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VIRASORO ACTIONS AND HARMONIC MAPS (AFTER SCHWARZ)

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Abstract

The actions of a half Virasoro algebra have appeared in many integrable systems. In this paper we show that there is an action of a (Half) Virasoro algebra on the space of (2+0) harmonic maps into a Lie group. This action is generated by a natural action on the frames. A similar calculation on the space-time (1+1) harmonic maps yields formulas generated by John Schwarz.

1. Introduction

The influence of theoretical physics on geometry and topology at this point in time is overwhelming. A great many mathematicians are working to verify conjectures made by physicists or suggested by physics. However, the most difficult problem facing mathematicians is to clarify or translate the intuition from quantum field theory or string theory which lies behind these ideas. In this article, we interpret a series of papers by John Schwarz, a leading originator and proponent of string theory, on Virasoro actions from a decade ago [**9**, **10**]. We hope to shed some light on the mathematical origins of Virasoro actions constructed in physics.

Our main result is that there is (formally) an infinitesimal action of a complex half-Virasoro algebra on the space of harmonic maps from a simply connected domain in $\mathbb{C}\cup(\infty)$ to the Lie group $SU(n)$. This action is an example of a family of actions defined on integrable systems. The action on KdV is probably the best known of these [**11**]. We spend some time describing the origin of these type of actions. They occur in the context of a loop group which is split via Riemann-Hilbert factorization. The half-Virasoro algebra acts infinitesimally on the group and restricts to one factor of the splitting. The action we are interested in is the derived action on the second factor. We give a description in terms of groups which is quite transparent and does not involve formulae. This description works well in the context of harmonic maps, and leads to the more complicated formulas in the Lie algebra setting.

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The plan of the paper is as follows. After this introduction, we review the structural background on harmonic maps into a Lie group which originates in a paper of one of us [**13**] and can be found in the text by Guest [**6**]. In Chapter 3 we review the definitions of triples of Lie groups and algebras and describe the loop groups factorization used for harmonic maps. Section 4 shows how the half-Virasoro actions arise in these factorizations. The main results in the paper are in 5.1 and the following corollaries. Section 6 outlines the results in the Wick rotated version of harmonic maps from $\mathbb{R}^{1,1}$ into $SU(n)$ and makes contact with the formulae of Schwarz.

This is a small part of a project on Virasoro actions on integrable systems which is joint with Chuu-Lian Terng. We thank Dan Freed for much needed inspiration and encouragement. We apologize for our non-inclusive reference list. The literature on harmonic maps, integrable systems and Virasoro actions is immense and quite splintered.

Our results are incomplete, since to obtain the full Virasoro action and the coupling to gravity proposed by Schwarz [**10**], it is necessary to extend the actions to include the L_{-1} generators. From comparison to the Virasoro actions on other integrable systems, these generators include second flows in their description, if we regard the harmonic maps as first flows. We hope to extend the description in this direction, as well as to treat the extension to harmonic maps in other contexts.

2. Background

We give a brief description following Guest [**6**] to Lie-group valued harmonic maps.

Definition 2.1. A *harmonic map* $s : \Omega \longrightarrow G$, (where Ω is a simply connected domain in $\mathbb{C} \cup \{\infty\}$ and G is a matrix Lie group) is a solution to the Euler-Lagrange equation:

$$\frac{\partial}{\partial x}\left(s^{-1}\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(s^{-1}\frac{\partial s}{\partial y}\right) = 0.$$

The map s satisfies the reality condition $s^{-1}(q) = s^*(q)$, if $G = SU(n)$.

Let $L(SU(n)) = \{s : \Omega \longrightarrow SU(n),\ s \text{ harmonic}\}$. The Euler-Lagrange equations are equivalent to the following:

$$(s^{-1}s_{\bar{z}})_z + (s^{-1}s_z)_{\bar{z}} = 0.$$

Write $A = s^{-1}s_z$ and $B = s^{-1}s_{\bar{z}}$; then the harmonic map equation becomes $A_{\bar{z}} + B_z = 0$, where $A, B : \mathbb{C} \longrightarrow g \otimes \mathbb{C}$, and $B = -A^*$. An equivalent description becomes:

Proposition 2.2. *The harmonic map equation is equivalent to the system:*

$$\begin{aligned} A_{\bar{z}} + B_z &= 0 \\ A_{\bar{z}} - B_z &= [A, B], \end{aligned}$$

where $A, B : \mathbb{C} \longrightarrow g \otimes \mathbb{C}$, *and* $B + A^* = 0$.

The generalized solution associated to the harmonic map equation can be constructed as follows. Let $\lambda \in \mathbb{C}$, $A_\lambda = \frac{1}{2}(1 - \lambda^{-1})A$, $B_\lambda = \frac{1}{2}(1-\lambda)B$, and $B = -A^*$. With this notation the harmonic map equation is equivalent to the equation:

$$(A_\lambda)_{\bar{z}} - (B_\lambda)_z = [A_\lambda, B_\lambda], \qquad \forall \lambda \in \mathbb{C} - \{0\}.$$

Note that λ is a spectral (or twistor) parameter and should be carefully distinguished from the spatial parameters z, $\bar{z}$. It is easy to see that this represents the flatness condition of the associated connections $D_\lambda = \{d + A(\lambda)\}$, which admit a frame of flat sections. Let E_λ be this frame.

