MOMENT MAPS AND NON-COMPACT COBORDISMS

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Abstract

We define an *abstract moment map* for a torus action on a smooth manifold *without* a two-form. Cobordisms of such structures are meaningful even if the manifolds are noncompact, as long as the abstract moment maps are proper. We prove that a compact manifold with a torus action and an abstract moment map is cobordant the normal bundle of its fixed point set. Two formulas follow easily: Guillemin's topological version of the abelian Jeffrey-Kirwan localization, and the Guillemin-Lerman-Sternberg formula for the Duistermaat-Heckman measure.

1. Introduction

This work is part of a joint project with Victor Guillemin and Viktor Ginzburg, in which we use cobordisms to study group actions on symplectic manifolds. Some of our results were announced and their proof sketched in [7].

In this paper we introduce a new, cleaner, cobordism technique. We treat non-compact manifolds directly, without "approximating" by compact manifolds as we did in [7]. Our basic theorem involves manifolds and not orbifolds; orbifolds only come up as reduced spaces. And we do not need to carry along a symplectic (or pre-symplectic) form for the theory to work.

Our basic object is a manifold with a torus action, a proper moment map, and an equivariant cohomology class. If (M, ω, Φ) is a symplectic manifold with a moment map in the usual sense, one can take the equivariant cohomology class represented by the sum $\omega + \Phi$. However, in our new approach we define an abstract moment map without a two-form (see section 2), and our equivariant cohomology class is unrelated to the

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abstract moment map. The normal bundle of every connected component of the fixed point set is a (non-compact) manifold with the same structure: a torus action, a proper abstract moment map (that needs to be chosen; see section 5), and an equivariant cohomology class. We prove in section 6 that the original manifold, if compact, is cobordant to the disjoint union of these normal bundles. The cobording manifold carries the same kind of structure, and all abstract moment maps are proper. In particular, if the torus acts with isolated fixed points, the manifold is cobordant to a disjoint union of vector spaces.

The motivation for this work comes from the formula of Guillemin-Lerman-Sternberg [9]. This formula expresses the Duistermaat-Heckman measure for a compact symplectic manifold with a torus action with isolated fixed points as the sum of Duistermaat-Heckman measures for the tangent spaces at the fixed points. Viktor Ginzburg observed several years ago [5] that the Duistermaat-Heckman measure is an invariant of (compact) cobordism (in the sense of [6]). This tempts one to assert that the manifold is cobordant to the disjoint union of its tangent spaces at the fixed points. In sections 2–6 we make sense of this assertion, which we prove in section 6. In sections 7 and 8 we state and prove that "cobordism commutes with reduction", a phenomenon first shown and used by Guillemin in the context of symplectic manifolds (see [7]). We use this in sections 9, 10, and 11 to deduce (generalizations of) the Guillemin-Jeffrey-Kirwan formula and the Guillemin-Lerman-Sternberg formula.

Another application of our cobordism technique, that will appear elsewhere (at least in [8]), is a new proof of the Atiyah-Bott-Berline-Vergne localization for equivariant differential forms. This uses the technique of Lerman cutting applied to spaces with proper abstract moment maps.

Acknowledgements

2. Moment maps

Let T be a torus and $\mathfrak{t}$ its Lie algebra. If H is a closed subgroup of T and $\mathfrak{h}$ is the Lie algebra of H, then the *H-component* of a map into $\mathfrak{t}^*$ is the composition of the map with the natural projection $\mathfrak{t}^* \longrightarrow \mathfrak{h}^*$.

Definition 2.0.1. Let M be a smooth manifold with a smooth T-action. An *abstract moment map* is a smooth map $\Phi : M \longrightarrow \mathfrak{t}^*$ with the following properties:

1. Φ is T-invariant.

2. For any subgroup H of T, the H-component of Φ is constant on each connected component of the fixed point set of H in M.

Remark. In property 2 it is enough to take circle subgroups H of T.

Example 2.0.2. A constant function is an abstract moment map.

Remark. A linear combination of abstract moment maps is an abstract moment map. Later on we will demand that our abstract moment maps be *proper*. Linear combinations of proper abstract moments maps might no longer be proper. However, if $\eta \in \mathfrak{t}$ is a vector and Φ_1 and Φ_2 are two abstract moment maps whose η-components, $\langle \Phi_i, \eta \rangle$, are proper and bounded from below, then so is every positive linear combination of Φ_1 and Φ_2.

Example 2.0.3. If T is a circle, and we identify $\mathfrak{t}^*$ with $\mathbb{R}$, then an abstract moment map is any real valued invariant function that is constant on each connected component of the set of fixed points.

Example 2.0.4. On a vector space on which the circle acts linearly and fixes only the origin, the function $\Phi(v) = ||v||^2$, with respect to any invariant metric, is an abstract moment map that is proper and bounded from below.

Example 2.0.5. If ξ_M is a vector field on M that generates a circle action, then the function $\Phi(x) = \langle \xi_x, \xi_x \rangle$, where $\langle , \rangle$ is any invariant Riemannian metric, is an abstract moment map.

Example 2.0.6. Suppose (M, ω) is pre-symplectic and Φ is a moment map in the traditional sense. This means that ω is a closed T-invariant two-form on M and $\Phi : M \longrightarrow \mathfrak{t}^*$ satisfies $\iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle$ for all $\xi \in \mathfrak{t}$, where ξ_M is the vector field on M that generates the action

of the one parameter subgroup $\exp(s\xi)$, $s \in \mathbb{R}$. Then Φ is an abstract moment map.

