A NOTE ON A COUNTEREXAMPLE OF DELGADO

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In this note we correct some incorrect analysis appearing in the paper of J. A. Delgado [1]. The example concerns two plane curves $\gamma_1, \gamma_2$, which both are regular and complete, and have nonnegative curvature $\kappa$, i.e., $\kappa(\gamma_1) \geq 0, \kappa(\gamma_2) \geq 0$.

In this example Delgado intended to show that $\gamma_1$ and $\gamma_2$ are internally tangent at 0 and that $\kappa(\gamma_1(t)) > \kappa(\gamma_2(s))$ whenever $N_1(t) = N_2(s)$ where $N_1$ (resp. $N_2$) is the unit outward normal of $\gamma_1$ (resp. $\gamma_2$). He also showed that $\gamma_1$ is not contained in the convex region formed by $\gamma_2$, thus showing that Blaschke's theorem does not apply to curves with nonnegative rather than positive curvature. However his analysis is incorrect. The example should go as follows:

$\gamma_1(t) = (pt, t^4), \ t \in \mathbb{R}, \ p > 1,$

$\gamma_2(s) = \begin{cases} (s, (s - 1)^4), & s \in \mathbb{R}, \ s \geq 1, \\ (s, 0), & s \in \mathbb{R}, \ |s| < 1, \\ (s, (s + 1)^4), & s \in \mathbb{R}, \ s \leq -1, \end{cases}$

$N_1(t) = \frac{1}{(p^2 + 16t^2)^{1/2}}(4t^3, -p),$

$N_2(s) = \begin{cases} \frac{1}{(1 + 16(s - 1)^6)^{1/2}}(4(s - 1)^3, -1), & \text{if } s \geq 1, \\ (0, -1), & \text{if } |s| < 1, \\ \frac{1}{(1 + 16(s + 1)^6)^{1/2}}(4(s + 1)^3, -1), & \text{if } s < -1. \end{cases}$

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Hence \( N_1(t) = N_2(s) \) iff \( s > 1 \), and \( t = \frac{s}{\sqrt{p}} (s - 1), -1 < s < 1 \) and \( t = 0 \) or \( s \leq -1 \) and \( t = \frac{s}{\sqrt{p}} (s + 1) \). We have

\[
\kappa(\gamma_1(t)) = \frac{12pt^2}{(p^2 + 16t^6)^{3/2}},
\]

\[
\kappa(\gamma_2(s)) = \begin{cases} 
\frac{12(s - 1)^2}{(1 + 16(s - 1)^6)^{3/2}}, & \text{if } s \geq 1, \\
0, & \text{if } |s| \leq 1, \\
\frac{12(s + 1)^2}{(1 + 16(s + 1)^6)^{3/2}}, & \text{if } s \leq -1,
\end{cases}
\]

(and not as appeared in [1]). So in fact we have

\[
\kappa(\gamma_1(0)) = \kappa(\gamma_2(s)) = 0, \quad |s| \leq 1,
\]

\[
\kappa(\gamma_1(t)) < \kappa(\gamma_2(s)) \quad \text{for } N_1(t) = N_2(s), \quad t \neq 0
\]

(and not \( \kappa(\gamma_1(t)) \geq \kappa(\gamma_2(s)) \) as appeared in [1]).

Hence it is no surprise that \( \gamma_1 \) eventually leaves the convex region formed by \( \gamma_2 \). However, looking at the conjecture the other way around we should have that \( \gamma_2 \) lies in the convex region formed by \( \gamma_1 \). In fact what we find is that in no neighborhood of the origin does it do so. Thus the conjecture fails rather strongly. The fact that \( \gamma_1 \) “cuts” \( \gamma_2 \) for points \( t \neq 0 \) is now irrelevant. This point is made even more clear by the fact that if \( p = 1 \) then \( \gamma_1 \cap \gamma_2 = \{0\} \) and the example still works.

References


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