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A SHARP ESTIMATE FOR THE BOTTOM OF THE SPECTRUM OF THE LAPLACIAN ON KÄHLER MANIFOLDS

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Abstract

On a complete noncompact Kähler manifold we prove that the bottom of the spectrum for the Laplacian is bounded from above by m^2 if the Ricci curvature is bounded from below by $-2(m+1)$. Then we show that if this upper bound is achieved then either the manifold is connected at infinity or it has two ends and in this case it is diffeomorphic to the product of the real line with a compact manifold and we determine the metric.

1. Introduction

In this paper we consider a complete noncompact Kähler manifold M with Ricci curvature bounded from below by a negative constant. Our goal is to prove a sharp upper bound estimate for the bottom of the spectrum of the Laplacian on M.

For Riemannian manifolds, a sharp upper bound estimate for λ_1 is well known from a theorem of S.Y. Cheng [**C**]. For Kähler manifolds, P. Li and J. Wang have recently proved a sharp upper bound estimate for λ_1 provided the more restrictive assumption of bisectional curvature bounded from below holds. We will improve their result here, using a different argument. Our technique will also be applied to study Kähler manifolds with maximal bottom of spectrum.

Let us state the results below. The proofs are given in the following sections. We first recall Cheng's upper bound estimate in the Riemannian setting, and a recent result of Li and Wang about the structure of manifolds with maximal bottom of spectrum.

Cheng's theorem states that the hyperbolic space $\mathbb{H}^n$ has the greatest bottom of spectrum among all Riemannian manifolds with Ricci curvature at least the Ricci curvature of $\mathbb{H}^n$. Thus, if a complete noncompact Riemannian manifold N^n of dimension n has Ricci curvature bounded below by $Ric_N \geq -(n-1)$, then the bottom of the spectrum of the

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Laplacian $\lambda_1(N)$ satisfies the sharp inequality $\lambda_1(N) \leq \frac{(n-1)^2}{4}$. Moreover, there are many other manifolds for which equality is achieved, e.g. see [**L**], [**S**].

Recently, Li and Wang [**L-W2**] have proved some remarkable results about the structure at infinity of manifolds with maximal λ_1. Assume that N^n with $n \geq 3$ is a complete Riemannian manifold such that $Ric_N \geq -(n-1)$ and $\lambda_1(N) = \frac{(n-1)^2}{4}$. Then either N is connected at infinity or it has two ends in which case it must either be

(1) a warped product $N = \mathbb{R} \times P$ with P compact and metric given by $ds_N^2 = dt^2 + \exp(2t)\, ds_P^2$, or

(2) if $n = 3$ a warped product $N = \mathbb{R} \times P$ with P compact and metric given by $ds_N^2 = dt^2 + \cosh^2(t)\, ds_P^2$.

We should also point out here that similar structure results were previously proved in [**L-W1**] for manifolds with infinite volume ends, where they generalized the work of Witten-Yau [**W-Y**], Cai-Galloway [**C-G**] and Wang [**W**].

In the Kähler category, a similar theory was developed in [**L-W3**] and [**L-W**], under the assumption of bisectional curvature lower bound. Consider M^m a complete noncompact Kähler manifold of complex dimension $m \geq 2$. Denote $ds^2 = h_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ the Kähler metric on M and let $Re\left(ds^2\right)$ be the Riemannian metric on M. Suppose $\{e_1, e_2, ..., e_{2m}\}$ with $e_{2k} = Je_{2k-1}$ for any $k \in \{1, ..., m\}$ is an orthonormal frame with respect to this Riemannian metric, then $\{v_1, ..., v_m\}$ is a unitary frame of $T_x^{1,0}M$, where

$$v_k = \frac{1}{2}\left(e_{2k-1} - \sqrt{-1} e_{2k}\right).$$

Recall that the bisectional curvature BK_M of M is defined by

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} = < R_{v_\alpha v_{\bar{\alpha}}} v_\beta, v_{\bar{\beta}} >$$

and we say that $BK_M \geq -1$ on M if for any α and β

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq -(1 + \delta_{\alpha\bar{\beta}}).$$

Note that for the space form $\mathbb{CH}^m$ we have $BK_{\mathbb{CH}^m} = -1$.

Theorem 1. ([**L-W3**]) *If M^m is a complete noncompact Kähler manifold of complex dimension $m \geq 2$ with $BK_M \geq -1$, then*

$$\lambda_1(M) \leq m^2 = \lambda_1\left(\mathbb{CH}^m\right).$$

Li-Wang proved this result using similar ideas to Cheng's proof in the Riemannian case. The bisectional curvature lower bound is used to deduce a Laplacian comparison theorem for Kähler manifolds. This is a powerful result that is more general than the upper bound estimate for $\lambda_1(M)$.

While it is not clear whether the Laplacian comparison is true for only Ricci curvature lower bound, the situation in the compact Kähler case motivates us to investigate if the sharp upper bound for $\lambda_1(M)$ remains true for Ricci curvature lower bound. Recall that in the compact Kähler case we have a version of Lichnerowicz's theorem. Namely, if for a compact Kähler manifold N^m the Ricci curvature has the lower bound $Ric_N \geq 2(m+1)$, then the first eigenvalue of the Laplacian has a sharp lower bound, $\lambda_1(N) \geq 4(m+1)$. We are grateful to Lei Ni for pointing out this result to us, for a simple proof of it see [**U**].

In this paper, our first goal is to show that indeed there is a sharp estimate for $\lambda_1(M)$ under Ricci curvature lower bound. To prove this, we will develop a new argument, that will prove a sharp integral estimate for the gradient of a certain class of harmonic functions. In fact, our argument can be localized on each end of the manifold.

Theorem 2. *Let M^m, $m \geq 2$ be a complete noncompact Kähler manifold such that the Ricci curvature is bounded from below by*

$$Ric_M \geq -2(m+1).$$

If E is an end of M and $\lambda_1(E)$ is the infimum of the Dirichlet spectrum of the Laplacian on E, then

$$\lambda_1(E) \leq m^2.$$

In particular, we have the sharp estimate

$$\lambda_1(M) \leq m^2.$$

Note that the condition on the Ricci curvature in Theorem 2 means

$$Ric(e_k, e_j) \geq -2(m+1)\,\delta_{kj}$$

for any $k, j \in \{1, .., 2m\}$, which is equivalent to

$$Ric_{\alpha\bar{\beta}} \geq -(m+1)\,\delta_{\alpha\bar{\beta}},$$

for the unitary frame $\{v_1, v_2, ..., v_m\}$. Let us point out that in the above theorem the Ricci curvature lower bound can be assumed to hold only on the end E. Theorem 2 will be proved in Section 2.

We now want to turn our attention to the question of connectedness at infinity of Kähler manifolds that have maximal bottom of spectrum. Using a beautiful argument involving the Buseman function Li and Wang have proved the following.

Theorem 3. ([**L-W**]) *If M^m, $m \geq 2$ is a complete noncompact Kähler manifold with $\lambda_1(M) = m^2$ and $BK_M \geq -1$ then either M has one end or M is diffeomorphic to the product of the real line with a compact manifold, $\mathbb{R} \times N$ with the metric*

$$ds_M^2 = dt^2 + e^{-4t}\omega_2^2 + e^{-2t}\left(\omega_3^2 + .. + \omega_{2m}^2\right),$$

where $\{\omega_2, .., \omega_{2m}\}$ is an orthonormal coframe for N. Moreover N is a compact quotient of the Heisenberg group and $\widetilde{M}$ is isometric to $\mathbb{CH}^m$ in the event that M has bounded curvature.

Our second goal in this paper is to obtain the same conclusion if equality is achieved in Theorem 2. This will be done by a more careful analysis of the estimates in Theorem 2 applied to a harmonic function that is defined by Li-Tam theory. The following result will be proved in Section 3.

Theorem 4. *Let M^m be a complete noncompact Kähler manifold of complex dimension $m \geq 2$ such that the Ricci curvature is bounded from below by*

$$Ric_M \geq -2\,(m+1)\,.$$

If $\lambda_1(M) = m^2$ then either M is connected at infinity or it is diffeomorphic to $\mathbb{R} \times N$ with the metric

$$ds_M^2 = dt^2 + e^{-4t}\omega_2^2 + e^{-2t}\left(\omega_3^2 + .. + \omega_{2m}^2\right),$$

where $\{\omega_2, .., \omega_{2m}\}$ is an orthonormal coframe for the compact manifold N. Moreover, $\widetilde{M}$ is isometric to $\mathbb{CH}^m$ and N is a compact quotient of the Heisenberg group provided M has bounded curvature.

