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# ISOTROPIC JACOBI FIELDS ON COMPACT 3-SYMMETRIC SPACES

José Carmelo González-Dávila

**Abstract**

We prove that a compact Riemannian 3-symmetric space is globally symmetric if every Jacobi field along a geodesic vanishing at two points is the restriction to that geodesic of a Killing field induced by the isotropy action or, in particular, if the isotropy action is variationally complete.

## 1. Introduction

A Jacobi field $V$ on a homogeneous Riemannian manifold $(M, g)$ which is the restriction of a Killing vector field along a geodesic is called *isotropic* [**10**]. It means that $V$ is the restriction of an infinitesimal motion of elements in the Lie algebra of the isometry group $I(M, g)$ of $(M, g)$. Moreover, if $V$ vanishes at a point $o$ of the geodesic then it is obtained as restriction of an infinitesimal $K$-motion, being $K$ the isotropy subgroup of $I(M, g)$ at $o \in M$. This particular situation was what originally motivated the term "isotropic" (see [**2**] and [**3**]).

The Jacobi equation on a symmetric space has simple solutions and one can directly show that all Jacobi field vanishing at two points is isotropic (see for example [**4**]). In the case of a naturally reductive space, the adapted canonical connection has the same geodesics and the Jacobi equation can be also written as a differential equation with constant coefficients (equation (2.7)). Using this fact, I. Chavel in [**2**] (see also [**3**]) proved that all simply connected normal Riemannian homogeneous space $(M = G/K, g)$ of rank one with the property that all Jacobi fields vanishing at two points are $G$-isotropic, i.e. restrictions of infinitesimal $G$-motions along geodesics, are homeomorphic to a rank one symmetric space. Afterwards, W. Ziller in [**10**] proposed to examine conjectures like:

*A naturally reductive space with the property that all Jacobi fields vanishing at two points are isotropic is locally symmetric.*

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In this paper, we consider a Riemannian 3-symmetric space $(M = G/K, \sigma, <,>)$, where $G$ is a compact connected Lie group acting effectively and the inner product $<,>$ determines an adapted naturally reductive metric on $M$, or equivalently, the canonical almost complex structure $J$ is nearly Kählerian [**5**]. Then, we find geodesics on non-symmetric spaces $(M = G/K, \sigma, <,>)$ admitting Jacobi fields vanishing at two points which are not $G$-isotropic. It allows us to prove the following theorem:

**Theorem 1.1.** *A compact Riemannian* 3*-symmetric space* $(M = G/K, \sigma, <,>)$ *with the property that all Jacobi fields vanishing at two points are* $G$*-isotropic is a symmetric space.*

When $(M = G/K, \sigma, <,>)$ is moreover simply connected, irreducible and not isometric to a symmetric space, $G$ coincides with the identity component $I_o(M, g)$ of $I(M, g)$ [**8**, Theorem 3.6]. Then, we have

**Corollary 1.2.** *A compact irreducible simply connected Riemannian* 3*-symmetric space is a symmetric space if and only if all Jacobi field vanishing at two points is isotropic.*

R. Bott and H. Samelson introduced in [**1**] the notion of *variationally complete action* and they obtained that the *isotropy action*, i.e. the action of an isotropy subgroup $K$ as subgroup of $G$, on a symmetric space of compact type is variationally complete (for the definition of variationally complete action, see section 2). Using Lemma 2.1, we can conclude

**Corollary 1.3.** *If the isotropy action of* $K$ *on a compact Riemannian* 3*-symmetric space* $(M = G/K, \sigma, <,>)$ *is variationally complete then it is a symmetric space.*

## 2. Variational completeness. Isotropic Jacobi fields

Let $(M, g)$ be a connected homogeneous *Riemannian* manifold. Then $(M, g)$ can be expressed as coset space $G/K$, where $G$ is a Lie group, which is supposed to be connected, acting transitively and effectively on $M$, $K$ is the isotropy subgroup of $G$ at some point $o \in M$ and $g$ is a $G$-invariant Riemannian metric. Moreover, we can assume that $G/K$ is a *reductive homogeneous space*, i.e., there is an $Ad(K)$-invariant subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of $G$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, $\mathfrak{k}$ being the Lie algebra of $K$. $(M = G/K, g)$ is said to be *naturally reductive*, or more precisely $G$*-naturally reductive*, if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ satisfying

$$< [X, Y]_{\mathfrak{m}}, Z > + < [X, Z]_{\mathfrak{m}}, Y >= 0 \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{m}$, where $[X, Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[X, Y]$ and $<,>$ is the metric induced by $g$ on $\mathfrak{m}$, or equivalently, $[X, \cdot]_{\mathfrak{m}} : \mathfrak{m} \to$

$\mathfrak{m}$ is skew-symmetric for all $X \in \mathfrak{m}$. When there exists a bi-invariant metric on $\mathfrak{g}$ whose restriction to $\mathfrak{m} = \mathfrak{k}^{\perp}$ is the metric $<,>$, the (naturally reductive) space $(M = G/K, g)$ is called *normal homogeneous*. Then, for all $X, Y, Z \in \mathfrak{g}$, we have

$$< [X, Y], Z > + < [X, Z], Y >= 0. \tag{2.2}$$

For each $X \in \mathfrak{g}$, the mapping $\psi : \mathbb{R} \times M \to M$, $(t, p) \in \mathbb{R} \times M \mapsto \psi_t(p) = (\exp tX)p$ is a one-parameter group of isometries and consequently, $\psi$ induces a Killing vector field $X^*$ given by

$$X^*_p = \frac{d}{dt}_{|t=0} (\exp tX)p, \qquad p \in M. \tag{2.3}$$

$X^*$ is called the *fundamental vector field* or the *infinitesimal $G$-motion* corresponding to $X$ on $M$. If $G = I_o(M, g)$, then all (complete) Killing vector field on $M$ is a fundamental vector field $X^*$, for some $X \in \mathfrak{g}$.

For any $a \in G$, we have

$$(a \exp tX)a^{-1} = \exp(tAd_a X).$$

This implies

$$(Ad_a X)^*_{ap} = a_{*p} X^*_p, \tag{2.4}$$

where $a_{*p}$ denotes the differential map of $a$ at $p \in M$.

