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CLOSED HYPERSURFACES OF PRESCRIBED MEAN CURVATURE IN LOCALLY CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

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Abstract

We prove the existence of smooth closed hypersurfaces of prescribed mean curvature homeomorphic to S^n for small $n, n \leq 6$, provided there are barriers.

0. Introduction

In a complete $(n+1)$-dimensional manifold N we want to find closed hypersurfaces M of *prescribed mean curvature*. To be more precise, let Ω be a connected open subset of N, $f \in C^{0,1}(\bar{\Omega})$. Then we look for a closed hypersurface $M \subset \Omega$ such that

$$H_{|M} = f(x) \qquad \forall\, x \in M, \tag{0.1}$$

where $H_{|M}$ is the mean curvature, i.e., the sum of the principal curvatures.

The existence of a generalized solution $M = \partial E$, where E is a Caccioppoli set minimizing an appropriate functional, is easily demonstrated if the boundary of Ω is supposed to consist of two components acting as barriers. For small n, $n \leq 6$, the generalized solution is also a classical one, since it is smooth, $M \in C^{2,\alpha}$, and hence a solution of (0.1); but nothing is known about its topological type.

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We shall prove that in the case where $n \le 6$ and N is locally conformally flat, or more precisely, when in Ω the metric is conformally flat, smooth solutions homeomorphic to S^n exist.

We make the following definition

Definition 0.1. Let M_1, M_2 be closed hypersurfaces in N homeomorphic to S^n and of class $C^{2,\alpha}$ which bound an open, connected, relatively compact subset Ω. M_1, M_2 are called *barriers* for (H, f) if

$$H_{|M_1} \le f \tag{0.2}$$

and

$$H_{|M_2} \ge f. \tag{0.3}$$

Here, the mean curvature of M_1 is calculated with respect to the normal that points outside of Ω while the mean curvature of M_2 is calculated with respect to the normal that points inside of Ω.

Remark 0.2. In view of the weak Harnack inequality the barriers do no t touch each other, unless both coincide and have prescribed mean curvature f. In this case Ω would be empty.

We shall consider such a region Ω bounded by barriers M_1, M_2 for (H, f), where $f \in C^{0,1}(\bar{\Omega})$ is given, and assume that Ω is conformally equivalent to an open, bounded set in $\mathbf{R}^{n+1}$. Furthermore, if we identify Ω with its image in $\mathbf{R}^{n+1}$ we suppose, that the barriers M_1, M_2 can be considered as graphs over S^n, i.e., after fixing the origin and having introduced Euclidean polar coordinates $(x^\alpha)_{0\le\alpha\le n}$, where $x^0 = r$ represents the radial distance, each M_i can be written as a graph

$$M_i = \operatorname{graph} u_{i|_{S^n}} = \{(x, x^0) \,:\, x^0 = u_i(x),\, x \in S^n\}, \tag{0.4}$$

where we use slightly ambiguous notation.

The polar coordinates can also be considered to be a coordinate system in N covering Ω; the metric in N then has the form

$$d\bar{s}_N^2 = e^{2\psi} d\bar{s}^2_{\mathbf{R}^{n+1}} = e^{2\psi}(dr^2 + r^2\sigma_{ij}dx^i dx^j), \tag{0.5}$$

where (σ_{ij}) is the standard metric on S^n.

Under these assumptions we shall prove

Theorem 0.3. *Let Ω, M_1, M_2, and f satisfy the assumptions stated above. Then the problem*

$$H_{|M} = f \tag{0.6}$$

has a solution $M \subset \bar{\Omega}$ of class $C^{2,\alpha}$ homeomorphic to S^n, if $n \le 6$.

Remark 0.4. Neither the function f nor its derivatives are supposed to satisfy any sign conditions. Even the assumption on the smoothness of f can be relaxed; if f is only bounded, then a solution M of class $H^{2,p}$ would exist for any finite p.

The problem of finding closed hypersurfaces of prescribed mean curvature has been considered by Bakelman and Kantor [1] and Treibergs and Wei [17] in the case where $N = \mathbf{R}^{n+1}$ assuming that f is positive and satisfies

$$\frac{\partial}{\partial r}(rf) \le 0, \tag{0.7}$$

r being the geodesic distance to some fixed origin. In [5] we proved the existence of a convex solution in space forms under the assumption

$$-K_N f \bar{g}_{\alpha\beta} + f_{\alpha\beta} \le 0 \qquad \text{in } \Omega, \tag{0.8}$$

where in addition f is supposed to be positive if $K_N > 0$. In all cases the existence of barriers is required.

The paper is organized as follows: In Section 1 we derive the basic equations for hypersurfaces in conformally flat spaces, in Section 2 we solve auxiliary problems, the solutions of which converge to the desired solution as is shown in Sections 3 and 4.

1. Notation and preliminary results

Let N be a complete $(n+1)$-dimensional manifold and M a closed hypersurface. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), (R_{ijkl}), etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N (resp. M) will be denoted by (x^α) (resp. (ξ^i)). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e., for a function u on N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but, e.g. the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon \tag{1.1}$$

with obvious generalizations to other quantities.

In local coordinates x^α and ξ^i the geometric quantities of the hypersurface M are connected through the following equations

$$x^\alpha_{ij} = -h_{ij}\nu^\alpha \tag{1.2}$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i,e.,

$$x^\alpha_{ij} = x^\alpha_{,ij} - \Gamma^k_{ij}x^\alpha_k + \bar\Gamma^\alpha_{\beta\gamma}x^\beta_i x^\gamma_j. \tag{1.3}$$

The comma indicates ordinary partial derivatives.

In this implicit definition (1.2) the *second fundamental form* (h_{ij}) is taken with respect to $-\nu$.

The second equation is the *Weingarten equation*

$$\nu^\alpha_i = h^k_i x^\alpha_k. \tag{1.4}$$

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar R_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k \tag{1.5}$$

and the *Gauß equation*

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \bar R_{\alpha\beta\gamma\delta}x^\alpha_i x^\beta_j x^\gamma_k x^\delta_l. \tag{1.6}$$

Assume now, that the metric in N is (locally) conformal to the metric in $\mathbf{R}^{n+1}$, i.e.,

$$d\bar s^2_N = e^{2\psi} d\bar s^2_{\mathbf{R}^{n+1}}\,, \tag{1.7}$$

or more precisely, assume that (1.7) is valid in the region Ω, where we shall consider Ω to be a subset of N as well as $\mathbf{R}^{n+1}$ without changing the notation. The same convention applies to hypersurfaces M contained in Ω where we distinguish the geometric quantities of M considered as a submanifold of $\mathbf{R}^{n+1}$ by using the notation $\hat h_{ij}$, $\hat g_{ij}$, $\hat\nu^\alpha$, etc. The connection with the corresponding quantities in N is given by

$$g_{ij} = e^{2\psi}\hat g_{ij}\,, \tag{1.8}$$

$$\nu^\alpha = e^{-\psi}\hat\nu^\alpha, \tag{1.9}$$

and

$$h_{ij}e^{-\psi} = \hat{h}_{ij} + \psi_\alpha \hat{\nu}^\alpha \hat{g}_{ij}\,. \tag{1.10}$$

Thus, the mean curvatures of M in N resp. $\mathbf{R}^{n+1}$ are related through

$$He^{\psi} = \hat{H} + n\psi_\alpha \hat{\nu}^\alpha. \tag{1.11}$$

Assume now, that M can be written as a graph over S^n, i.e., after introducing polar coordinates (x^α) in $\mathbf{R}^{n+1}$, where $x^0 = r$,

$$M = \operatorname{graph} u_{|_{S^n}} = \big\{\, (r, x) \;:\; r = u(x),\ x \in S^n \,\big\}. \tag{1.12}$$

The metric in $\mathbf{R}^{n+1}$ is then expressed as

$$d\bar{s}^2_{\mathbf{R}^{n+1}} = dr^2 + r^2\sigma_{ij}dx^i dx^j, \tag{1.13}$$

where (σ_{ij}) is the metric of S^n; the induced metric of M is

$$\hat{g}_{ij} = u_i u_j + u^2\sigma_{ij} = u^2(\varphi_i\varphi_j + \sigma_{ij}), \tag{1.14}$$

where $\varphi = \log u$, and its inverse is given by

$$\hat{g}^{ij} = u^{-2}\Big(\sigma^{ij} - \frac{\varphi^i}{v}\frac{\varphi^j}{v}\Big), \tag{1.15}$$

where $v = \sqrt{1 + \sigma^{ij}\varphi_i\varphi_j} \equiv \sqrt{1 + |D\varphi|^2}$, σ^{ij} is the inverse of σ_{ij}, and $\varphi^i = \sigma^{ij}\varphi_j$.

