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HYPERBOLIC MANIFOLDS WITH NEGATIVELY CURVED EXOTIC TRIANGULATIONS IN DIMENSIONS GREATER THAN FIVE

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Rigidity results state that, under certain conditions, two homotopically equivalent manifolds are isomorphic (e.g. diffeomorphic, PL homeomorphic, homeomorphic). Among those results are, for example, Mostow's rigidity theorem, and the topological rigidity of closed non-positively curved manifolds (Farrell and Jones). Also, in [1], examples were given showing the lack of differentiable rigidity for negatively curved manifolds of dimensions larger than six, and in [3], examples were given of the lack of PL rigidity (that implies lack of differentiable rigidity) for negatively curved manifolds of dimension six. Explicitly, the following theorem appears in [3]:

1. Theorem. *There are closed real hyperbolic manifolds M of dimension 6, such that the following holds. Given $\epsilon > 0$, M has a finite cover $\tilde{M}$ that supports an exotic (smoothable) PL structure that admits a Riemannian metric with sectional curvatures in the interval* $(-1-\epsilon, -1+\epsilon)$.

(By an exotic PL structure on a hyperbolic manifold we mean a PL structure not PL homeomorphic to the PL structure induced by the differentiable structure of the hyperbolic manifold.)

In this short paper we generalize this result to all dimensions greater than five:

2. Theorem. *There are closed real hyperbolic manifolds M in every dimension n, $n > 5$, such that the following holds. Given $\epsilon > 0$, M has*

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a finite cover $\tilde{M}$ that supports an exotic (smoothable) PL structure that admits a Riemannian metric with all sectional curvatures in the interval $(-1-\epsilon, -1+\epsilon)$.

We assume all notation from [3]. To prove Theorem 2 we first prove a version of Theorem 3.1. of [3]:

3. Theorem. *Consider the following data. For each $k = 1, 2, 3, \ldots$ we have closed orientable hyperbolic manifolds $M_0(k), M_1(k), M_2(k), M_3(k), M_4(k)$ such that*

(a) $dim M_0(k) = n$, $dim M_1(k) = n-1$, $dim M_2(k) = n-3$, $dim M_3(k) = 3$, $dim M_4(k) = n-2$.

(b) $M_2(k) \subset M_4(k) \subset M_1(k) \subset M_0(k)$ and $M_3(k) \subset M_0(k)$. All the inclusions are totally geodesic.

(c) $M_2(k)$ and $M_3(k)$ intersect at exactly one point and here transversally.

(d) For each k there is a finite covering map $p(k) : M_0(k) \to M_0(1)$ such that

$$p(k)(M_i(k)) = M_i(1), \text{ for } i = 0, 1, 2, 3, 4.$$

(e) $M_1(k)$ has a tubular neighborhood in $M_0(k)$ of width $r(k)$ and $r(k) \to \infty$ as $k \to \infty$.

Then, given $\epsilon > 0$, there is a K such that all $M_0(k)$, $k \geq K$, have exotic (smoothable) triangulations admitting Riemannian metrics with all sectional curvatures in the interval $(-1-\epsilon, -1+\epsilon)$.

The difference with Theorem 3.1 of [3], besides dropping dimension six, is that now we require one more manifold, $M_4(k)$, of codimension 2 that fits between $M_2(k)$ and $M_1(k)$: $M_2(k) \subset M_4(k) \subset M_1(k)$. We also demand all manifolds to be orientable (a condition that the manifolds constructed in the second part of [3] satisfy anyway).

Proof of Theorem 3. The proof of Theorem 3 is the same as 3.1 of [3], except for the smoothability of the exotic triangulation. We explain this part.

Recall that there is a one to one correspondance between the set of concordance classes of PL structures on a PL manifold M (dim M > 5) and $[M, TOP/PL]$, the set of homotopy classes of maps from M to TOP/PL, and because TOP/PL is a $K(\mathbb{Z}_2, 3)$, $[M, TOP/PL]$ is in one to one correspondance with $H^3(M, \mathbb{Z}_2)$. This last correspondance is given in the following way. Let $\psi \in H^3(TOP/PL, \mathbb{Z}_2) \cong \mathbb{Z}_2$ denote the generator (i.e., the non-zero element). Then the correspondance is given by: if $f \in [M, TOP/PL]$ then $f \mapsto f^*\psi \in H^3(M, \mathbb{Z}_2)$. (See essay IV of [2].)

