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GEOMETRIC CRITERION FOR GIESEKER-MUMFORD STABILITY OF POLARIZED MANIFOLDS

HUAZHANG LUO

In Geometric Invariant Theory, the notion of stability for any polarized projective variety is introduced. However to check the stability is usually a difficult problem; see [13], [7] and [21]. It is therefore very interesting to describe the meaning of stability by geometric data of the polarized projective varieties. In this paper we will in particular show that the Gieseker-Mumford stability of a polarized smooth projective variety (as used by them in [7], [13]) is related to the existence of a special metric on the polarized line bundle.

In early 80's, Yau conjectured the relation between notions of stability of manifolds and existence of special metrics such as Kähler-Einstein metrics. In this paper, we are working towards this direction and going to deal with the case of Gieseker-Mumford stability. Similar problems have been studied before. In Tian's recent work ([18], [19]), he dealt with the relation between Kähler-Einstein metric and stability. The notion of stability used by Tian is different from those used by Gieseker and Mumford. However we will see with modifications his methods can still be used in the study of the stability of polarized manifolds in the sense of Gieseker and Mumford.

Another motivation comes from the work on Mumford stability of vector bundles by Donaldson ([4],[5]), and by Uhlenbeck and Yau ([20]). They proved that Mumford stability is equivalent to the existence of

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Hermitian-Einstein metric. So the meaning of stability of a vector bundle is clearly described by its geometry. We would like to call this correspondence the HKDUY correspondence (Hitchin-Kobayashi-Donaldson-Uhlenbeck-Yau).

Our first main result is an interesting geometric criterion for a smooth projective subvariety of $\mathbb{C}P^N$ to be GIT stable in the Hilbert scheme.

Theorem 0.1. *Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety. Assume there exists $\sigma \in SL(N+1, \mathbb{C})$ such that*

$$\frac{1}{vol(M)} \int_{\sigma(M)} \left(\frac{z_i \cdot \overline{z}_j}{|z_0|^2 + ... + |z_N|^2} \right) \omega_{FS}^n = \frac{1}{N+1} \delta_{ij},$$

where ω_{FS} is the Fubini-Study metric, and $[z_0, \cdots, z_N]$ is the homogeneous coordinates of $\mathbb{C}P^N$. Then the Hilbert point of M is (GIT) stable if its stabilizer with respect to the action of $SL(N+1, \mathbb{C})$ is finite.

Applying this theorem, we can study the stability of polarized manifolds. For any polarized manifold (M, L) with fixed Hilbert polynomial, choose a large number k (depending on Hilbert polynomial), we can embed M into some $\mathbb{C}P^N$ by L^k. Then we can talk about the stability of (M, L) by considering the GIT stability of the corresponding Hilbert point. This is the stability notion for polarized manifolds used by Gieseker and Mumford, a more precise definition will be given in Chapter 1.

Theorem 0.2. *Let (M, L) be a polarized manifold. For any large number k, suppose that there exists a metric g (depending on k) on L such that $B_k(z) = B_k(z, g, Ric(g))$ is a pointwise constant function on M. Then the k-th Hilbert point of (M, L) is (GIT) stable if it has finite stabilizer, and consequently (M, L) is Gieseker-Mumford stable.*

Here $B_k(z)$ is the limiting function (as the time goes to infinity) of heat kernel for Hermitian line bundle L^k. The definition of $B_k(z)$ is given in Chapter 3. We do not know whether the converse of this theorem is still true. However at least for a large class of polarized manifolds, the converse is true.

The proof of these two theorems will occupy the next three chapters. In Chapter 1, we introduce the Gieseker-Mumford stability and will also try to reduce the problem of checking stability. The definition of Gieseker-Mumford stability depends on the Hilbert scheme and the universal family which is usually "very singular". In Chapter 2, we will deal with this difficulty, and by using singular Riemann-Roch, logarithmic Green Currents and some intersection theory. By secondary characteristic classes type computations, we will also introduce a functional D_M which is closely related to Gieseker-Mumford stability. Actually the definition of D_M is motivated by K-energy in the study of Kähler-Einstein metric, and also by Donaldson functional in the study of stability of vector bundles. In Chapter 3, we will prove these two theorems and relate the Gieseker-Mumford stability to the existence of good metrics on polarized line bundles.

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1. Gieseker-Mumford stability

In algebraic geometry people frequently need to consider the moduli problem of polarized varieties, *i.e.*, we consider moduli functor

$$\Im : \quad \text{Schemes}/\mathbb{C} \longrightarrow \text{Sets}$$

for the following objects:

$$\Im(\mathbb{C}) = \{(\Gamma, \mathcal{H}) \mid \Gamma \text{ is a projective variety}, \mathcal{H} \text{ ample line bundle on } \Gamma\}.$$

$\mathcal{H}$ is called a polarization of Γ, and $(\Gamma, \mathcal{H})$ is a polarized variety. (M, L) is called a polarized manifold when Γ is smooth. If the canonical sheaf ω_Γ is an ample line bundle, then usually people choose $\mathcal{H}$ to be ω_Γ, and Γ is called canonically polarized. Also we identify $(\Gamma, \mathcal{H})$ with $(\Gamma', \mathcal{H}')$ if there isomorphism $\tau : \Gamma \to \Gamma'$ such that $\tau^*(\mathcal{H}') \cong \mathcal{H}$.

It turns out we should fix some numerical invariants (Hilbert polynomial) first, in order to "split" $\Im$ into smaller pieces. Recall for any line bundle $\mathcal{H}$ over Γ, the Euler-Poincaré characteristic $\chi(\Gamma, \mathcal{H}^m)$ is a polynomial of m. Fixing a polynomial $h'(T) \in \mathbb{Q}[T]$ of degree n, we can consider the moduli problem for

$$\Im_{h'}(\mathbb{C}) = \{(\Gamma, \mathcal{H}) \mid (\Gamma, \mathcal{H}) \in \Im, \, \chi(\Gamma, \mathcal{H}^m) = h'(m) \quad \forall m \geq 1\}.$$

People are interested in proving the existence of moduli space. If using Geometric Invariant Theory, then the essential point is the study of stability.