The product $u\hat{H}$ can be represented as

$$u\hat{H} = -D_i(a^i(D\varphi)) + \frac{n}{v}, \tag{1.16}$$

where $a^i(D\varphi) = v^{-1}\varphi^i$ and the divergence is calculated with respect to σ_{ij}.

2. Existence of solutions to an auxiliary problem

As in [6, Section 4] we first find solutions to an auxiliary problem that will converge to the final solution.

Definition 2.1. A hypersurface M_0 is called a supersolution for (H, f) if

$$H_{|_{M_0}} \geq f. \tag{2.1}$$

We, furthermore, recall our assumption that the boundary of Ω consists of barriers M_i, $i = 1, 2$, for (H, f) that can be written as graphs over S^n, $M_i = \operatorname{graph} u_i$.

We can now formulate the auxiliary problem.

Theorem 2.2. *Let $M_0 = \operatorname{graph} u_0$, where $u_1 \le u_0 \le u_2$, and u_0 is a supersolution for (H, f) with $f \in C^{0,1}(\bar{\Omega})$. Then the problem*

$$\begin{cases} H = f - \gamma e^{-\mu u}[u - u_0], \\ u_1 \le u \le u_0 \end{cases} \tag{2.2}$$

has a solution $u \in C^{2,\alpha}(S^n)$ provided γ and μ are sufficiently large, $\mu = \mu(\Omega, f)$ and $\gamma = \gamma(\mu, \Omega, f)$. Here, the reference that a term depends on Ω should also indicate that geometrical quantities of the ambient space and the barriers are involved.

2.1. $\mathbf{C^1}$-estimates

Let $u \in C^{3,\alpha}(S^n)$ be a solution of (2.2), where we first assume a slightly higher degree of smoothness so that the classical maximum principle can be applied to estimate Du, or equivalently, the quantity $v = \sqrt{1 + |D\varphi|^2}$ in (1.15).

Lemma 2.3. *Let $u \in C^{3,\alpha}(S^n)$ be a solution of* (2.2). *Then $v = \sqrt{1 + |D\varphi|^2}$ is estimated by*

$$v \le \operatorname{const}(|Du_0|, |Df|, \gamma, \mu), \tag{2.3}$$

provided γ and μ are sufficiently large.

Proof. We transfer the equation in (2.2) into $\mathbf{R}^{n+1}$, i.e., we consider

$$\hat{H} = e^{\psi} H - n\psi_\alpha \hat{\nu} = e^{\psi}\{f - \gamma e^{-\mu u}[u - u_0]\} - n\psi_\alpha \hat{\nu}^\alpha. \tag{2.4}$$

Let x be the embedding vector for the hypersurface. Then we define

$$\chi = \langle x, \hat{\nu}\rangle^{-1} = u^{-1}\langle Dr, \hat{\nu}\rangle^{-1} = u^{-1} v, \tag{2.5}$$

and we shall prove a priori estimates for χ.

We choose local coordinates and compute the first and the second covariant derivatives of χ

$$\chi_i = -\langle x, \hat{\nu}\rangle^{-2}\langle x, \hat{\nu}_i\rangle = -\chi^2 \hat{h}_i^k \langle x_k, x\rangle, \tag{2.6}$$

$$\chi_{ij} = 2\chi^{-1}\chi_i\chi_j - \chi^2 \hat{h}^k_{i;j}\langle x_k, x\rangle + \chi \hat{h}_i^k \hat{h}_{kj} - \chi^2 \hat{h}_{ij}. \tag{2.7}$$

Hence, we conclude in view of the Codazzi equations,

$$-\Delta\chi = -\hat{g}^{ij}\chi_{ij} = -2\chi^{-1}\|D\chi\|^2 - |\hat{A}|^2\chi + \hat{H}\chi^2 + \hat{H}_k u^k u \chi^2. \tag{2.8}$$

Here, all indices are raised with respect to the induced metric, and we used the abbreviations

$$\|D\chi\|^2 = \hat{g}^{ij}\chi_i\chi_j \qquad \text{and} \qquad |\hat{A}|^2 = \hat{h}_i^k \hat{h}_k^i. \tag{2.9}$$

The crucial terms are those which are quadratic in χ, they have to add up to something negative, if χ is large.

To compute $\hat{H}_k$ it is convenient to introduce polar coordinates (x^α) in $\mathbf{R}^{n+1}$ and to decompose $D_\alpha f$ into its radial part and the tangential components with respect to S^n

$$\dot{f} = \frac{\partial f}{\partial r} \qquad \text{and} \qquad f_{,i} = \frac{\partial f}{\partial x^i}\,. \tag{2.10}$$

We then obtain from (2.4)

$$\begin{aligned} \hat{H}_k &= e^{\psi}\big\{f - \gamma e^{-\mu u}[u-u_0]\big\}\psi_\alpha x_k^\alpha - n\psi_{\alpha\beta}\hat{\nu}^\alpha x_k^\beta - n\psi_\alpha \hat{h}_k^i x_i^\alpha \\ &\quad + e^{\psi}\big\{\dot{f}u_k + f_{,k} + \gamma\mu e^{-\mu u}[u-u_0]u_k - \gamma e^{-\mu u}[u_k - u_{0,k}]\big\}. \end{aligned} \tag{2.11}$$

Using the relations

$$u^k = u^{-2}\sigma^{ik}u_i v^{-2}, \quad \|Du\|^2 = \frac{|D\varphi|^2}{v^2} \tag{2.12}$$

and

$$\chi_i = -\chi^2 \hat{h}_i^k u_k u \tag{2.13}$$

we conclude

$$\begin{aligned} \hat{H}_k u^k u &\le c|f| + c|Df| - c\gamma e^{-\mu u}[u-u_0]\big[\|Du\|^2 + \chi^{-2}\big] \\ &\quad + c\chi^{-1} + \chi^{-3}\|D\chi\|^2 + \gamma\mu e^{-\mu u}[u-u_0]\|Du\|^2 u e^{\psi} \\ &\quad - \frac{1}{2}\gamma e^{-\mu u}\big[\|Du\|^2 - u^{-2}v^{-2}\sigma^{ij}D_i u_0 D_j u_0\big] u e^{\psi}, \end{aligned} \tag{2.14}$$

where $c = c(\Omega)$. Therefore, the righthand-side in (2.8) can be estimated from above by

$$\begin{aligned} & c\big[|f| + |Df| + 1\big]\chi^2 \\ &\quad + (\mu - c)\|Du\|^2\gamma e^{-\mu u}[u-u_0]ue^{\psi}\chi^2 \\ &\quad - \frac{1}{2}\gamma e^{-\mu u}\big[\|Du\|^2 - u^{-2}v^{-2}\sigma^{ij}D_i u_0 D_j u_0\big] u e^{\psi}\chi^2 \end{aligned} \tag{2.15}$$

at points where $\|Du\|^2 \ge \frac{1}{2}$. We now choose μ larger than $2c$, so that the second term in (2.15) is non-positive. Then, we choose γ such that

$$c\big[|f| + |Df| + 1\big] \le \frac{1}{4}\gamma e^{-\mu u} u e^{\psi} \tag{2.16}$$

and from the maximum principle deduce that

$$v \le c(\Omega, Du_0), \tag{2.17}$$

where the constant is determined by the relation

$$|D\varphi|^2 - u^{-2}\sigma^{ij} D_i u_0 D_j u_0 \le \frac{1}{2} v^2. \tag{2.18}$$

q.e.d.