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2. Proof of the sharp estimate

In this section we prove Theorem 2. Let us outline the main steps of the proof first.

1) We first establish a sharp integral gradient estimate for a class of harmonic functions on any given end E. These functions have different constructions and properties, depending on the case when E has infinite or finite volume. We therefore discuss these two cases separately. We prove the integral gradient estimate in Lemma 1, where the goal is to estimate from above and from below an integral involving the complex Hessian of a harmonic function. The estimate from above is more technical and it is based on repeated integration by parts and use of the Ricci identities for Kähler manifolds. The estimate from below will be obtained from the Cauchy-Schwarz inequality.
2) We then prove another integral estimate using the Poincaré inequality for $\lambda_1(E)$. This is the converse to the integral gradient estimate and will be proved in Lemma 2.
3) Finally, Theorem 2 will follow from the two steps described above.

We now discuss the proof in detail. Let E be an end of M. Without loss of generality we will henceforth assume that $\lambda_1(E) > 0$. Recall that $\lambda_1(E)$ is the infimum of the Dirichlet spectrum of the Laplacian on E, hence $\lambda_1(E)$ satisfies the following Poincaré inequality:

$$\lambda_1(E)\int_E h^2 \leq \int_E |\nabla h|^2\,,$$

for all $C^\infty(E)$ functions h with compact support in E.

First, consider the case when E has infinite volume. Since $\lambda_1(E) > 0$, this is equivalent to E being nonparabolic, i.e. there exists a positive symmetric Green's function on E satisfying the Neumann boundary condition on ∂E, see [**L-W1**].

We will use a construction of Li and Tam ([**L-T**]) that defines a bounded harmonic function and with finite Dirichlet integral f on E. This is done as follows.

Let f_R be the harmonic function with Dirichlet boundary conditions: $f_R = 1$ on ∂E, $f_R = 0$ on $\partial E_p(R)$, where $E_p(R) = E \cap B_p(R)$.

Then it can be showed that f_R admits a subsequence convergent uniformly on compact sets to a harmonic function f, with the properties: $0 < f < 1$ on E, $f = 1$ on ∂E and f has finite Dirichlet integral. Moreover, since $\lambda_1(E) > 0$, we know by a theorem of Li and Wang that ([**L-W1**])

$$\int_{E_p(R+1)\setminus E_p(R)} f^2 \leq c_1 \exp\left(-2\sqrt{\lambda_1(E)}R\right).$$

Integration on the level sets of f and the co-area formula will play an important role in our proofs, and for this let us recall the following property of f ([**L-W4**]).

For $t, a, b < 1$ let us consider

$$l(t) = \{x \in E \mid f(x) = t\}$$

and define the set

$$L(a,b) = \{x \in E \mid a < f(x) < b\}.$$

Notice that in general $L(a,b)$ or $l(t)$ might not be compact subsets of E. However, it can be proved that there exists a constant C such that for almost all $t < 1$

$$\int_{l(t)} |\nabla f| = C < \infty$$

and we have:

$$\int_{L(a,b)} |\nabla f|^2 = (b-a)\int_{l(t_0)} |\nabla f|\,.$$

That $\int_{l(t)} |\nabla f|$ is finite and independent of t follows from the fact that f is harmonic and $\int_E |\nabla f|^2 < \infty$, see [**L-W4**] for details. The second property follows using co-area formula.

Let us now denote

$$L = L\left(\frac{1}{2}\delta\varepsilon, 2\varepsilon\right),$$

where $\delta, \varepsilon > 0$ are sufficiently small fixed numbers to be chosen later.

Since we will use integration by parts on L let us construct a cut-off ϕ with compact support in L. Define $\phi = \psi\varphi$ with ψ depending on the distance function

$$\psi = \begin{cases} 1 & \text{on} \quad E_p(R-1) \\ R-r & \text{on} \quad E_p(R) \backslash E_p(R-1) \\ 0 & \text{on} \quad E \backslash E_p(R) \end{cases}$$

and φ defined on the level sets of f

$$\varphi = \begin{cases} (\log 2)^{-1}\left(\log f - \log\left(\frac{1}{2}\delta\varepsilon\right)\right) & \text{on } \ L\left(\frac{1}{2}\delta\varepsilon, \delta\varepsilon\right) \\ 1 & \text{on } L(\delta\varepsilon, \varepsilon) \\ (\log 2)^{-1}(\log 2\varepsilon - \log f) & \text{on } \ L(\varepsilon, 2\varepsilon) \\ 0 & \text{otherwise.} \end{cases}$$

For convenience, let us assume $R = \frac{1}{\delta\varepsilon}$. We have the following result:

Lemma 1. *The following inequality holds for any ε and δ positive:*

$$\frac{1}{(-\log\delta)} \int_L \frac{|\nabla f|^4}{f^3}\phi^2 \leq 4m^2 \int_{l(t_0)} |\nabla f| + \frac{c}{(-\log\delta)^{\frac{1}{2}}},$$

where c is a constant not depending on δ or ε.

Proof. Note that the gradient and the Laplacian satisfy:

$$\begin{aligned} \nabla f \cdot \nabla h &= 2\left(f_\alpha h_{\bar{\alpha}} + f_{\bar{\alpha}} h_\alpha\right) \\ \Delta f &= 4 f_{\alpha\bar{\alpha}}. \end{aligned}$$

Everywhere in the paper we use the Einstein summation convention and the formulas are with respect to any unitary frame $\{v_\alpha\}$.

Let $u = \log f$, then a simple computation shows that

$$u_{\alpha\bar{\beta}} = f^{-1} f_{\alpha\bar{\beta}} - f^{-2} f_\alpha f_{\bar{\beta}}.$$

Consider now

$$\int_L f \left|u_{\alpha\bar{\beta}}\right|^2 \phi^2$$

which we estimate from above and from below to prove our claim. First, we pause to explain this choice of function. The goal is to obtain a gradient estimate for f, and in this sense it is standard to look for estimates of $|\nabla u|^2$ from above, where $u = \log f$. Certainly, we want to use the Kähler structure of M, thus we observe that $|\nabla u|^2 = -\Delta u$ and then estimate

$$(\Delta u)^2 \leq 16m \left|u_{\alpha\bar{\beta}}\right|^2,$$

using Cauchy-Schwarz inequality. This justifies the need to estimate the norm of the complex Hessian of u from above. Using co-area formula, it is more convenient to work on level sets of f than on geodesic balls on M, and this philosophy is used throughout the paper. The quantity

$f\left|u_{\alpha\bar\beta}\right|^2$ is preferred instead of $\left|u_{\alpha\bar\beta}\right|$ because it is more suitable for an argument based on integration by parts.

We now start our estimates. To begin with, notice that

$$\begin{aligned}\int_L f\left|u_{\alpha\bar\beta}\right|^2\phi^2 &= \int_L f^{-1}\left|f_{\alpha\bar\beta}\right|^2\phi^2 - 2\int_L f^{-2}(f_{\alpha\bar\beta}f_{\bar\alpha}f_\beta)\phi^2 \\ &\quad +\frac{1}{16}\int_L f^{-3}\left|\nabla f\right|^4\phi^2.\end{aligned}$$

In view of our discussion above, the objective is to estimate the right hand side in terms only of the gradient of f. The first term is computed as follows:

$$\begin{aligned}\int_L f^{-1}\left|f_{\alpha\bar\beta}\right|^2\phi^2 &= \int_L f^{-1}(f_{\alpha\bar\beta}\cdot f_{\bar\alpha\beta})\phi^2 = -\int_L f_\alpha\left(f^{-1}f_{\bar\alpha\beta}\phi^2\right)_{\bar\beta} \\ &= \int_L f^{-2}(f_{\bar\alpha\beta}f_\alpha f_{\bar\beta})\phi^2 - \int_L f^{-1}f_\alpha f_{\bar\alpha\beta\bar\beta}\phi^2 \\ &\quad - \int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^2\right)_{\bar\beta}\end{aligned}$$

and using the Ricci identities and $\Delta f = 0$ we see that $f_{\bar\alpha\beta\bar\beta} = 0$. It also shows that the last integral needs to be a real number.