1.1. Moduli space and Gieseker-Mumford stability

Let's introduce the approach Gieseker and Mumford used to study moduli space (see [7], [13]). By Matsusaka's Big Theorem, $\Im_{h'}$ is bounded, so we can choose a large number $\mu_0 \geq 1$ depending only on h', such that for all $(\Gamma, \mathcal{H}) \in \Im_{h'}(\mathbb{C})$ we have

$$\begin{aligned} &\mathcal{H}^{\mu} \text{ is very ample, for all } \mu \geq \mu_0, \\ &H^i(\Gamma, \mathcal{H}^{\mu}) = 0, \quad \text{for all } i \geq 1, \mu \geq \mu_0. \end{aligned} \tag{1}$$

Therefore for all $(\Gamma, \mathcal{H}) \in \Im_{h'}(\mathbb{C})$ we can use $\mathcal{H}^{\mu}$ $(\mu \geq \mu_0)$ to embedd Γ as a closed subvariety of a fixed projective space $\mathbb{C}P^N$, for $N = h'(\mu) - 1$. This embedding is not canonical; it depends on the choice of a basis of $H^0(\Gamma, \mathcal{H}^{\mu})$. Let $h(T) = h'(\mu T)$ be a polynomial in $\mathbb{Q}[T]$. Grothendieck proved that there exists a scheme $Hilb_h$ (the so called Hilbert scheme) parametrize all the subschemes of $\mathbb{C}P^N$ with fixed Hilbert polynomial h, and over $Hilb_h$ there is a universal family $Univ_h$ given as:

$$\begin{array}{ccc} Univ_h & \xrightarrow{\subset} & Hilb_h \times \mathbb{C}P^N \\ {\scriptstyle g}\big\downarrow & & \\ Hilb_h & & \end{array} \tag{2}$$

Definition 1.1. For any projective subvariety $X \subset \mathbb{C}P^N$ with Hilbert polynomial $h \in \mathbb{Q}[T]$, the Hilbert point of X is the corresponding point $[X] \in Hilb_h$. For any polarized variety $(\Gamma, \mathcal{H}) \in \Im_{h'}(\mathbb{C})$, let $\mu \geq \mu_0$ and consider an embedding $e_{\mu} : \Gamma \to \mathbb{C}P^N$ by $\mathcal{H}^{\mu}$. Then the Hilbert point of $e_{\mu}(\Gamma) \subset \mathbb{C}P^N$ is called (one of) the μ-th Hilbert point of $(\Gamma, \mathcal{H})$.

Group $G = SL(N+1, \mathbb{C})$ acts on $\mathbb{C}P^N$ naturally, and consequently G will acts on $Univ_h$ and $Hilb_h$ equivariantly. Let ν_0 be a large number depending on Hilbert polynomial h, Grothendieck proved on $Hilb_h$ there is an ample line bundle given by

$$\mathfrak{L} = det(g_*(\pi_2^* \mathcal{O}(\nu))) \qquad \nu \geq \nu_0, \tag{3}$$

where $\pi_2 : Univ_h \to \mathbb{C}P^N$ is the projection. $\mathfrak{L}$ is G-linearized; by this we mean the action of G on $Hilb_h$ can be lifted to the geometric bundle $\mathfrak{L}$. Therefore we may apply GIT to Hilbert scheme with respect to the action of group G and the line bundle $\mathfrak{L}$. Let's recall the definition of stable points from [13], [22].

Definition 1.2. A point $x \in H = Hilb_h$ is called (GIT) stable with respect to $G, \mathfrak{L}$ and the given linearisation, or $x \in H(\mathfrak{L})^s$, if x has finite stabilizer and for some $m \geq 1$, there exists a section $t \in \Gamma(Hilb_h, \mathfrak{L}^m)^G$ such that:

1) $H_t = H - V(t)$ is affine, where V(t) denotes the zero locus of t,
2) $x \in H_t$, or in other terms, $t(x) \neq 0$,
3) the induced action of G on H_t is closed.

Moreover, $(\Gamma, \mathcal{H})$ is called Gieseker-Mumford stable if when μ is very large, there exists $\nu_0 \geq 1$ such that for any $\nu \geq \nu_0$, the μ-th Hilbert points of $(\Gamma, \mathcal{H})$ in $Hilb_h$ is (GIT) stable with respect to G and $\mathfrak{L} = det(g_* \pi_2^* \mathcal{O}(\nu))$.

As we see "stability" depends on the choice of a G-linearized line bundle. Actually do not like Gieseker and Mumford, Viehweg choosed a different ample line bundle on some quasi-projective subscheme of Hilbert scheme. Our formulation of stability is the same as used by Gieseker and Mumford ([7], [13]), therefore we call it Gieseker-Mumford stability.

1.2. Simple propositions for stability

Now picking up a polarized manifold (M, L) in $\Im_{h'}(\mathbb{C})$, we will try to understand the stability of M from the differential geometric view point. Notice Hilbert scheme $Hilb_h$ and the universal family $Univ_h$ are usually singular, and this will be one of the difficulties for us to apply differential geometric method later. Let's do some reduction first, in order to simplify the problem a little.

Since the Hilbert scheme $H = Hilb_h$ is complete and $\mathfrak{L}$ is ample line bundle over H, we can give a more geometric description for stable points on H. Assume $\mathfrak{L}^m$ is very ample for some $m \geq 1$. Then we embed H into a projective space $\mathbb{C}P^M$, such that $\mathfrak{L}^m = \mathcal{O}_{\mathbb{C}P^M}(1)|_H$. Since $\mathfrak{L}$ is G-linearized, it follows that G acts on $\mathbb{C}P^M$ by a rational representation $G \to SL(\mathbb{C}^{M+1})$, and the embedding is G equivariant. Let $\theta : \mathbb{C}^{M+1} - \{0\} \to \mathbb{C}P^M$ be the projection, and $\hat{H}$ be the affine cone over H, *i.e.*, the closure of $\theta^{-1}(H)$ in $\mathbb{C}^{M+1}$.

Proposition 1.1. *$x \in H(\mathfrak{L})^s$ if and only if for all points $\hat{x} \in \theta^{-1}(x)$, the orbit of $\hat{x}$ in $\hat{H}$ is closed and the stablizer of x is finite.*

This proposition is well known, so we omit its proof. This proposition can be transformed into better versions for doing analysis later. Give a Hermitian metric $\| \cdot \|$ on $\mathcal{O}_{\mathbb{C}P^M}(1)$ over $\mathbb{C}P^M$. For a fixed point

$x \in H$, define a function $F_x : G \to \mathbb{R}$ by

$$F_x(\sigma) = -log(\|\sigma(\hat{x})\|^{\frac{2}{m}}), \qquad \text{for} \quad \sigma \in G, \tag{4}$$

where $\hat{x}$ is a fixed lifting of x to the fiber of $\mathcal{O}_{\mathbb{C}P^M}(1)$ at x. Then Proposition 1.1 is the same as the following.

Proposition 1.2. *$x \in H(\mathfrak{L})^s$ if and only if F_x is a proper function on G, i.e., for any $c_1, c_2 \in \mathbb{R}$ the set*

$$\{\sigma \in G \mid c_1 \leq F(\sigma) \leq c_2\}$$

is a compact subset of G with respect to Hausdorff topology.