2.2. Existence of solutions

We are still looking for hypersurfaces in $\mathbf{R}^{n+1}$, i.e., we want to solve equation (2.4) with the side-conditions

$$u_1 \le u \le u_0, \tag{2.19}$$

or equivalently, we can solve

$$-D_i(a^i(D\varphi)) + \frac{n}{v} = u\hat{H} = ue^{\psi}\{f - \gamma e^{-\mu u}[u - u_0]\} - un\psi_\alpha \hat{\nu}^\alpha, \tag{2.20}$$

where $\varphi = \log u$; cf. equation (1.16).

Let us denote the lower order terms in the preceding equation by $a(x, \varphi, D\varphi)$, and let $\varphi_1 = \log u_1$, $\varphi_0 = \log u_0$. Then we have to solve

$$\begin{cases} -D_i(a^i(D\varphi)) + a(x, \varphi, D\varphi) = 0, \\ \varphi_1 \le \varphi \le \varphi_0. \end{cases} \tag{2.21}$$

Here, a^i is a strictly monotone vectorfield, the lower order term and its derivatives are bounded in $\varphi_1 \le \varphi \le \varphi_0$, i.e.,

$$|a| + \Big|\frac{\partial a}{\partial x^i}\Big| + \Big|\frac{\partial a}{\partial \varphi}\Big| + \Big|\frac{\partial a}{\partial p^i}\Big| \le \text{const}, \tag{2.22}$$

and moreover

$$\frac{\partial a}{\partial \varphi} \ge \epsilon_0 > 0 \qquad \text{in} \qquad \varphi_1 \le \varphi \le \varphi_0, \tag{2.23}$$

due to our choice of μ and γ, where we increase μ and γ a bit in view of the presence of the additional factor e^{φ}.

To solve (2.21) we first assume that f is of class $C^{1,\alpha}$ and u_0 of class $C^{3,\alpha}$. Extend $f_0 = H_{|M_0}$, where $M_0 = \operatorname{graph} u_0$, to $\bar{\Omega}$ by setting

$$f_0(x, r) = f_0(x) \;, \qquad x \in S^n, \tag{2.24}$$

and consider the convex combination

$$f_t = tf + (1-t)f_0 \;, \qquad 0 \le t \le 1. \tag{2.25}$$

Then, we look at the problems

$$\begin{cases} -D_i(a^i(D\varphi_t)) + a(x, \varphi_t, D\varphi_t) = 0, \\ \ \varphi_1 \le \varphi_t \le \varphi_0, \end{cases} \tag{2.26}$$

where f is replaced by f_t, and we have a slight ambiguity in the notation for $t = 1$. The lower order term also depends explicitly on t, but since the estimates (2.22) and (2.23) are independent of t, if we choose γ sufficiently large—at the moment γ also depends on f_0—, we do not indicate it specifically.

We shall use the continuity method to prove that (2.26) has a solution for all $0 \le t \le 1$. Let us treat (2.26) as a variational inequality

$$\begin{cases} \langle -D_i(a^i(D\varphi_t)) + a(x, \varphi_t, D\varphi_t), \eta - \varphi_t \rangle \ge 0 \qquad \forall\, \eta \in K, \\ K = \big\{\, \eta \in C^{0,1}(S^n) \;:\; \varphi_1 \le \eta \le \varphi_0 \,\big\}. \end{cases} \tag{2.27}$$

It can easily be shown that the obstacles φ_1, φ_0 act as barriers, i.e., they are sub- resp. supersolutions for any value of t, $0 \le t \le 1$, so that any solution of the variational inequality is actually a solution of the corresponding equation. However, our proof of the solvability of (2.27) is valid for arbitrary $C^{2,\alpha}$ obstacles.

Define Λ through

$$\Lambda = \big\{\, t \in [0,1] \;:\; (2.27) \text{ has a solution } \varphi_t \,\big\}. \tag{2.28}$$

Then, we conclude:

(i) $\Lambda \ne \emptyset$, since $0 \in \Lambda$.

(ii) Λ *is closed.* It is well known that any solution of the variational inequality is of class $H^{2,p}(S^n)$ for any finite p. Therefore, the solution does not touch the obstacles at points, where the gradient is larger than the gradients of the obstacles, and it is a solution of the equation there. At those points the solution is also of class $C^{3,\alpha}$ because f, f_0 are of class $C^{1,\alpha}$, and Lemma 2.3 is thus applicable. We conclude further that uniform $H^{2,p}$-estimates are valid and hence uniform $C^{1,\alpha}$-estimates, which proves the closedness of Λ.

(iii) Λ *is open.* To prove the openness we argue as in a former paper [8]. Let $t_0 \in \Lambda$ and let φ_{t_0} be the corresponding solution of (2.27) with

$$|D\varphi_{t_0}| \le c_0. \tag{2.29}$$

Let $\tilde{a}^i = \tilde{a}^i(p)$ be a uniformly monotone vectorfield that coincides with a^i for $|p| \le c_0{+}1$. The existence of such a vectorfield has been shown in [8, Appendix II]. Then, the corresponding variational inequality

$$\begin{cases} \langle -D_i(\tilde{a}^i(D\tilde{\varphi}_t)) + a(x, \tilde{\varphi}_t, D\tilde{\varphi}_t), \eta - \tilde{\varphi}_t \rangle \ge 0 & \forall\, \eta \in \widetilde{K}, \\ \widetilde{K} = \{\, \eta \in H^{1,2}(S^n) \;:\; \varphi_1 \le \eta \le \varphi_0 \,\} \end{cases} \tag{2.30}$$

has a solution $\tilde{\varphi}_t \in H^{2,p}(S^n)$ for any t, $0 \le t \le 1$, since the differential operator is uniformly elliptic in $\widetilde{K}$, and there exists $\lambda > 0$ such that the operator

$$A\varphi = -D_i(\tilde{a}^i(D\varphi)) + a(x, \varphi, D\varphi) + \lambda\varphi \tag{2.31}$$

is uniformly monotone, i.e., there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 \|\varphi - \eta\|^2_{1,2} \le \langle A\varphi - A\eta, \varphi - \eta \rangle \qquad \forall\, \varphi, \eta \in \widetilde{K}, \tag{2.32}$$

where the norm on the left-hand side is the norm in $H^{1,2}(S^n)$. The operator

$$-D_i(\tilde{a}^i(D\varphi)) + a(x, \varphi, D\varphi) \tag{2.33}$$

is therefore *pseudomonotone* and *coercive* in $\widetilde{K}$, and the existence of solutions for the problem (2.30) follows from the general theory for solutions of variational inequalities; cf. [2].

As we shall show below, the solutions of (2.30) are unique for each t, hence, they depend continuously on t in the $C^{1,\alpha}$-norm, and we conclude that for small $\epsilon > 0$

$$(2.34) \qquad |D\tilde{\varphi}_t| \le c_0 + \frac{1}{2} \qquad \forall\, t \in B_\epsilon(t_0),$$

since $\tilde{\varphi}_{t_0} = \varphi_{t_0}$, and we deduce further, that these $\tilde{\varphi}_t$ are also solutions of (2.27), i.e., Λ is open.

We have thus proved that Λ coincides with the whole interval, so we have especially proved the existence of a solution for the crucial value $t = 1$. In this case, $f_t = f$ and the obstacles are barriers, so that we deduce with the help of the weak Harnack inequality, that the solution of the variational inequality is actually a solution of the equation. For details see the uniqueness proof in Lemma 2.4 below.