The $K$-orbit $O_p(K) = \{kp \mid k \in K\}$, for each $p \in M$, is a regular submanifold of $M$ and, from (2.3), its tangent space $T_pO_p(K)$ at $p$ is given by

$$T_pO_p(K) = \{A^*_p \mid A \in \mathfrak{k}\}.$$

A geodesic $\gamma = \gamma(t)$ of $(M = G/K, g)$ is called *$K$-transversal* if for each $t \in \mathbb{R}$ the tangent vector $\gamma'(t)$ is orthogonal to the $K$-orbit $O_{\gamma(t)}(K)$ at $\gamma(t)$. Since a geodesic which is orthogonal to a Killing vector field at one of its points it is orthogonal to it at all of points, this condition is equivalent to require only the existence of a $t_o \in \mathbb{R}$ such that $\gamma'(t_o)$ is orthogonal to $O_{\gamma(t_o)}(K)$ at $\gamma(t_o)$. A Jacobi field $V$ along $\gamma$ is said to be *$K$-transversal* if it is derived from a geodesic variation $\phi$ of $\gamma$ in which all geodesic $t \to \phi(t, s_o)$ is $K$-transversal. Any restriction of a Killing field to a $K$-transversal geodesic induced by the isotropy action is $K$-transversal [**1**, Proposition 6.6].

The action of $K$ on $M = G/K$, as subgroup of $G$, is said to be *variationally complete* [**1**] if every $K$-transversal Jacobi field $V$ along a (transversal) geodesic $\gamma$ with $V(t_0) = 0$, for some $t_0 \in \mathbb{R}$, and for which there exists $t_1 \neq t_0$ such that $V(t_1)$ is tangent to the $K$-orbit of $\gamma(t_1)$ is $G$-isotropic.

Let $K_p = \{a \in G \mid a(p) = p\}$ be the isotropy subgroup at a point $p \in M$. In particular, $K_o = K$, where $o$ denotes the origin of $G/K$.

Let $a \in G$ such that $p = a(o)$. The elements of $K_p$ are obtained by conjugation of elements of $K$ by $a$, i.e.,

$$K_p = aKa^{-1}$$

and hence, the Lie algebra $\mathfrak{k}_p$ of $K_p$ is given by

$$\mathfrak{k}_p = Ad_a\mathfrak{k}. \tag{2.5}$$

From (2.4), a geodesic $\gamma$ on $(M, g)$ is $K$-transversal if and only if $a \circ \gamma$ is $K_p$-transversal and $V$ is a $K$-transversal (resp. $G$-isotropic) Jacobi field along $\gamma$ if and only if $a_*V$ is $K_p$-transversal (resp. $G$-isotropic) along $a \circ \gamma$. Hence, taking into account that any Jacobi field $V$ along a geodesic $\gamma$ starting at a point $p \in M$ with $V(0) = 0$ is always $K_p$-transversal, we directly obtain

**Lemma 2.1.** *If the isotropy action of $K$ on $M = G/K$ is variationally complete, then:*

(i) *the action of the isotropy subgroup $K_p$ at $p$, for all $p \in M$, is variationally complete;*
(ii) *all Jacobi field vanishing at two points is $G$-isotropic.*

Next, let $\tilde{T}$ denote the torsion tensor and $\tilde{R}$ the curvature tensor of the *canonical connection* $\tilde{\nabla}$ of $(M, g)$ adapted to the reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ [**7**, I, p.110]. Because $\tilde{\nabla}$ is a $G$-invariant affine connection, these tensors (under the canonical identification of $\mathfrak{m}$ with the tangent space $T_oM$ of the origin $o$) are given by

$$\tilde{T}_o(X, Y) = -[X, Y]_{\mathfrak{m}} \quad , \quad \tilde{R}_o(X, Y) = \mathrm{ad}_{[X,Y]_{\mathfrak{k}}} \tag{2.6}$$

for $X, Y \in \mathfrak{m}$, where $[X, Y]_{\mathfrak{k}}$ denotes the $\mathfrak{k}$-component of $[X, Y]$.

On naturally reductive spaces, $\nabla$ and $\tilde{\nabla}$ have the same geodesics and, consequently, the same Jacobi fields (see [**10**]). Such geodesics are orbits of one-parameter subgroups of $G$ of type $\exp tu$ where $u \in \mathfrak{m}$. Then, taking into account that $\tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = 0$ and the parallel translation with respect to $\tilde{\nabla}$ of tangent vectors at the origin $o$ along $\gamma(t) = (\exp tu)o$, $u \in \mathfrak{m}$, $\|u\| = 1$, coincides with the differential of $\exp tu \in G$ acting on $M$, it follows that the Jacobi equation can be expressed as the differential equation

$$X'' - \tilde{T}_uX' + \tilde{R}_uX = 0 \tag{2.7}$$

in the vector space $\mathfrak{m}$, where $\tilde{T}_uX = \tilde{T}(u, X) = -[u, X]_{\mathfrak{m}}$ and $\tilde{R}_uX = \tilde{R}(u, X)u = [[u, X]_{\mathfrak{k}}, u]$. The operator $\tilde{T}_u$ is skew-symmetric with respect to $<, >$, $\tilde{R}_u$ is self-adjoint and they satisfy [**4**]

$$R_u = \tilde{R}_u - \frac{1}{4}\tilde{T}_u^2. \tag{2.8}$$

A Jacobi field $V$ along $\gamma(t) = (\exp tu)o$ with $V(0) = 0$ is *$G$-isotropic* if and only if there exists an $A \in \mathfrak{k}$ such that (see [**4**])

$$V'(0) = [A, u]. \tag{2.9}$$

Then, $V = A^* \circ \gamma$.

We shall need the following characterization for $G$-isotropic Jacobi fields on normal homogeneous spaces $(M = G/K, g)$.

**Lemma 2.2.** *A Jacobi field $V$ along $\gamma(t) = (\exp tu)o$ on a normal homogeneous space $(M = G/K, g)$ with $V(0) = 0$ is $G$-isotropic if and only if $V'(0) \in (\mathrm{Ker}\ \tilde{R}_u)^\perp$.*

*Proof.* From (2.9), we have to show that

$$(\mathrm{Ker}\ \tilde{R}_u)^\perp \cap \mathfrak{m} = [u, \mathfrak{k}]. \tag{2.10}$$

For each $v \in \mathfrak{m}$, using (2.2), we get

$$< \tilde{R}_u v, v >=< [[u, v]_\mathfrak{k}, u], v >=< [u, v]_\mathfrak{k}, [u, v]_\mathfrak{k} > .$$

Hence, $v \in \mathrm{Ker}\ \tilde{R}_u$ if and only if $[u, v]_\mathfrak{k} = 0$. But,

$$[u, v]_\mathfrak{k} = 0 \Leftrightarrow 0 =< [u, v]_\mathfrak{k}, \mathfrak{k} >= - < [u, \mathfrak{k}], v >\Leftrightarrow v \in [u, \mathfrak{k}]^\perp.$$

Then, we obtain (2.10) and it gives the result. q.e.d.

Next, we give conditions to obtain Jacobi fields along $\gamma$ vanishing at two points which are or not $G$-isotropic.