For some technique reason, let's reduce this proposition a little further. For any $x \in H$ we have morphism

$$\tau_x : G \to Hilb_h \tag{5}$$

given by $\tau_x(\sigma) = \sigma(x)$. Notice the Hilbert scheme $Hilb_h$ is complete, so we can choose $\bar{G}$, a smooth compactification of G, such that τ_x extends to a morphism

$$\tau : \bar{G} \to Hilb_h. \tag{6}$$

Use τ to pull back the universal family $Univ_h$ over $Hilb_h$, then we get a flat family of varieties $\bar{\Sigma}$ over $\bar{G}$:

$$\begin{array}{ccc} \bar{\Sigma} & \xrightarrow{\subset} & \bar{G} \times \mathbb{C}P^N \\ {\scriptstyle f}\downarrow & & \\ \bar{G} & & \end{array} \tag{7}$$

Let $i : \bar{\Sigma} \to \bar{G} \times \mathbb{C}P^N$ be the inclusion, and use $\bar{\pi}_1, \bar{\pi}_2$ to denote the projection of $\bar{G} \times \mathbb{C}P^N$ to $\bar{G}$ and $\mathbb{C}P^N$ respectively. In general τ is not a flat morphism, however since the family $Univ_h$ is bounded (see [22], for example), we know if ν_0 is very large, then for all fibers Γ of $g : Univ_h \to Hilb_h$ we have

$$H^i(\Gamma, \ \mathcal{O}_\Gamma(\nu)) = 0 \qquad \text{for} \quad i \geq 1, \ \nu \geq \nu_0. \tag{8}$$

Therefore we can apply Cohomology and Base Change Theorem to get, for all $\nu \geq \nu_0$,

$$\tau^*(\mathfrak{L}) = \tau^*(\det(g_*(\pi_2^*\mathcal{O}(\nu)))) = \det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(\nu))). \tag{9}$$

Consequently Proposition 1.2 now becomes the following proposition which will be used to check the Gieseker-Mumford stability of polarized manifolds.

Proposition 1.3. *Let $(M, L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be given as in (1). Then for any $\mu \geq \mu_0$, the μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = \det(g_*(\pi_2^*\mathcal{O}(v)))$ $(\nu \geq \nu_0)$, if and only if F_M is a proper function on G, where $F_M : G \to \mathbb{R}$ is defined by*

$$F_M(\sigma) = -log(\|\sigma(\hat{x})\|^2),$$

and $\|\cdot\|$ is any Hermitian metric on $\mathfrak{L}_0 = \det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(v)))$ over $\bar{G}$.

The difference between this proposition and Proposition 1.2 is that the definition of F_M depends now only on the family $f : \bar{\Sigma} \to \bar{G}$ as given in (7), however in Proposition 1.2, F_x depends on the line bundle $\mathfrak{L}$ which is defined from the universal family over the Hilbert scheme. So in some sense, we are able to "forget" about Hilbert scheme and the universal family which are usually very singular, and pay attention only to the subfamily $f : \bar{\Sigma} \to \bar{G}$. There are still singularities on $\bar{\Sigma}$, but notice all the singular points are contained in $f^{-1}(\bar{G} - G)$.

2. Singular Riemann-Roch

In order to study the behavior of F_M, we are going to use Riemann-Roch to relate the information on $\bar{G}$ to each fiber of the family $\bar{\Sigma}$ (see [4], [18], [19]). By this way we will introduce a functional D_M which is similar to Donaldson functional in the study of stability of vector bundles ([4], [5]), and also similar to K-energy in the study of Kähler-Einstein metrics ([18], [19]). It is defined on the set of Kähler metrics on M, and unlike F_M, the definition of this functional D_M depends only on the geometry of M. We will prove that F_M can be bounded from below by D_M (see Lemma 2.6), and thus the properness of D_M will imply Gieseker-Mumford stability. We will prove this estimate by the differential geometric method.

2.1. Deal with singular fibers: some intersection theory

In our situation, the family $f : \bar{\Sigma} \to \bar{G}$ has still singular fibers. This forces us to use Singular Riemann-Roch of Baum-Fulton-MacPherson.

We expect the singualr fibers will play a minor role. First let's recall singular Riemann-Roch theorem from [6], which tells us for any variety X, we can associate a homomorphism from the Grothendick group of coherent sheaves to the Chow group on X:

$$\tau_X : K_0(X) \to A_*(X)_{\mathbb{Q}}.$$

This homomorphism will in particular satisfy the following properties:

1) (Covariance). If $f : X \to Y$ is proper, $\alpha \in K_0(X)$, then $f_*\tau_X(\alpha) = \tau_Y f_!(\alpha)$.

2) (Module). If $\alpha \in K_0(X), \beta \in K^0(X)$ (Grothendick group of locally free sheaves), then $\tau_X(\beta \otimes \alpha) = ch(\beta) \cap \tau_X(\alpha)$.

3) (Top Term) If V is a closed subvariety of X, with $dim(V) = n$, then

$$\tau_X(\mathcal{O}_V) = [V] + (\text{terms of dimension} < n).$$

Using the homomorphism τ, we then know the Todd class for a general variety X can be defined by

$$Td(X) = \tau_X(\mathcal{O}_X) \in A_*(X)_{\mathbb{Q}}, \tag{10}$$

and for any $\beta \in K^0(X)$, $\tau_X(\beta)$ can be writen as

$$\tau_X(\beta) = ch(\beta) \cap Td(X).$$

Let's return to our case, consider the family of varieties $f : \bar{\Sigma} \to \bar{G}$ given in (7). Recall $\mathcal{L}_0$ is the determinant line bundle $\det(f_*(i^*\bar{\pi}_2^*\mathcal{O}(v)))$. For simplicity, we denote the line bundle $\bar{\pi}_2^*\mathcal{O}(1)$ over $\bar{G} \times \mathbb{C}P^N$ by L. Applying the covariance of Riemann-Roch to $f : \bar{\Sigma} \to \bar{G}$ then gives

$$f_*\tau_{\bar{\Sigma}}(i^*(L^\nu)) = \tau_{\bar{G}}(f_! i^*(L^\nu)). \tag{11}$$

From the vanishing results (8), when $\nu \geq \nu_0$ the right-hand side of the above equation can be simplified to

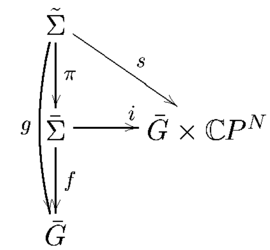
$$f_!(i^*(L^\nu)) = f_*(i^*(L^\nu)). \tag{12}$$

Also by the properties of Riemann-Roch, the left-hand side of (11) can be writen as

$$\begin{aligned} f_*\tau_{\bar{\Sigma}}(i^*(L^\nu)) &= f_*(ch(i^*(L^\nu)) \cap \tau_{\bar{\Sigma}}(\mathcal{O}_{\bar{\Sigma}})) \qquad \text{(Module)} \\ &= f_*(ch(i^*(L^\nu)) \cap ([\bar{\Sigma}] \\ &\qquad + \text{terms of lower dimension})) \quad \text{(Top Term)}. \end{aligned} \tag{13}$$

Now let $\tilde{\Sigma}$ be a desingularization of $\bar{\Sigma}$. We want to write down Riemann-Roch by using smooth varieties $\tilde{\Sigma}$ and $\bar{G} \times \mathbb{C}P^N$ instead of $\bar{\Sigma}$ since we need to do some analysis later:

(14)



Notice that we have the following simple relation after desingularization

$$[\bar{\Sigma}] = \pi_*[\tilde{\Sigma}].$$

Since $i^*(L^\nu)$ is a line bundle over $\bar{\Sigma}$, by the Projection Formular for Chow groups we get

$$\pi_*(ch(s^*(L^\nu)) \cap [\tilde{\Sigma}]) = ch(i^*(L^\nu)). \cap \pi_*[\tilde{\Sigma}]. \tag{15}$$