Let us point out, that at the moment the parameter γ also depends on f_0, and hence on the second derivatives of u_0. However, γ should only depend on $|u_0|$ and the other quantities mentioned in Lemma 2.3. To achieve this result, we more or less repeat the argument just given in the first part of the existence proof.

Let γ_0 be a constant such that the gradient estimate in Lemma 2.3 and the relation (2.23) are valid for $\gamma \ge \gamma_0$. Let $\bar{\gamma} \ge \gamma_0$ be arbitrary and define Λ through

$$(2.35) \qquad \Lambda = \{\, \gamma \ge \bar{\gamma} \ :\ (2.21){=}\text{ has a solution } \}.$$

Λ is not empty as we have just proved. Let $\gamma^\star = \inf \gamma$. By repeating the arguments which we used to prove that the variational inequality has a solution, we conclude that $\gamma^\star = \bar{\gamma}$, i.e., the existence of a solution to the auxiliary problem is guaranteed provided $\gamma \ge \gamma_0$.

Before we prove the uniqueness of the solution to the variational inequality, let us remove the additional assumptions regarding the smoothness of f and u_0. We assumed in the proof $f \in C^{1,\alpha}(S^n)$ and $u_0 \in C^{3,\alpha}(S^n)$, so that the solutions φ_t of (2.27) are of class $C^{3,\alpha}$ at points where they do not touch the obstacles in order to apply the classical maximum principle to estimate the C^1-norm. But the gradient estimate for the final solution, when $t = 1$, only depends on the C^1-norms of f and u_0, hence, we obtain solutions under the weaker assumptions by approximation.

To complete the proof of the theorem, let us now show

Lemma 2.4. *Let $\varphi \in K$ be a solution of the variational inequality*

$$\begin{cases} \langle -D_i(a^i(D\varphi)) + a(x,\varphi,D\varphi), \eta - \varphi\rangle \geq 0 & \forall\, \eta \in K, \\ K = \{\, \eta \in C^{0,1}(S^n) \;:\; \varphi_1 \leq \eta \leq \varphi_0 \,\}. \end{cases} \tag{2.36}$$

Then φ is uniquely determined, where we assume that the condition (2.23) *is valid and a^i, a are of class C^1 in their arguments.*

Proof. Let φ, $\tilde\varphi$ be two solutions of (2.36). Then we have to show $\varphi = \tilde\varphi$, or by symmetry, $\varphi \geq \tilde\varphi$.

We know that φ, $\tilde\varphi$ are of class $H^{2,p}(S^n)$ for any finite p. Suppose

$$G = \{\, x \;:\; \varphi < \tilde\varphi \,\} \neq \emptyset. \tag{2.37}$$

Then G is open, and in G we have

$$-D_i(a^i(D\varphi)) + a(x,\varphi,D\varphi) \geq 0 \qquad \text{because} \qquad \varphi < \varphi_2, \tag{2.38}$$

and

$$-D_i(a^i(D\tilde\varphi)) + a(x,\tilde\varphi,D\tilde\varphi) \leq 0 \qquad \text{because} \qquad \tilde\varphi > \varphi_1. \tag{2.39}$$

Hence, we infer

$$-D_i(a^i(D\varphi)) + D_i(a^i(D\tilde\varphi)) + a(x,\varphi,D\varphi) - a(x,\tilde\varphi,D\tilde\varphi) \geq 0\,, \tag{2.40}$$

or, by setting $\varphi_t = t\varphi + (1-t)\tilde\varphi$, $0 \leq t \leq 1$, and using the main theorem of calculus

$$-D_i(\hat a^{ij} D_j(\varphi - \tilde\varphi)) + \frac{\partial \hat a}{\partial \varphi}(\varphi - \tilde\varphi) + \frac{\partial \hat a}{\partial p_i} D_i(\varphi - \tilde\varphi) \geq 0\,, \tag{2.41}$$

where

$$\begin{aligned} \hat a^{ij} &= \int_0^1 a^{ij}(D\varphi_t)\,, \\ \frac{\partial \hat a}{\partial \varphi} &= \int_0^1 \frac{\partial a}{\partial \varphi}(x,\varphi_t,D\varphi_t)\,, \\ \frac{\partial \hat a}{\partial p_i} &= \int_0^1 \frac{\partial a}{\partial p_i}(x,\varphi_t,D\varphi_t)\,, \end{aligned} \tag{2.42}$$

and we conclude that

$$-D_i(\hat a^{ij} D_j(\varphi - \tilde\varphi)) + \frac{\partial \hat a}{\partial p_i} D_i(\varphi - \tilde\varphi) \geq -\frac{\partial \hat a}{\partial \varphi}(\varphi - \tilde\varphi) \tag{2.43}$$

in G.

Now, by assumption

$$m_0 = \inf(\varphi - \tilde{\varphi}) < 0\,, \tag{2.44}$$

and the infimum is attained at a point $x_0 \in G$. Define $\eta = \varphi - \tilde{\varphi} - m_0$. Then $\eta \geq 0$, and (2.43) yields

$$-D_i(\hat{a}^{ij} D_j \eta) + \frac{\partial \hat{a}}{\partial p_i} D_i \eta > 0 \tag{2.45}$$

contradicting the weak Harnack inequality that would demand $\eta \equiv 0$. Thus, we deduce $m_0 \geq 0$, and the uniqueness is proved. q.e.d.

3. Almost minimal solutions

We now apply the existence result of Theorem 2.2 successively. Let u_2 be *the* upper barrier; if u_{k-1} is already defined for $k \geq 3$, then let $u_k \in C^{2,\alpha}(S^n)$ be the solution of

$$\begin{cases} H = f - \gamma e^{-\mu u_k}[u_k - u_{k-1}], \\ u_1 \leq u_k \leq u_{k-1}. \end{cases} \tag{3.1}$$

The solutions (u_k) form a bounded monotone decreasing sequence, which converges pointwise to a function u. The mean curvatures of the graphs converge pointwise to $f(x,u)$, since γ and μ are fixed; hence, graph u would be a solution of our problem, if the u_k's would satisfy uniform C^1-estimates. But unfortunately, we cannot prove this, it might even be false. Gradient estimates for graphs depend on the Lipschitz constant of the mean curvature, i.e., $|Du_k|$ depends on $|Du_{k-1}|$.

However, the regularity results of De Giorgi, Massari, and Tamanini for almost minimal hypersurfaces imply uniform $C^{1,\frac{1}{2}}$-estimates for the *hypersurfaces* provided the hypersurfaces are almost minimal, their mean curvatures uniformly bounded, and $n \leq 6$; cf. [15], [16].

To apply these results, we shall prove that the hypersurfaces $M_k = \operatorname{graph} \log u_k$ are almost minimal in the *metric product* $S^n \times \mathbf{R}$.

We adopt the view point and the notation from Section 2, i.e., we consider the hypersurfaces as submanifolds of $\mathbf{R}^{n+1}$ and look at their diffeomorphic images in $S^n \times \mathbf{R}$ under the diffeomorphism $\Phi(x,r) = (x, \log r)$. Then, each $\varphi_k = \log u_k$ satisfies the equation in (2.21) on S^n. For notational reasons we drop the index k, having in mind that it is fixed. Furthermore, we consider the lower order term $a(x,\varphi,D\varphi)$ to depend only on x without changing the symbol, i.e., φ is a solution of

$$-D_i(a^i(D\varphi)) + a(x) = 0 \qquad \text{in } S^n, \tag{3.2}$$

where $a(x)$ is uniformly bounded.