3. Equivariant cohomology

We will work with triples (M, Φ, c) where M is an oriented manifold acted upon by a torus T, $\Phi : M \longrightarrow \mathfrak{t}^*$ is a proper abstract moment map, and c is an equivariant cohomology class on M (see below). Our cohomology is always over $\mathbb{R}$. We keep track of the torus action on M although it is not explicit in our notation.

Recall, an equivariant differential form is a polynomial on the Lie algebra $\mathfrak{t}$ whose values are differential forms on M. The equivariant exterior derivative is defined by $(d_T\alpha)(\xi) = d(\alpha(\xi)) + \iota(\xi_M)\alpha(\xi)$ for all $\xi \in \mathfrak{t}$, ξ_M being the corresponding vector field on M. The equivariant cohomology (over $\mathbb{R}$) of a manifold is the quotient $\ker(d_T)/\mathrm{image}(d_T)$. Good references on equivariant cohomology and equivariant differential forms are [1] and [2].

For example, if (M, ω) is a symplectic manifold with a torus action and Φ is a moment map, the sum $\omega + \Phi$ is a closed equivariant form. However, an important point is that *one can work with equivariant forms that have nothing to do with the moment map.*

4. Cobordism

For cobordism to be meaningful one must make some compactness assumption. Otherwise, any manifold M would be cobordant to zero via the non-compact cobordism $M \times (0, 1]$. However, we can drop the compactness assumption if we demand the cobordism to carry a proper abstract moment map:

Definition 4.0.1. Let M, M' be smooth oriented manifolds with actions of a torus T, let $\Phi : M \longrightarrow \mathfrak{t}^*$ and $\Phi' : M \longrightarrow \mathfrak{t}^*$ be proper abstract moment maps, and let c and c' be equivariant cohomology classes on M and M' respectively. A *cobordism* between the triples (M, Φ, c) and (M', Φ', c') is a triple $(W, \tilde{\Phi}, \tilde{c})$, where W is an oriented manifold-with-boundary with a T-action, $\tilde{\Phi} : W \longrightarrow \mathfrak{t}^*$ is a proper abstract moment map, and $\tilde{c}$ is an equivariant cohomology class on W, and an equivariant diffeomorphism of the boundary ∂W with the disjoint union $M \sqcup M'$ that carries $\tilde{\Phi}$ to $\Phi \sqcup \Phi'$, $\tilde{c}$ to $c \sqcup c'$, and the boundary orientation on

∂W to the given orientation of M and the opposite orientation on M'.

Example 4.0.2. Every ordinary cobordism of compact oriented manifolds is also a cobordism in the sense of Definition 4.0.1 when we take the trivial T-actions, zero abstract moment maps, and zero cohomology classes.

Example 4.0.3. Suppose that M is a manifold with a circle action and that $\Phi_i : M \longrightarrow \mathbb{R}$, $i = 0, 1$, are abstract moment maps that are proper and bounded from below and that take the same values on the fixed points. Then (M, Φ_0) is cobordant to (M, Φ_1); take the cobording manifold $M \times [0, 1]$ with the abstract moment map

$$(m, t) \mapsto (1 - t)\Phi_0(m) + t\Phi_1(m).$$

Example 4.0.4. A space with a trivial torus action and a proper abstract moment map must be compact, because its abstract moment map must be constant. Two such spaces are cobordant in the sense of Definition 4.0.1 if and only if they are cobordant in the usual (compact) sense. This is because on a connected manifold-with-boundary, if a torus action acts trivially on the boundary (or, in fact, on any submanifold of codimension one), it acts trivially everywhere.

Remark. If two compact manifolds with torus actions and proper abstract moment maps are cobordant in the sense of Definition 4.0.1 via a stable complex non-compact cobordism, they are also cobordant in the sense of Definition 4.0.1 via a compact cobordism. Thus our notion of cobordism refines the usual notion of equivariant cobordism for compact manifolds with torus actions. To show this, first use Lerman's cutting procedure [11] to replace a non-compact cobordism by a compact orbifold cobordism, then get rid of the orbifold singularities by successive blowups along singular strata (with respect to a complex structure on the normal bundles to the strata). These blowups are T-equivariant, and the cohomology class pulls back via the blowups. Details will appear in [8]. In this paper we do not use this result.

Remark. The above Examples show that our notion of cobordism is non-trivial. Note that it is absolutely crucial for the abstract moment maps to be proper; otherwise, every manifold M would be cobordant to the empty set via the non-compact cobordism $M \times (0, 1]$, where the T action and abstract moment maps are induced from those on M.

Also notice that it is crucial to assume Condition 2 in Definition 2.0.1 of the abstract moment map. If we drop this condition and only work

with, say, manifolds equipped with proper functions to $\mathbb{R}$, any compact manifold M would be cobordant to the empty set via the non-compact cobordism $M \times (0,1]$ and the proper function $\tilde{\Phi}(m,t) = \Phi(m) + 1/t - 1$.

Lemma 4.0.5. *Cobordism is an equivalence relation.*

Proof. For smooth manifolds with no extra structure, this is a standard fact in differential topology. See, e.g., [3]. It is not hard to adapt the standard proof to manifolds equipped with a T-action, an abstract moment map, and an equivariant cohomology class. We omit the details; we might include them in [8].   q.e.d.

5. Linear actions

The most useful non-compact manifolds are vector spaces.