**Proposition 2.3.** *Let $u, v$ be orthonormal vectors in $\mathfrak{m}$ such that $[[u, v], u] = \lambda v$, for some $\lambda > 0$. We have:*

(i) *If $[u, v]_\mathfrak{m} = 0$, then the vector fields $V(t)$ along $\gamma(t) = (\exp tu)o$ given by*

$$V(t) = (\exp tu)_{*o} \left( A \sin \sqrt{\lambda} t v \right),$$

*for $A$ constant, are $G$-isotropic Jacobi fields with $V(\frac{p\pi}{\sqrt{\lambda}}) = 0$, for all $p \in \mathbb{Z}$.*

(ii) *If $[u, v] \in \mathfrak{m}\backslash\{0\}$, then the vector fields $V(t)$ along $\gamma(t) = (\exp tu)o$ given by*

$$\begin{aligned} V(t) &= (\exp tu)_{*o}\Big(\Big(A \sin \sqrt{\lambda} t + B(1 - \cos \sqrt{\lambda} t)\Big)v \\ &\quad + \Big( - A(1 - \cos \sqrt{\lambda} t) + B \sin \sqrt{\lambda} t\Big) w\Big), \end{aligned}$$

*for $A, B$ constants with $w = \frac{1}{\sqrt{\lambda}}[u, v]$, are Jacobi fields such that $V(\frac{2p\pi}{\sqrt{\lambda}}) = 0$, for all $p \in \mathbb{Z}$, which are not $G$-isotropic.*

*Proof.* (i) From (2.6) we get $\tilde{T}_u v = 0$ and $\tilde{R}_u v = \lambda v$. Then it is easy to see that $X(t) = A \sin \sqrt{\lambda} t v$ is a solution of (2.7) with $X(0) = 0$. Because $\tilde{R}_u$ is self-adjoint, $v \in (\mathrm{Ker}\ \tilde{R}_u)^{\perp}$ and from Lemma 2.2, $V(t) = (\exp tu)_{*o} X(t)$ is $G$-isotropic.

(ii) Here, we obtain

$$\tilde{T}_u v = -\sqrt{\lambda} w, \quad \tilde{T}_u w = \sqrt{\lambda} v, \quad \tilde{R}_u v = \tilde{R}_u w = 0$$

and, from (2.2), $\|[u,v]\| = \sqrt{\lambda}$. Then the solutions $X(t) = X^1(t) v + X^2(t) w$ of (2.7) satisfy

$$\begin{cases} Y^{1\prime}(t) - \sqrt{\lambda} Y^2(t) = 0, \\ Y^{2\prime}(t) + \sqrt{\lambda} Y^1(t) = 0, \end{cases}$$

where $Y^i(t) = X^{i\prime}(t)$, $i = 1, 2$. Hence $X(t)$ with $X(0) = 0$ is given by

$$X(t) = \Big(A \sin \sqrt{\lambda} t + B(1 - \cos \sqrt{\lambda} t)\Big) v + \Big(-A(1 - \cos \sqrt{\lambda} t) + B \sin \sqrt{\lambda} t\Big) w,$$

for $A, B$ constants. Hence, $V(t) = (\exp tu)_{*o} X(t)$ are Jacobi fields along $\gamma$ verifying $V(0) = V(\frac{2p\pi}{\sqrt{\lambda}}) = 0$ and $V'(0) = X'(0)$. Because $X'(0) \in \mathbb{R}\{v, w\}$ and $[u,v]_{\mathfrak{k}} = [u,w]_{\mathfrak{k}} = 0$, we have $\tilde{R}_u V'(0) = 0$ and Lemma 2.2 implies that these Jacobi fields $V$ are not $G$-isotropic. q.e.d.

## 3. Compact irreducible Riemannian 3-symmetric spaces

We recall that a connected Riemannian manifold $(M, g)$ is called a 3-*symmetric space* [**5**] if it admits a family of isometries $\{\theta_p\}_{p \in M}$ of $(M, g)$ satisfying

(i) $\theta_p^3 = I$,
(ii) $p$ is an isolated fixed point of $\theta_p$,
(iii) the tensor field $\Theta$ defined by $\Theta = (\theta_p)_{*p}$ is of class $C^\infty$,
(iv) $\theta_{p*} \circ J = J \circ \theta_{p*}$,

where $J$ is the *canonical almost complex structure* associated with the family $\{\theta_p\}_{p \in M}$ given by $J = \frac{1}{\sqrt{3}}(2\Theta + I)$. Riemannian 3-symmetric spaces are characterized by a triple $(G/K, \sigma, <,>)$ satisfying the following conditions:

(1) $G$ is a connected Lie group and $\sigma$ is an automorphism of $G$ of order 3,
(2) $K$ is a closed subgroup of $G$ such that $G_o^\sigma \subseteq K \subseteq G^\sigma$, where $G^\sigma = \{x \in G \mid \sigma(x) = x\}$ and $G_o^\sigma$ denotes its identity component,
(3) $<,>$ is an $Ad(K)$- and $\sigma$-invariant inner product on the vector space $\mathfrak{m} = (\mathfrak{m}^+ \oplus \mathfrak{m}^-) \cap \mathfrak{g}$, where $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are the eigenspaces of $\sigma$ on the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ corresponding to the eigenvalues $\varepsilon$ and $\varepsilon^2$, respectively, where $\varepsilon = e^{2\pi\sqrt{-1}/3}$.

Here and in the sequel, $\sigma$ and its differential $\sigma_*$ on $\mathfrak{g}$ and on $\mathfrak{g}_{\mathbb{C}}$ are denoted by the same letter $\sigma$.

The inner product $<,>$ induces a $G$-invariant Riemannian metric $g$ on $M = G/K$ and $(G/K, g)$ becomes into a Riemannian 3-symmetric space. Then, it is a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, where the algebra of Lie $\mathfrak{k}$ of $K$ is $\mathfrak{g}^\sigma = \{X \in \mathfrak{g} \mid \sigma X = X\}$. Moreover, the canonical almost structure $J$ on $G/K$ is $G$-invariant, $(M = G/K, g, J)$ is quasi-Kählerian and it is nearly Kählerian if and only if $(G/K, g)$ is a naturally reductive homogeneous space with adapted reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$. In this case $g$ is said to be an *adapted naturally reductive metric* for $M$.