Instead of $a(x)$ let us consider the modified lower order term

$$a_{\epsilon_0}(x,t) = a(x) + \epsilon_0(t-\varphi(x)) \tag{3.3}$$

with $\epsilon_0 > 0$. Then, $a_{\epsilon_0}(x,\varphi) = a(x)$ and therefore, we have

$$-D_i(a^i(D\varphi)) + a_{\epsilon_0}(x,\varphi) = 0 \tag{3.4}$$

with

$$\frac{\partial a_{\epsilon_0}}{\partial \varphi} = \epsilon_0 > 0. \tag{3.5}$$

We shall prove that the boundary of the subgraph

$$E = \operatorname{sub}\varphi = \{\, (x,t) \ : \ t < \varphi(x), \ x \in S^n \,\} \tag{3.6}$$

is almost minimal in the metric product $S^n \times \mathbf{R}$, or more precisely, that it solves the variational problem of minimizing the so-called *perimeter* plus an additional prescribed mean curvature term in $S^n \times \mathbf{R}$.

We first show that E is minimal compared with other subgraphs.

Lemma 3.1. *The solution φ of* (3.4) *is also a solution of the variational problem*

$$J(\eta) = \int_{S^n} \sqrt{1+|D\eta|^2} + \int_{S^n}\int_0^\eta a_{\epsilon_0}(x,t) \to \min \qquad \forall\, \eta \in BV(S^n). \tag{3.7}$$

$BV(S^n)$ is the space of functions of bounded variation, i.e., functions whose derivatives are bounded measures. For $\eta \in BV(S^n)$ the area term in (3.7) is defined by

$$\begin{aligned}\int_{S^n} &\sqrt{1+|D\eta|^2} \\ &= \sup\Big\{ \int_{S^n} (\gamma^0 + \eta D_i\gamma^i) \ : \ \gamma^\alpha \in C^\infty(S^n), \ |\gamma^0|^2 + \sigma_{ij}\gamma^i\gamma^j \le 1 \Big\}.\end{aligned} \tag{3.8}$$

It coincides with the usual definition if η is Lipschitz continuous

Proof of Lemma 3.1 The functional in (3.7) consists of the standard area for graphs plus a mean curvature term; the corresponding Euler-Lagrange equation is exactly the equation in (3.4). Thus, it is not surprising that φ should also solve (3.7), since from Lemma 2.4 we know that the solutions of (3.4) are uniquely determined.

Let c_0 be an arbitrary constant such that

$$|D\varphi| < c_0. \tag{3.9}$$

Then, solve the variational problem

$$\begin{cases} J(\eta) \to \min & \forall\, \eta \in K, \\ K = \{\, \eta \in C^{0,1}(S^n) \;:\; |D\eta| \le c_0 \,\}. \end{cases} \tag{3.10}$$

Let $\tilde{\varphi}$ be a solution of (3.10), the existence of which can easily be proved in view of (3.5). Then $\tilde{\varphi}$ solves the variational inequality

$$\langle -D_i(a^i(D\tilde{\varphi})) + a(x,\tilde{\varphi}), \eta - \tilde{\varphi} \rangle \ge 0 \qquad \forall\, \eta \in K. \tag{3.11}$$

On the other hand, since φ is a solution of the equation and belongs to K, from the strict monotonicity of the operator we deduce that $\varphi = \tilde{\varphi}$.

Thus, φ is a solution of the unconstrained variational problem

$$J(\eta) \to \min \qquad \forall\, \eta \in C^{0,1}(S^n). \tag{3.12}$$

Since c_0 is arbitrary, and by approximation we conclude, that φ also minimizes the functional in $BV(S^n)$. q.e.d.

Let $\widetilde{N} = S^n \times \mathbf{R}$ be the metric product of S^n and $\mathbf{R}$, so that the metric in $\widetilde{N}$ is given by

$$ds^2_{\tilde{N}} = dt^2 + \sigma_{ij}dx^i dx^j. \tag{3.13}$$

We also use x^0 instead of t when appropriate.

The *perimeter* of a measurable set $E \subset \widetilde{N}$ with respect to an open set $\widetilde{\Omega} \subset \widetilde{N}$ is defined by

$$\begin{aligned} &\int_{\tilde{\Omega}} |D\chi_E| \\ &\quad = \sup\Big\{ \int_{\tilde{\Omega}} \chi_E D_\alpha \gamma^\alpha \;:\; \gamma^\alpha \in C_c^\infty(\widetilde{\Omega}),\ |\gamma^0|^2 + \sigma_{ij}\gamma^i\gamma^j \le 1 \Big\}, \end{aligned} \tag{3.14}$$

i.e., E has finite perimeter in $\widetilde{\Omega}$ iff χ_E belongs to $BV(\widetilde{\Omega})$. Sets of finite perimeter are also called *Caccioppoli* sets. It is well known that the perimeter of subgraphs is equal to the area of the boundary.

Lemma 3.2. *Let $\Omega \subset S^n$ be open, $\eta \in BV(\Omega)$ and $E = \operatorname{sub} \eta_{|\Omega}$. Then*

$$\int_\Omega \sqrt{1+|D\eta|^2} = \int_{\Omega\times\mathbf{R}} |D\chi_E|. \tag{3.15}$$

The proof is the same as in the case where E is a subgraph in $\mathbf{R}^{n+1}$; cf. [9, Theorem 14.6]. Moreover, we only need the relation when η is of class C^1 and then (3.15) follows immediately from the divergence theorem.

The demonstration of the next lemma is also identical to the proof of its Euclidean counterpart which is due to Miranda, cf. [14], but for the convenience of the reader we shall repeat a version of the proof that can be found in [9, Lemma 14.7].

Lemma 3.3. *Let $\Omega \subset S^n$ be open, and $F \subset \Omega \times \mathbf{R}$ be measurable such that*

$$\Omega \times (-\infty, -T) \subset F \subset \Omega \times (-\infty, T). \tag{3.16}$$

For $x \in \Omega$ define

$$\psi(x) = \lim_{k\to\infty} \Big\{ \int_{-k}^{k} \chi_F(x,t) - k \Big\}. \tag{3.17}$$

Then, there holds

$$\int_\Omega \sqrt{1+|D\psi|^2} \le \int_{\Omega\times\mathbf{R}} |D\chi_F|. \tag{3.18}$$

Proof. We note that $\partial F \cap \Omega \times \mathbf{R} \subset \Omega \times (-T, T)$. Let

$$\psi_k = \int_{-k}^{k} \chi_F(x,t) - k. \tag{3.19}$$

Then ψ_k is stationary for $k \ge T$, for

$$\int_{-k}^{k} \chi_F(x,t) = \int_{-T}^{T} \chi_F(x,t) + \int_{-k}^{-T} 1 = \int_{-T}^{T} \chi_F(x,t) - T + k, \tag{3.20}$$

i.e.,

$$\psi = \int_{-T}^{T} \chi_F(x,t) - T, \tag{3.21}$$

and $-T \le \psi \le T$.

Consider now $\gamma^\alpha \in C_c^\infty(\Omega)$, $0 \le \alpha \le n$, satisfying $|\gamma^0|^2 + \sigma_{ij}\gamma^i\gamma^j \le 1$, and a smooth real function η such that $0 \le \eta \le 1$ and

$$\begin{cases} \eta(t) = 0\ , & |t| \ge T+1, \\ \eta(t) = 1\ , & |t| \le T. \end{cases} \tag{3.22}$$

Then, we derive

$$\int_{-\infty}^{\infty} \dot\eta \chi_F = \int_{-\infty}^{-T} \dot\eta = \eta(-T) = 1, \tag{3.23}$$

and

$$\int_{-\infty}^{\infty} \eta\chi_F = \int_{-T-1}^{-T} \eta + \int_{-T}^{T} \chi_F = \psi + T + \int_{-T-1}^{-T} \eta \equiv \psi + c \tag{3.24}$$

with $c = \text{const}$, from which we infer

$$\begin{aligned} \int_{\Omega\times\mathbf{R}} |D\chi_F| &\ge \int_{\Omega\times\mathbf{R}} \chi_F D_\alpha(\eta\gamma^\alpha) \\ &= \int_\Omega \int_{-\infty}^{\infty} \chi_F \dot\eta\gamma^0 + \int_\Omega \int_{-\infty}^{\infty} \chi_F \eta D_i\gamma^i \\ &= \int_\Omega \gamma^0 + \int_\Omega (\psi + c) D_i\gamma^i, \end{aligned} \tag{3.25}$$

and hence

$$\int_{\Omega\times\mathbf{R}} |D\chi_F| \ge \int_\Omega \sqrt{1 + |D\psi|^2}. \tag{3.26}$$

q.e.d.