We will now show that a moment map for a linear torus action is determined up to cobordism by its value at the origin and its behavior at infinity, provided that some component of the moment map is proper.

Recall that a complex representation of the torus T is determined up to isomorphism by its *weights*, which are elements of ℓ^*, the integral weight lattice in $\mathfrak{t}^*$. On the underlying real representation, the weights are defined only up to sign; we call these elements of $\ell^*/\pm 1$ *real weights*. A representation of a torus on a real vector space V is determined up to isomorphism by the dimension of the subspace, V^T, of fixed vectors, and by the real weights for the torus action on the quotient V/V^T.

Let V be a real vector space with a linear action of the torus T that fixes only the origin. Let $\pm\alpha_1, \ldots, \pm\alpha_m$ be the real weights for the action (without repetitions). Let $V = V_1 \oplus \ldots \oplus V_m$ be the decomposition of V into invariant subspaces such that T acts on V_i by the real weight $\pm\alpha_i$; any vector $v \in V$ can be expressed uniquely as a sum, $v = \sum v_i$, where v_i is in V_i. Let $\eta \in \mathfrak{t}$ be a "polarizing vector": a vector such that $\langle \alpha_i, \eta \rangle \neq 0$ for all i. Let $\alpha_1^\eta, \ldots, \alpha_m^\eta$ be the "polarized weights": $\alpha_i^\eta \in \{\alpha_i, -\alpha_i\}$, with the sign chosen so that $\langle \alpha_i^\eta, \eta \rangle > 0$. Choose any invariant metric on V. Let a be any element of $\mathfrak{t}^*$. With these choices, define a map $\Phi : V \longrightarrow \mathfrak{t}^*$ by

$$\Phi(v) = a + \sum_{i=1}^m ||v_i||^2 \alpha_i^\eta,$$

where $v = \sum_i v_i$, $v_i \in V_i$. This map is an abstract moment map whose value at the origin is a and whose η-component, $\langle \Phi, \eta \rangle$, is proper and

bounded from below.

Lemma 5.0.1. *Any other abstract moment map* $\Phi' : V \longrightarrow \mathfrak{t}^*$, *whose value at the origin is a and whose η-component is proper and bounded from below, is cobordant to Φ, i,e., the triple (V, Φ', c) is then cobordant to the triple (V, Φ, c) for any equivariant cohomology class c on V.*

Proof. Let I be the closed interval $[0, 1]$. On the product $I \times V$, take the T-action induced from that on V. The function

$$\tilde{\Phi}(t, v) := (1 - t)\Phi(v) + t\Phi'(v)$$

is proper because its η-component is proper. It is an abstract moment map because the fixed point set of any subgroup of the torus contains the origin, 0, and because $\Phi(0) = \Phi'(0)$. The pair $(I \times V, \tilde{\Phi})$ provides a cobordism between (V, Φ) and (V, Φ'). An equivariant cohomology class c on V can be simply pulled back to $I \times V$. q.e.d.

A similar result holds for vector bundles. Let E be a real vector bundle over a compact connected manifold F. Suppose that the torus T acts linearly on the fibers of E, fixing only the zero section. Let $\pm\alpha_i \in \mathfrak{t}^*$ be the real weights for this action. Let $E = E_1 \oplus \ldots \oplus E_m$ be the decomposition of E into sub-vector-bundles such that the torus T acts on the fibers of E_i by the real weight $\pm\alpha_i$. Fix a polarizing vector, $\eta \in \mathfrak{t}$, and let α_i^η be the polarized weights. Fix an invariant inner product on the fibers of E. Let a be any element of $\mathfrak{t}^*$. Define

$$\Phi(v) = a + \sum_{i=1}^{n} ||v_i||^2 \alpha_i^\eta,$$

where $v = \sum v_i$, $v_i \in E_i$. Then Φ is an abstract moment map on E whose value on the zero section is a and whose η-component is proper and bounded from below. Any other abstract moment map with these properties is cobordant to Φ.

We will need an equivariant version of this result. Let $\pi : E \longrightarrow F$ be a real vector bundle over a compact connected manifold. Suppose that the torus T acts on E by bundle maps and that $H \subset T$ is a subgroup whose fixed point set is exactly the zero section. Let $\pm\alpha_i \in \mathfrak{h}^*$ be the real weights for the action of the identity component, H_0, of H on the fibers of E. Fix a vector $\eta \in \mathfrak{h}$ such that $\langle \alpha_i, \eta \rangle \neq 0$ for all i. Let $\alpha_i^\eta = \pm\alpha_i$ be the polarized weights, with $\langle \alpha_i^\eta, \eta \rangle > 0$. For $v \in E$, let $v = \sum v_i$ be the decomposition such that H_0 acts on v_i with weight α_i.

Then

$$\Phi_1(v) = \sum_{i=1}^{n} ||v_i||^2 \alpha_i^\eta$$

is an H-abstract moment map on the fibers of E. For every embedding $i : \mathfrak{h}^* \hookrightarrow \mathfrak{t}^*$ whose composition with the restriction $\mathfrak{t}^* \longrightarrow \mathfrak{h}^*$ is the identity, the composition $i \circ \Phi_1 : E \longrightarrow \mathfrak{t}^*$ is a T-abstract moment map.