Then, the harmonic map equations are equivalent to the following system for the flat frame E_λ of the associated connection:

$$\begin{aligned} E_\lambda{}^{-1}\bar{\partial}E_\lambda = E_\lambda{}^{-1}(E_\lambda)_z &= \frac{1}{2}(1 - \lambda^{-1})A, \\ E_\lambda{}^{-1}\partial E_\lambda = E_\lambda{}^{-1}(E_\lambda)_{\bar{z}} &= \frac{1}{2}(1 - \lambda)B. \end{aligned} \tag{2.1}$$

The harmonic map can be easily reconstructed from the flat frame E_λ as $s(z) = E_{-1}(z)$. More precisely, for the case of $SU(n)$, which we treat in the rest of the paper, we have:

Theorem 2.3 ([**13**]). *If s is harmonic and $s(p) \equiv I$, then there exists a unique $E : \mathbb{C}^* \times \Omega \longrightarrow SL(n, \mathbb{C})$ satisfying equations* (2.1) *with*

(a) $E_1 \equiv I$,
(b) $E_{-1} = s$,
(c) $E_\lambda(p) = I$.
(d) $E^{-1}_{\lambda^{-1}} = E^*_{\bar{\lambda}}$.

Moreover, E is analytic and holomorphic in $\lambda \in \mathbb{C}^*$. Note that E_λ is unitary for $|\lambda| = 1$.

Theorem 2.4 ([**13**]). *Suppose $E : \mathbb{C}^* \times \Omega \longrightarrow SL(n, \mathbb{C})$ is analytic and holomorphic in the first variable, satisfying the reality condition $E^{-1}_{\lambda^{-1}} = E^*_{\bar{\lambda}}$, $E_1 \equiv I, E(p) = I$, and the expressions*

$$\frac{E_\lambda^{-1}(E_\lambda)_{\bar{z}}}{1 - \lambda}, \qquad \frac{E_\lambda^{-1}(E_\lambda)_z}{1 - \lambda^{-1}}$$

are constant in λ. Then $s = E_{-1}$ is harmonic.

Thus, from the harmonic map s we have obtained the "extended solution" $E : \Omega \longrightarrow \Omega G$, where ΩG is the loop group of G.

Remark 2.5. The choice of basepoint affects the extended solution, and hence the Virasoro actions. It is standard in integrable systems for the choice of basepoint to affect the constructions.

3. Manin triples of groups and Riemann-Hilbert Factorization

Let (X^+, X^-, X) be a triple of Lie groups with $X^\pm \subset X$ and $\mu : X^+ \times X^- \longrightarrow X$ a diffeomorphism, where $\mu(s_+, s_-) = s_+ s_-$. Then (X^+, X^-, X) is a (*Manin*) *triple of Lie groups.* We say that (X^+, X^-, X) is a *local* (*Manin*) *triple* if

$$\mu : X^+ \times X^- \simeq \tilde{X} \subset X$$

is a diffeomorphism onto an open dense subset $\tilde{X} \subset X$.

We define projections

$$P^\pm(s) = (\mu^{-1}(s))_\pm, \qquad P^\pm : \tilde{X} \longrightarrow X^\pm, \tag{3.1}$$

from the open set $\tilde{X}$ onto the subgroups $X^\pm$.

The standard example of a Manin triple is

$$(X^+, X^-, X) = (SU(n), \Delta(n), SL(n, \mathbb{C})),$$

where $\Delta(n)$ is the group of upper triangular matrices with real diagonal elements. The projection $P^+ : SL(n, \mathbb{C}) \longrightarrow SU(n)$ is realized by applying the Gramm-Schmidt process to the columns of a matrix.

In general, the projections $P^\pm$ do not have nice formulae. However, at the Lie algebra level, the projection operators are easily described. The triple of groups (X^+, X^-, X) gives rise to a triple of Lie algebras $(\mathfrak{X}^+, \mathfrak{X}^-, \mathfrak{X})$. At the Lie algebra level $\mathfrak{X}^\pm \subset \mathfrak{X}$ and $\mathfrak{X} = \mathfrak{X}^+ + \mathfrak{X}^-$. The projection operators $\Pi^\pm : \mathfrak{X} \longrightarrow \mathfrak{X}^\pm$ exist everywhere, even when the group triple is only local. Note that

$$dP_{s^+}(s^+ V) = s^+ \Pi^+ V, \tag{3.2}$$

for $s^+ \in X^+, V \in \mathfrak{X}$. These are infinitesimal formulae which appear later in the paper.

The examples of interest to us are local triples of groups where the projections $P^\pm$ are realized by the Riemann-Hilbert factorization. We refer to Guest [**6**] and Pressley-Segal [**8**] for a more detailed analysis of the sketch we give of such factorizations.