This proves that we have the following formula

$$\begin{aligned}\int_L f\left|u_{\alpha\bar\beta}\right|^2\phi^2 &= -\int_L f^{-2}(f_{\alpha\bar\beta}f_{\bar\alpha}f_\beta)\phi^2 \\ &\quad +\frac{1}{16}\int_L f^{-3}\left|\nabla f\right|^4\phi^2 - \int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^2\right)_{\bar\beta}.\end{aligned}\tag{1}$$

We will now use again integration by parts to see that

$$\begin{aligned}-\int_L f^{-2}(f_{\alpha\bar\beta}f_{\bar\alpha}f_\beta)\phi^2 &= \int_L f_\alpha\left(f^{-2}f_{\bar\alpha}f_\beta\phi^2\right)_{\bar\beta} \\ &= -2\int_L f^{-3}f_\alpha f_{\bar\alpha}f_\beta f_{\bar\beta}\phi^2 + \int_L f^{-2}f_{\bar\alpha\bar\beta}f_\alpha f_\beta\phi^2 \\ &\quad +\int_L f^{-2}f_\alpha f_{\bar\alpha}f_\beta\left(\phi^2\right)_{\bar\beta}.\end{aligned}$$

Similarly, one finds

$$\begin{aligned}-\int_L f^{-2}(f_{\alpha\bar\beta}f_{\bar\alpha}f_\beta)\phi^2 &= -\int_L f^{-2}(f_{\bar\alpha\beta}f_\alpha f_{\bar\beta})\phi^2 \\ &= \int_L f_{\bar\alpha}\left(f^{-2}f_\alpha f_{\bar\beta}\phi^2\right)_\beta \\ &= -2\int_L f^{-3}f_\alpha f_{\bar\alpha}f_\beta f_{\bar\beta}\phi^2 + \int_L f^{-2}f_{\alpha\beta}f_{\bar\alpha}f_{\bar\beta}\phi^2 \\ &\quad +\int_L f^{-2}f_\alpha f_{\bar\alpha}f_{\bar\beta}\left(\phi^2\right)_\beta.\end{aligned}$$

Combining the two identities we get

$$-\int_L f^{-2}(f_{\alpha\bar{\beta}}f_{\bar{\alpha}}f_{\beta})\phi^2 = -\frac{1}{8}\int_L f^{-3}\left|\nabla f\right|^4\phi^2 \tag{2}$$
$$+\int_L f^{-2}Re\left(f_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right)\phi^2 + \frac{1}{4}\int_L f^{-2}\left|\nabla f\right|^2 Re(f_{\bar{\beta}}\left(\phi^2\right)_{\beta}).$$

Note that the following inequality holds on E:

$$\left|f_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right| \le \frac{1}{4}\left|f_{\alpha\beta}\right|\left|\nabla f\right|^2 \tag{3}$$

We want to include the proof of this inequality because it will matter when we study the manifolds with $\lambda_1(M) = m^2$. Since the two numbers in (3) are independent of the unitary frame, let us choose an orthonormal frame at the fixed point $x \in E$ such that

$$e_1 = \frac{1}{|\nabla f|}\nabla f.$$

Certainly, we need $|\nabla f|(x) \neq 0$ which we assume without loss of generality because if $|\nabla f|(x) = 0$ there is nothing to prove.

Then one can see that

$$f_{e_1} = |\nabla f|,\ f_{e_2} = 0, ..., f_{e_{2m}} = 0$$

or, in the unitary frame

$$f_1 = f_{\bar{1}} = \frac{1}{2}|\nabla f|, \quad f_\alpha = f_{\bar{\alpha}} = 0 \text{ if } \alpha > 1.$$

This proves the inequality because

$$\left|f_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right| = \frac{1}{4}\left|\nabla f\right|^2\left|f_{11}\right| \le \frac{1}{4}\left|f_{\alpha\beta}\right|\left|\nabla f\right|^2.$$

Moreover, we learn that equality holds in (3) if and only if

$$f_{\alpha\beta} = 0 \text{ for } (\alpha,\beta) \neq (1,1),$$

with respect to the unitary frame chosen above.

Since the following holds:

$$Re\left(f_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right) \le \left|f_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right| \le \frac{1}{4}\left|f_{\alpha\beta}\right|\left|\nabla f\right|^2,$$

we get using the arithmetic mean inequality that

$$2\int_L f^{-2}Re\left(f_{\bar{\alpha}\bar{\beta}}f_{\alpha}f_{\beta}\right)\phi^2 \tag{4}$$
$$\le \int_L 2\left(f^{-1/2}\left|f_{\alpha\beta}\right|\phi\right)\left(\frac{1}{4}f^{-3/2}\left|\nabla f\right|^2\phi\right)$$
$$\le \frac{m}{m+1}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2 + \frac{m+1}{16m}\int_L f^{-3}\left|\nabla f\right|^4\phi^2.$$

Moreover, again integrating by parts we have

$$\begin{aligned}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2 &= \int_L f^{-1} f_{\alpha\beta} f_{\bar\alpha\bar\beta}\phi^2 = -\int_L f_\alpha\left(f^{-1}f_{\bar\alpha\bar\beta}\phi^2\right)_\beta \\ &= \int_L f^{-2} f_{\bar\alpha\bar\beta} f_\alpha f_\beta\phi^2 - \int_L f^{-1} f_\alpha f_{\bar\alpha\bar\beta\beta}\phi^2 \\ &- \int_L f^{-1} f_\alpha f_{\bar\alpha\bar\beta}(\phi^2)_\beta\end{aligned}$$

and on the other hand

$$\begin{aligned}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2 &= \int_L f^{-1} f_{\alpha\beta} f_{\bar\alpha\bar\beta}\phi^2 = -\int_L f_{\bar\alpha}\left(f^{-1}f_{\alpha\beta}\phi^2\right)_{\bar\beta} \\ &= \int_L f^{-2} f_{\alpha\beta} f_{\bar\alpha} f_{\bar\beta}\phi^2 - \int_L f^{-1} f_{\bar\alpha} f_{\alpha\beta\bar\beta}\phi^2 \\ &- \int_L f^{-1} f_{\bar\alpha} f_{\alpha\beta}\left(\phi^2\right)_{\bar\beta}\end{aligned}$$

so that combining the two identities we get

$$\begin{aligned}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2 &= \int_L f^{-2} Re(f_{\bar\alpha\bar\beta} f_\alpha f_\beta)\phi^2 - \int_L f^{-1} f_\alpha f_{\bar\alpha\bar\beta\beta}\phi^2 \\ &\quad -\int_L f^{-1} Re(f_\alpha f_{\bar\alpha\bar\beta}(\phi^2)_\beta).\end{aligned}$$

Note that the Ricci identities imply

$$f_{\bar\alpha\bar\beta\beta} = f_{\bar\beta\bar\alpha\beta} = f_{\bar\beta\beta\bar\alpha} + Ric_{\beta\bar\alpha} f_{\bar\beta} = Ric_{\beta\bar\alpha} f_{\bar\beta}$$

and therefore we have proved that

$$\begin{aligned}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2 &\leq \int_L f^{-2} Re(f_{\bar\alpha\bar\beta} f_\alpha f_\beta)\phi^2 + \frac{m+1}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^2 \\ &\quad -\int_L f^{-1} Re(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^2\right)_\beta).\end{aligned}$$

Plug this inequality into (4) and it follows that

$$\begin{aligned}&\frac{m+2}{m+1}\int_L f^{-2} Re\left(f_{\bar\alpha\bar\beta} f_\alpha f_\beta\right)\phi^2 \leq \frac{m}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^2 \\ &+\frac{m+1}{16m}\int_L f^{-3}\left|\nabla f\right|^4\phi^2 - \frac{m}{m+1}\int_L f^{-1} Re(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^2\right)_\beta).\end{aligned}$$

Now use this inequality in (2) and obtain

$$\begin{aligned}(5)\quad &-\int_L f^{-2}(f_{\alpha\bar\beta} f_{\bar\alpha} f_\beta)\phi^2 \leq \left(-\frac{1}{8} + \frac{(m+1)^2}{16m(m+2)}\right)\int_L f^{-3}\left|\nabla f\right|^4\phi^2 \\ &+\frac{m\,(m+1)}{4\,(m+2)}\int_L f^{-1}\left|\nabla f\right|^2\phi^2 - \frac{m}{m+2}\int_L f^{-1} Re(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^2\right)_\beta) \\ &+\frac{1}{4}\int_L f^{-2}\left|\nabla f\right|^2 Re(f_{\bar\beta}\left(\phi^2\right)_\beta).\end{aligned}$$

Summing up, we have thus proved that

$$\begin{aligned}(6)\qquad \int_L f\left|u_{\alpha\bar\beta}\right|^2\phi^2 &\le \frac{1}{16}\frac{1}{m\,(m+2)}\int_L f^{-3}\left|\nabla f\right|^4\phi^2\\ &+\frac{m\,(m+1)}{4\,(m+2)}\int_L f^{-1}\left|\nabla f\right|^2\phi^2+\frac{1}{4}\int_L f^{-2}\left|\nabla f\right|^2 Re(f_{\bar\beta}\left(\phi^2\right)_\beta)\\ &-\int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^2\right)_{\bar\beta}-\frac{m}{m+2}\int_L f^{-1}Re(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^2\right)_\beta).\end{aligned}$$

To finish the upper estimate of $\int_L f\left|u_{\alpha\bar\beta}\right|^2\phi^2$ we need to estimate the terms involving $\left(\phi^2\right)_\beta$. We will prove that they can be bounded from above by a constant multiple of $(-\log\delta)^{1/2}$.