Recall that we use $\bar{\pi}_1, \bar{\pi}_2$ to denote the projections of $\bar{G} \times \mathbb{C}P^N$ to $\bar{G}$ and $\mathbb{C}P^N$ respectively. By combining the results of (13), (15), and noticing $g_* = \bar{\pi}_{1*}s_*$, the left-hand side of (11) becomes

$$\begin{aligned} f_*(\tau_{\bar{\Sigma}}(i^*(L^\nu))) &= g_*(ch(s^*(L^\nu)) \cap ([\tilde{\Sigma}] \\ &\qquad + \text{terms of lower dimension}) \\ &= g_*(ch(s^*(L^\nu)) \cap [\tilde{\Sigma}]) + \bar{\pi}_{1*}(ch(L^\nu) \cap [Z]), \end{aligned} \tag{16}$$

where $[Z]$ is a cycle of $\bar{G} \times \mathbb{C}P^N$ surpported in $\bar{\Sigma}$, and

$$\dim(Z) \leq n + r - 1, \qquad r = \dim(\bar{G}). \tag{17}$$

Here $n = \dim(\bar{\Sigma}) - \dim(\bar{G})$ is the dimension of generic fiber.

Now $\mathbb{C}P^N$ has a filtration $\mathbb{C}P^N \supset \mathbb{C}P^{N-1} \supset \cdots \supset \mathbb{C}P^1$ by linear subspaces, and each $\mathbb{C}P^k - \mathbb{C}P^{k-1} = \mathbb{C}^k$ is affine. This means $\mathbb{C}P^N$ has a cellular decomposition. It follows (see [6], for example) that for any m, we have a surjective morphism of Chow groups

$$\bigoplus_{k+l=m} A_k(\bar{G}) \otimes A_l(\mathbb{C}P^N) \to A_m(\bar{G} \times \mathbb{C}P^N).$$

In particular, this implies

$$[Z] = [C_1] \times [D_1] + \cdots + [C_r] \times [D_r], \tag{18}$$

where $[C_i]$'s are cycles on $\bar{G}$, and $[D_i]$'s are cycles on $\mathbb{C}P^N$. Assume among $[C_1], \cdots, [C_r]$, only $[C_1], \cdots, [C_s]$ are in $Z_{r-1}(\bar{G}), r = \dim(\bar{G})$. From (11), (12), (16) and (18), by comparing the corresponding parts in $A_{r-1}(\bar{G})$, we get

$$\begin{aligned} &\frac{1}{(n+1)!} g_*(c_1(s^* L^\nu)^{n+1}) + \bar{\pi}_{1*}(ch(L^\nu) \cap \sum_{k=1}^{s}([C_k] \times [D_k]))_{r-1} \\ &= c_1(det(f_* i^*(L^\nu))) + \frac{1}{2} c_1(\bar{G}) \\ &= c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}), \end{aligned} \tag{19}$$

where $(\cdot)_{r-1}$ means the $(r-1)$-dimensional part of this cycle.

Now notice $\tilde{\Sigma}, \bar{G}$ are smooth varieties, and $g, \bar{\pi}_1$ are holomorphic maps, so we can compute the terms in this equation by using differential geometric methods. Of course then we will have to deal with those $[C_k]$ and $[D_k]$ terms. In the following lemma, we have a simple but useful observation about those $[C_k]$ terms.

Lemma 2.1. *There are cycles $[D_k](1 \leq k \leq s)$ on $\mathbb{C}P^N$, and $(r-1)$-dimensional cycles $[C_k](1 \leq k \leq s)$ on $\bar{G}$, such that*

$$\frac{1}{(n+1)!} g_*(c_1(s^* L^\nu)^{n+1}) + \bar{\pi}_{1*}(ch(L^\nu) \cap \sum_{k=1}^{s}([C_k] \times [D_k]))_{r-1} = c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}), \tag{20}$$

and we may choose $C_k(1 \leq k \leq s)$ to be divisors of $\bar{G}$ surpported in $\bar{G} - G$.

Proof. Assume $[D_k]$ is b_k-dimensional cycle of $\mathbb{C}P^N$. Notice that for all $0 \leq i \leq N$, $A_i(\mathbb{C}P^N)$ is a free abelian group generated by $i-$dimensional linear subspace $\mathbb{C}P^i$ of $\mathbb{C}P^N$. Therefore in (19) we may assume b_k is different from each other, and $b_1 < b_2 < \cdots < b_s$. Notice

$$
\begin{aligned}
&\bar{\pi}_{1*}\left(ch(L^\nu) \cap \sum_{k=1}^{s} [C_k] \times [D_k]\right)_{r-1} \\
&= \sum_{k=1}^{s} \bar{\pi}_{1*} \left(ch(L^\nu) \cap [D_k]\right)_0 [C_k] \\
&= \sum_{k=1}^{s} \frac{\nu^{b_k}}{b_k!} \left(c_1(L)^{b_k} \cap [D_k]\right)_0 [C_k] \\
&= \sum_{k=1}^{s} \lambda_k \nu^{b_k} [C_k],
\end{aligned}
\tag{21}
$$

where λ_k are some constants. Now by (19), we get

$$
\frac{\nu^{n+1}}{(n+1)!} g_*(c_1(s^*L)^{n+1}) + \sum_{k=1}^{s} \lambda_k \nu^{b_k} [C_k] = c_1(\mathcal{L}_0) + \frac{1}{2} c_1(\bar{G}). \tag{22}
$$

Notice when restricted on G, $\mathfrak{L}_0$ and $K_{\bar{G}}$ are trivial line bundles because of the G action. By the following exact sequence

$$
A_*(\bar{G} - G) \to A_*(\bar{G}) \to A_*(G) \to 0
$$

we conclude that there exist divisors Y and Y_0, such that they are supported in $\bar{G} - G$, and

$$
\frac{\nu^{n+1}}{(n+1)!} g_*(c_1(s^*L)^{n+1}) + \sum_{k=1}^{s} \lambda_k \nu^{b_k} [C_k] = [Y] + \frac{1}{2} [Y_0]. \tag{23}
$$

Also recall, by (17), we know

$$
(r-1) + b_k = \dim(C_k \times D_k) \leq (n + r - 1).
$$

Therefore $b_1 < b_2 < \cdots < b_s < (n+1)$. Choose $\nu = 1, 2, \cdots, s+1$ in (23). By solving a non-degenerate $(s+1) \times (s+1)$ system of linear equations we find for every k, $\lambda_k [C_k]$ may be represented by divisors with surpport in $\bar{G} - G$, *i.e.*, we may assume C_k is a divisor with surpport in $\bar{G} - G$. This proves Lemma 2.1.