We shall need some general results of complex simple Lie algebras. See [**6**] for more details. Let $\mathfrak{g}_{\mathbb{C}}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}_{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\Delta$ denote the set of non-zero roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a system of simple roots or a basis of $\Delta$. Because the restriction of the Killing form $B$ of $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{C}} \times \mathfrak{h}_{\mathbb{C}}$ is nondegenerate, there exists a unique element $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ such that

$$B(H, H_\alpha) = \alpha(H),$$

for all $H \in \mathfrak{h}_{\mathbb{C}}$. Moreover, we have $\mathfrak{h}_{\mathbb{C}} = \sum_{\alpha\in\Delta} \mathbb{C} H_\alpha$ and $B$ is strictly positive definite on $\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha\in\Delta} \mathbb{R} H_\alpha$. Put $< \alpha, \beta >= B(H_\alpha, H_\beta)$. We choose root vectors $\{E_\alpha\}_{\alpha\in\Delta}$, such that for all $\alpha, \beta \in \Delta$, we have

$$(3.1) \quad \begin{cases} [E_\alpha, E_{-\alpha}] = H_\alpha, & [H, E_\alpha] = \alpha(H) E_\alpha \quad \text{for } H \in \mathfrak{h}_{\mathbb{C}}; \\ [E_\alpha, E_\beta] = 0 & \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Delta; \\ [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta, \end{cases}$$

where the constants $N_{\alpha,\beta}$ satisfy

$$(3.2) \qquad N_{\alpha,\beta} = -N_{-\alpha,-\beta}, \quad N_{\alpha,\beta} = -N_{\beta,\alpha}$$

and, if $\alpha, \beta, \gamma \in \Delta$ and $\alpha + \beta + \gamma = 0$, then

$$(3.3) \qquad N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}.$$

Moreover, given an $\alpha$-series $\beta + n\alpha$ $(p \leq n \leq q)$ containing $\beta$, then

$$(3.4) \qquad (N_{\alpha,\beta})^2 = \frac{q(1-p)}{2} < \alpha, \alpha > .$$

For this choice, $E_\alpha$ and $E_\beta$ are orthogonal under $B$ if $\alpha + \beta \neq 0$, $B(E_\alpha, E_{-\alpha}) = 1$ and we have the orthogonal direct sum

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \sum_{\alpha\in\Delta} \mathbb{C} E_\alpha.$$

Denote by $\Delta^+$ the set of positive roots of $\Delta$ with respect to some lexicographic order in $\Pi$. Then the $\mathbb{R}$-linear subspace $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ given by

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta^+} (\mathbb{R}\, U_\alpha + \mathbb{R}\, V_\alpha)$$

| Type | $\sigma$ | $m_i$ | $\Pi(H)$ |
|---|---|---|---|
| I | $Ad_{\exp\frac{2\pi\sqrt{-1}}{3}H_i}$ | 1 | $\{\alpha_k \in \Pi \mid k \neq i\}$ |
| II | $Ad_{\exp 2\pi\sqrt{-1}\frac{(H_i+H_j)}{3}}$ | $m_i = m_j = 1$ | $\{\alpha_k \in \Pi \mid k \neq i,\ k \neq j\}$ |
| III | $Ad_{\exp\frac{4\pi\sqrt{-1}}{3}H_i}$ | 2 | $\{\alpha_k \in \Pi \mid k \neq i\}$ |
| IV | $Ad_{\exp 2\pi\sqrt{-1}H_i}$ | 3 | $\{\alpha_k \in \Pi \mid k \neq i\} \cup \{-\mu\}$ |

Table I

is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{h} = \sum_{\alpha\in\Delta} I\!R\sqrt{-1}H_\alpha$ and $U_\alpha = E_\alpha - E_{-\alpha}$ and $V_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Here, we get

(3.5) $\quad [U_\alpha, \sqrt{-1}H_\beta] = -<\alpha,\beta> V_\alpha, \qquad [V_\alpha, \sqrt{-1}H_\beta] = <\alpha,\beta> U_\alpha.$

Next, we shall describe automorphisms of order 3 on the compact Lie algebra $\mathfrak{g}$ (or on $\mathfrak{g}_{\mathbb{C}}$) which do not preserve any proper ideals. First, suppose that $\sigma$ is an *inner* automorphism.

(A) $\sigma$ is an inner automorphism

Because $\mathfrak{g}$ decomposes into a direct sum of an abelian ideal and simple ideals, we can assume that $\mathfrak{g}$ is simple. Let $\mu = \sum_{i=1}^{l} m_i\alpha_i$ be the *maximal root* of $\Delta$ and consider $H_i \in \mathfrak{h}_{\mathbb{C}}$, $i = 1,\dots,l$, defined by

$$\alpha_j(H_i) = \frac{1}{m_i}\delta_{ij}, \quad i,j = 1,\dots,l.$$

Following [**9**, Theorem 3.3], each inner automorphism of order 3 on $\mathfrak{g}_{\mathbb{C}}$ is conjugate in the inner automorphism group of $\mathfrak{g}_{\mathbb{C}}$ to some $\sigma = Ad_{\exp 2\pi\sqrt{-1}H}$, where $H = \frac{1}{3}m_iH_i$ with $1 \leq m_i \leq 3$ or $H = \frac{1}{3}(H_i + H_j)$ with $m_i = m_j = 1$. Then there are four types of $\sigma = Ad_{\exp 2\pi\sqrt{-1}H}$ with corresponding simple root systems $\Pi(H)$ for $\mathfrak{g}_{\mathbb{C}}^{\sigma}$ given in Table I. Denote by $\Delta^+(H)$ the positive root system generated by $\Pi(H)$. Then, we have $\mathfrak{h} \subset \mathfrak{k} = \mathfrak{g}^\sigma$ and

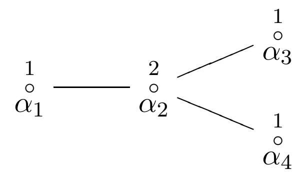
$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha\in\Delta^+(H)} (I\!R\, U_\alpha + I\!R\, V_\alpha).$$

Because $B(U_\alpha, U_\beta) = B(V_\alpha, V_\beta) = -2\delta_{\alpha\beta}$ and $B(U_\alpha, V_\beta) = 0$, it follows that $\{U_\alpha, V_\alpha \mid \alpha \in \Delta^+ \setminus \Delta^+(H)\}$ becomes into an orthonormal basis for $\left(\mathfrak{m}, <,> = -\frac{1}{2}B_{|\mathfrak{m}}\right)$.