Start with

$$\begin{aligned}2\int_L f^{-2}\left|\nabla f\right|^2 Re(f_{\bar\beta}\left(\phi^2\right)_\beta) &\le \frac{1}{2}\int_L f^{-2}\left|\nabla f\right|^3\left|\nabla\phi^2\right|\\ &\le \int_L f^{-2}\left|\nabla f\right|^3\left|\nabla\varphi\right|\psi+\int_L f^{-2}\left|\nabla f\right|^3\left|\nabla\psi\right|\varphi.\end{aligned}$$

Now it is easy to see that by the gradient estimate and co-area formula

$$\begin{aligned}\int_L f^{-2}\left|\nabla f\right|^3\left|\nabla\varphi\right| &\le c_2\left(\int_{L(\frac{1}{2}\delta\varepsilon,\delta\varepsilon)} f^{-1}\left|\nabla f\right|^2+\int_{L(\varepsilon,2\varepsilon)} f^{-1}\left|\nabla f\right|^2\right)\\ &\le c_{3,}\end{aligned}$$

while by the decay rate of f^2 and the fact that $R=\frac{1}{\delta\varepsilon}$ we get

$$\begin{aligned}\int_L f^{-2}\left|\nabla f\right|^3\left|\nabla\psi\right| &\le c_4\int_{L\cap(E_p(R)\setminus E_p(R-1))} f\le\frac{2c_4}{\delta\varepsilon}\int_{E_p(R)\setminus E_p(R-1)} f^2\\ &\le\frac{2c_1c_4}{\delta\varepsilon}\exp\left(-2\sqrt{\lambda_1(E)}R\right)\le c_5.\end{aligned}$$

Clearly, the constants so far do not depend on the choice of δ or ε.

To estimate the other terms one proceeds similarly. For example,

$$\begin{aligned}-2\int_L f^{-1}Re(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^2\right)_\beta) &\le \int_L f^{-1}\left|f_{\bar\alpha\bar\beta}\right|\left|\nabla f\right|\phi\left|\nabla\phi\right|\\ &\le\left(\int_L f^{-1}\left|\nabla f\right|^2\left|\nabla\phi\right|^2\right)^{\frac{1}{2}}\left(\int_L f^{-1}\left|f_{\bar\alpha\bar\beta}\right|^2\phi^2\right)^{\frac{1}{2}}\\ &\le c_6\left(\int_L f^{-1}\left|f_{\bar\alpha\bar\beta}\right|^2\phi^2\right)^{\frac{1}{2}}.\end{aligned}$$

However, using an inequality proved above we get

$$\begin{aligned}
\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2 &\leq \int_L f^{-2}Re(f_{\bar\alpha\bar\beta}f_\alpha f_\beta)\phi^2+\frac{m+1}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^2\\
&\quad -\int_L f^{-1}Re(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^2\right)_\beta)\\
&\leq \frac{1}{4}\int_L f^{-2}\left|f_{\alpha\beta}\right|\left|\nabla f\right|^2\phi^2+\frac{m+1}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^2\\
&\quad +\frac{1}{2}\int_L f^{-1}\left|f_{\alpha\beta}\right|\left|\nabla f\right|\phi\left|\nabla\phi\right|\\
&\leq \frac{1}{8}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2+\frac{1}{8}\int_L f^{-3}\left|\nabla f\right|^4\phi^2\\
&\quad +\frac{m+1}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^2\\
&\quad +\frac{1}{4}\int_L f^{-1}\left|f_{\alpha\beta}\right|^2\phi^2+\frac{1}{4}\int_L f^{-1}\left|\nabla f\right|^2\left|\nabla\phi\right|^2,
\end{aligned}$$

which shows there exists constants c_7 and c_8 such that:

$$\begin{aligned}
\int_L f^{-1}\left|f_{\bar\alpha\bar\beta}\right|^2\phi^2 &\leq c_7\int_L f^{-1}\left|\nabla f\right|^2\phi^2+c_8\int_L f^{-1}\left|\nabla f\right|^2\left|\nabla\phi\right|^2\\
&\leq c_9\left(-\log\delta\right).
\end{aligned}$$

We have proved that

$$\int_L f^{-1}\left|f_{\bar\alpha\bar\beta}\right|\left|\nabla f\right|\phi\left|\nabla\phi\right|\leq c_{10}(-\log\delta)^{\frac{1}{2}}.$$

It should be noted that in fact this estimate can be improved to bound the left hand side simply by a constant, thus eliminating any dependence on δ. However, the estimate that we proved above is sufficient for our needs in this paper. Let us gather the information we have so far:

$$\begin{aligned}
\int_L f\left|u_{\alpha\bar\beta}\right|^2\phi^2 &\leq \frac{1}{16}\frac{1}{m\left(m+2\right)}\int_L f^{-3}\left|\nabla f\right|^4\phi^2\\
&\quad +\frac{m\left(m+1\right)}{4\left(m+2\right)}\int_L f^{-1}\left|\nabla f\right|^2\phi^2+c(-\log\delta)^{\frac{1}{2}}.
\end{aligned}$$

The estimate from below is easier to get:

$$\left|u_{\alpha\bar\beta}\right|^2\geq\sum_\alpha\left|u_{\alpha\bar\alpha}\right|^2\geq\frac{1}{m}\left|\sum_\alpha u_{\alpha\bar\alpha}\right|^2=\frac{1}{16m}f^{-4}\left|\nabla f\right|^4. \tag{7}$$

Hence, this shows that

$$\begin{aligned}
\frac{1}{16}\frac{m+1}{m\left(m+2\right)}\int_L f^{-3}\left|\nabla f\right|^4\phi^2 &\leq \frac{m\left(m+1\right)}{4\left(m+2\right)}\int_L f^{-1}\left|\nabla f\right|^2\phi^2\\
&\quad +c(-\log\delta)^{\frac{1}{2}},
\end{aligned}$$

which by co-area formula proves the Lemma. q.e.d.

In the following Lemma, we will estimate $\int_L f^{-3} |\nabla f|^4 \phi^2$ from below. To serve our purpose, we need this estimate to depend on $\lambda_1(E)$ and this is done using the variational principle. Recall that E is an infinite volume end, $\lambda_1(E) > 0$ and we set $L = L\left(\frac{1}{2}\delta\varepsilon, 2\varepsilon\right)$ for δ, ε sufficiently small.

Lemma 2. *The following inequality holds for any δ and ε positive:*

$$\frac{1}{(-\log \delta)} \int_L f^{-3} |\nabla f|^4 \phi^2 \geq 4\lambda_1(E) \int_{l(t_0)} |\nabla f| - \frac{c_0}{(-\log \delta)^{\frac{1}{2}}}.$$

Proof. By the variational principle for $\lambda_1(E)$,

$$\lambda_1(E) \int_E f\phi^2 \leq \int_E \left|\nabla\left(\phi f^{\frac{1}{2}}\right)\right|^2,$$

which means that

$$\begin{aligned} \lambda_1(E) \int_L f\phi^2 &\leq \frac{1}{4} \int_L f^{-1} |\nabla f|^2 \phi^2 + \int_L f |\nabla \phi|^2 + \int_L \phi |\nabla f| |\nabla \phi| \\ &\leq \frac{1}{4} \int_{L(\delta\varepsilon,\varepsilon)} f^{-1} |\nabla f|^2 + c_{11}, \end{aligned}$$

based on estimates similar to what we did in Lemma 1.