2.2. Logarithmic green current

Now let's begin to use differential geometric method. Give the Hermitian metric on $s^*(L)$ over $\tilde{\Sigma}$ and the Hermitian metric on L over $\bar{G} \times \mathbb{C}P^N$ by using the standard Euclidean metric on the hyperplane bundle over $\mathbb{C}P^N$. Let ω_{FS} be Fubini-Study metric on $\mathbb{C}P^N$. Then the curvature of $s^*(L)$ is $s^*\bar{\pi}_2^*(\omega_{FS})$, and the curvature of L is $\bar{\pi}_2^*(\omega_{FS})$. We will also fix a Hermitian metric $\|\cdot\|$ on $\mathcal{L}_0$. Denote the curvature of this Hermitian line bundle by $R(\|\cdot\|)$. Fix a Hermitian metric $\|\cdot\|_{\bar{G}}$ on $K_{\bar{G}}$ and its curvature is denoted by $R(\|\cdot\|_{\bar{G}})$. Assume $[C_k]$ is Poincaré dual to a smooth differential form α_k on $\bar{G}$, and $[D_k]$ is Poincaré dual to a smooth differential form β_k on $\mathbb{C}P^N$.

We want to write the equation in Lemma 2.1 as equality of currents. For this purpose, let's recall the Green current which was used by Gillet-Soulé ([8]) in their study of Arakelov geometry. If X is any n-dimensional smooth projective (complex) variety, and $Y \subset X$ a closed irreducible subvariety of codimension p, then there exist a $(p-1, p-1)$-current ψ (the so called Green current), and a smooth closed (p,p)-form ω on X, such that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\psi) + \delta_Y = \omega.$$

Here δ_Y is the current representing integration on Y. What important is that we can choose ψ to be given by smooth differential form on $X - Y$ which is of logarithmic type along Y. By Hironaka's theorem on the resolution of sigularities, there exists a proper morphism

$$\pi : \quad \tilde{X} \longrightarrow X$$

such that $\tilde{X}$ is smooth, $E = \pi^{-1}(Y)$ is a divisor with normal crossings, and when restricted on $\tilde{X} - E$, π is an isomorphism. Then ψ is of logarithmic type along Y means near each $x \in \tilde{X}$, if $z_1 \cdots z_k = 0$ $(0 \leq k \leq n)$ is the local equation of E then there exist ∂ and $\bar{\partial}$ closed smooth forms α_i and a smooth form β such that

$$\pi^*(\psi) = \sum_{i=1}^{k}(\log|z_i|^2)\alpha_i + \beta.$$

Thus ψ is called the logarithmic Green current of the subvariety $Y \subset X$. Using this kind of logarithmic Green current, we can write the results in Lemma 2.1 in terms of equalities of currents.

Lemma 2.2. *There is a measurable function θ_ν (depending on ν), such that as currents*

$$\frac{\nu^{n+1}}{(n+1)!}g_*(s^*\bar{\pi}_2^*(\omega_{FS})^{n+1}) + \bar{\pi}_{1*}(exp(\bar{\pi}_2^*\omega_{FS}) \wedge \sum_{k=1}^{s}\alpha_k \wedge \beta_k)_{r-1} = \frac{\sqrt{-1}}{2\pi}R(\|\cdot\|) + \frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}}) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\theta_\nu, \tag{24}$$

θ_ν is a smooth function when restrict on G, and is bounded from above by a constant on G. Here $(\cdot)_{r-1}$ means the $(r-1, r-1)$ part of a differential form.

Proof. Let $[Z] \in A_{n+r-1}(\tilde{\Sigma})$ and $[Y] \in A_{r-1}(\bar{G})$ be cycles such that $[Z] = c_1(s^*L^\nu)^{n+1}$, and $[Y] = (c_1(\mathcal{L}_0) + \frac{1}{2}c_1(\bar{G}))$. By Lemma 2.1 we get an equality between cycles:

$$\frac{\nu^{n+1}}{(n+1)!}g_*[Z] + \sum_{k=1}^{s}\lambda_k\nu^{b_k}[C_k] = [Y].$$

This equation, in terms of currents, is

$$\frac{\nu^{n+1}}{(n+1)!}g_*(\delta_Z) + \sum_{k=1}^{s}\lambda_k\nu^{b_k}\delta_{C_k} = \delta_Y.$$

Let ψ_Z, ψ_Y and ψ_{C_k} be the logarithmic Green currents of $Z \subset \tilde{\Sigma}$, $Y \subset \bar{G}$ and $C_k \subset \bar{G}$ respectively. Then we find (24) is true for some measurable function θ_ν given by

$$\theta_\nu = \frac{\nu^{n+1}}{(n+1)!}g_*(\psi_Z) + \sum_{k=1}^{s}\lambda_k\nu^{b_k}\psi_{C_k} - \psi_Y.$$

Here θ_ν is smooth on $G - g_*(Z) - Y - (C_1 + \cdots + C_s)$ and has at most logarithmic growth along $Y + g_*(Z) + (C_1 + \cdots + C_s) + (\bar{G} - G)$. However, every term other than $\partial\bar{\partial}\theta_\nu$ in (24) is a smooth differential form on G. Therefore by the regularity of $\bar{\partial}$ operator, θ_ν can be extended to be a smooth function on G. In (24), $g_*(s^*\bar{\pi}_2^*((\omega_{FS})^{n+1}))$ is a positive $(1,1)$ current on $\bar{G}$, and other terms except $\partial\bar{\partial}\theta_\nu$ are smooth differential forms on $\bar{G}$. Therefore for example by Green's Formular

$$\theta_\nu(x) = \frac{1}{V}\int_{\bar{G}}\theta_\nu(y)\omega_y^r - \frac{1}{V}\int_{\bar{G}}G(x,y)\Delta\theta_\nu(y)\omega_y^r \tag{25}$$

we can show θ_ν is bounded from above on G. Notice here we have used the fact that L^1 norm of θ_ν on $\bar{G}$ is finite, since θ_ν has at most logarithmic growth along $\bar{G} - G$.

2.3. Secondary characteristic classes type computations

The right-hand side of the equation in Lemma 2.2 will contain $\partial\bar{\partial}F_M$ term when restricted on G; thus we may expect to recover some information about F_M from this equation.

Fix a reference point $0 \in G \subset \bar{G}$, let M_0 be the fiber of $g : \tilde{\Sigma} \to \bar{G}$ over 0. Then M_0 is isomorphic to M. Let us identify M_0 with M, and let $\omega = s^*\bar{\pi}_2^*(\omega_{FS})|_{M_0}$. Denoted by $P(M,\omega)$ the set of all Kähler metrics on M in the same cohomology class as ω. We will define a functional on $P(M,\omega)$.

Definition 2.1. D_M is defined to be a functional from $P(M,\omega)$ to $\mathbb{R}$. For any $\omega' \in P(M,\omega)$, let $\omega' = \omega + \partial\bar{\partial}\varphi$ for some smooth function φ. Then $D_M(\omega')$ is defined by

$$D_M(\omega') = \int_0^1 \int_M \dot{\varphi}_t \omega_t^n \wedge dt. \tag{26}$$

Here $\omega_t = \omega + \partial\bar{\partial}\varphi_t$ $(0 \le t \le 1)$ is a smooth path from ω to ω' in $P(M,\omega)$.