(B) $\sigma$ is an outer automorphism

Let $\sigma$ be an outer automorphism of order 3 on a compact Lie algebra $\mathfrak{g}$ such that there is no proper $\sigma$-invariant ideal in $\mathfrak{g}$. Then $\mathfrak{g}$ must be semisimple [**9**, Theorem 5.10]. First, suppose that $\mathfrak{g}$ is simple. Then the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is of type

$\mathfrak{d}_4$ :




and the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}^{\sigma}$, the set of fixed points of $\sigma$ on $\mathfrak{g}_{\mathbb{C}}$, is either of type $\mathfrak{g}_2$, where a Weyl basis is given by [**9**, Theorem 5.5]

$$\begin{aligned}&\big\{H_{\alpha_2}, H_{\alpha_1}+H_{\alpha_3}+H_{\alpha_4}; E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, E_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)},\\ &E_{\pm\alpha_1}+E_{\pm\alpha_3}+E_{\pm\alpha_4}, E_{\pm(\alpha_1+\alpha_2)}+E_{\pm(\alpha_2+\alpha_3)}+E_{\pm(\alpha_2+\alpha_4)},\\ &E_{\pm(\alpha_1+\alpha_2+\alpha_3)}+E_{\pm(\alpha_2+\alpha_3+\alpha_4)}+E_{\pm(\alpha_1+\alpha_2+\alpha_4)}\big\},\end{aligned}$$

or of type $\mathfrak{a}_2$, being a Weyl basis $\big\{H_{\beta_1}, H_{\beta_2}; F_{\pm\beta_1}, F_{\pm\beta_2}, F_{\pm(\beta_1+\beta_2)}\big\}$, where

$$\begin{aligned}H_{\beta_1} &= H_{\alpha_1}+H_{\alpha_3}+H_{\alpha_4},\\ H_{\beta_2} &= -3H_{\alpha_2}-2(H_{\alpha_1}+H_{\alpha_3}+H_{\alpha_4}),\\ F_{\pm\beta_1} &= E_{\pm\alpha_1}+E_{\pm\alpha_3}+E_{\pm\alpha_4},\\ F_{\beta_2} &= E_{-(\alpha_1+\alpha_2+\alpha_3)}+\varepsilon^2E_{-(\alpha_2+\alpha_3+\alpha_4)}+\varepsilon E_{-(\alpha_1+\alpha_2+\alpha_4)},\\ F_{-\beta_2} &= E_{\alpha_1+\alpha_2+\alpha_3}+\varepsilon E_{\alpha_2+\alpha_3+\alpha_4}+\varepsilon^2E_{\alpha_1+\alpha_2+\alpha_4},\\ F_{\beta_1+\beta_2} &= E_{-(\alpha_1+\alpha_2)}+\varepsilon^2E_{-(\alpha_2+\alpha_3)}+\varepsilon E_{-(\alpha_2+\alpha_4)},\\ F_{-(\beta_1+\beta_2)} &= E_{\alpha_1+\alpha_2}+\varepsilon E_{\alpha_2+\alpha_3}+\varepsilon^2E_{\alpha_2+\alpha_4}.\end{aligned}$$

Finally, if $\mathfrak{g}$ is semisimple but not simple then $\mathfrak{g}=\mathfrak{L}\oplus\mathfrak{L}\oplus\mathfrak{L}$ with $\mathfrak{L}$ simple and $\mathfrak{k}=\mathfrak{g}^{\sigma}$ is $\mathfrak{L}$ embedded diagonally.

## 4. Proof of Theorem 1.1

This will require some previous propositions and lemmas. We start considering, as in above section, Riemannian 3-symmetric spaces $(M = G/K, \sigma, <,>)$ where $G$ is a compact connected Lie group acting effectively, the automorphism $\sigma$ on the Lie algebra $\mathfrak{g}$ of $G$ does not preserve any proper ideals and $<,>$ determines a *naturally reductive* Riemannian metric adapted to $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$. According with [**5**], the inner product $<,>$ is then the restriction to $\mathfrak{m}$ of a bi-invariant product on $\mathfrak{g}$. Because $\mathfrak{g}$ is semisimple, we take $<,>= -\frac{1}{2}B_{|\mathfrak{m}}$, where $B$ denotes the Killing form of $\mathfrak{g}$. Then, we have

**Lemma 4.1.** *$(M = G/K, \sigma, <,>)$ is a normal homogeneous space.*

*Proof.* We only have to prove that $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$ is orthogonal with respect to $B$. Let $B_{\mathfrak{g}_{\mathbb{C}}}$ be the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Because $B_{\mathfrak{g}_{\mathbb{C}}}$ is invariant under automorphisms, we get that the subspaces $\mathfrak{g}_{\mathbb{C}}^{\sigma}$, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are orthogonal in $(\mathfrak{g}_{\mathbb{C}}, B_{\mathfrak{g}_{\mathbb{C}}})$. Then the result follows taking into account that $B_{\mathfrak{g}_{\mathbb{C}}}(X,Y) = B(X,Y)$, for all $X, Y \in \mathfrak{g}$. q.e.d.

In the following, we look for geodesics on $(M = G/K, \sigma, <,>)$ with non-$G$-isotropic Jacobi fields vanishing at two points.

(A) $\sigma$ is an inner automorphism

**Lemma 4.2.** *If $\sigma$ is of Type I then $(M = G/K, \sigma, <,>)$ is one of the following irreducible Hermitian symmetric spaces of compact type:*

$$SU(n)/(S(U(r)\times U(n-r))), \qquad SO(n)/(SO(n-2)\times SO(2)),$$
$$Sp(n)/U(n), \qquad SO(2n)/U(n),$$
$$E_6/(SO(10)\times SO(2)), \qquad E_7/(E_6\times SO(2)).$$

*Proof.* (i) Put $H = \frac{1}{3}H_i$, for some $i \in \{1,\dots,l\}$ with $m_i = 1$. Then each $\alpha \in \Delta^+ \setminus \Delta^+(H)$ may be written as

$$\alpha = \sum_{j=1}^{l} n_j \alpha_j,$$

where $n_j \in \mathbb{Z}$, $n_j \geq 0$, and $n_i = 1$. It implies that $\alpha + \beta \notin \Delta$ and $\alpha - \beta \notin \Delta \setminus \Delta(H)$, for all $\alpha, \beta \in \Delta^+ \setminus \Delta^+(H)$. Hence, using (3.1) and (3.2), we get that $[U_\alpha, U_\beta]$ and $[V_\alpha, V_\beta]$ are collinear with $U_{\alpha-\beta}$ and $[U_\alpha, V_\beta]$ with $V_{\alpha-\beta}$. Then, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$ and consequently $(M = G/K, g)$ must be locally symmetric. Because rank $G$ = rank $K$ and the center of $G$ is trivial, it follows from [**9**, Theorem 6.4] that it is moreover globally symmetric. For the list of these spaces, we use [**9**, Theorem 3.3].

q.e.d.

**Remark 4.3.** Notice that on above compact symmetric spaces $M = G/K$, the action of $G$ is almost effective but not necessarily effective.

As a consequence of the following results, we will prove that a compact irreducible Riemannian 3-symmetric space $(M = G/K, \sigma, <,>)$ is a Hermitian symmetric space if and only if $\sigma$ is an inner automorphism of Type I.