This implies that

$$\frac{1}{(-\log \delta)} \int_L f\phi^2 \leq \frac{1}{4\lambda_1(E)} \int_{l(t_0)} |\nabla f| + \frac{c_{11}}{(-\log \delta)}.$$

Finally, using the Cauchy-Schwarz inequality and the co-area formula we get

$$\begin{aligned} \int_{l(t_0)} |\nabla f| &= \frac{1}{(-\log \delta)} \int_{L(\delta\varepsilon,\varepsilon)} f^{-1} |\nabla f|^2 \\ &\leq \frac{1}{(-\log \delta)} \int_L f^{-1} |\nabla f|^2 \phi^2 \\ &+ \frac{1}{(-\log \delta)} \int_{L\cap(E\setminus E_p(R-1))} f^{-1} |\nabla f|^2 \\ &\leq \left(\frac{1}{(-\log \delta)} \int_L f^{-3} |\nabla f|^4 \phi^2\right)^{\frac{1}{2}} \left(\frac{1}{(-\log \delta)} \int_L f\phi^2\right)^{\frac{1}{2}} \\ &\quad + \frac{c_{12}}{(-\log \delta)} \frac{1}{\delta\varepsilon} \exp\left(-2\sqrt{\lambda_1(E)}R\right) \\ &\leq \left(\frac{1}{(-\log \delta)} \int_L f^{-3} |\nabla f|^4 \phi^2\right)^{\frac{1}{2}} \times \\ &\quad \times \left(\frac{1}{4\lambda_1(E)} \int_{l(t_0)} |\nabla f| + \frac{c_{11}}{(-\log \delta)}\right)^{\frac{1}{2}} + \frac{c_{13}}{(-\log \delta)}, \end{aligned}$$

which proves the Lemma. q.e.d.

The results we proved above hold for any infinite volume end E.

Let us now discuss the case when our end has finite volume. Let F be a finite volume end with $\lambda_1(F) > 0$. This is known to be equivalent to F being parabolic.

A theorem of Nakai ([**N**], see also [**N-R**]) states that there exists an exhaustion function f on $\bar{F}$ which is harmonic on F and $f = 0$ on ∂F. In this case we consider for $T, \gamma > 0$ fixed

$$\phi = \begin{cases} (\log 2)^{-1}\left(\log f - \log\left(\frac{1}{2}T\right)\right) & \text{on } L\left(\frac{1}{2}T, T\right) \\ 1 & \text{on } L(T, \gamma T) \\ (\log 2)^{-1}(\log(2\gamma T) - \log f) & \text{on } L(\gamma T, 2\gamma T) \\ 0 & \text{otherwise,} \end{cases}$$

where the level sets are now defined on F. Since f is proper, there is no need for a cut-off depending on the distance function because now the level sets of f are compact in F. Our point now is that Lemma 1 and Lemma 2 hold for this choice of ϕ also, the proofs are identical. Note that if

$$\tilde{L} = L\left(\frac{1}{2}T, 2\gamma T\right)$$

then the following inequalities hold on $\tilde{L}$:

$$\begin{aligned} \frac{1}{\log\gamma}\int_{\tilde{L}} \frac{|\nabla f|^4}{f^3}\phi^2 \quad &\leq \quad 4m^2 \int_{l(t_0)} |\nabla f| \\ &\quad + \frac{\widetilde{c}}{(\log\gamma)^{\frac{1}{2}}}, \end{aligned}$$

and

$$\frac{1}{\log\gamma}\int_{\tilde{L}} f^{-3}|\nabla f|^4 \phi^2 \geq 4\lambda_1(F)\int_{l(t_0)} |\nabla f| - \frac{\widetilde{c}_0}{(\log\gamma)^{\frac{1}{2}}}.$$

We are now in position to prove Theorem 2.

Proof of Theorem 2. We know from Lemma 1 and Lemma 2 that

$$\lambda_1(E)\int_{l(t_0)} |\nabla f| \leq m^2 \int_{l(t_0)} |\nabla f| + \frac{C}{(-\log\delta)^{\frac{1}{2}}},$$

inequality that holds for any $\delta > 0$. The claim then follows by making $\delta \to 0$.

This proves Theorem 2 for an infinite volume end E. The proof for a finite volume end F is similar. q.e.d.

3. Manifolds with maximal λ_1

We now prove Theorem 4.

Suppose that M has more than one end. We know ([**L-W**]) that if $\lambda_1(M) > \frac{m+1}{2}$ the manifold has only one infinite volume end. Hence let us denote this infinite volume end by E and, consequently, $F = M\backslash E$ is a finite volume end.

The construction of Li-Tam implies that there exists a harmonic function $f : M \to (0, \infty)$ with the following properties:

1. On E the function has the decay rate
$$\int_{E_p(R)\backslash E_p(R-1)} f^2 \leq c_1 \exp\left(-2\sqrt{\lambda_1(M)}R\right),$$
2. On F the function is proper.
3. We have:
$$\sup_{x\in F} f(x) = \infty, \quad \inf_{x\in E} f(x) = 0.$$

Let us highlight some facts about the proofs of Lemma 1 and Lemma 2. In the two lemmata, the function f was defined only on a single end, which was first assumed to be of infinite volume, and then we observed that the proofs still work on a finite volume end. In the framework of Theorem 4, we know that f is defined on the whole manifold, so now $L = L(b_0, b_1) = \{x \in M \mid b_0 < f(x) < b_1\}$ makes sense for any $0 < b_0 < b_1$. One can see that the computations proved in Lemma 1 are true for L and moreover we may replace ϕ^2 with ϕ^3 everywhere. In fact, the argument in Lemma 1 is based on repeated integration by parts on a set L where a cut-off function is given, therefore taking ϕ^3 does not change the argument.

With this in mind, let us fix $b_0 = \delta\varepsilon$, $b_1 = \gamma T$, where $0 < \delta\varepsilon < \varepsilon < T < \gamma T$ and for convenience choose $\gamma = \frac{1}{\delta}$. Hence, everywhere in this proof
$$L = L(\delta\varepsilon, \gamma T).$$
The proof of this theorem is based on a more detailed study of the inequalities in Lemma 1 and Lemma 2. We want to prove that $\lambda_1(M) = m^2$ forces all the inequalities to become equalities on $L(\varepsilon, T)$. Since ε, T are arbitrary, it will follow that we need to have equalities everywhere on M. Then studying these equalities we infer that the structure of M is as stated in Theorem 4. Choose $\phi = \varphi\psi$, where
$$\psi = \begin{cases} 1 & \text{on} \quad E_p(R-1) \cup F \\ R - r & \text{on} \quad E_p(R)\backslash E_p(R-1) \\ 0 & \text{on} \quad E\backslash E_p(R) \end{cases}$$
and
$$\varphi = \begin{cases} (-\log\delta)^{-1}(\log f - \log(\delta\varepsilon)) & \text{on} \quad L(\delta\varepsilon, \varepsilon) \\ 0 & \text{on} \quad L(0, \delta\varepsilon) \cup (L(\gamma T, \infty) \cap F) \\ (\log\gamma)^{-1}(\log(\gamma T) - \log f) & \text{on} \quad L(T, \gamma T) \cap F \\ 1 & \text{otherwise.} \end{cases}$$
Let us emphasize again that we need to consider ψ on the infinite volume end because $L \cap E$ might not be compact in E, whereas on F we can

take $\psi = 1$ because $L \cap F$ is compact in F as it follows from Nakai's theorem.

Recall that by (6) and (7) we have

$$
\begin{aligned}
(8) \qquad \frac{1}{16}\frac{m+1}{m(m+2)}\int_L f^{-3}|\nabla f|^4\phi^3 &\le \frac{m(m+1)}{4(m+2)}\int_L f^{-1}|\nabla f|^2\phi^3 \\
&+\frac{1}{4}\int_L f^{-2}|\nabla f|^2 Re(f_{\bar\beta}(\phi^3)_\beta) - \int_L f^{-1} f_{\bar\alpha\beta} f_\alpha (\phi^3)_{\bar\beta} \\
&-\frac{m}{m+2}\int_L f^{-1} Re(f_\alpha f_{\bar\alpha\bar\beta}(\phi^3)_\beta).
\end{aligned}
$$

On the other hand, Cauchy-Schwarz inequality implies

$$
(9) \qquad \left(\int_L f^{-1}|\nabla f|^2\phi^3\right)^2 \le \left(\int_L f^{-3}|\nabla f|^4\phi^3\right)\left(\int_L f\phi^3\right),
$$

and by the variational principle it follows that

$$
\begin{aligned}
\lambda_1(M)\int_L f\phi^3 &\le \int_L\left|\nabla\left(f^{\frac12}\phi^{\frac32}\right)\right|^2 \\
&= \frac{1}{4}\int_L f^{-1}|\nabla f|^2\phi^3 + \frac{9}{4}\int_L f\phi|\nabla\phi|^2 \\
&\quad +\frac{3}{2}\int_L \phi^2\nabla f\cdot\nabla\phi.
\end{aligned}
$$

Our point now is that a careful study of the two $\nabla\phi-$terms shows that they converge to zero as $\gamma\to\infty$ (and $\delta=\frac{1}{\gamma}\to 0$) and $R\to\infty$.