Let $\Phi_2 : F \longrightarrow \mathfrak{t}^*$ be any T-abstract moment map on the base space, F, identified with the zero section of E. (Since H acts trivially and F is connected, the image of Φ_2 is contained in a translation of the annihilator, $\mathfrak{h}^0$, of $\mathfrak{h}$ in $\mathfrak{t}^*$.) The composition $\Phi_2 \circ \pi : E \longrightarrow \mathfrak{t}^*$ is also a T-abstract moment map, because the projection π is T-equivariant.

The sum $i \circ \Phi_1 + \Phi_2 \circ \pi$ is a T-abstract moment map on E whose η-component is proper and bounded from below.

Lemma 5.0.2. *Any two T-moment maps on E whose η-components are proper and bounded from below and whose restrictions to the zero section coincide are cobordant.*

Proof. If Φ and Φ' are such abstract moment maps, take $I \times E$ with the abstract moment map $\tilde{\Phi}(t, v) = (1 - t)\Phi(v) + t\Phi'(v)$.   q.e.d.

6. T-fixed points

We now state our basic theorem.

Theorem 1. *Let M be an oriented manifold with an action of a torus T, let Φ be an abstract moment map, and let c be an equivariant cohomology class on M. Suppose that the vector $\eta \in \mathfrak{t}$ is such that the η-component of the abstract moment map,*

$$\langle \Phi, \eta \rangle : M \longrightarrow \mathbb{R},$$

is proper and bounded from below. Denote by M^η the zero set of the vector field on M corresponding to η. Then the triple (M, Φ, c) is cobordant to the disjoint union, over all the components F of the set M^η, of the triples (NF, Φ_F^η, c_F), where NF is the normal bundle of F in M with the T-action induced from that on M, c_F is the pullback of the cohomology class c via the maps $NF \longrightarrow F \hookrightarrow M$, and Φ_F^η is the T-abstract moment map

$$(6.0.1) \qquad \Phi_F^\eta(v) = \Phi(p) + \sum_i ||v_i||^2 \alpha_{i,F}^\eta, \quad v \in N_p F,$$

as constructed above Lemma 5.0.2 with $E = NF$, $\Phi_2 = \Phi|_F$, and H the closure of $\{\exp(t\eta),\ t \in \mathbb{R}\}$ in T.

In particular, if M is compact and T acts with isolated fixed points, M is cobordant to the disjoint union of the vector spaces T_pM, over all fixed points p in M, with appropriate abstract moment maps.

Proof of a special case of Theorem 1. To convey the idea of the proof let us first deal with a circle action with isolated fixed points. Let $\Phi : M \longrightarrow \mathbb{R}$ be an abstract moment map that is proper and bounded from below. We will show that the pair (M, Φ) is cobordant to the union of the pairs (T_pM, Φ_p) over the fixed points, p, of the circle action, where $\Phi_p : T_pM \longrightarrow \mathbb{R}$ is defined by $\Phi_p(v) = \Phi(p) + ||v||^2$ for some invariant norm on T_pM.

Take the product of M with the half-open interval:

$$M \times (0, 1],$$

with the circle action induced from that on M. This provides a non-compact cobordism of M with the empty set. The function $(m, x) \mapsto \Phi(m)$ is an abstract moment map whose restriction to the boundary, $M \times \{1\}$, is Φ. However, this map is not proper. We fix this by adding a function $\rho(x)$, $0 < x \le 1$, that approaches ∞ as x approaches 0, and that vanishes for $\epsilon \le x \le 1$. If there are no fixed points, the sum

$$\tilde{\Phi}(m, x) = \Phi(m) + \rho(x)$$

is a proper abstract moment map whose restriction to the boundary, $M \times \{1\}$, is Φ, and we are done.

If there are fixed points, this does not work: the components of the fixed point set in $M \times (0, 1]$ have the form $F \times (0, 1]$ where F is a component of the fixed point set in M. The function $\tilde{\Phi}$ is constant on $F \times [\epsilon, 1]$ but not on $F \times (0, \epsilon]$, so it is not constant on the fixed components, so it is not an abstract moment map. However, in this case we can take the same function $\tilde{\Phi}$ on the manifold

$$(M \times (0, 1]) \smallsetminus \sqcup_p B_p,$$

where B_p is an ϵ-neighborhood of $(p, 0)$ in $M \times [0, 1]$ with respect to some invariant metric on M. The union is over the (isolated) fixed points p in M. Here ϵ is the same as in the definition of $\tilde{\Phi}$, and the metric on M is chosen so that ϵ-neighborhoods of fixed points are disjoint from each other, and so that an ϵ-neighborhood of the fixed point p

is equivariantly diffeomorphic to T_pM. The removal of B_p creates a boundary component that is equivariantly diffeomorphic to a ball in M around p via the projection map $(m, x) \mapsto m$. This is, further, equivariantly diffeomorphic to T_pM. The function $\tilde{\Phi}$ on this boundary component transforms to a function $\Phi_p(v)$ on T_pM that takes the value $\Phi(p)$ at the origin and that approaches ∞ as we approach infinity in T_pM. By Lemma 5.0.1, this function is cobordant to Φ_p.   q.e.d.

Proof of Theorem 1.   We return to the full generality of torus actions. Let $H \subseteq T$ be the closure of the one-parameter subgroup $\{\exp(s\eta), s \in \mathbb{R}\}$ of T. Since the set M^η coincides with the fixed point set, M^H, of H in T, it consists of a disjoint union of submanifolds of M.