**Proposition 4.4.** *Let $\alpha, \beta \in \Delta \setminus \Delta(H)$ such that $\alpha - \beta \neq 0$, $\alpha - \beta \notin \Delta$ and $2\alpha + \beta \neq 0$, $2\alpha + \beta \notin \Delta$. We have:*

(i) *If $\alpha + \beta \in \Delta(H)$ then $\gamma(t) = (\exp tU_\alpha)o$ on $(M = G/K, \sigma, <,>)$ admits $G$-isotropic Jacobi fields $V$ with $V(\frac{\sqrt{2}p\pi}{\|\alpha\|}) = 0$, $p \in \mathbb{Z}$.*

(ii) *If $\alpha + \beta \in \Delta \setminus \Delta(H)$ then $\gamma(t) = (\exp tU_\alpha)o$ on $(M = G/K, \sigma, <,>)$ admits Jacobi fields $V$ with $V(\frac{2\sqrt{2}p\pi}{\|\alpha\|}) = 0$, $p \in \mathbb{Z}$, which are not $G$-isotropic.*

*Proof.* Since $\alpha + \beta \in \Delta$, we get from (3.1) and (3.2)

$$[U_\alpha, U_\beta] = N_{\alpha,\beta} U_{\alpha+\beta}.$$

Then, taking into account that the $\alpha$-series containing $\beta$ is $\{\beta, \beta + \alpha\}$, (3.3) and (3.4) imply

$$[[U_\alpha, U_\beta], U_\alpha] = N_{\alpha,\beta} N_{-(\alpha+\beta),\alpha} U_\beta = (N_{\alpha,\beta})^2 U_\beta = \frac{<\alpha,\alpha>}{2} U_\beta.$$

Hence, taking $u = U_\alpha$, $v = U_\beta$ and $\lambda = \frac{<\alpha,\alpha>}{2}$ in Proposition 2.3, we obtain the result. q.e.d.

Next, we put

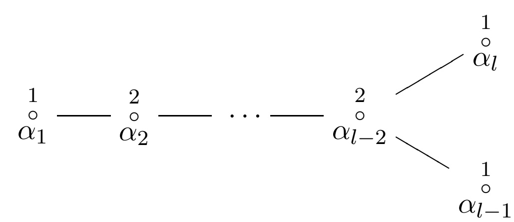
$$\alpha_{ij} = \alpha_i + \cdots + \alpha_j \quad (1 \le i \le j \le l), \qquad \tilde{\mu} = \sum_{j=1}^{l} (m_j - 1)\alpha_j.$$

It is easy to see by a case-by-case check the following.

**Lemma 4.5.** *We have:*

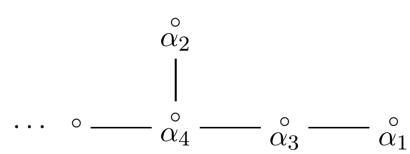
(a) $\alpha_{ij} \in \Delta$ *except if* $(i,j) = (l-1, l)$ *in*

$\mathfrak{d}_l$ $(l \ge 4)$ :




*or* $(i,j) = (1,2)$, $(1,3)$ *and* $(2,3)$ *in*

$\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ :




(b) $\tilde{\mu} \in \Delta$ *for* $\mathfrak{g}_{\mathbb{C}} \ne \mathfrak{a}_l$. *In* $\mathfrak{a}_l$, $\tilde{\mu}$ *is zero.*

Then, we can conclude

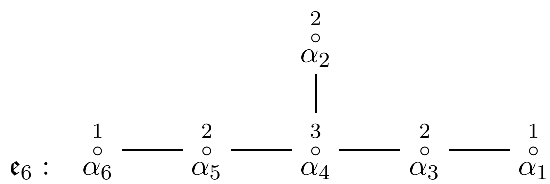
**Proposition 4.6.** *Let* $(M = G/K, \sigma, <,>)$ *be a Riemannian* 3-*symmetric space where* $G$ *is a compact simple Lie group acting effectively on* $M$ *and* $\sigma$ *is an inner automorphism on the Lie algebra* $\mathfrak{g}$ *of* $G$. *If all Jacobi field vanishing at two points is* $G$-*isotropic then* $(M, g)$ *is a symmetric space.*

*Proof.* From Lemma 4.2, we only need to show that there exist $\alpha, \beta \in \Delta \setminus \Delta(H)$ satisfying the hypothesis of Proposition 4.4 (ii) for $\sigma$ of Type II, III and IV.

If $\sigma$ is of Type II then $m_i = m_j = 1$ ($H = \frac{1}{3}(H_i + H_j)$), for some $i, j \in \{1, \ldots, l\}$, $i < j$. So, the complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{a}_l$ ($l \geq 2$), $\mathfrak{d}_l$ ($l \geq 4$) or $\mathfrak{e}_6$. Take

$$\alpha = \alpha_{1l}, \quad \beta = -\alpha_{(i+1)l}.$$

Then $\alpha + \beta = \alpha_{1i}$. Because $i \neq l - 2$ for $\mathfrak{g}_{\mathbb{C}}$ of type $\mathfrak{d}_l$ and $(i, j) = (1, 6)$ for

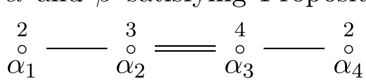



it follows from Lemma 4.5 that $\alpha, \beta$ and $\alpha + \beta$ belongs to $\Delta \setminus \Delta(H)$. Moreover, taking into account that $m_i = m_j = 1$, we easily see that $\alpha - \beta$ and $2\alpha + \beta$ are not roots.

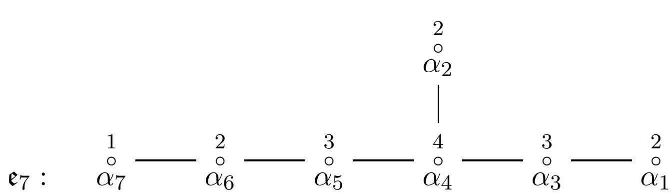
If $\sigma$ is of Type III then $m_i = 2$ ($H = \frac{2}{3}H_i$), for some $i = 1, \ldots, l$. It implies that $\mathfrak{g}_{\mathbb{C}}$ is one of the following: $\mathfrak{b}_l$, ($l \geq 2$), $\mathfrak{c}_l$, ($l \geq 2$), $\mathfrak{d}_l$, ($l \geq 4$), $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. Here, we put

$$\alpha = \mu, \quad \beta = -\tilde{\mu}.$$

Then $\alpha + \beta = \alpha_{1l}$ and from Lemma 4.5, $\beta$ and $\alpha + \beta$ are non-zero roots of $\mathfrak{g}_{\mathbb{C}}$. Moreover, $\alpha, \beta, \alpha + \beta$ belong to $\Delta \setminus \Delta(H)$. Because $\mu$ is the maximal root, we have that $\alpha - \beta$ and $2\alpha + \beta$ are not roots.