It is straightforward to check that $D_M(\omega')$ is well defined, *i.e.*, it is independent of the choice of a path ω_t in $P(M,\omega)$.

Since we have indentified M with M_0, M becomes a subvariety in $\mathbb{C}P^N$. We know for any $\sigma \in G$, $g^{-1}(\sigma(0))$ can be indentified with $\sigma(M) \subset \mathbb{C}P^N$. Thus we let

$$\omega_\sigma = \sigma^*(\omega_{FS}|_{\sigma(M)}). \in P(M,\omega) \tag{27}$$

For convenience, let give another simple definition though it is not essential.

Definition 2.2. Bergman metrics of $M \subset \mathbb{C}P^N$ is defined by

$$\mathrm{Berg}(M) = \{\omega_\sigma | \sigma \in G\} \subset P(M,\omega).$$

Now using Bergman metrics, D_M can be considered as a functional defined on $\mathrm{Berg}(M)$. D_M can also be considered as a functional on G by

$$D_M(\sigma) = D_M(\omega_\sigma), \qquad \text{for any } \sigma \in G. \tag{28}$$

Eventually we will show that in order to prove $F_M(\sigma)$ is proper it is enough to show $D_M(\sigma)$ is proper.

Now we try to derive information of D_M from (24). Let's do some computation first. The following lemma is in fact Bott-Chern secondary characteristic classes type arguments.

Lemma 2.3. *For any smooth $2(r-1)$-form ϕ with compact surpport in G*

$$\int_{\tilde{\Sigma}} s^*\bar{\pi}_2^*(\omega_{FS})^{n+1} \wedge g^*(\phi) = \int_G \frac{\sqrt{-1}}{2\pi}(n+1)D_M(\sigma) \wedge \partial\bar{\partial}\phi. \tag{29}$$

Proof. Let $G(M) = g^{-1}(G) \subset G \times \mathbb{C}P^N$. Now define $\psi : G \times M \to G(M) \subset \tilde{\Sigma}$ by sending (σ, x) to $(\sigma, \sigma(x))$. Let H be the Hermitian metric on $\psi^*s^*(L)$ by pulling back the Hermitian metric on $s^*(L)$, and the curvature is denoted by $R(H)$. Let $H_0 = pr_2^*(H|_M)$ be another Hermitian metric on $\psi^*s^*(L) \cong pr_2^*(L|_M)$, where $pr_2 : G \times M \to M$ is the projection. Define a path of Hermitian metrics $H_t (0 \leq t \leq 1)$ on $\psi^*s^*(L)$ over $G \times M$ from H_0 to H, such that

$$H_t = e^{\varphi_t} H_0 \qquad \text{and} \quad \varphi_t = t \cdot \log(\frac{H(\sigma,x)}{H_0(\sigma,x)}).$$

Then by straightforward computation we have

$$\begin{aligned}
&\text{LHS of (29)} \\
&= \int_{G\times M} (\frac{\sqrt{-1}}{2\pi} R(H_t))^{n+1} \wedge pr_1^*(\phi) \\
&= \int_{G\times M} \int_0^1 \frac{\sqrt{-1}}{2\pi}(n+1)\partial\bar{\partial}\dot{\varphi}_t \wedge (\frac{\sqrt{-1}}{2\pi} R(H_t))^n \wedge pr_1^*(\phi) \\
&= \int_{G\times M} \int_0^1 \frac{\sqrt{-1}}{2\pi}(n+1)\dot{\varphi}_t \wedge (\frac{\sqrt{-1}}{2\pi} R(H_t))^n \wedge pr_1^*(\partial\bar{\partial}\phi) \\
&= \int_G \frac{\sqrt{-1}}{2\pi}(n+1)D_M(\sigma) \wedge \partial\bar{\partial}\phi.
\end{aligned} \tag{30}$$

Therefore the lemma is proved.

Note $K_{\bar{G}}$ is trivial on G, so we may pick up a meromorphic section s_0 of $K_{\bar{G}}$, and when restricted on G, s_0 is a nonzero holomorphic section of K_G. By Poincaré-Lelong lemma we have

$$\frac{\sqrt{-1}}{2\pi} R(\|\cdot\|_{\bar{G}}) = \delta_{Y_0} - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\|s_0\|^2_{\bar{G}}), \tag{31}$$

where Y_0 is a divisor of $\bar{G}$ supported in $\bar{G}-G$. Similarly we can choose a divisor Y (depends on ν) supported in $\bar{G}-G$ too, such that

$$\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|) = \delta_Y + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}F_M(\sigma). \tag{32}$$

Here F_M is defined in Proposition 1.3.

Lemma 2.4. *For any smooth $2(r-1)$ form ϕ with compact support in G,*

$$\begin{aligned}\int_{\bar{G}}(\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|) + \frac{\sqrt{-1}}{4\pi}R(\|\cdot\|_{\bar{G}}))\wedge\phi \\ = \int_G(\frac{\sqrt{-1}}{2\pi}F_M(\sigma) - \frac{\sqrt{-1}}{4\pi}log(\|s_0\|^2_{\bar{G}}))\wedge\partial\bar{\partial}\phi.\end{aligned} \tag{33}$$

Proof. (31) is true in the sense of current, so we get

$$\begin{aligned}\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|_{\bar{G}})\wedge\phi = \int_{\bar{G}}\left(\delta_{Y_0} - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\|s_0\|^2_{\bar{G}})\right)\wedge\phi \\ = -\int_G\frac{\sqrt{-1}}{2\pi}\log(\|s_0\|^2_{\bar{G}})\wedge\partial\bar{\partial}\phi.\end{aligned} \tag{34}$$

By (32), similar arguments show

$$\int_{\bar{G}}\frac{\sqrt{-1}}{2\pi}R(\|\cdot\|)\wedge\phi = \int_G\frac{\sqrt{-1}}{2\pi}F_M(\sigma)\wedge\partial\bar{\partial}\phi. \tag{35}$$

Add these two results together; then the lemma will follow.

Using Lemma 2.1, we can prove something similar to Lemma 2.3 and Lemma 2.4. Fix a Hermitian metric $\|\cdot\|$ on $\mathcal{O}_{\bar{G}}(C_k)$; then by Poincaré-Lelong Lemma again as what we did before we can deduce the following lemma.

Lemma 2.5. *For any smooth $2(r-1)$-form ϕ with compact support in G*

$$\begin{aligned}\int_{\bar{G}\times\mathbb{C}P^N} exp\,(\bar{\pi}_2^*(\omega_{FS}))\wedge\alpha_k\wedge\beta_k\wedge\bar{\pi}_1^*(\phi) \\ = -\int_G\frac{\sqrt{-1}}{2\pi}\lambda_k\nu^{b_k}log(\|s_k\|^2)\wedge\partial\bar{\partial}\phi,\end{aligned} \tag{36}$$

where s_k is the section of $\mathcal{O}_{\bar{G}}(C_k)$ defining C_k, and λ_k is the constant given by 21.