As in [3], denote by $\tau(k)$ the PL structure on $M_0(k)$ that corresponds to the cohomology class in $H^3(M, \mathbb{Z}_2)$ dual to the homology class in $H_{n-3}(M, \mathbb{Z}_2)$ represented by the submanifold $M_2(k) \subset M_0(k)$ (the correspondance here sends the PL structure induced by the hyperbolic structure to $0 \in H^3(M, \mathbb{Z}_2)$). Denote also by $\theta : M_0(k) \to TOP/PL$ a map that corresponds to $\tau(k)$.

In [3] we required dimension six so that $\tau(k)$ is smoothable (in dimension six all PL structures are smoothable). Now we use the new hypothesis. In fact, because $M_0(k)$ is smoothable and the PL structure induced by the hyperbolic structure corresponds to the constant map in $[M_0(k), TOP/PL]$, the following lemma implies that $\tau(k)$ is smoothable.

4. Lemma. *We can assume that $\theta : M_0(k) \to TOP/PL$ factors through TOP/O.*

Proof. Because $M_2(k) \subset M_4(k) \subset M_1(k) \subset M_0(k)$ are all orientable and with dimensions $n-3, n-2, n-1, n$, respectively, we have that $M_2(k)$ has trivial normal bundle in $M_4(k)$, and $M_4(k)$ has trivial normal bundle in $M_1(k)$ and $M_1(k)$ has trivial normal bundle in $M_0(k)$. Consequently $M_2(k)$ has trivial normal bundle in $M_0(k)$, so that we can assume that

$$M_2(k) \times D^3$$

is embedded in $M_0(k)$, where D^3 is the three dimensional closed disc. We claim that we can take θ as the following map:

$$\begin{aligned} M_0(k) &\to M_0(k) \,/\, [M_0(k) - M_2(k) \times Int\, D^3] \\ &\cong M_2(k) \times D^3 \,/\, M_2(k) \times \partial D^3 \to D^3 \,/\, \partial D^3 \xrightarrow{\alpha} TOP/PL, \end{aligned}$$

where the first maps are collapsing maps and $\alpha : D^3 \,/\, \partial D^3 \to TOP/PL$ is the generator of $\pi_3(TOP/PL) \cong \mathbb{Z}_2$. Write $\theta = \alpha\, c$ where $c : M_0(k) \to D^3 \,/\, \partial D^3$ is the collapsing map. To see that the claim is true, just note that $\theta^*\psi = (\alpha\, c)^*\psi = c^*(\alpha^*\psi)$ and this cohomology class is dual to $M_2(k)$. (Recall that ψ is the generator of

$$H^3(TOP/PL, \mathbb{Z}_2) \cong \pi_3(TOP/PL) \cong \pi_3(K(\mathbb{Z}_2, 3)) \cong \mathbb{Z}_2;$$

see p.200 of [2]. Also note that $\alpha^*\psi$ is the non-zero element $\eta \in H^3(\mathbb{S}^3, \mathbb{Z}_2) \cong \mathbb{Z}_2$, where $\mathbb{S}^3$ is the three-sphere, and observe that if

$f : M \to \mathbb{S}^3$ is differentiable, then the submanifold $f^{-1}(p) \subset M$, where p is a regular value of f, is dual to $f^*(\eta) \in H^3(M, \mathbb{Z}_2)$.) This proves the claim.

But $\pi_3(TOP/PL) \cong \pi_3(TOP/O)$ (see p.200 of [2]) and the isomorphism is induced by the natural map $TOP/O \to TOP/PL$. This means that we can factor α through TOP/O, so that we can also factor $\alpha\, c = \theta$ through TOP/O. This proves the lemma and Theorem 3.

Now, to prove Theorem 2 we have to show that there are manifolds satisfying the hypothesis of Theorem 3. For this we proceed as in the second part of [3] (section 3.2.) but we make one more definition (see p. 15 of [3]):

$$G_4 = \{g \in G_0 \ : \ ge_i = e_i \ \ i = 1, 2\}$$

and we have $G_2 \subset G_4 \subset G_1 \subset G_0$. The remaining part of the construction follows word by word.

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