Finally, we consider $\sigma$ of Type IV. Then, $m_i = 3$, for some $i = 1, \ldots, l$, and $\mathfrak{g}_{\mathbb{C}}$ is one of the exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$. Except for $\mathfrak{g}_2$, in each one of these algebras we can find, using again Lemma 4.5, $\alpha$ and $\beta$ satisfying Proposition 4.4 (ii). Concretely, for the case $\mathfrak{f}_4$ : 

, we take $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ and $\beta = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4)$. In $\mathfrak{e}_6$, we can choose $\alpha = \tilde{\mu}$ and $\beta = -\alpha_{24}$. In



we take $\alpha = \tilde{\mu}$ and $\beta = -\alpha_{35}$ and, $\alpha = \tilde{\mu} - \alpha_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$ and $\beta = -\alpha_{27}$, in

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & & \overset{3}{\circ} & & & & & \\
 & & & & & & & & & \alpha_2 & & & & & \\
 & & & & & & & & & | & & & & & \\
\mathfrak{e}_8 : & \underset{\alpha_8}{\overset{2}{\circ}} & — & \underset{\alpha_7}{\overset{3}{\circ}} & — & \underset{\alpha_6}{\overset{4}{\circ}} & — & \underset{\alpha_5}{\overset{5}{\circ}} & — & \underset{\alpha_4}{\overset{6}{\circ}} & — & \underset{\alpha_3}{\overset{4}{\circ}} & — & \underset{\alpha_1}{\overset{2}{\circ}} & .
\end{array}
$$

The corresponding Riemannian 3-symmetric space for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_2$ : $\underset{\alpha_1}{\overset{3}{\circ}} \equiv \underset{\alpha_2}{\overset{2}{\circ}}$ is the sphere $S^6 = G_2/SU(3)$ equipped with the usual metric of constant curvature. q.e.d.

**Remark 4.7.** There exist geodesics on $(M = G/K, \sigma, <,>)$ with isotropically conjugate points and admitting Jacobi fields vanishing at these points which are not isotropic. This is the case of the geodesic $\gamma(t) = (\exp tU_\alpha)o$ in $M = G/K$, where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}_l$ $(l > 2)$ and $\alpha = \alpha_{1l}$. From the proof of above Proposition, $\gamma$ admits Jacobi fields vanishing at the origin and at $p = \gamma(\frac{2\sqrt{2}\pi}{\|\alpha\|})$ which are not $G$-isotropic and, taking $\beta = -\alpha_{1j}$, $j < l$, it follows from Proposition 4.4 (i) that $o$ and $p$ are moreover $G$-isotropically conjugate points.

(B) $\sigma$ is an outer automorphism

**Proposition 4.8.** *Let $(M = G/K, \sigma, <,>)$ be a Riemannian 3-symmetric space where $G$ is a compact Lie group acting effectively on $M$ and $\sigma$ is an outer automorphism on the Lie algebra $\mathfrak{g}$ of $G$ such that there is no proper $\sigma$-invariant ideal in $\mathfrak{g}$. Then there exist Jacobi fields vanishing at two points which are not $G$-isotropic.*

*Proof.* Suppose that $\mathfrak{g}$ is simple. If $\mathfrak{g}^\sigma_{\mathbb{C}}$ is of type $\mathfrak{g}_2$, then the corresponding real form $\mathfrak{k} = \mathfrak{g}^\sigma$ is generated by

$$
\begin{aligned}
&\{\sqrt{-1}H_{\alpha_2}, \sqrt{-1}(H_{\alpha_1} + H_{\alpha_3} + H_{\alpha_4}); U_{\alpha_2}, V_{\alpha_2},\\
&U_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, V_{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)},\\
&U_{(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}, V_{(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}, U_{\alpha_1} + U_{\alpha_3} + U_{\alpha_4}, V_{\alpha_1} + V_{\alpha_3} + V_{\alpha_4},\\
&U_{(\alpha_1+\alpha_2)} + U_{(\alpha_2+\alpha_3)} + U_{(\alpha_2+\alpha_4)}, V_{(\alpha_1+\alpha_2)} + V_{(\alpha_2+\alpha_3)} + V_{(\alpha_2+\alpha_4)},\\
&U_{(\alpha_1+\alpha_2+\alpha_3)} + U_{(\alpha_2+\alpha_3+\alpha_4)} + U_{(\alpha_1+\alpha_2+\alpha_4)},\\
&V_{(\alpha_1+\alpha_2+\alpha_3)} + V_{(\alpha_2+\alpha_3+\alpha_4)} + V_{(\alpha_1+\alpha_2+\alpha_4)}\}.
\end{aligned}
$$

Since the set $\Delta^+$ of the positive roots of $\mathfrak{d}_4$ is given by

$$
\begin{aligned}
\Delta^+ \;=\; &\{\alpha_i \ (1 \le i \le 4), \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3,\\
&\alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\\
&\mu = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}
\end{aligned}
$$

and the simple roots $\alpha_1, \dots, \alpha_4$ satisfy

$$(4.1) \quad <\alpha_i, \alpha_i> = \frac{1}{6}, \qquad <\alpha_1, \alpha_2> = <\alpha_2, \alpha_3> = <\alpha_2, \alpha_4> = -\frac{1}{12},$$

where $1 \le i \le 4$, and the other inner products are zero, we obtain the following basis for $\mathfrak{m} = (\mathfrak{g}^\sigma)^\perp$ in $\mathfrak{g}$ :

$$\begin{aligned}
&\{\sqrt{-1}(H_{\alpha_1} - H_{\alpha_3}), \sqrt{-1}(H_{\alpha_1} + H_{\alpha_3} - 2H_{\alpha_4}); U_{\alpha_1} - U_{\alpha_3}, V_{\alpha_1} - V_{\alpha_3},\\
&U_{\alpha_1} + U_{\alpha_3} - 2U_{\alpha_4}, V_{\alpha_1} + V_{\alpha_3} - 2V_{\alpha_4},\\
&U_{(\alpha_1+\alpha_2)} - U_{(\alpha_2+\alpha_3)}, V_{(\alpha_1+\alpha_2)} - V_{(\alpha_2+\alpha_3)},\\
&U_{(\alpha_1+\alpha_2)} + U_{(\alpha_2+\alpha_3)} - 2U_{(\alpha_2+\alpha_4)}, V_{(\alpha_1+\alpha_2)} + V_{(\alpha_2+\alpha_3)} - 2V_{(\alpha_2+\alpha_4)},\\
&U_{(\alpha_1+\alpha_2+\alpha_3)} - U_{(\alpha_2+\alpha_3+\alpha_4)}, V_{(\alpha_1+\alpha_2+\alpha_3)} - V_{(\alpha_2+\alpha_3+\alpha_4)},\\
&U_{(\alpha_1+\alpha_2+\alpha_3)} + U_{(\alpha_2+\alpha_3+\alpha_4)} - 2U_{(\alpha_1+\alpha_2+\alpha_4)},\\
&V_{(\alpha_1+\alpha_2+\alpha_3)} + V_{(\alpha_2+\alpha_3+\alpha_4)} - 2V_{(\alpha_1+\alpha_2+\alpha_4)}\}.
\end{aligned}$$