For the general setting, we have a contour (not necessarily connected) $\Gamma \subset S^2 = \mathbb{C} \cup \{\infty\}$ and a sequence of open sets $\mathcal{O}_\epsilon^\pm \subset \mathcal{O}_\delta^\pm$ for $0 < \delta < \epsilon$, where

$$\begin{aligned} S^2 &\subset \mathcal{O}_\epsilon^+ \cup \mathcal{O}_\delta^-, \quad \forall \epsilon, \delta \\ \Gamma &= \bigcap_\epsilon \mathcal{O}_\epsilon^+ \cap \mathcal{O}_\epsilon^-. \end{aligned}$$

Regard $\mathcal{O}_\epsilon^-$ as a thickening of the interior of Γ and $\mathcal{O}_\epsilon^+$ as a thickening of the exterior. We define:

$$\begin{aligned} X &= \{Q : \Gamma \longrightarrow SL(n,\mathbb{C}) \text{ analytic}\} \\ &= \{Q : \mathcal{O}_\epsilon^+ \cap \mathcal{O}_\epsilon^- \longrightarrow SL(n,\mathbb{C}) \text{ holomorphic}\}, \\ X_c^+ &= \{E : \mathcal{O}_\epsilon^+ \longrightarrow SL(n,\mathbb{C}),\ \text{holomorphic for some } \epsilon > 0, \\ &\qquad\qquad E(p) = 1,\ \text{for some } p \in \mathcal{O}_\epsilon\}, \\ X^- &= \{F : \mathcal{O}_\epsilon^- \longrightarrow SL(n,\mathbb{C}),\ \text{holomorphic for some } \epsilon > 0\}. \end{aligned}$$

The Riemann-Hilbert problem is to factor $Q \in X$ into a product $Q = E \cdot F$, where $E \in X_c^+, F \in X^-$. This can be done on a big cell.

Theorem 3.1. *(X_c^+, X^-, X) is a local Manin triple of groups.*

The usual Riemann-Hilbert factorization scheme is given by the following choices:

$$\begin{aligned} \Gamma &= \{\lambda : |\lambda| = 1\} \\ \mathcal{O}_\epsilon^+ &= \{\lambda : |\lambda^{-1}| \le 1 + \epsilon\} \\ \mathcal{O}_\epsilon^- &= \{\lambda : |\lambda| \le 1 + \epsilon\} \end{aligned} \tag{3.3}$$

with $p = \infty \in \mathcal{O}_\epsilon^+$ chosen as the normalizing point.

It is useful to keep in mind that the projection operators at the Lie algebra level are given by Cauchy integral formulae. We have for $V \in \mathfrak{X} = \{V : S^1 \longrightarrow sl(n,\mathbb{C}) \text{ analytic}\}$

$$\Pi^+ V \in \mathfrak{X}^+ = \{W : \mathcal{O}_\epsilon^+ \longrightarrow sl(n,\mathbb{C}) \text{ holomorphic }, W(\infty) = 0\}.$$

For $|\lambda| < 1$,

$$\Pi^+ V(\lambda) = W(\lambda) = \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{V(\xi)}{\lambda - \xi}\, d\xi. \tag{3.4}$$

Note that $\Pi^+ V$ extends to a neighborhood of Γ when V is analytic on $|\lambda| = 1$.

We now procede to describe the more complicated Riemann-Hilbert problem useful for harmonic maps. Now we have

$$\begin{aligned} \Gamma_\epsilon &= \{|\lambda| = \epsilon\} \cup \{|\lambda^{-1}| = \epsilon\} \\ \mathcal{O}^+ &= \mathbb{C} - \{0\} \\ \mathcal{O}_\epsilon^- &= \{|\lambda| < \epsilon\} \cup \{|\lambda^{-1}| \le \epsilon\}. \end{aligned} \tag{3.5}$$

The normalization point is chosen as $1 \in \mathcal{O}^+$. We also have a reality condition.

Definition 3.2. A map $Q : \mathcal{O} \longrightarrow SL(n, \mathbb{C})$ satisfies the *harmonic map reality condition* (HMRC) if

(a) $\lambda \longrightarrow \bar{\lambda}^{-1}$ maps $\mathcal{O} \longrightarrow \mathcal{O}$,
(b) $Q(\lambda) = \big(Q(\bar{\lambda}^{-1})^*\big)^{-1}$.

This condition is compatible with the domains $\mathcal{O}^+, \mathcal{O}^-_\epsilon$ and Γ_ϵ, as well as the notion of holomorphicity. Let

(3.6)

$$Y^- = \{F : \mathcal{O}^-_\epsilon \longrightarrow SL(n, \mathbb{C}) \text{ holomorphic, } F \text{ satisfies the HMRC}\}$$
$$Y^+ = \{E : \mathbb{C} - \{0\} \longrightarrow SL(n, \mathbb{C}) \text{ holomorphic, } E \text{ satisfies the HMRC}\}$$
$$Y = \{Q : \mathcal{O}^-_\epsilon - (\{0\} \cup \{\infty\}) \longrightarrow SL(n, \mathbb{C}) \text{ holomorphic,} \quad Q \text{ satisfies the HMRC}\}.$$

As a special case of factorization, we have

Theorem 3.3. (Y^+, Y^-, Y) *is a local Manin triple.*

In the infinitesimal version, we have Lie algebras $(\mathfrak{Y}^+, \mathfrak{Y}^-, \mathfrak{Y})$, where $SL(n, \mathbb{C})$ is replaced by $sl(n, \mathbb{C})$. The reality condition becomes

$$W(\lambda) = -W(\bar{\lambda}^{-1})^*.$$

For $W \in \mathfrak{Y}$, the formula for the projection at the Lie algebra level is

$$\Pi^+ W(\lambda) = \frac{1}{2\pi i} \int_{|\gamma|=\delta \cup |\gamma|=\delta^{-1}} \frac{W(\lambda)(\lambda - 1)}{(\lambda - \gamma)(\gamma - 1)}\, d\gamma \qquad \text{where } \delta < \lambda < \delta^{-1},\ \delta < \epsilon, \tag{3.7}$$

and

$$\mathfrak{Y} = \{W : \mathcal{O}^-_\epsilon - (\{0\} \cup \{\infty\}) \longrightarrow sl(n, \mathbb{C}), W(\lambda) = -W(\bar{\lambda}^{-1})^*\}.$$

Since $W(\xi)$ is holomorphic in $\mathcal{O}^-_\epsilon - (\{0\} \cup \{\infty\})$, the domain of analyticity of $\Pi^+ W$ extends to $\mathbb{C} - \{\infty\}$.