It is clear that $\frac{9}{4}\int_L f\phi|\nabla\phi|^2 \le \frac{c_1}{\log\gamma}$, while

$$
\begin{aligned}
\int_L \phi^2\nabla f\cdot\nabla\phi &= \frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}|\nabla f|^2\phi^2 \\
&\quad -\frac{1}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-1}|\nabla f|^2\phi^2.
\end{aligned}
$$

The integral on F is readily found by the co-area formula:

$$
\begin{aligned}
\frac{1}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-1}|\nabla f|^2\phi^2 &= \left(\int_{l(t_0)}|\nabla f|\right)\times \\
&\quad \times\int_T^{\gamma T}\frac{1}{t}\frac{(\log(\gamma T)-\log t)^2}{(\log\gamma)^3}dt \\
&= \frac{1}{3}\int_{l(t_0)}|\nabla f|.
\end{aligned}
$$

It is clear that the same formula holds on E if we integrate against φ^2 and therefore:

$$\begin{aligned}\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}\left|\nabla f\right|^2\phi^2 &\leq \frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} \varphi^2\left|\nabla f\right|^2 f^{-1} \\ &= \frac{1}{3}\int_{l(t_0)}\left|\nabla f\right|.\end{aligned}$$

For later use, observe that a converse of the latter inequality also holds:

$$\begin{aligned}\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}\left|\nabla f\right|^2\phi^2 &\geq \frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}\left|\nabla f\right|^2\varphi^2 \\ &\quad -\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)\cap(E\setminus E_p(R-1))} f^{-1}\left|\nabla f\right|^2\varphi^2 \\ &\geq \frac{1}{3}\int_{l(t_0)}\left|\nabla f\right| - \frac{c_2}{(-\log\delta)}.\end{aligned}$$

In particular, from the above estimates it follows that

$$\int_L \phi^2\nabla f\cdot\nabla\phi \leq 0.$$

We have thus proved that

$$\lambda_1(M)\int_L f\phi^3 \leq \frac{1}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^3 + \frac{c_1}{\log\gamma},$$

which plugged into (9) yields

$$\begin{aligned}\int_L f^{-3}\left|\nabla f\right|^4\phi^3 &\geq 4\lambda_1(M)\frac{\left(\int_L f^{-1}\left|\nabla f\right|^2\phi^3\right)^2}{\int_L f^{-1}\left|\nabla f\right|^2\phi^3 + \frac{c_3}{\log\gamma}} \\ &= 4\lambda_1(M)\int_L f^{-1}\left|\nabla f\right|^2\phi^3 - \frac{c_4}{\log\gamma}\frac{\int_L f^{-1}\left|\nabla f\right|^2\phi^3}{\int_L f^{-1}\left|\nabla f\right|^2\phi^3 + \frac{c_3}{\log\gamma}} \\ &\geq 4\lambda_1(M)\int_L f^{-1}\left|\nabla f\right|^2\phi^3 - \frac{c_4}{\log\gamma}.\end{aligned}$$

Now let's return to (8) and use this lower bound and that $\lambda_1(M) = m^2$. It follows that

$$\begin{aligned}0 &\leq \frac{c_5}{\log\gamma} + \frac{1}{4}\int_L f^{-2}\left|\nabla f\right|^2 Re\left(f_{\bar\beta}\left(\phi^3\right)_\beta\right) \\ &\quad -\int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^3\right)_{\bar\beta} - \frac{m}{m+2}\int_L f^{-1}Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^3\right)_\beta\right).\end{aligned} \tag{10}$$

We will argue now that, as $\gamma\to\infty$, the right hand side of (10) is convergent to zero.

Claim: There exists a constant $c \geq 0$ such that

$$\frac{1}{4}\int_L f^{-2}|\nabla f|^2 Re\left(f_{\bar\beta}\left(\phi^3\right)_\beta\right)-\int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^3\right)_{\bar\beta}$$
$$-\frac{m}{m+2}\int_L f^{-1}Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^3\right)_\beta\right)\le\frac{c}{(\log\gamma)^{\frac{1}{2}}}.$$

Proof of the claim. Let us study each of the three terms in the left hand side.

I. We have:

$$\begin{aligned}\frac{1}{4}\int_L f^{-2}|\nabla f|^2 Re\left(f_{\bar\beta}\left(\phi^3\right)_\beta\right) &= \frac{3}{16}\int_L \phi^2 f^{-2}|\nabla f|^2\nabla f\cdot\nabla\phi\\ &= \frac{3}{16}\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-3}|\nabla f|^4\phi^2\\ &\quad -\frac{3}{16}\frac{1}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-3}|\nabla f|^4\phi^2.\end{aligned}$$

As we stressed above, the estimates in Lemma 1 and Lemma 2 are true on any end. We want to apply Lemma 1 to estimate from above

$$\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-3}|\nabla f|^4\phi^2.$$

We can extend ϕ on $L(\varepsilon,2\varepsilon)$ as in Lemma 1, and notice that there is no need to consider a cut-off φ on $L\left(\frac{1}{2}\delta\varepsilon,\delta\varepsilon\right)$, because ϕ already is zero there. Let us denote $\widetilde\phi$ this new cut-off, hence $\widetilde\phi=\widetilde\varphi\psi$, where ψ is the same as in Lemma 1 and

$$\widetilde\varphi=\begin{cases}(-\log\delta)^{-1}(\log f-\log\delta\varepsilon) & \text{on } L(\delta\varepsilon,\varepsilon)\\ (\log 2)^{-1}(\log 2\varepsilon-\log f) & \text{on } L(\varepsilon,2\varepsilon)\\ 0 & \text{otherwise.}\end{cases}$$

We can apply the computations in Lemma 1 to our setting here, and it is easy to see that formula (6) holds true if we replace ϕ with $\widetilde\phi$. We want to prove that

$$\frac{1}{4}\int_{\widetilde L} f^{-2}|\nabla f|^2 Re\left(f_{\bar\beta}\left(\widetilde\phi^2\right)_\beta\right)-\int_{\widetilde L} f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\widetilde\phi^2\right)_{\bar\beta}$$
$$-\frac{m}{m+2}\int_{\widetilde L} f^{-1}Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\widetilde\phi^2\right)_\beta\right)\le\widetilde C(-\log\delta)^{\frac{1}{2}},$$

where $\widetilde L=L(\delta\varepsilon,2\varepsilon)$. Observe that

$$\begin{aligned}\int_{L(\delta\varepsilon,\varepsilon)} f^{-2}|\nabla f|^2 Re\left(f_{\bar\beta}\left(\widetilde\varphi^2\right)_\beta\right) &\le \frac{1}{2}\int_{L(\delta\varepsilon,\varepsilon)} f^{-2}|\nabla f|^3|\nabla\widetilde\varphi|\\ &\le \frac{\widetilde c_2}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}|\nabla f|^2\\ &= \widetilde c_3,\end{aligned}$$

where in the second line we used the gradient estimate for f. A similar argument works to bound the other terms on $L(\delta\varepsilon,\varepsilon)$. Clearly, the estimates on $L(\varepsilon,2\varepsilon)$ involving $\nabla\widetilde{\varphi}$ or the estimates involving $\nabla\psi$ follow as in Lemma 1. Consequently, using (6) and (7) we get that:

$$\begin{aligned}\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-3}|\nabla f|^4\phi^2 &\leq 4m^2\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}|\nabla f|^2\phi^2 \\ &\quad +\frac{c_6}{(-\log\delta)^{\frac{1}{2}}} \\ &\leq \frac{4}{3}m^2\int_{l(t_0)}|\nabla f| + \frac{c_6}{(-\log\delta)^{\frac{1}{2}}}.\end{aligned}$$

Similarly, applying Lemma 2 for $L(T,\gamma T)\cap F$ it follows that

$$\frac{1}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-3}|\nabla f|^4\phi^2 \geq \frac{4}{3}m^2\int_{l(t_0)}|\nabla f| - \frac{c_7}{(\log\gamma)^{\frac{1}{2}}}.$$

Combining the two estimates, it results

$$\frac{1}{4}\int_L f^{-2}|\nabla f|^2 Re(f_{\bar\beta}\left(\phi^3\right)_\beta) \leq \frac{c_8}{(\log\gamma)^{\frac{1}{2}}}.$$