Let F be a connected component of the set M^η. Since the η-component of Φ is proper and is constant on F, F is compact. There exist an invariant Riemannian metric on M and an $\epsilon > 0$ such that the ϵ-neighborhood of F in M is equivariantly diffeomorphic to the normal bundle, NF, of F in M, and such that these neighborhoods are disjoint for different F's.

Let B_F denote the ϵ-neighborhood of $\{0\} \times F$ in $[0, 1] \times M$. As before, $W := (M \times (0, 1]) \smallsetminus \sqcup_F B_F$ provides a non-compact equivariant cobordism between M and $\sqcup_F NF$. Our cobording abstract moment map will now have the form $\tilde{\Phi}(m, x) = \Phi(m) + \rho(m, x)$.

We first define ρ locally: let U be an invariant neighborhood of m in M that admits an equivariant retraction to the orbit $T \cdot m$. On the product $U \times (0, 1]$, define

$$\rho_m(m', x) = \alpha_m w(x),$$

where $w : (0, 1] \longrightarrow \mathbb{R}$ is such that $w(x)$ approaches infinity as x approaches 0 and vanishes for $\epsilon \leq x \leq 1$, and where $\alpha_m \in \mathfrak{t}^*$ is chosen in the following way. If $m \in M^\eta$, we set $\alpha_m = 0$. If $m \notin M^\eta$, the vector η is not in the Lie algebra, $\mathfrak{g}_m$, of the stabilizer of m. We then choose $\alpha_m \in \mathfrak{t}^*$ to be an element of the annihilator, $\mathfrak{g}_m^0$, of $\mathfrak{g}_m$, such that $\langle \alpha, \eta \rangle > 0$.

We patch the functions ρ_m together, using a partition of unity, to obtain an abstract moment map ρ on $W = M \times (0, 1] \smallsetminus \sqcup_F B_F$ satisfying $\langle \rho(m, x), \eta \rangle \longrightarrow \infty$ as $x \longrightarrow 0$ and $\rho(m, x) = 0$ for $\epsilon \leq x \leq 1$.

Finally, take

$$\tilde{\Phi}(m, x) := \Phi(m) + \rho(m, x).$$

This is an abstract moment map on $(M \times (0,1]) \setminus \sqcup_F B_F$. The component $\langle \tilde{\Phi}, \eta \rangle$ is proper, so $\tilde{\Phi}$ itself is proper. On the boundary component $M \times \{1\}$ we get the original abstract moment map:

$$\tilde{\Phi}(m,1) = \Phi(m).$$

The new boundary component that we get by removing the open set B_F can be identified with an ϵ-neighborhood of F via $(m,x) \mapsto m$, and further with the normal bundle NF. Via these identifications, the abstract moment map on this boundary component becomes an abstract moment map Φ'_F on NF that coincides with the map Φ at the points of the zero section (identified with the points of the base F), and whose η-component, $\langle \Phi'_F(v), \eta \rangle$, approaches infinity as v approaches infinity. By Lemma 5.0.2, this is further cobordant to the abstract moment map Φ^η_F that is given by the formula (6.0.1).

Finally, the pullback of the equivariant cohomology class c under the projection $M \times (0,1] \longrightarrow M$ restricts to an equivariant cohomology class $\tilde{c}$ on the cobording manifold W, whose pullback to the boundary component $\partial B_F \cong NF$ is c_F.   q.e.d.

7. Reduction

If (M, ω) is a symplectic manifold with a Hamiltonian torus action and Φ is a moment map in the usual sense, the torus acts locally freely on the regular level sets of Φ. The same holds for abstract moment maps; the proof is only slightly harder:

Lemma 7.0.1. *Let M be a manifold with an action of a torus T, and let Φ be an abstract moment map. If a is a regular value of Φ, the T-action on the level set $\Phi^{-1}(a)$ is locally free, i.e., all stabilizers are discrete.*

Proof. First let us assume that T is a circle. Let F be a component of its fixed point set. Since Φ is constant on F, the restriction of $d\Phi$ to the tangent bundle of F is zero. Since Φ is invariant under the circle action, the restriction of $d\Phi$ to the normal bundle of F is zero. This shows that every fixed point for the circle action is a critical point for the abstract moment map Φ.

Now let the torus T be of any dimension. We need to show that any point $m \in M$ whose stabilizer has positive dimension is a critical point for Φ. Let H be a circle subgroup of the stabilizer of m. The previous

paragraph, applied to the action of H, implies that m is a critical point for the H-component of Φ, hence for Φ.   q.e.d.

We denote by $M_{\mathrm{red}} = M_{\mathrm{red}}^{\Phi}(a)$ the quotient $\Phi^{-1}(a)/T$. By Lemma 7.0.1, if a is a regular value of Φ, M_{red} is an orbifold. If Φ is proper, M_{red} is compact.

An orientation on M induces an orientation on M_{red} in the following way. At any point $p \in \Phi^{-1}(a)$ we have a (non-canonical) splitting: $T_pM = T_{\pi(p)}M_{\mathrm{red}} \oplus \mathfrak{t} \oplus \mathfrak{t}^*$, where the $\mathfrak{t}$-piece is the tangent to the orbit, and the $\mathfrak{t}^*$-piece is the normal to the level set $\Phi^{-1}(a)$, identified with $\mathfrak{t}^*$ via $d\Phi$. We demand that the orientation of M_{red}, followed by the natural orientation on $\mathfrak{t} \oplus \mathfrak{t}^*$, give the orientation of M.