4. Derived Group Actions and (Half) Virasoro Actions

We now turn to a useful observation on passing automorphisms of X, which restrict to automorphisms of X^-, to diffeomorphisms of X^+. Assume (X^+, X^-, X) is a local triple of groups.

Definition 4.1. We call $G \subset \mathrm{Hom}(X, X)$ a negative automorphism group if $G\big|_{X^-} \subset \mathrm{Hom}(X^-, X^-)$. For $f \in G$, define $f^\# \in \mathrm{Diff}\, X^+$ by

$$f^\#(s_+) = P^+ f(s_+). \tag{4.1}$$

Theorem 4.2. $\# : G \longrightarrow \mathrm{Diff}\, X^+$ *is a (local) homomorphism.*

Proof. Note that by local we mean $f^{\#}g^{\#} = (f \cdot g)^{\#}$, where $f^{\#}$ and $g^{\#}$ are defined.We expect $f^{\#}$ and $g^{\#}$ to be well-defined close to $1 \in G$.

$$\begin{aligned} (g \circ f)(s_+) &= (g \circ f)^{\#}(s_+)r_1 \\ f(s_+) &= f^{\#}(s_+)r_2 \\ g\left(f^{\#}(s_+)\right) &= g^{\#}\left(f^{\#}(s_+)\right)r_3, \end{aligned}$$

where $r_1, r_2, r_3 \in X^-$. Since g is a group homorphism,

$$\begin{aligned} g\left(f(s_+)\right) &= g\left(f^{\#}(s_+)r_2\right) \\ &= g\left(f^{\#}(s_+)\right)g(r_2) \\ &= g^{\#}\left(f^{\#}(s_+)\right)r_3 g(r_2). \end{aligned}$$

By uniqueness, $r_1 = r_3 g(r_2)$ and $g^{\#}\left(f^{\#}(s_+)\right) = (g \circ f)^{\#}(s_+)$. q.e.d.

Corollary 4.3. *Let $\mathfrak{g}$ and $(\mathfrak{X}^+, \mathfrak{X}^-, \mathfrak{X})$ be the Lie algebras of G and (X^+, X^-, X). Then the infinitesimal generator of the group action for $\sigma \in \mathfrak{g}$ is given by*

$$\sigma^{\#}(s_+) = s_+\Pi^+\left(s_+^{-1}\sigma(s_+)\right), \tag{4.2}$$

where

$$\# : X^+ \times \mathfrak{g} \longrightarrow T(X^+).$$

Moreover, # maps Lie brackets of elements in $\mathfrak{g}$ to the Lie brackets of the vector-fields of the image.

Proof. This is the infinitesimal version of the group action. Hence, the Lie bracket formula is a consequence of the composition law. To see the correctness of the formula for the infinitesimal generator, choose $\sigma \in \mathfrak{g}$. Hence, for small t,

$$e^{\sigma t}s_+ = s_+(t)s_-(t)$$

where $(s_+(t), s_-(t)) = \mu^{-1}\left(e^{\sigma t}s_+\right)$. But

$$\sigma(s_+) = \sigma^{\#}(s_+)s_-(0) + s_+(0)\frac{d}{dt}\bigg|_{t=0} s_-(t).$$

Since $s_-(0) = 1$ and $\frac{d}{dt}\Big|_{t=0} s_-(t) = V_- \in \mathfrak{X}^-$. Notice also that $s_+(0) = s_+$, so we have

$$s_+^{-1}\sigma(s_+) = s_+^{-1}\sigma^{\#}(s_+) + V_-,$$

which implies

$$\Pi^+\left(s_+^{-1}\sigma(s_+)\right) = s_+^{-1}\sigma^{\#}(s_+).$$

Finally,

$$\sigma^{\#}(s_+) = s_+\Pi^+\left(s_+^{-1}\sigma(s_+)\right).$$

Note that one may start with this formula and prove directly that # is consistent with the Lie brackets if one wishes. q.e.d.

The familiar example of a derived group action is the dressing action of X^- on X^+. Here the group action is

$$\begin{aligned} \operatorname{Ad} X &\subseteq \operatorname{Hom}(X, X) \\ \operatorname{Ad} X^- &\subseteq \operatorname{Hom}(X^+, X^+). \end{aligned}$$

The formula for the dressing action of $s_- \in X^-$ on $s_+ \in X^-$ is

$$\begin{aligned} s_-^{\#} s_+ &= P^+(s_- s_+) = P^+(s_- s_+ s_-^{-1}) \\ &= P_+ (\operatorname{Ad} s_-(s_+)) . \end{aligned}$$

The action can be thought of as derived either from left multiplication or an Ad action. For more on dressing actions, see either the book of Guest [**6**] or the lecture notes of Terng [**11**].