II. Start with

$$\begin{aligned}-\int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^3\right)_{\bar\beta} &= -\frac{3}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-2}(f_{\bar\alpha\beta}f_\alpha f_{\bar\beta})\phi^2 \\ &\quad +\frac{3}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-2}(f_{\bar\alpha\beta}f_\alpha f_{\bar\beta})\phi^2.\end{aligned}$$

From (1) and (7) we have:

$$\begin{aligned}\frac{1}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-2}(f_{\bar\alpha\beta}f_\alpha f_{\bar\beta})\phi^2 &\leq \left(\frac{1}{16}-\frac{1}{16m}\right)\frac{1}{\log\gamma}\times \\ &\times\int_{L(T,\gamma T)\cap F} f^{-3}|\nabla f|^4\phi^2 + \frac{c_9}{(\log\gamma)^{\frac{1}{2}}} \\ &\leq \left(\frac{1}{16}-\frac{1}{16m}\right)\frac{4}{3}m^2\int_{l(t_0)}|\nabla f| + \frac{c_{10}}{(\log\gamma)^{\frac{1}{2}}},\end{aligned}$$

while from (5) we know that

$$
\begin{aligned}
-\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-2}(f_{\bar\alpha\beta}f_\alpha f_{\bar\beta})\phi^2 &\le \left(-\frac{1}{8}+\frac{(m+1)^2}{16m\,(m+2)}\right)\times \\
&\times\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-3}\left|\nabla f\right|^4\phi^2 \\
&+\frac{m\,(m+1)}{4\,(m+2)}\frac{1}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-1}\left|\nabla f\right|^2\phi^2+\frac{c_{11}}{(-\log\delta)^{\frac{1}{2}}} \\
&\le\left(\left(-\frac{1}{8}+\frac{(m+1)^2}{16m\,(m+2)}\right)\frac{4}{3}m^2+\frac{m\,(m+1)}{4\,(m+2)}\frac{1}{3}\right)\int_{l(t_0)}\left|\nabla f\right| \\
&+\frac{c_{12}}{(-\log\delta)^{\frac{1}{2}}} \\
&=-\frac{1}{12}m\,(m-1)\int_{l(t_0)}\left|\nabla f\right|+\frac{c_{12}}{(-\log\delta)^{\frac{1}{2}}},
\end{aligned}
$$

using the estimates in I. Adding the two estimates proved above it follows that

$$
-\int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^3\right)_{\bar\beta}\le\frac{c_{13}}{(\log\gamma)^{\frac{1}{2}}}.
$$

Note also that in a similar fashion it can be proved that

$$
\int_L f^{-1}f_{\bar\alpha\beta}f_\alpha\left(\phi^3\right)_{\bar\beta}\le\frac{c_{14}}{(\log\gamma)^{\frac{1}{2}}}.
$$

III. Finally, by (2) one has:

$$
\begin{aligned}
-\int_L f^{-1}Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^3\right)_\beta\right) &= -\frac{3}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-2}Re(f_{\bar\alpha\bar\beta}f_\alpha f_\beta)\phi^2 \\
&+\frac{3}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-2}Re(f_{\bar\alpha\bar\beta}f_\alpha f_\beta)\phi^2 \\
\le -\frac{3}{8(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-3}\left|\nabla f\right|^4\phi^2 &+\frac{3}{(-\log\delta)}\int_{L(\delta\varepsilon,\varepsilon)} f^{-2}(f_{\alpha\bar\beta}f_{\bar\alpha}f_\beta)\phi^2 \\
+\frac{3}{8\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-3}\left|\nabla f\right|^4\phi^2 &-\frac{3}{\log\gamma}\int_{L(T,\gamma T)\cap F} f^{-2}(f_{\alpha\bar\beta}f_{\bar\alpha}f_\beta)\phi^2 \\
&+\frac{c_{15}}{(\log\gamma)^{\frac{1}{2}}}.
\end{aligned}
$$

By I and II it can be showed that

$$
\int_L f^{-1}Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^3\right)_\beta\right)\le\frac{c_{16}}{(\log\gamma)^{\frac{1}{2}}}.
$$

This proves the claim. q.e.d.

Using this result in (10) we infer that

$$0 \leq \frac{c_5}{\log \gamma} + \frac{1}{4}\int_L f^{-2}\left|\nabla f\right|^2 Re\left(f_{\bar\beta}\left(\phi^3\right)_\beta\right) \tag{11}$$
$$-\int_L f^{-1} f_{\bar\alpha\beta} f_\alpha \left(\phi^3\right)_{\bar\beta} - \frac{m}{m+2}\int_L f^{-1} Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^3\right)_\beta\right) \leq \frac{C}{(\log\gamma)^{\frac{1}{2}}}.$$

Since γ (and $\delta = \frac{1}{\gamma}$) is arbitrary it follows that, for ε and T fixed, the above inequality becomes an equality as $\gamma \to \infty$.

From (11) we are able to draw the conclusion that the following formulas are valid on M:

$$\begin{array}{rcl} Ric_{1\bar{1}} & = & -(m+1) \\ |\nabla f| & = & 2\sqrt{\lambda_1(M)}f \\ u_{\alpha\bar\beta} & = & -m\delta_{\alpha\bar\beta} \\ u_{\alpha\beta} & = & m\delta_{1\alpha}\delta_{1\beta} \end{array} \tag{12}$$

with respect to the frame

$$\begin{array}{rcl} v_\alpha & = & \frac{1}{2}\left(e_{2\alpha-1} - \sqrt{-1}Je_{2\alpha-1}\right), \\ e_1 & = & \frac{1}{|\nabla f|}\nabla f, \;\; Je_{2k-1} = e_{2k}. \end{array}$$

Note that in view of (12) this frame is globally defined on M.

Let us prove that indeed we have these relations on M.

Suppose that there exists a point $x_0 \in M$ and a positive η_0 such that:

$$Ric_{1\bar1}(x_0) \geq -(m+1) + \eta_0.$$

Let us choose ε and T such that $x_0 \in L(\varepsilon, T)$.

Recall that $L = L(\delta\varepsilon, \gamma T)$, for arbitrary γ and for $\delta = \frac{1}{\gamma}$. Then one can see that there exists $\eta_1 > 0$ such that

$$-\int_L f^{-1} f_\alpha f_{\bar\alpha\bar\beta\beta}\phi^3 \leq \frac{m+1}{4}\int_L f^{-1}\left|\nabla f\right|^2\phi^3 - \eta_1.$$

It is easy to check that (11) gives

$$\begin{array}{rcl} 0 & < & \eta_1 \leq \frac{c_5}{\log\gamma} + \frac{1}{4}\int_L f^{-2}\left|\nabla f\right|^2 Re\left(f_{\bar\beta}\left(\phi^3\right)_\beta\right) \\ & & -\int_L f^{-1} f_{\bar\alpha\beta} f_\alpha\left(\phi^3\right)_{\bar\beta} - \frac{m}{m+2}\int_L f^{-1} Re\left(f_\alpha f_{\bar\alpha\bar\beta}\left(\phi^3\right)_\beta\right) \\ & \leq & \frac{C}{(\log\gamma)^{\frac{1}{2}}}, \end{array}$$

which leads to a contradiction if we let $\gamma \to \infty$.

Next, let us focus on the Cauchy-Schwarz inequality (9). Suppose for contradiction that there exists no constant $a \neq 0$ such that

$$|\nabla f|(x) = af(x) \text{ for any } x \in U,$$

where $U \subset L(\varepsilon, T)$ is a fixed open set. It is clear that if

$$h = f^{-\frac{3}{2}} |\nabla f|^2 \phi^{\frac{3}{2}}, \quad g = f^{\frac{1}{2}} \phi^{\frac{3}{2}},$$

then there exists no $a \in \mathbb{R}$ such that $g = ah$ on U, which implies that

$$\eta_0 := \min_{a \in \mathbb{R}} \int_U (g - ah)^2 > 0.$$

This shows that

$$\begin{aligned} \eta_0 &\leq a^2 \int_U h^2 - 2a \int_U gh + \int_U g^2, \\ 0 &\leq a^2 \int_{L \backslash U} h^2 - 2a \int_{L \backslash U} gh + \int_{L \backslash U} g^2, \end{aligned}$$

for any $a \in \mathbb{R}$. As a consequence, the following inequality is true for any $a \in \mathbb{R}$:

$$0 \leq a^2 \int_L h^2 - 2a \int_L gh + \left(\int_L g^2 - \eta_0 \right).$$

It follows that

$$\left(\int_L gh \right)^2 \leq \left(\int_L h^2 \right) \left(\int_L g^2 - \eta_0 \right).$$