Let us return to the solution φ of (3.4) that also solves the variational problem (3.7). We are going to prove that $E = \operatorname{sub}\varphi$ locally minimizes the functional

$$\mathcal{F}(F, \widetilde\Omega) = \int_{\tilde\Omega} |D\chi_F| + \int_{\tilde\Omega} \chi_F a_{\epsilon_0}, \tag{3.27}$$

which is defined for any $\widetilde{\Omega} \Subset \widetilde{N}$ and any Caccioppoli set $F \subset \widetilde{N}$.

Definition 3.4. A Caccioppoli set $E \subset \widetilde{N}$ is said to be a local minimizer for the functional $\mathcal{F}$ if for any open $\widetilde{\Omega} \Subset \widetilde{N}$ and any Caccioppoli set F with $F \vartriangle E \Subset \widetilde{\Omega}$ we have

$$\mathcal{F}(E, \widetilde{\Omega}) \le \mathcal{F}(F, \widetilde{\Omega}). \tag{3.28}$$

We are now ready to prove

Theorem 3.5. *Let φ be the solution of the variational problem* (3.7) *and $E = \operatorname{sub}\varphi$. Then E is a local minimizer for the functional $\mathcal{F}$.*

Proof. Let $\widetilde{\Omega} \Subset \widetilde{N}$ and let F be a Caccioppoli set with $F \vartriangle E \Subset \widetilde{\Omega}$. Then F satisfies the conditions of Lemma 3.3 since φ is bounded—actually any $BV(S^n)$ solution of (3.7) is bounded as it is well known—, and where we choose $\Omega = S^n$.

Define ψ as in Lemma 3.3 and set $F^\star = \operatorname{sub}\psi$; then from (3.7), (3.15), and (3.18) we deduce

$$\int_{\tilde{\Omega}} |D\chi_E| \le \int_{\tilde{\Omega}} |D\chi_F| + \int_{S^n} \int_\varphi^\psi a_{\epsilon_0}(x,t). \tag{3.29}$$

We now observe that for arbitrary but fixed $x \in S^n$

$$\int_\varphi^\psi a_{\epsilon_0}(x,t) = \int_{-k}^k a_{\epsilon_0}(x,t)[\chi_{F^\star} - \chi_E], \tag{3.30}$$

where $k \ge |\varphi| + |\psi|$, and we claim furthermore, that

$$\int_{-k}^k a_{\epsilon_0}(x,t)[\chi_{F^\star} - \chi_F] \le 0, \tag{3.31}$$

since $a_{\epsilon_0}(x,\cdot)$ is monotone increasing.

To verify (3.31), we first notice that

$$\int_{-k}^k \chi_{F^\star} = \psi + k = \int_{-k}^k \chi_F, \tag{3.32}$$

and hence

$$
\begin{aligned}
\int_{-k}^{k} a_{\epsilon_0}(x,t)[\chi_{F^\star}-\chi_F] &= \int_{-k}^{k}[a_{\epsilon_0}(x,t)-a_{\epsilon_0}(x,\psi)][\chi_{F^\star}-\chi_F] \\
&= \int_{-k}^{\psi}[a_{\epsilon_0}(x,t)-a_{\epsilon_0}(x,\psi)][1-\chi_F] \\
&\quad + \int_{\psi}^{k}[a_{\epsilon_0}(x,t)-a_{\epsilon_0}(x,\psi)][0-\chi_F].
\end{aligned}
\tag{3.33}
$$

But both integrals are non-positive due to the monotonicity of $a_{\epsilon_0}(x,\cdot)$, and Theorem 3.5 is proved. q.e.d.

The function a_{ϵ_0} is locally bounded in $\tilde{N}$—in fact we could modify it so that it would be globally bounded—, from which we immediately infer that the boundary of any local minimizer E of $\mathcal{F}$ is *almost minimal*, i.e., for any $\tilde{\Omega} \Subset \tilde{N}$ there exists $R > 0$ and a constant c, such that

$$
\int_{B_\rho(x)} |D\chi_E| \le \int_{B_\rho(x)} |D\chi_F| + c\rho^{n+1} \tag{3.34}
$$

for any $x \in \tilde{\Omega}$, any $0 < \rho < R$, and any F with $F \bigtriangleup E \Subset B_\rho(x)$.

This definition is a special case of a more general one, where the second term on the right-hand side of (3.34) is supposed to grow with exponent $(n+2\alpha)$, $0<\alpha<1$; cf. [15].

We note, that almost minimal boundaries in $\tilde{N}$—or any other $(n+1)$-dimensional Riemannian space—are also almost minimal in $\mathbf{R}^{n+1}$, hence the regularity results proved in Euclidean space apply, i,e., the reduced boundary of an almost minimal hypersurface is of class $C^{1,\alpha}$, thus, in our case of class $C^{1,\frac{1}{2}}$, and the singular set is empty if $n \le 6$.

4. Proof of the main theorem

The $C^{1,\alpha}$-estimates for almost minimal boundaries yield uniform a priori estimates in the case of a sequence of almost minimal boundaries satisfying the condition (3.34) or its more general variant uniformly. Moreover, assuming that (3.34) holds uniformly for a sequence of Caccioppoli sets $E_k \subset \tilde{N}$ which converge locally to some limit set E, then, for any convergent sequence $x_k \in \partial E_k$ with $x = \lim x_k$ we have $x \in \partial E$; if in addition $x \in \partial^\star E$ (the reduced boundary), then there exists k_0, such that $x_k \in \partial^\star E_k$ for any $k \ge k_0$ and the unit normals at x_k converge to the unit normal at x, cf. [15, Theorem 1].

In view of our assumption $n \le 6$, there are no singular points, i.e., $\partial^* E = \partial E$, and we conclude that the subgraphs $E_k = \operatorname{sub} \varphi_k$, where $\varphi_k = \log u_k$, converge to $E = \operatorname{sub} \varphi$, $\varphi = \log u$; ∂E is almost minimal and of class $C^{1,\frac{1}{2}}$, and the mean curvature of $M = \operatorname{graph} u$ in N is equal to f; cf. Theorem 4.2 below. Hence, M and ∂E are of class $C^{2,\alpha}$ for any $0 < \alpha < 1$. We emphasize that only M and not necessarily u is smooth.

To complete the proof of Theorem 0.3, we have to show that M is homeomorphic to S^n and that the mean curvature of M is equal to f. or the verification of the spherical type of M we observe that each $M_k = \operatorname{graph} u_k$ is homeomorphic to S^n and that we have

Proposition 4.1. *For large k the hypersurfaces ∂E_k are graphs over ∂E.*

Proof. Each point $x \in \partial E$ is the limit of a sequence of points $x_k \in \partial E_k$, and the corresponding unit normals ν_k converge uniformly to ν. ∂E is therefore oriented and, by construction, all ∂E_k lie on one side of ∂E. Let d be the signed distance function of ∂E; it is of class $C^{2,\alpha}$ in a small tubular neighbourhood $\mathcal{U}$ of ∂E. Then, only finitely many ∂E_k's are not completely contained in $\mathcal{U}$. Fix k such that $\partial E_k \subset \mathcal{U}$. Then for any $y \in \partial E_k$ there is exactly one $x \in \partial E$ such that

$$\operatorname{dist}(y, x) = d(y). \tag{4.1}$$

We claim furthermore, that, if $\mathcal{U}$ is chosen small enough, any normal geodesic starting at an arbitrary point $x \in \partial E$—and pointing in the right direction—, intersects ∂E_k at exactly one point, which together with (4.1) yields that ∂E_k is a graph over ∂E.

