{g'=0}^{g}\sum_{j=0}^{2k-2}(-1)^j\langle\langle\tau_j(T_a)\rangle\rangle_{g'}\langle\langle\tau_{2k-2-j}(T^a)\rangle\rangle_{g-g'}=0$$

for $k\geq g$.

Conjecture 5.2. *Let $k>g$. Then*

$$\sum_{j=0}^{2k}(-1)^j\langle\langle\tau_j(T_a)\tau_{2k-j}(T^a)\rangle\rangle_g^X=0. \tag{4}$$

We also have

$$\frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\langle\tau_j(T_a)\tau_{2g-2-j}(T^a)\rangle\rangle_{g-1}=\frac{(2g)!}{B_{2g}}\langle\langle\mathrm{ch}_{2g-1}(\mathbb{E})\rangle\rangle_g. \tag{5}$$

Similar vanishing conjectures 5.1 and 5.2 can also be made about Witten's r-spin intersection numbers [**26**]. Thus these vanishing identities should be regarded as some universal topological recursion relations (TRR) valid in all genera.

Note that by the Chern character formula of Faber and Pandharipande [**4**] and the fact $\mathrm{ch_k}(\mathbb{E})=0$, $k>2g$, we have the equivalence

$$\text{Conjecture 5.1 } (r = s = 0) \Longleftrightarrow \text{identities (4) and (5)}$$

Recently, X. Liu and R. Pandharipande [**15**] give a proof of the above Conjectures 5.1 and 5.2. Their proof uses virtual localization to get topological recursion relations in the tautological ring of moduli spaces of curves, which are translated into universal equations for Gromov-Witten invariants by the splitting axiom and cotangent line comparison equations.

Earlier, X. Liu [**14**] proves the case $r = s = 0$ of Conjecture 5.1 and Conjecture 5.2 (4) both for $g \leq 2$ using topological recursion relations in low genus, which is tour de force, since the number of terms in TRR increase very rapidly with g. For example, Getzler's TRR in $g = 2$ contains 15 terms.

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