Consider the inclusion map i and the quotient map π in the following diagram:

$$(7.0.2) \qquad \begin{array}{ccc} \Phi^{-1}(a) & \overset{i}{\hookrightarrow} & M \\ \pi \downarrow & & \\ M_{\mathrm{red}} & & \end{array}$$

An equivariant cohomology class on M descends to an ordinary cohomology class on M_{red} in the following way. An ordinary cohomology class on M_{red} can be viewed as an equivariant cohomology class when M_{red} is taken with the trivial circle action. As such, the class c_{red} on M_{red} that is associated to an equivariant class c on M is characterized by the equality

$$(7.0.3) \qquad\qquad \pi^*(c_{\mathrm{red}}) = i^*(c)$$

of equivariant classes on the level set $\Phi^{-1}(a)$. Here we have used the fact that if a group acts locally freely on a space, the equivariant cohomology (over $\mathbb{R}$) is equal to the ordinary cohomology of the quotient.

The map $c \mapsto c_{\mathrm{red}}$ that we have just described is a ring homomorphism from the equivariant cohomology of M to the ordinary cohomology of M_{red}. This is the famous *Kirwan map*.

Definition 7.0.4. Let M be a manifold with an action of a torus T, Φ be an abstract moment map on M, and a be a regular value of Φ. The *reduction* of the triple (M, Φ, c) at the value a is the pair $(M_{\mathrm{red}}, c_{\mathrm{red}})$, where M_{red} is the compact oriented orbifold $\Phi^{-1}(a)/T$, and c_{red} is the ordinary (real) cohomology class on M_{red} satisfying (7.0.3).

To avoid ambiguity we may keep track of Φ and a in the notation by writing $M_{\mathrm{red}}(a)$ and $c_{\mathrm{red}}(a)$, or $M_{\mathrm{red}}^{\Phi}(a)$ and $c_{\mathrm{red}}^{\Phi}(a)$.

8. Cobordism of reduced spaces

Reduction of a triple (M, Φ, c), consisting of a manifold with a T-action, a proper abstract moment map, and an equivariant cohomology class, produced a pair $(M_{\mathrm{red}}, c_{\mathrm{red}})$ where M_{red} is a compact orbifold and c_{red} is an ordinary cohomology class on M_{red}.

Definition 8.0.1. A cobordism between pairs (M, c) and (M', c'), where M and M' are compact oriented orbifolds, and c and c' are cohomology classes, is a pair $(W, \tilde{c})$ where W is a compact oriented orbifold-with-boundary and $\tilde{c}$ an ordinary cohomology class on W, and a diffeomorphism of $M \sqcup M'$ with the boundary ∂W, that sends $\tilde{c}$ to $c \sqcup c'$, respects the orientation on M, and flips the orientation on M'.

The integral $\int_M c$ is an invariant of cobordism of the pair (M, c); this follows from Stokes's theorem.

Lemma 8.0.2. *Let (M, Φ, c) be a manifold with a T-action, a proper abstract moment map, and an equivariant cohomology class. Let a and b be two values in $\mathfrak{t}^*$ such that all values in the closed interval $[a, b]$ are regular values for Φ. Then $M_{\mathrm{red}}(a)$ is cobordant to $M_{\mathrm{red}}(b)$.*

Proof. The quotient $\Phi^{-1}([a, b])/T$ provides the desired cobordism. The cobording cohomology class is the one whose pullback to $\Phi^{-1}([a, b])$ is the restriction of c. q.e.d.

Remark. Under the assumptions of Lemma 8.0.2, the reduced spaces are diffeomorphic and not just cobordant. (Without the torus action and cohomology class, this follows from Ehresmann's fibration theorem.)

Lemma 8.0.2 shows that, up to cobordism, the reduced space depends in a robust way on the value at which we reduce. We will now see that the dependence on the abstract moment map is also robust.

Guillemin was the first to note that "cobordism commutes with reduction" is true and useful for symplectic manifolds. Here is the statement for abstract moment maps:

Lemma 8.0.3. *Let (M, Φ, c) and (M', Φ', c') be cobordant oriented manifolds with T-actions, proper abstract moment maps, and equivariant cohomology classes. Let $a \in \mathfrak{t}^*$ be a value that is regular for both Φ and Φ'. Then the corresponding reductions, $(M_{\mathrm{red}}, c_{\mathrm{red}})$ and $(M'_{\mathrm{red}}, c'_{\mathrm{red}})$, are cobordant.*

Proof. Let $(W, \tilde{\Phi}, \tilde{c})$ be a cobording manifold, proper abstract mo-

ment map, and equivariant cohomology class. Since Φ and Φ' are proper, their sets of regular values are open, so any value b that is close enough to a is also a regular value for Φ and Φ'. For such b, the reduced spaces at b are cobordant to the reduced spaces at a, by Lemma 8.0.2. If we choose such a b that is also a regular value for $\tilde{\Phi}$, the quotient $\tilde{\Phi}^{-1}(b)/T$ provides a cobordism between the reduced spaces at b. The cobording cohomology class is the one whose pullback to $\tilde{\Phi}^{-1}(b)$ is the restriction of $\tilde{c}$.   q.e.d.