To verify that claim, let us consider normal Gaussian coordinates (x^α) relative to ∂E, where x^0 represents the coordinate axis normal to ∂E. We also suppose that the unit normal ν of ∂E has coordinates $\nu = (1, 0, \dots, 0)$. Since the unit normals ν_k of ∂E_k converge uniformly to ν we conclude, that ν_k^0 is as close to 1 as we wish for large k, but then ∂E_k is at least locally a graph over $\{x^0 = 0\}$, e.g., $\partial E_k = \operatorname{graph} \eta$ (locally), cf. [9, Proposition 4.9]. But then

$$\eta(x) = d(y). \tag{4.2}$$

q.e.d.

Finally, let us verify that the mean curvature in N of the corresponding limiting hypersurface M is equal to f which is not so obvious.

Theorem 4.2. *The mean curvature of M is equal to f.*

To prove the theorem we need the following lemmata.

Lemma 4.3. *Let $K \subset S^n$ be compact with $H^{n-1}(K) < \infty$. Then*

$$H^n(\partial E \cap (K \times \mathbf{R})) = 0. \tag{4.3}$$

Proof. We consider ∂E and ∂E_k as submanifolds of $\widetilde{N}$. For a subset $U \subset S^n$ we define $\widehat{U} = U \times \mathbf{R}$, where we use polar coordinates. We also denote by μ (resp. μ_k) the measures $|D\chi_E|$ (resp. $|D\chi_{E_k}|$), where E (resp. E_k) are the subgraphs defined above.

Now, let $\Omega = S^n \setminus K$; then Ω is a Caccioppoli set, since $H^{n-1}(\partial\Omega)$ is finite. For $\epsilon > 0$, let $\xi = (\xi^\alpha) \in H^{1,p}(\widetilde{N})$ be a vectorfield such that

$$|\xi| \leq 1 \qquad \text{and} \qquad |\xi - \nu|_{\partial E} \leq \epsilon = 2E \tag{4.4}$$

Here, we use the fact that we already know ∂E and ∂E_k to be uniformly of class $H^{2,p}$ for any finite p, since their mean curvatures are uniformly bounded, and we have established a prioiri estimates in $C^{1,1/2}$.

Let $Q = S^n \times [t_1, t_2]$ such that $\partial E_k \subset Q$ for all k. Then we deduce

$$\begin{aligned} 0 &= \int_Q D_\alpha[\xi^\alpha(\chi_{E_k} - \chi_E)\chi_{\widehat{\Omega}}] \\ &= \int_Q D_\alpha\xi^\alpha(\chi_{E_k} - \chi_E)\chi_{\widehat{\Omega}} + \int_Q \xi^\alpha D_\alpha\chi_{E_k}\chi_{\widehat{\Omega}} \\ &\quad - \int_Q \xi^\alpha D_\alpha\chi_E\chi_{\widehat{\Omega}} + \int_Q \xi^\alpha(\chi_{E_k} - \chi_E)D_\alpha\chi_{\widehat{\Omega}} \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.5}$$

I_1 tends to zero if k goes to infinity, since χ_{E_k} converges pointwise to χ_E. The same argument applies to I_4, while I_3 is estimated by

$$|I_3| \leq \mu(\widehat{\Omega}). \tag{4.6}$$

Thus, we conclude

$$\begin{aligned} \overline{\lim}\, \mu_k(\widehat{\Omega}) &= \overline{\lim} \int_Q \chi_{\widehat{\Omega}} |D\chi_{E_k}| \\ &\leq \mu(\widehat{\Omega}) + c\,|\xi - \nu|_M \leq \mu(\widehat{\Omega}) + c\,\epsilon, \end{aligned} \tag{4.7}$$

because of the uniform convergence of ν_k to ν.

On the other hand, we have

$$\mu(\widehat{\Omega}) \le \underline{\lim}\, \mu_k(\widehat{\Omega}), \tag{4.8}$$

and hence

$$\mu(\widehat{\Omega}) = \lim \mu_k(\widehat{\Omega}). \tag{4.9}$$

Finally, we observe that $\mu_k(\widehat{K}) = 0$ and

$$\mu(\widetilde{N}) = \lim \mu_k(\widetilde{N}), \tag{4.10}$$

yielding the desired result. q.e.d.

In the next lemma we consider a Caccioppoli set $E \subset \widetilde{N}$ which is the subgraph of a function $\varphi \in BV(S^n)$.

Lemma 4.4. *Let $\varphi \in BV(S^n)$, $E = \operatorname{sub}\varphi$, and (x,t) be an interior point of E. Then, the line $\{(x,\tau) : -\infty < \tau < t\}$ does not intersect the measure-theoretical boundary of E, i.e., the set of all points z such that*

$$0 < |E \cap B_\rho(z)| < |B_\rho(z)| \qquad \forall\, 0 < \rho < \rho(z). \tag{4.11}$$

Here, $B_\rho(z)$ is the geodesic ball of radius ρ and center z, and $|B_\rho(z)|$ its volume. We note that due to the metric product structure of $\widetilde{N}$

$$|B_\rho(x,t)| = |B_\rho(x,\tau)| \quad \text{and} \quad \chi_{B_\rho(x,t)}(\cdot, \cdot + \tau) = \chi_{B_\rho(x,t-\tau)}(\cdot,\cdot) \tag{4.12}$$

for arbitrary t and τ.

Proof of the Lemma. We denote the measure-theoretical boundary of E by ∂E, since in the case we have in mind, the measure-theoretical and the topological boundary coincide.

First, let us observe that the partial derivative of χ_E with respect to $-\frac{\partial}{\partial x^0}$, $-D_0\chi_E$, or more precisely, $-\langle \frac{\partial}{\partial x^0}, D\chi_E\rangle$, is a non-negative measure. For let $\eta \in C_c^\infty(\widetilde{N})$, we have

$$\int_{S^n} \eta(x,\varphi(x)) = \int_{S^n} \int_{-\infty}^{\varphi(x)} D_0\eta(x,t) = -\int_{\widetilde{N}} \eta D_0 \chi_E. \tag{4.13}$$

Secondly, let $(x,\tau) \in \partial E$ with $\tau < t$. Then, we claim

$$|E \cap B_\rho(x,\tau)| - |E \cap B_\rho(x,t)| = -\int_0^{t-\tau} ds \int_{B_\rho(x,t-s)} D_0\chi_E. \tag{4.14}$$

The proof of this relation is exactly the same as that of its Euclidean counterpart; cf. [9, Lemma 4.5].

Now, the right-hand side of (4.14) is non-negative, while the left-hand side is strictly negative for small ρ, a contradiction. q.e.d.

We return to our original meaning of φ and define

Definition 4.5. Let $[\varphi](x)$ be the jump of φ at x, i.e.,

$$[\varphi](x) = \overline{\lim}_{y\to x}\varphi(y) - \underline{\lim}_{y\to x}\varphi(y) \equiv \varphi^+(x) - \varphi^-(x). \tag{4.15}$$

We have of course $\varphi^+(x) = \varphi(x)$ since φ is u.s.c.

An immediate corollary of Lemma 4.4 is

Lemma 4.6. *Let $[\varphi](x) > 0$. Then the whole line segment $x \times [\varphi^-(x), \varphi^+(x)]$ belongs to ∂E.*

The proof is straightforward since $(x, \varphi^-(x)) \in \partial E$.

Lemma 4.7. *Let $\tau > 0$ and*

$$\Lambda_\tau = \{x \in S^n \,:\, [\varphi](x) \geq \tau\}. \tag{4.16}$$

Then, Λ_τ is compact and $H^{n-1}(\Lambda_\tau) \leq c\,\tau^{-1}$.