We turn now to Virasoro actions. We persist in describing the actions at the group level for conceptual simplicity, although we are ultimately interested in infinitesimal formulae. The Virasoro algebra is described in terms of generators

$$V = \operatorname{span}\{\dots, L_{-j}, \dots, L_{-1}, L_0, L_1, \dots\}$$

with the bracket operation $[L_j, L_k] = (k-j)L_{j+k}$. These are suggestively written as $L_j = \lambda^{j+1}\dfrac{\partial}{\partial\lambda}$. The span can be over the complex numbers, yielding $\mathbb{V}_{\mathbb{C}}$, or the real numbers, giving $\mathbb{V}_{\mathbb{R}}$. The algebras we will discover are half-Virasoro algebras:

$$\begin{aligned} \mathbb{V}_{\mathbb{C}}^+ &= \operatorname{span}_{\mathbb{C}}\{L_{-1}, L_0, \dots, L_j\} \\ \mathbb{V}_{\mathbb{C},0}^+ &= \operatorname{span}_{\mathbb{C}}\{L_0, \dots, L_j, \dots\} \\ \mathbb{V}_{\mathbb{C}}^- &= \operatorname{span}_{\mathbb{C}}\{\dots, L_{-j}, \dots, L_0, L_1\} \\ \mathbb{V}_{\mathbb{C},\infty}^- &= \operatorname{span}_{\mathbb{C}}\{\dots, L_{-j}, \dots, L_0\}. \end{aligned}$$

We will meet the corresponding real Virasoro algebras $\mathbb{V}_{\mathbb{R}}^{\pm}$, etc., in Section 5.

The Virasoro algebra is not the Lie algebra for a Lie group. However, it can be considered as infinitesimal generators of holomorphic mappings, which have composition properties modulo difficulties in keeping track of domains and images. Since we are interested in holomorphic mappings which are near the identity mapping, it is possible to keep track of this, although we will be sloppy about it. Choose a large set $\mathcal{O} = \{\lambda : |\lambda| \le N\}$ and let

$$G_{\text{hol}}^+ = \{f : f(\lambda) = \lambda + v(\lambda), v \text{ small and holomorphic in } \mathcal{O}\}.$$

If $f, g \in G^+_{\text{hol}}$, $f \circ g$ is defined and holomorphic, with possibly a smaller domain. Let

$$G^+_{\text{hol},0} = \{f \in G^+_{\text{hol}}, f(0) = 0\}.$$

Then $\mathbb{V}^+_{\mathbb{C}}$ is the infinitesimal algebra for G^+_{hol}, and $\mathbb{V}^+_{\mathbb{C},0}$ is the infinitesimal algebra for $G^+_{\text{hol},0}$.

Recall the usual triple (X^+_c, X^-, X) for the Riemann-Hilbert splitting, with the domains given in (3.3).

Lemma 4.4. G^+_{hol} *acts by composition as a negative family of automorphisms on* (X^+_c, X^-, X).

Proof. This is straightforward, except there is a real problem, which we do not try to solve, of keeping track of domains. Note that we have the notation

$$L_j F = \left.\frac{d}{dt}\right|_{t=0} F\left(\lambda + t\lambda^{j+1}\right) = \lambda^{j+1}\frac{\partial}{\partial\lambda}F \tag{4.3}$$

is the infinitesimal action of generators on X^- or X. q.e.d.

Corollary 4.5. G^+_{hol} *has a derived action on* X^+_c, *where for* $f \in G^+_{\text{hol}}, E \in X^+_c$

$$f^{\#}E = P^+(E \circ f).$$

For $V = v\dfrac{\partial}{\partial\lambda} \in \mathfrak{g}^+_{\text{hol}}$,

$$V^{\#}E = E\Pi^+\left(E^{-1}v\frac{\partial}{\partial\lambda}E\right).$$

Proof. This follows from (2.1), (3.2), and (4.2). Note that we have the explicit formula from (3.4).

$$V^{\#}E(\lambda) = \frac{1}{2\pi i}E\oint_{|\gamma|=1}\frac{E^{-1}(\gamma)v(\gamma)\frac{\partial}{\partial\gamma}E(\gamma)}{(\lambda-\gamma)}\,d\gamma \tag{4.4}$$

q.e.d.

We are now ready to construct the more complicated version of the Virasoro action used for harmonic maps. First we construct $G^+_{\text{hol},0}$ and $\mathbb{V}^+_{\mathbb{C},0}$ on (Y^+_c, Y^-, Y) defined in (3.6). The domains $\mathcal{O}^-_\epsilon$ consist of two pieces, a small neighborhood about 0, $\{|\lambda| \le \epsilon\}$ and a small neighborhood about ∞, $\{|\lambda|^{-1} \le \epsilon\}$. $G^+_{\text{hol},0}$ acts on $\{|\lambda| \le \epsilon\}$. We induce the action on $\{|\lambda|^{-1} \le \epsilon\}$ by $f(\lambda) = \left(\overline{f\left(\bar\lambda^{-1}\right)}\right)^{-1}$. Now the action of f is compatible with the HMRC (2.2).

Theorem 4.6. $G^+_{\text{hol},0}$ *acts by composition as a negative family of automorphisms on* (Y^+_c, Y^-, Y).