Since the proof is similar as before, we omit it. Remember we proved in Lemma 2.1 that C_k is supported in $\bar{G}-G$.

2.4. Analytic criterion to check stability

Now from the computations of last section and Lemma 2.2, we conclude that there is a holomorphic function R on G, such that

$$\begin{aligned} F_M(\sigma) - \frac{\nu^{n+1}}{n!} D_M(\sigma) + \sum_{k=1}^{s} \lambda_k \nu^{b_k} \log(\|s_k\|^2) \\ - \frac{1}{2}\log(\|s_0\|^2) + \theta_\nu = \log|R|^2. \end{aligned} \tag{37}$$

We can show that R is actually a constant function. This follows from an observation of Tian in [18]. For reader's convenience, let's write out the detail. Let $\{[z_{ij}, \omega] | 0 \leq i, j \leq n\}$ denote the homogenious coordinates of $\mathbb{C}P^{(N+1)^2}$. Then we can use W, a projective subvariety of $\mathbb{C}P^{(N+1)^2}$, to compactify $G = SL(N+1, \mathbb{C})$ natually. Where W is given by

$$W = \{[z_{ij}, \omega]_{0 \leq i,j \leq N} | det(z_{ij}) = \omega^{N+1}\}.$$

Then by the definition of D_M and Lemma 2.2, some easy computation shows that R has at most polynomial growth near $W \setminus G$, *i.e.*, there exist constants $l > 0, C > 0$, such that

$$|R(\sigma)| \leq C \cdot d(\sigma, W \setminus G)^{-l},$$

where $d(\sigma, W \setminus G)$ denotes the distance from σ to $W \setminus G$ with respect to the Study-Fubini metric of $\mathbb{C}P^{(N+1)^2}$. Therefore R extends to be a meromorphic function on W. Notice that W is normal and $W \setminus G$ is irreducible. Since R is nonzero everywhere in G, it follows that R has to be a constant, otherwise the divisor $W \setminus G$ will be linearly equivalent to zero. Also recall, we already showed that θ_ν is bounded from above, and consequently from (37) we get the following lemma.

Lemma 2.6. *There are constants $C' > 0$ such that for ν large enough*

$$F_M(\sigma) \geq \frac{\nu^{n+1}}{n!} D_M(\sigma) - \sum_{k=1}^{s} \lambda_k \nu^{b_k} log(\|s_k\|^2) + \frac{1}{2} log(\|s_0\|^2) - C'. \tag{38}$$

Here λ_k and $0 \leq b_k \leq n$ are constants.

Therefore eventually we can establish an analytic criterion for the stability of a smooth subvariety.

Proposition 2.1. *Let $(M, L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be given as in (1). For any $\mu \geq \mu_0$, if D_M is a proper function on $Berg(M)$, then the μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable with respect to G and $\mathfrak{L} = \det(g_*(i^*\bar{\pi}_2^*\mathcal{O}(v)))$ for very large ν.*

Proof. By (29) D_M is a pluri-subharmonic function on G. And from its definition D_M has logarithmic growth along $\bar{G} - G$. If we know D_M is proper, then there will exist constants $\delta > 0$ and $C > 0$, such that

$$D_M(\sigma) \geq \delta \cdot \log(d(\sigma, W \setminus G)^{-1}) - C.$$

Thus by (38) we know for ν large enough, $F_M(\sigma)$ will be proper. Consequently by Proposition 1.3 the μ-th Hilbert point $x \in Hilb_h$ of (M, L) is (GIT) stable.

Notice this functional D_M is closely related to the K-energy functional defined by Mabuchi for the study of Kähler-Einstein metric. Recall the K-energy ν_ω is a functional from $P(M, \omega)$ to $\mathbb{R}$, and in the case when $c_1(M) > 0$, for any $\omega' \in P(M, \omega)$ we define

$$\nu_\omega(\omega') = \int_0^1 \int_M \dot{\varphi}_t(s(\omega_t) - n)\omega_t^n \wedge dt,$$

where $\omega_t = \omega + \partial\bar{\partial}\varphi_t$ $(0 \leq 1)$ is a path from ω to ω' in $P(M, \omega)$. Actually from [18], [19] the properness of this K-energy will imply the existence of Kähler-Einstein metric. In our case the Gieseker-Mumford stability of variety does not relate to Kähler-Einstein metric directly.

This functional D_M is also very similar to Donaldson functional for vector bundles; see [5] where the relation between Donaldson functional and family index theorem is explained.

3. Heat kernel and Gieseker-Mumford stability

Recall HKDUY correspondence ([4], [5], [20]) says Mumford stability of a complex vector bundle is equivalent to the existence of Hermitian-Einstein metric on this vector bundle. Suggested by this correspondence, we will also try to relate the Gieseker-Mumford stability of a polarized manifold (M, L) with the existence of some good metric. Due to some technique difficulty, up to now we can only succeed to show one side of this story is true, *i.e.*, the existence of a good mertic implies the

Gieseker-Mumford stability. Along the way, we also get an interesting criterion for the Hilbert point of a smooth projective subvariety of $\mathbb{C}P^N$ to be (GIT) stable.

3.1. Criterion for stability of subvariety of $\mathbb{C}P^N$

First let us try to find the equation satisfied by the critical points of D_M. Let $\sigma \in G$ be a critical point of D_M. Let $s(t)(-\epsilon < t < \epsilon)$ be a path in $G = SL(N+1,\mathbb{C})$ and $s(0) = \sigma$. We will denote $D_M(s(t))$ by $D_M(t)$, and denote $\omega_{s(t)}$ by ω_t when there is no confusion. Recall

$$\omega_t = \omega + \partial\bar{\partial}\varphi_t, \tag{39}$$

where φ_t is function on M, and for any $z = [z_0, \cdots, z_N] \in M$

$$\varphi_t(z) = \log\left(\frac{\|s(t)\cdot z\|^2}{\|z\|^2}\right).$$

From the definition of D_M in (26), by straightforward computation, we get

$$D_M(\tau) = D_M(0) + \int_0^\tau \int_M \frac{\partial}{\partial t}(\varphi_t).\omega_t^n \wedge dt. \tag{40}$$

Consequently by simple computation, we find that σ is a critical point of D_M on G if and only if it satisfies the following equation

$$\frac{1}{Vol(M)}\int_{\sigma(M)}\left(\frac{z_i\cdot\bar{z}_j}{|z_0|^2+\cdots+|z_N|^2}\right)\omega_{FS}^n = \frac{1}{N+1}\delta_{ij}. \tag{41}$$

Lemma 3.1. *Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety, and suppose that its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $G = SL(N+1,\mathbb{C})$. If D_M has a critical point, then D_M is a proper function on G, and there exist constant $\delta > 0$ and $C > 0$ such that*

$$D_M(s) \geq \delta \cdot log(d(s, \bar{G}\setminus G)^{-1}) - C. \tag{42}$$

Here $d(s, \bar{G}\setminus G)$ is the distance of s to $\bar{G}\setminus G$ with respect to a smooth metric on $\bar{G}$.