Similarly, one can see that there exists an $\eta_1 > 0$ such that

$$\left(\int_L gh \right)^2 \leq \left(\int_L g^2 \right) \left(\int_L h^2 - \eta_1 \right).$$

Adding these two inequalities and using the arithmetic mean inequality we get that there exists $\eta_2 > 0$ with the property

$$\left(\int_L gh + \eta_2 \right)^2 \leq \int_L g^2 \int_L h^2.$$

We have thus proved that there exists a constant $\eta_2 > 0$ depending on U but not on γ (and δ) such that

$$\left(\int_L f^{-1} |\nabla f|^2 \phi^3 + \eta_2 \right)^2 \leq \left(\int_L f^{-3} |\nabla f|^4 \phi^3 \right) \left(\int_L f \phi^3 \right).$$

This inequality will be used instead of (9) in the argument that followed. Consequently,

$$\begin{aligned} \int_L f^{-3} |\nabla f|^4 \phi^3 &\geq 4\lambda_1(M) \frac{\left(\int_L f^{-1} |\nabla f|^2 \phi^3 + \eta_2 \right)^2}{\int_L f^{-1} |\nabla f|^2 \phi^3 + \frac{c_3}{\log \gamma}} \\ &\geq 4\lambda_1(M) \int_L f^{-1} |\nabla f|^2 \phi^3 + 8\lambda_1(M) \eta_2 - c_{17} \frac{1}{\log \gamma}. \end{aligned}$$

However, using the same reasoning as in the proof that $Ric_{1\bar{1}} - (m+1)$ one can see that this yields a contradiction.

Summing up, we have proved that there exists a constant $a > 0$ such that $|\nabla f| = af$ on M. Using Lemma 1 and Lemma 2 one can see that $a = 2m$.

The proofs for the remaining two formulas use the same ideas. Note that in (7) we need to have equality everywhere on M, therefore there exists a function μ on M such that

$$u_{\alpha\bar{\beta}} = \mu\delta_{\alpha\bar{\beta}}.$$

However, taking the trace and using that f is harmonic one can show that $\mu = -m$.

Finally, we pointed out that if equality holds in (3) then

$$f_{\alpha\beta} = 0 \;\text{ for } (\alpha,\beta) \neq (1,1)\,,$$

and, on the other hand, equality holds in (4) if and only if

$$\begin{aligned} |f_{\alpha\beta}| &= \frac{m+1}{m}\frac{|\nabla f|^2}{4f} = m(m+1)f, \\ Re(f_{11}) &= |f_{11}|\,. \end{aligned}$$

This means that

$$f_{11} = m(m+1)f,$$

or in terms of u one has

$$u_{\alpha\beta} = m\delta_{1\alpha}\delta_{1\beta}$$

as claimed.

Now we are ready to complete the proof of Theorem 4. Let us compute the real Hessian of

$$B := \frac{1}{2m}u.$$

We have:

$$\begin{aligned} B_{e_1e_1} &= B_{11} + B_{\bar{1}\bar{1}} + 2B_{1\bar{1}} = 1 - 1 = 0, \\ B_{e_2e_2} &= -\left(B_{11} + B_{\bar{1}\bar{1}} - 2B_{1\bar{1}}\right) = -2, \\ B_{e_{2k-1}e_{2k-1}} &= B_{kk} + B_{\bar{k}\bar{k}} + 2B_{k\bar{k}} = -1, \\ B_{e_{2k}e_{2k}} &= -B_{kk} - B_{\bar{k}\bar{k}} + 2B_{k\bar{k}} = -1, \\ B_{e_ke_j} &= 0 \;\; if \;\; k \neq j, \end{aligned}$$

for $k \in \{2, ..., m\}$. Also, notice that $|\nabla B| = 1$ on M.

Since all the computations from now on will be done in the real frame $\{e_1, ..., e_{2m}\}$ with $Je_{2k-1} = e_{2k}$ and $e_1 = \frac{1}{|\nabla f|}\nabla f$, we will drop the e_k index for convenience and use only k in the formulas for the real Hessian and the curvature.

Also, let us make the convention that Roman letters i, j, k run from 1 to $2m$ and Greek letters α, β, γ run from 3 to $2m$.

We have proved that there exists a smooth function B on M with real Hessian

$$(B_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 & . & . & 0 \\ 0 & -2 & 0 & 0 & . & . & 0 \\ 0 & 0 & -1 & 0 & . & . & 0 \\ 0 & 0 & 0 & -1 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & -1 \end{pmatrix}$$

and with unit length gradient, $|\nabla B| = 1$ on M.

Note that our function B satisfies the same properties as the Buseman function β in [**L-W**]. The advantage of our argument is that we are able to deduce this information using just the Ricci curvature lower bound. For the rest of the proof, we use the same argument as in [**L-W**].

Denote the level set of B by

$$N_t = \{x \in M \mid B(x) = t\}.$$

Since $|\nabla B| = 1$, M is diffeomorphic to $\mathbb{R} \times N_0$ and $e_1 = \nabla B$ is the unit normal to N_t for any t. If N_0 is noncompact, then M will have one end, which contradicts our assumption that M has more than one end.

Consequently, N_0 is compact, and this implies that M has two ends. For the remainder of this proof M has two ends, and we want to find the metric of N_t depending on the metric of N_0.

Knowing B_{ij} is equivalent to knowing the second fundamental form of N_t, which implies that if

$$\nabla e_i = \omega_{ik} e_k,$$

then one can find

$$\omega_{i1}(e_j) = \begin{cases} 0 & \text{for } i \neq j \\ 2 & \text{for } i = j = 2 \\ 1 & \text{for } 3 \leq i = j \leq 2m. \end{cases}$$

Also, using the Kähler property we know that

$$\omega_{1k} J e_k = J \nabla e_1 = \nabla J e_1 = \nabla e_2 = \omega_{2k} e_k,$$

which implies

$$\omega_{\alpha 2}(e_j) = \begin{cases} 0 & \text{for } j = 1 \; or \; j = 2 \\ -1 & \text{for } \alpha = 2p+1, \; j = 2p+2 \\ 1 & \text{for } \alpha = 2p+2, \; j = 2p+1. \end{cases}$$

It is clear that the flow $\phi_t : M \to M$ generated by e_1 is a geodesic flow. Since

$$\nabla_{e_1} e_2 = \nabla_{e_1} J e_1 = J \nabla_{e_1} e_1 = 0$$

we can conclude that e_2 is parallel along the geodesic τ defined by e_1. We will consider the rest of the frame so that it is also parallel along this geodesic.

The next step is to prove that

$$\begin{aligned} V_2(t) &= e^{-2t}e_2 \\ V_\alpha(t) &= e^{-t}e_\alpha \end{aligned}$$

are the Jacobi fields along the geodesic τ with initial conditions

$$\begin{aligned} V_2(0) &= e_2,\ \ V_2'(0) = -2e_2 \\ V_\alpha(0) &= e_\alpha,\ \ V_\alpha'(0) = -e_\alpha. \end{aligned}$$

This is true because the information on ω_{i1} and $\omega_{\alpha 2}$ allows to find sufficient values for the curvature tensor. Using the second structural equations one can show that

$$\begin{aligned} R_{1212} &= -4,\ \ R_{121\alpha} = 0, \\ R_{1\alpha 1\beta} &= -\delta_{\alpha\beta}, \end{aligned}$$

and this indeed shows that $V_k(t)$ are Jacobi fields for $k \in \{2, ..., 2m\}$.

However, $d\phi_t(e_k)$ for $k \geq 2$ are also Jacobi fields with the same initial conditions as $V_k(t)$, so they must coincide.

The conclusion is that the metrics on N_t viewed as one parameter of metrics on N_0 are

$$ds_t^2 = e^{-4t}\omega_2^2(0) + e^{-2t}\left(\omega_3^2(0) + ... + \omega_{2m}^2(0)\right),$$

where $\{\omega_1, ..., \omega_{2m}\}$ is the dual frame of $\{e_1, ..., e_{2m}\}$.

This shows that indeed the manifold is diffeomorphic to $\mathbb{R} \times N$ with the metric described in Theorem 4.

The last part in Theorem 4 to be proved is that $\widetilde{M}$ is isometric to $\mathbb{CH}^m$ if M has bounded curvature. This follows from the following argument. Since we know the metric, we can use the structural equations and the Gauss formula to compute the curvature of M. The curvature of M will depend exponentially in t on the curvature of N_0, so if we ask that it is bounded, this implies that M must have constant holomorphic bisectional curvature, hence covered by $\mathbb{CH}^m$. The details of this proof can be found in [**L-W**] and will not be included here. q.e.d.

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