Corollary 8.0.4. *For every regular value $a \in \mathfrak{t}^*$, the integral $\int_{M_{\mathrm{red}}(a)} c_{\mathrm{red}}(a)$ is an invariant of cobordism of the triple (M, Φ, c).*

9. Hamiltonian manifolds

Corollary 8.0.4 implies that the Duistermaat-Heckman measure of a Hamiltonian T-manifold is an invariant of cobordism. We first recall some definitions:

Definition 9.0.1. Let (M, ω) and (M', ω') be two symplectic manifolds acted upon by a torus T with proper moment maps Φ and Φ'. A *cobordism* between them is an oriented manifold-with-boundary W with a T-action, a closed two-form $\tilde{\omega}$, and a proper moment map $\tilde{\Phi} : W \longrightarrow \mathfrak{t}^*$ (*i.e.*, a map satisfying $d\langle \tilde{\Phi}, \xi \rangle = -\iota(\xi_W)\tilde{\omega}$ for all $\xi \in \mathfrak{t}$, when ξ_W is the vector field on W corresponding to ξ), and an equivariant diffeomorphism of $M \sqcup M'$ with the boundary ∂W that carries $\tilde{\omega}$ to $\omega \sqcup \omega'$ and $\tilde{\Phi}$ to $\Phi \sqcup \Phi'$, and that respects orientation on M and flips orientation on M'.

Remark. In the above definition there is no reason to assume that the ω and ω' are nondegenerate. They can be just closed two-forms. In fact, any two closed two-forms on the same manifold that belong to the same cohomology class are cobordant, so we can even work with cohomology classes instead of two-forms.

Definition 9.0.2. *Liouville measure* on a symplectic manifold (M, ω) is defined on open sets by integrating the volume form $\omega^n/n!$ ($n = \dim M$) with respect to the symplectic orientation. In the presence of a Hamiltonian T-action, the corresponding *Duistermaat-Heckman measure* is the measure on $\mathfrak{t}^*$ obtained as the push-forward of Liouville measure by the moment map.

Let the torus T act on a symplectic manifold (M, ω) with a proper

moment map Φ. At any regular value a, the Marsden-Weistein reduced space is the compact symplectic orbifold $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ defined by $M_{\mathrm{red}} = \Phi^{-1}(a)/T$ and $\pi^*\omega_{\mathrm{red}} = i^*\omega$, where i and π are the inclusion and quotient maps as in (7.0.2).

Definition 9.0.3. The *Duistermaat-Heckman function* associates to a regular value of the moment map the symplectic volume of the reduced space: $a \mapsto \int_{M_{\mathrm{red}}} \omega_{\mathrm{red}}^d/d!$, $M_{\mathrm{red}} = \Phi^{-1}(a)/T$, $d = \dim M_{\mathrm{red}}$.

The Duistermaat-Heckman measure is absolutely continuous with respect to Lebesgue measure on $\mathfrak{t}^*$, provided that the torus action is effective. It is well known, and is not hard to prove, that the Radon-Nikodym derivative of the Duistermaat-Heckman measure is the Duistermaat-Heckman function.

The Kirwan map (see §7) sends the equivariant cohomology class $[\omega + \Phi]$ on M to the ordinary cohomology class $[\omega_{\mathrm{red}} + a]$ (of mixed degree) on $M_{\mathrm{red}}(a)$.

Finally, Corollary 8.0.4 applied to the equivariant cohomology classes of $(\omega + \Phi - a)^d/d!$ on M implies the following:

Proposition 9.0.4. *The Duistermaat-Heckman measure is an invariant of cobordism.*

Remark. The fact that the Duistermaat-Heckman measure is invariant under (compact) cobordisms was observed by V. Ginzburg several years ago. This was one of the main motivations for introducing cobordism techniques into equivariant symplectic geometry.

10. Application: the Jeffrey-Kirwan localization formula

Theorem 1, together with the facts that "cobordism commutes with reduction" (Lemma 8.0.3) and that the integral of a reduced class is an invariant of cobordism (Corollary 8.0.4), yields a formula for integrals over the reduced space in terms of fixed point data:

Proposition 10.0.1. *Under the assumptions and notation of Theorem 1, let $a \in \mathfrak{t}^*$ be a value that is regular for Φ and all the Φ_F^η's. Let $M_{\mathrm{red}} = M_{\mathrm{red}}^\Phi(a)$ and $(NF)_{\mathrm{red}} = (NF)_{\mathrm{red}}^{\Phi_F^\eta}(a)$ be the corresponding reduced spaces, and let $c_{\mathrm{red}} = c_{\mathrm{red}}^\Phi(a)$ and $(c_F)_{\mathrm{red}} = (c_F)_{\mathrm{red}}^{\Phi_F^\eta}(a)$ be the*

corresponding cohomology classes on these reduced spaces. Then

$$(10.0.2) \qquad \int_{M_{\mathrm{red}}} c_{\mathrm{red}} = \sum_F \int_{(NF)_{\mathrm{red}}} (c_F)_{\mathrm{red}}.$$

Notice that for a generic η, the sum in (10.0.2) is over the components of the fixed point set, M^T. Also notice that the F-summand is nonzero only if $\langle \Phi(F), \eta \rangle < \langle a, \eta \rangle$.

Proposition 10.0.1 generalizes Guillemin's topological form for the Jeffrey-Kirwan localization theorem in the abelian case. Jeffrey and Kirwan consider Hamiltonian actions of compact nonabelian groups on compact symplectic manifolds. Every cohomology class on the reduced space at 0 comes via "Kirwan's map" from an equivariant cohomology class on the manifold (as in (7.0.3)). Jeffrey and Kirwan express the integral of such a class over the reduced space by an explicit formula that only depends on data at the set of points that are fixed under the action of a maximal torus. Shaun Martin showed, by topological means, that their formula would follow from a localization formula for abelian groups. Guillemin interpreted this abelian localization formula as Formula (10.0.2).