Proof. For any $s \in G = SL(N+1,\mathbb{C})$, let $s^*s = U^*\Lambda^2 U$, where U is a unitary matrix and Λ is a real diagonal matrix. Then by the definition, $D_M(s) = D_M(\Lambda \cdot U)$. Let $\phi : \mathbb{C}^N \times U(N+1, \mathbb{C}) \to G$ be a surjective map such that for any $(z_1, \cdots, z_N, U) \in \mathbb{C}^N \times U(N+1, \mathbb{C})$,

$$\phi(z_1, \cdots, z_N, U) = \Lambda \cdot U, \quad \text{for } \Lambda = \text{diag}(z_0, \cdots, z_N),$$

where $z_0 = (z_1 \cdots z_N)^{-1}$. Then we need only to prove the pull back function $\phi^*(D_M)$ on $\mathbb{C}^N \times U(N+1, \mathbb{C})$ is a proper function. Fix any $U \in U(N+1, \mathbb{C})$, and let $\varphi = \phi^*(D_M)|_{\mathbb{C}^N \times \{U\}}$. Then by (29) φ is a pluri-subharmonic function on $\mathbb{C}^N$. What's more, notice that the complex Hessian of φ is nonzero everywhere and φ is invariant under the obvious action of torus $S^1 \times \cdots \times S^1$ on $\mathbb{C}^N$. Simple computation shows φ is a strict convex function of $(\log|z_1|, \cdots, \log|z_N|)$, *i.e.*, for all $(z_1, \cdots, z_N) \in \mathbb{C}^N$,

$$\left(\frac{\partial^2 \varphi}{\partial \log|z_i| \partial \log|z_j|} \right) > 0.$$

Consequently since φ has a critical point, straightforward computation shows that there exist constant $\delta > 0$ and $C > 0$ such that

$$\varphi(z_1, \cdots, z_N) \geq \delta \cdot \log(|z_1|^2 + \cdots + |z_N|^2) - C.$$

Thus the lemma is proved.

Now we get our first main theorems stated in the introduction.

Theorem 3.1. *Let $M \subset \mathbb{C}P^N$ be a smooth projective subvariety, and its Hilbert point $[M] \in Hilb_h$ has only finite stabilizer with respect to the action of $SL(N+1, \mathbb{C})$. Then $[M] \in Hilb_h$ is (GIT) stable if there exists $\sigma \in SL(N+1, \mathbb{C})$, such that (41) holds.*

This theorem says for the Hilbert point of $M \subset \mathbb{C}P^N$ to be (GIT) stable, M must have a lot of symmetries.

3.2. Relate Gieseker-Mumford stability to heat kernel

Now let's try to translate the results to be the existence of a good metric. In order to characterize the metric we need a definition.

Definition 3.1. Let (M, ω) be a compact Kähler manifold, and let L be a holomorphic line bundle with a Hermitian metric g. Then we define $B_k(z) = B_k(z, g, \omega)$ to be a function on M, and for any $z \in M$

$$B_k(z, g, \omega) = \sum_{i=0}^{N} \|s_i(z)\|_g^2. \tag{43}$$

Here $s_0, \cdots, s_N$ is any orthonormal frame of $H^0(M, L^k)$.

It is easy to check that $B_k(z)$ is independent of choice of the orthonormal frame $s_0, \cdots, s_N$. Also we should point out that $B_k(z)$ is closely related to the so called distortion function discussed before by Kempf and Ji.

Now let $(M, L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold. Consider an embedding $e_k : M \to \mathbb{C}P^N$ such that $e_k^* \mathcal{O}(1) = L^k$ for some $k \geq \mu_0$ (μ_0 is given in (1)). Assume (41) is true for some $\sigma \in SL(N+1, \mathbb{C})$. Then the k-th Hilbert point of (M, L) is (GIT) stable. Notice if we pull back the standard Euclidean metric on $\mathcal{O}(1)$ over $\mathbb{C}P^N$ by the mapping

$$\sigma \cdot e_k : M \longrightarrow \mathbb{C}P^N,$$

then we get a Hermitian metric $\| \cdot \|$ on L^k, and the curvature of this metric is given by

$$\frac{\sqrt{-1}}{2\pi} R(\| \cdot \|) = e_k^* \sigma^*(\omega_{FS}).$$

Now let us choose $g = \| \cdot \|^{\frac{1}{k}}$ to be the Hermitian metric on L, and $\omega = Ric(g)$ to be the Kähler metric on M. Then $\{e_k^* \sigma^*(z_i) \mid 0 \leq i \leq N\}$ will be holomorphic sections of L^k on M. What's more, we can check that

$$\|z_i\|_g = \frac{|z_i|^2}{|z_0|^2 + \cdots + |z_N|^2}. \tag{44}$$

Therefore by (41), $\{e_k^* \sigma^*(z_i) \mid 0 \leq i \leq N\}$ is an orthonormal frame of $H^0(M, L^k)$ with respect to g and $\omega = Ric(g)$. Consequently from the explicit expression of g we conclude

$$B_k(z, g, Ric(g)) = \sum_{i=0}^{N} \frac{|z_i|^2}{|z_0|^2 + \cdots + |z_N|^2} = 1.$$

In particular $B_k(z, g, Ric(g)$ is a pointwise constant function on M.

Conversely assume there exists Hermitian metric g for L such that for an othonormal basis $s_0, \cdots, s_N$ of $H^0(M, L^k)$, $B_k(z) = B_k(z, g, Ric(g))$ is a pointwise constant function. Then we have a cannonical embedding of M into $\mathbb{C}P^N$ by

$$z \to [s_0(z), \cdots, s_N(z)].$$

We can check that (41) is satisfied when $\sigma = id$. Therefore the k-th Hilbert point of (M, L) is stable in the Hilbert scheme $Hilb_h$. So we have established one of our main theorems of this paper.

Theorem 3.2. *Let $(M,L) \in \Im_{h'}(\mathbb{C})$ be a polarized manifold, and μ_0 be a large number given by (1). For any $k \geq \mu_0$, if there exists a Hermitian metric g (depends on k) on L over M such that $B_k(z) = B_k(z, g, Ric(g))$ is pointwise constant function on M, then the k-th Hilbert point of (M,L) is (GIT) stable with respect to G, and $\mathfrak{L} = \det(g_*(\pi_2^*\mathcal{O}(v)))$ for all large enough ν as long as the stabilizer of the Hilbert point is finite. And consequently, (M,L) is Gieseker-Mumford stable.*

This kind of metric deserves a further study. In fact, we should point out the function $B_k(z) = B_k(z, g, \omega)$ is related to the heat kernel. If we denote $H_t(z,w)$ to be the heat kernel with respect to the $\bar{\partial}$-Laplacian operator on $C^\infty(M, L^k)$, then $B_k(z)$ is precisely the limit function of $H_t(z,z)$ when the time t goes to infinity.

We still do not know if the converse of this theorem is true or not. The main reason is in the estimate (38), we only used the highest order term (with respect to ν). However, certainly for a large class of polarized manifolds, the converse of this theorem is true.

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M.I.T., CAMBRIDGE