We put,

$$u = \frac{1}{\sqrt{6}}(U_{\alpha_1} + U_{\alpha_3} - 2U_{\alpha_4}), \quad v = \sqrt{-6}(H_{\alpha_1} - H_{\alpha_3}).$$

Then, $u$ and $v$ are orthonormal vectors in $\mathfrak{m}$ and using (3.1), (3.5) and (4.1), we obtain

$$[u, v] = -\frac{1}{6}(V_{\alpha_1} - V_{\alpha_3}), \qquad [[u, v], v] = \frac{1}{18}v.$$

Hence, Lemma 4.1 and Proposition 2.3 (ii) imply that $\gamma(t) = (\exp tu)o$ admits Jacobi fields vanishing at the origin and at $\gamma(6\sqrt{2}p\pi)$, $p \in \mathbb{Z}$, which are not $G$-isotropic.

Next, we suppose that $\mathfrak{g}^\sigma_{\mathbb{C}}$ is of type $\mathfrak{a}_2$. Then, a basis for $\mathfrak{k} = \mathfrak{g}^\sigma$ is given by

$$\left\{\sqrt{-1}H_{\beta_1}, \sqrt{-1}H_{\beta_2}; \tilde{U}_{\beta_1}, \tilde{V}_{\beta_1}, \tilde{U}_{\beta_2}, \tilde{V}_{\beta_2}, \tilde{U}_{\beta_1+\beta_2}, \tilde{V}_{\beta_1+\beta_2}\right\},$$

where

$$\begin{aligned}
&\tilde{U}_{\beta_i} = F_{\beta_i} - F_{-\beta_i}, && \tilde{V}_{\beta_i} = \sqrt{-1}\left(F_{\beta_i} + F_{-\beta_i}\right), \quad i = 1, 2,\\
&\tilde{U}_{\beta_1+\beta_2} = F_{\beta_1+\beta_2} - F_{-(\beta_1+\beta_2)}, && \tilde{V}_{\beta_1+\beta_2} = \sqrt{-1}\left(F_{\beta_1+\beta_2} + F_{-(\beta_1+\beta_2)}\right).
\end{aligned}$$

Put,

$$u = \sqrt{-6}(H_{\alpha_1} - H_{\alpha_3}), \qquad v = \frac{1}{\sqrt{2}}(U_{\alpha_1} - U_{\alpha_3}).$$

Then, using (3.1) and (4.1), we can check that $u, v \in \mathfrak{m}$ and they are orthonormal. Moreover, we get

$$[u, v] = \frac{\sqrt{3}}{6}(V_{\alpha_1} + V_{\alpha_3}), \qquad [[u, v], u] = \frac{1}{6}v$$

and $V_{\alpha_1} + V_{\alpha_3} \in \mathfrak{m}$. Hence, $u, v$ satisfy the hypothesis of Proposition 2.3 (ii).

Finally, suppose that $\mathfrak{g}$ is semisimple but not simple, then $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{L} \oplus \mathfrak{L}$ with $\mathfrak{L}$ simple and $\mathfrak{k} = \mathfrak{g}^\sigma$ is $\mathfrak{L}$ embedded diagonally. Let $\alpha$ be a root of $\mathfrak{L}_{\mathbb{C}}$. Take in $\mathfrak{g}$,

$$u = \frac{1}{\sqrt{6}}(U_\alpha, U_\alpha, -2U_\alpha), \qquad v = \frac{\sqrt{-1}}{\|\alpha\|}(H_\alpha, -H_\alpha, 0).$$

Then $u, v$ are orthogonal to $\mathfrak{k}$ and, from (3.1) and (3.5),

$$[u, v] = -\frac{\|\alpha\|}{\sqrt{6}}(V_\alpha, -V_\alpha, 0), \quad [[u, v], u] = \frac{< \alpha, \alpha >}{3} v$$

and, consequently they again satisfy the hypothesis of Proposition 2.3 (ii). q.e.d.

The proof of Theorem 1.1 is now easy. Following [**9**, Theorem 6.4], compact 3-symmetric spaces $(M = G/K, \sigma, <, >)$ are given by

$$M = (M_0 \times M_1 \times \cdots \times M_r)/\Gamma = \{(G_0 \times G_1 \times \cdots \times G_r)/\Gamma\}/K,$$

where

(i) $M_0$ is a complex Euclidean space, $G_0$ is its translation group and $K_0 = \{I\} \subset G_0$;
(ii) $M_i = G_i/K_i$, $1 \le i \le r$, is a simply connected 3-symmetric space, $G_i$ is a compact connected Lie group acting effectively and $\sigma_i = \sigma_{|\mathfrak{g}_i}$ does not preserve any proper ideals;
(iii) $\Gamma$ is any discrete subgroup of $G_0 \times Z_1 \times \cdots \times Z_r$, being $Z_i$ the center of $G_i$ and $\Gamma \cap G_0$ a lattice in $G_0$;
(iv) $K$ is the image of $(K_0 \times K_1 \times \cdots \times K_r)$ in $(G_0 \times G_1 \times \cdots \times G_r)/\Gamma$.

Hence, the subspace $\mathfrak{m} = \mathfrak{k}^\perp$ of $\mathfrak{g}$ can be expressed as

$$\mathfrak{m} = \mathfrak{g}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r,$$

where $\mathfrak{m}_i = \mathfrak{k}_i^\perp$ on $(\mathfrak{g}_i, B_i)$, $i = 1, \ldots, r$. Then $(\exp tu)o$ with $u \in \mathfrak{m}_i$ is geodesic on $(M = G/K, g)$ and we can again apply Propositions 4.6 and 4.8 to obtain that $(M = G/K, g)$ must be locally symmetric. In this case, rank $G_i$ = rank $K_i$ and $Z_i$ is trivial, for $i = 1, \ldots r$, which implies that $(M = G/K, g)$ is moreover symmetric. Concretely, $(M, g)$ is given by

$$M = T \times M_1 \times \cdots \times M_r,$$

where $(T, g_0)$ is a complex flat torus and $(M_i, g_i)$, is one of the irreducible symmetric spaces given in Lemma 4.2 and the metric $g$ on $M$ is the product metric $g = g_0 + g_1 + \cdots + g_r$.

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Departamento de Matemática Fundamental
Sección de Geometría y Topología
Universidad de La Laguna
La Laguna, Spain
*E-mail address*: jcgonza@ull.es