Corollary 4.7. *$G^+_{\mathrm{hol},0}$ has a derived action on Y^+_c, where for $f \in G^+_{\mathrm{hol},0}$,*

$$f^\# E = EP^+\left(E^{-1}(E \circ f)\right),$$

and for $V \in \mathbb{V}^+_{\mathrm{hol},0}, V = v\dfrac{\partial}{\partial\lambda}$,

$$V^\# E = E\Pi^+\left(E^{-1} v \frac{\partial}{\partial\lambda} E\right).$$

Proof. The proof is the same as (2.1) and (3.2), despite the difficulties of keeping track of domains. However, we haven't really finished, as we are interested in explicit formulas. $V = v\dfrac{\partial}{\partial\lambda}$ is the correct expression at $\lambda = 0$. However, at $\lambda = \infty$, the transformation $v(\lambda) = -\overline{v\left(\bar{\lambda}^{-1}\right)}$ gives $\overline{v\left(\bar{\lambda}^{-1}\right)}\lambda^2\dfrac{\partial}{\partial\lambda}$. q.e.d.

Proposition 4.8. *If $V = v(\lambda)\dfrac{\partial}{\partial\lambda}$, then*

$$V^\# E(\lambda) = \frac{1}{2\pi i} E\left[\oint_{|\gamma|=\epsilon} \frac{E^{-1}(\gamma)v(\gamma)\frac{\partial}{\partial\gamma}E(\gamma)(\lambda-1)}{(\lambda-\gamma)(\gamma-1)}\,d\gamma + \oint_{|\gamma|=\epsilon^{-1}} \frac{E^{-1}(\gamma)\overline{v(\bar{\gamma}^{-1})}\gamma^2\frac{\partial}{\partial\gamma}E(\gamma)(\lambda-1)}{(\lambda-\gamma)(\gamma-1)}\,d\gamma\right].$$

Notice that the change of variable gives a correspondence between $\mathbb{V}^+_{\mathbb{C},0}$ and $\mathbb{V}^-_{\mathbb{C},\infty}$:

$$\sum_{j=0}^{\infty} c_j \lambda^{j+1}\frac{\partial}{\partial\lambda} \longrightarrow -\sum_{j=0}^{\infty} \bar{c}_j \lambda^{-j+1}\frac{\partial}{\partial\lambda}.$$

We are, in fact, choosing the graph of this representation in $\mathbb{V}^+_{\mathbb{C},0} \times \mathbb{V}^-_{\mathbb{C},\infty}$, where the first Virasoro factor acts at $\lambda = 0$ and the second at $\lambda = \infty$. The full algebra $\mathbb{V}^+_{\mathbb{C},0} \times \mathbb{V}^-_{\mathbb{C},\infty}$ would act on harmonic maps $s : \Omega \longrightarrow SL(n, \mathbb{C})$, i.e., maps without the reality condition $s^* = s^{-1}$.

We give the formula for the generators. Take note that constants multiply the second formula by the complex conjugate, so the formula is a bit misleading. For $j \geq 0$, we have

$$L_j(E)(\lambda) = \frac{1}{2\pi i}\left[\oint_{|\gamma|=\epsilon} \frac{E^{-1}(\gamma)\gamma^{j+1}\frac{\partial}{\partial\gamma}E(\gamma)(\lambda-1)}{(\lambda-\gamma)(\gamma-1)}\,d\gamma + \oint_{|\gamma|=\epsilon^{-1}} \frac{E^{-1}(\gamma)\gamma^{-j+1}\frac{\partial}{\partial\gamma}E(\gamma)(\lambda-1)}{(\lambda-\gamma)(\gamma-1)}\,d\gamma\right]. \tag{4.5}$$

Note that the singularities in the contour integral are at $(0, \infty, 1, \lambda)$. Hence, there are many deformations of the contour possible if we wish to compute $L_j(E)(\lambda)$ for $|\lambda| = 1$ only.

5. Virasoro Actions on Harmonic Maps

We associate to every harmonic map $s : \Omega \longrightarrow SU(n)$ the extended harmonic map $E : \mathbb{C} - \{0\} \times \Omega \longrightarrow SL(n, \mathbb{C})$ as in Theorem 2.3. The Virasoro action acts on the extended harmonic map via its dependence on the variable $\lambda \in \mathbb{C} - \{0\}$. The special variable $z = x + iy$ is carried along as an auxiliary variable.

Theorem 5.1. *Let $f \in G^+_{\mathrm{hol},0}$ be a holomorphic map near the identity, which we extend to $\mathcal{O}^-_\epsilon$ by $f(\lambda) = \left(\overline{f\left(\bar{\lambda}^{-1}\right)}\right)^{-1}$. Define*

$$\widehat{E} = f^* E = P^+(E \circ f). \tag{5.1}$$

Let $\widehat{\Omega} = \{z \in \Omega : f^\# E_\bullet(z) \text{ is defined}\}$. Then $\widehat{E}$ is an extended harmonic map on $\widehat{\Omega}$.

Proof. We need to show that $\widehat{E}$ satisfies the conditions of Theorem 2.4. We have used Corollary 4.5 to define $\widehat{E}$.