Proof. We use a Besicovitch type covering argument. Let $0 < \delta < \tau$. Then there exists a sequence of pairwise disjoint balls $B_{\rho_i}(x_i)$ in $\mathbf{R}^{n+1}$, with centers $x_i \in \Lambda_\tau$ and radii $\rho_i < \delta/3$, such that the balls $B_{3\rho_i}(x_i)$ cover Λ_τ; see e.g. [9, Lemma 2.2]. We also choose δ small enough, such that the volume of a geodesic ball in $\widetilde{N}$ of radius $\rho < \delta$ and center in a compact set is uniformly bounded from below and above by a multiple of ρ^{n+1}. Consider the pairwise disjoint cylinders

$$Q_{\rho_i}(x_i) = (B_{\rho_i}(x_i) \cap S^n) \times \mathbf{R}. \tag{4.17}$$

Then, $Q_{\rho_i}(x_i) \cap \partial E$ contains the line segment $x_i \times [\varphi^-(x_i), \varphi^+(x_i)]$, and we can find N_i disjoint geodesic balls $B_{\rho_i/2}(y_{i,m})$, $1 \leq m \leq N_i$, with centers

$$y_{i,m} \in x_i \times [\varphi^-(x_i), \varphi^+(x_i)], \tag{4.18}$$

where N_i can be estimated by

$$N_i \geq \frac{1}{4\rho_i}[\varphi](x_i) \geq \frac{\tau}{4\rho_i}. \tag{4.19}$$

Hence, we deduce

$$\mu(Q_{\rho_i}(x_i)) \ge \sum_{m=1}^{N_i} \mu(B_{\rho_i/2}(y_{i,m})) \ge c \sum_{m=1}^{N_i} \rho_i^n \ge \frac{c}{4}\tau\rho_i^{n-1}, \tag{4.20}$$

where we use the well-known estimate for almost minimal boundaries

$$\mu(B_{\rho_i/2}(y_{i,m})) \ge c\rho_i^n, \tag{4.21}$$

if $y_{i,m} \in \partial E$, with a uniform positive constant c.

We infer further

$$\sum_{i=1}^{\infty} \rho_i^{n-1} \le c\,\tau^{-1} \sum_{i=1}^{\infty} \mu(Q_{\rho_i}(x_i)) \le c\,\tau^{-1}\mu(\widetilde{N}), \tag{4.22}$$

and conclude that the spherical $(n-1)$-dimensional measure of Λ_τ is bounded by a multiple of $\tau^{-1}\mu(\widetilde{N})$, but this is equivalent to

$$H^{n-1}(\Lambda_\tau) \le c\,\tau^{-1}\mu(\widetilde{N}) \tag{4.23}$$

with a different constant.

Λ_τ is also closed, for let $x_m \in \Lambda_\tau$ be a sequence converging to $x_0 \in S^n$, then the line segments $x_m \times [\varphi^-(x_m), \varphi^+(x_m)]$ in ∂E converge to a line segment over x_0 of length at least τ. q.e.d.

Combining Lemma 4.3 and Lemma 4.7 we deduce

Lemma 4.8. *For each $\tau > 0$, $H^{n-1}(\Lambda_\tau) = 0$ and φ is H^{n-1}-a.e. continuous.*

Proof. We observe that for any Borel set $U \subset S^n$

$$\mu(\widehat{U}) = \int_U \sqrt{1 + |D\varphi|^2}. \tag{4.24}$$

Hence, we have

$$\int_{\Lambda_\tau} \sqrt{1 + |D\varphi|^2} = 0, \tag{4.25}$$

and for any $\epsilon > 0$ there is an open set Ω, $\Lambda_\tau \subset \Omega \subset S^n$, such that

$$\mu(\widehat{\Omega}) = \int_\Omega \sqrt{1 + |D\varphi|^2} < \epsilon. \tag{4.26}$$

In the proof of Lemma 4.7 we can now choose the covering $B_{3\rho_i}(x_i)$ such that

$$B_{3\rho_i}(x_i) \cap S^n \subset \Omega, \tag{4.27}$$

and instead of the estimate (4.23) we obtain

$$H^{n-1}(\Lambda_\tau) \le c\,\tau^{-1}\mu(\widehat{\Omega}) \le c\,\tau^{-1}\epsilon, \tag{4.28}$$

i.e., $H^{n-1}(\Lambda_\tau) = 0$.

The set where φ is discontinuous is given by

$$\bigcup_{k=1}^{\infty} \Lambda_{1/k}, \tag{4.29}$$

which is an H^{n-1} null set. q.e.d.

We are now able to prove Theorem 4.2. The proof will be achieved, if we can show

$$\lim \int_{\partial E_k} (e^{\varphi_{k-1}} - e^{\varphi_k}) = 0. \tag{4.30}$$

For large k we can write ∂E_k as a graph over ∂E

$$\partial E_k = \{(\xi, \eta_k) : \ \xi \in \partial E\}, \tag{4.31}$$

where the η_k's are uniformly of class C^1 and converge to $\eta = 0$, which corresponds to ∂E in this setting. An integral of the form

$$\int_{\partial E_k} f\ , \tag{4.32}$$

f, defined in $\widetilde{N}$, can then be expressed as

$$\int_{\partial E} f(\xi, \eta_k)\sqrt{1 + |D\eta_k|^2}\, b(\xi, \eta_k), \tag{4.33}$$

where the continuous volume forms $\sqrt{1 + |D\eta_k|^2}\, b(\xi, \eta_k)$ converge to $b(\xi, 0)$, the volume form for ∂E. This can be readily seen by introducing normal Gaussian coordinates in a tubular neighbourhood of ∂E.

We extend φ_{k-1} to $\widetilde{N}$ by the definition $\varphi_{k-1} \circ x$, x being the projection on S^n, and observe that φ_k is equal to $x^0_{|\partial E_k}$, where we still use polar coordinates (x^i, x^0) in $\widetilde{N}$.

Thus, we have

$$\begin{aligned}\int_{\partial E_k}&(e^{\varphi_{k-1}}-e^{\varphi_k})\\ &=\int_{\partial E}[e^{\varphi_{k-1}\circ x(\xi,\eta_k)}-e^{x^0(\xi,\eta_k)}]\sqrt{1+|D\eta_k|^2}\,b(\xi,\eta_k).\end{aligned}\tag{4.34}$$

Let $l \geq k_0$ be large, so that ∂E_l is a graph over ∂E, and have in mind that the sequence φ_k is monotone falling. Then, we have for $k \geq l-1$

$$\begin{aligned}\int_{\partial E}&[e^{\varphi_{k-1}\circ x(\xi,\eta_k)}-e^{x^0(\xi,\eta_k)}]\sqrt{1+|D\eta_k|^2}\,b(\xi,\eta_k)\\ &\leq\int_{\partial E}[e^{\varphi_l\circ x(\xi,\eta_k)}-e^{x^0(\xi,\eta_k)}]\sqrt{1+|D\eta_k|^2}\,b(\xi,\eta_k)\end{aligned}\tag{4.35}$$

and hence

$$\overline{\lim}\int_{\partial E_k}(e^{\varphi_{k-1}}-e^{\varphi_k})\leq\int_{\partial E}[e^{\varphi_l\circ x(\xi,0)}-e^{x^0(\xi,0)}]\,b(\xi,0),\tag{4.36}$$

or, if we let l tend to infinity,

$$\overline{\lim}\int_{\partial E_k}(e^{\varphi_{k-1}}-e^{\varphi_k})\leq\int_{\partial E}[e^{\varphi\circ x(\xi,0)}-e^{x^0(\xi,0)}]\,b(\xi,0).\tag{4.37}$$

But

$$\varphi\circ x(\xi,0)-x^0(\xi,0)\leq[\varphi](x(\xi,0)),\tag{4.38}$$

and in view of the preceding lemmata we know that μ-a.e. the jump of φ is zero, i.e.,

$$\lim\int_{\partial E_k}(e^{\varphi_{k-1}}-e^{\varphi_k})=0.\tag{4.39}$$

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