11. Application: the Guillemin-Lerman-Sternberg formula

Proposition 10.0.1 implies a generalization of the formula of Guillemin, Lerman, and Sternberg for the Duistermaat-Heckman measure (whose definition was recalled in §9.)

Proposition 11.0.1. *Let (M, ω) be a pre-symplectic oriented manifold with a Hamiltonian action of the torus T and a moment map $\Phi : M \longrightarrow \mathfrak{t}^*$ (in the sense of Example 2.0.6). Let $\eta \in \mathfrak{t}$ be a vector such that the function $\langle \Phi, \eta \rangle : M \longrightarrow \mathbb{R}$ is proper and bounded from below. Let M^η be the set of zeros of the vector field on M that corresponds to η.*

For every component F of the set M^η, let NF be the normal bundle of F in M, ω_F be a pre-symplectic form on NF, and Φ_F^η be its moment map, such that its restriction to the zero section is $\Phi|_F$ and its η-component is proper and bounded from below. Then

$$(11.0.2) \qquad DH(M) = \sum_F DH(NF),$$

where $DH(\cdot)$ are the corresponding Duistermaat-Heckman measures.

Remark. There exist symplectic forms on the fibers of NF for which the map Φ_F^η, given by (6.0.1), is a moment map in the usual, symplectic, sense. With these forms NF becomes a symplectic vector bundle. By "minimal coupling" (see, e.g., [10]) we can construct a closed two-form, ω_F, on the total space of the bundle NF whose restriction to the fibers is given by the symplectic forms just mentioned, and whose pullback to F via the zero section coincides with the pullback of ω under the inclusion of F into M. (ω_F need not be symplectic on NF.) Then $\Phi_F^\eta : NF \longrightarrow \mathfrak{t}^*$ is a moment map for (NF, ω_F) whose restriction to the zero section is $\Phi|_F$ and whose η-component is proper and bounded from below.

The Guillemin-Lerman-Sternberg formula was originally asserted for compact symplectic manifolds and torus actions with isolated fixed points. The vector $\eta \in \mathfrak{t}$ was chosen to be generic, in the sense that the zero set M^η was required to be equal to M^T, the fixed point set for T in M. The right-hand side of (11.0.2) in this case is the sum of the Duistermaat-Heckman measures, $\mathrm{DH}(T_pM)$, for the linear torus actions on the tangent spaces at the fixed point. The original formula of Guillemin-Lerman-Sternberg [9] involves an explicit formula for the measure $\mathrm{DH}(T_pM)$ as the push-forward of Lebesgue measure on a positive orthant in $\mathbb{R}^n$, $n = \frac{1}{2}\dim M$, via a linear projection to $\mathfrak{t}^*$. See [10]. The formula was extended by Canas da Silva and Guillemin [4] to manifolds with non-isolated fixed points (and to orbifolds). The formula of Duistermaat and Heckman, whose "dual" is the Guillemin-Lerman-Sternberg formula, was extended by Prato and Wu [12] to non-compact manifolds with finitely many fixed points and in which the moment map has a component that is proper and bounded from below. Our version is yet more general: we allow a non-compact manifold with infinitely many components of fixed points that need not be isolated, and we obtain formulas for polarizing vectors η that are not generic; see Example 11.0.4.

Example 11.0.3. Take $\mathbb{CP}^2$ with the Fubini-Study symplectic form, whose pullback to $S^5 \subset \mathbb{C}^3$ is the standard two-form $\sum_{j=1}^3 dx_j \wedge dy_j$. The linear action of $T = S^1 \times S^1$ on $\mathbb{C}^3$ by scalar multiplication on the first two coordinates induces an action on $\mathbb{CP}^2$. The image of the moment map is Lebesgue measure on a triangle. The action on $\mathbb{CP}^2$ has three isolated fixed points; their images are the vertices of the triangle. Each of the summands $\mathrm{DH}(T_pM)$ is Lebesgue measure on the region between two rays and zero outside. The Guillemin-Lerman-Sternberg

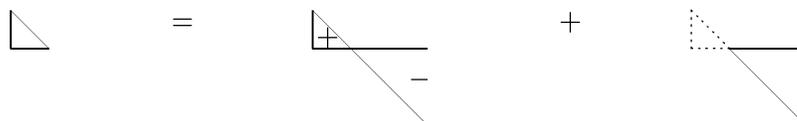FIGURE 1. The Guillemin-Lerman-Sternberg formula for $\mathbb{CP}^2$

FIGURE 2. The (generalized) Guillemin-Lerman-Sternberg formula for $\mathbb{CP}^2$, with a non-generic choice of "polarizing vector"

formula exhibits the triangle as a combination of three such "wedges"; see Figure 1. Notice that this involves a choice of direction in which the infinite rays are pointing; this is the choice of the vector η.

Example 11.0.4. In Example 11.0.3, if η is chosen non-generically, the zero set M^η consists of one copy of $\mathbb{CP}^1$ and one isolated fixed point. The (generalized) Guillemin-Lerman-Sternberg formula then gives the combination illustrated in Figure 2. Here the first summand is the Duistermaat-Heckman measure for the normal bundle of $\mathbb{CP}^1$ in $\mathbb{CP}^2$. This summand is a *signed* measure; this reflects the fact that the cohomology class on this normal bundle does not have an invariant symplectic representative.

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