Certainly $\widehat{E_1}(z) = I$ and $\widehat{E_\lambda}(p) = I$ by construction. The HMRC ensures that $\widehat{E_\lambda}(z) = \left(\widehat{E}_{\bar{\lambda}^{-1}}(z)\right)^{*-1}$. It is sufficient to show that $\widehat{E}^{-1}_\lambda \frac{\partial}{\partial z} E_\lambda$ has a simple pole at 0 (and no poles at ∞). Then $\widehat{E}^{-1}_\lambda \frac{\partial}{\partial z} E_\lambda = \alpha + \lambda^{-1}\beta$, but $\beta = -\alpha$ since $\widehat{E}^{-1}_1 \frac{\partial}{\partial z} E_1 = 0$. The HMRC gives a relationship between $\widehat{E}^{-1}_\lambda \frac{\partial}{\partial z} E_\lambda$ and $\widehat{E}^{-1}_\lambda \frac{\partial}{\partial \bar{z}} E_\lambda$ which finishes the proof.

First, by construction, $\widehat{E}_\lambda(z) \in Y^+_c$ is holomorphic in $\lambda \in \mathbb{C} - \{0\}$. Hence $\frac{\partial}{\partial z} \widehat{E}_\lambda$ and $\widehat{E}^{-1}_\lambda$ and their products are holomorphic for $\lambda \in \mathbb{C} - \{0\}$. We need only worry about the singularity at 0 and ∞. To handle this case, we look at the factors

$$E_{f(\lambda)}(z) = \widehat{E}_\lambda(z) R_\lambda(z),$$

where $R_\lambda(z) \in Y^-$ will be smooth in z. Hence, we can write $\widehat{E}_\lambda = E_{f(\lambda)} R^{-1}_\lambda$ and

$$\widehat{E}^{-1}_\lambda \frac{\partial}{\partial z} \widehat{E}_\lambda = R_\lambda \frac{\partial}{\partial z} R^{-1}_\lambda + R_\lambda E^{-1}_{f(\lambda)} \frac{\partial}{\partial z} E_{f(\lambda)} R^{-1}_\lambda.$$

The terms $R_\lambda, R^{-1}_\lambda, \frac{\partial}{\partial z} R^{-1}_\lambda$ are all holomorphic at 0 and ∞. Hence the singularities at 0 and ∞ come from the term

$$E^{-1}_{f(\lambda)} \frac{\partial}{\partial z} E_{f(\lambda)} = \left(1 - f(\lambda)^{-1}\right) \alpha.$$

This expression is holomorphic at ∞. At $\lambda = 0$, since $f(0) = 0$ and f is "close to the identity", we have $E_{f(\lambda)}^{-1} \frac{\partial}{\partial z} E_{f(\lambda)}$ has a simple pole at $\lambda = 0$. The result follows. q.e.d.

Corollary 5.2. *Let* $V = v\frac{\partial}{\partial \lambda} \in \mathbb{V}^+_{\mathbb{C},0}$. *Then*

$$V^{\#} E_\lambda(z) = \frac{1}{2\pi i} E_\lambda(z) \left[\oint_{|\gamma|=\epsilon} \frac{E_\gamma(z)^{-1} v(\gamma) \frac{\partial}{\partial \gamma} E_\gamma(z)(\lambda - 1)}{(\lambda-\gamma)(\gamma-1)}\, d\gamma + \oint_{|\gamma|=\epsilon^{-1}} \frac{E_\gamma(z)^{-1} \overline{v(\bar\gamma)} \frac{\partial}{\partial \gamma} E_\gamma(z)(\lambda - 1)}{(\lambda-\gamma)(\gamma-1)}\, d\gamma \right] \tag{5.2}$$

is tangent to the space of extended harmonic maps.

Proof. While for a holomorphic map $f \in G^+_{\mathrm{hol},0}$, the factorization defining the new extended harmonic map cannot always be done, if $f(\lambda) = \lambda + tv(\lambda)$, the factorization can be done if t is sufficiently small. Since $f_t^{\#} E$ is an extended harmonic map for small t,

$$V^{\#} E = \frac{d}{dt}\bigg|_{t=0} f_t^{\#} E$$

is tangent to the space of extended harmonic maps. q.e.d.

Corollary 5.3. *For* $V \in \mathbb{V}^+_{\mathrm{hol},0}$ *the map* $V \longrightarrow V^{\#}$ *given in* (5.2) *is a representation of* $\mathbb{V}^+_{\mathrm{hol},0}$ *on vector fields tangent to extended harmonic maps.*

Proof. This is an application of Corollary 4.5 and Corollary 5.2. q.e.d.

Corollary 5.4. *Let*

$$L_j(E_\lambda)(z) = \frac{1}{2\pi i} E_\lambda(z) \left[\oint_{|\gamma|=\epsilon} \frac{E_\gamma^{-1}(z) \gamma^{j+1} \frac{\partial}{\partial \gamma} E_\gamma(z)(\lambda - 1)}{(\lambda-\gamma)(\gamma-1)}\, d\gamma + \oint_{|\gamma|=\epsilon^{-1}} \frac{E_\gamma^{-1}(z) \gamma^{-j+1} \frac{\partial}{\partial \gamma} E_\gamma(z)(\lambda - 1)}{(\lambda-\gamma)(\gamma-1)}\, d\gamma \right]$$

generates a representation of $\mathbb{V}^+_{\mathrm{hol},0}$ *on the space of vector fields tangent to extended harmonic maps.*

Proof. This is a result of Corollary 5.3 in terms of specific generators. Note multiplication by a constant acts via multiplication by itself on the first factor, and to complex conjugate on the second. q.e.d.

Corollary 5.4 is the main result of the paper. The reader is invited to prove it directly and to decide whether the route we have taken sheds light on the formula.

6. The Results of Schwarz

We now turn to the Virasoro action on harmonic maps from $\mathbb{R}^{1,1}$ to $SU(n)$ treated by John Schwarz [**9**]. The results are stated without proof, as the construction is identical, except for the different reality conditions.

Theorem 6.1. *Let* $E : \mathbb{C}^* \times \mathbb{R}^{1,1} \longrightarrow SL(n, \mathbb{C})$ *be holomorphic in* $\lambda \in \mathbb{C}^*$ *and smooth for* $(\xi, \eta) \in \mathbb{R}^{1,1}$. *Assume* $E_\lambda(0) = I$ *and*

(a) $E_\lambda = \left(E_{\bar{\lambda}}^{-1}\right)^*$.

(b) $E_1 = I$.

(c) $E_\lambda^{-1} \frac{\partial}{\partial \xi} E_\lambda$ *and* $E_\lambda^{-1} \frac{\partial}{\partial \eta} E_\lambda$ *have simple pols at* 0 *and* ∞, *respectively.*

Then $s = E_{-1} : \mathbb{R}^{1,1} \longrightarrow SU(n)$ *is harmonic. Moreover, any harmonic map* $s : \mathbb{R}^{1,1} \longrightarrow SU(n)$ *with* $s(0) = I$ *has an unique extended harmonic map associated with it.*

The different reality condition $f(\lambda) = \overline{f(\bar{\lambda})}$ results in the decoupling of the Virasoro action at 0 and ∞, as well as a new restriction that the Virasoro actions are real.

Theorem 6.2. *Let* $W = w \frac{\partial}{\partial \lambda} \in \mathbb{V}^+_{\mathbb{R},0}$ *and* $V = v \frac{\partial}{\partial \lambda} \in \mathbb{V}^-_{\mathbb{R},\infty}$ *be the elements of the product of two half-Virasoro algebras. Then*

$$\delta_{v,w}(E_\lambda) = \frac{1}{2\pi i} E_\lambda \left[\oint_{|\gamma|=\epsilon} \frac{E_\gamma^{-1} w(\gamma) \frac{\partial}{\partial \gamma} E_\gamma (\lambda - 1)}{(\lambda - \gamma)(\gamma - 1)}\, d\gamma \right.$$
$$\left. + \oint_{|\gamma|=\epsilon^{-1}} \frac{E_\gamma^{-1} v(\gamma) \frac{\partial}{\partial \gamma} E_\gamma(z) (\lambda - 1)}{(\lambda - \gamma)(\gamma - 1)}\, d\gamma \right]$$

is a representation of $\mathbb{V}^+_{\mathbb{R},0} \times \mathbb{V}^-_{\mathbb{R},\infty}$ *on the vector fields tangent to the space of extended harmonic maps.*

Now Schwarz's formulae are quite different from these formulae. However, a simple transformation $t = \frac{\lambda - 1}{\lambda + 1}$ and $\tau = \frac{\gamma - 1}{\gamma + 1}$ will transform our integrals into his integrals. However, because he is working with a different complex parameter, the natural choice of Virasoro generators are $L_j = t^{j+1} \frac{\partial}{\partial t}$, which are in principle allowable, since the expression in λ is holomorphic at $t = \pm 1$, or $\lambda = 0, \infty$.

Proposition 6.3. $\mathbb{V}_{\mathbb{R}} \subset \mathbb{V}^+_{\mathbb{R}} \times \mathbb{V}^-_{\mathbb{R}}$.

Proof. This embedding is achieved by expressing the generators $t^j \frac{\partial}{\partial t}$ in terms of the coordinates adapted to ± 1, or $t = \frac{\lambda - 1}{\lambda + 1}$. The algebra $\mathbb{V}^+_{\mathbb{R}} \times$

$\mathbb{V}_{\mathbb{R}}^{-}$ is actually larger, as the linear fractional transformations $L_j, j = -1, 0, 1$ correspond to 3 generators in $\mathbb{V}_{\mathbb{R}}$ and 6 generators in $\mathbb{V}_{\mathbb{R}}^{+} \times \mathbb{V}_{\mathbb{R}}^{-}$.
q.e.d.

Unfortunately, Schwarz fails to obtain a representation of the full Virasoro algebra for the same reasons that we fail. We generate $\mathbb{V}_{\mathbb{R},0}^{+} \times \mathbb{V}_{\mathbb{R},\infty}^{-} \subset \mathbb{V}_{\mathbb{R}}^{+} \times \mathbb{V}_{\mathbb{R}}^{-}$. A careful check of the conditions of Schwarz shows that he also imposes the constraint that the vector fields fix $(0, \infty) \sim (1, -1)$ and hence miss a full realization. If we transform the description to one in terms of scattering data, the harmonic maps correspond to first flows. The missing $L_{-1} \in \mathbb{V}_{\mathbb{R}}^{+}$ and $L_1 \in \mathbb{V}_{\mathbb{R}}^{-}$ will be written in terms of second flows.

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