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GEODESIC LENGTH FUNCTIONS AND TEICHMÜLLER SPACES

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Abstract

Given a compact orientable surface with finitely many punctures Σ, let $\mathcal{S}(\Sigma)$ be the set of isotopy classes of essential unoriented simple closed curves in Σ. We determine a complete set of relations for a function from $\mathcal{S}(\Sigma)$ to $\mathbf{R}$ to be the geodesic length function of a hyperbolic metric with geodesic boundary and cusp ends on Σ. As a consequence, the Teichmüller space of hyperbolic metrics with geodesic boundary and cusp ends on Σ is reconstructed explicitly from an intrinsic $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$ structure on $\mathcal{S}(\Sigma)$.

0. Introduction

Let $\Sigma = \Sigma^s_{g,r}$ be a compact oriented surface of genus g with r boundary components and s punctures, i.e., a surface of signature (g, r, s) where $(g, r, s) \geq 0$. The Teichmüller space of isotopy classes of hyperbolic metrics with geodesic boundary and cusp ends on Σ is denoted by $T^s_{g,r} = T(\Sigma)$, and the isotopy classes of essential simple closed unoriented curves in Σ is denoted by $\mathcal{S} = \mathcal{S}(\Sigma)$. A simple loop in Σ is called *parabolic* if it is homotopic into an end of Σ. The set of isotopy classes of essential parabolic simple loops in Σ is denoted by $P(\Sigma)$. For each $m \in T(\Sigma)$ and $\alpha \in \mathcal{S}(\Sigma)$, let $l_m(\alpha)$ be the length of the geodesic representing α if $\alpha \notin P(\Sigma)$ and let $l_m(\alpha) = 0$ if $\alpha \in P(\Sigma)$. The goal of the paper is to characterize the geodesic length function l_m in terms of an intrinsic $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$ structure on $\mathcal{S}(\Sigma)$.

Theorem 1. *For surface $\Sigma^s_{g,r}$ of negative Euler number, a function $f : \mathcal{S}(\Sigma^s_{g,r}) \to \mathbf{R}$ is a geodesic length function if and only if $f|_{\mathcal{S}(\Sigma')}$ is a geodesic length function for each incompressible subsurface $\Sigma' \cong \Sigma^0_{1,1}$, $\Sigma^s_{0,r}$ $(r + s = 4)$ in $\Sigma^s_{g,r}$. Furthermore, geodesic length functions on*

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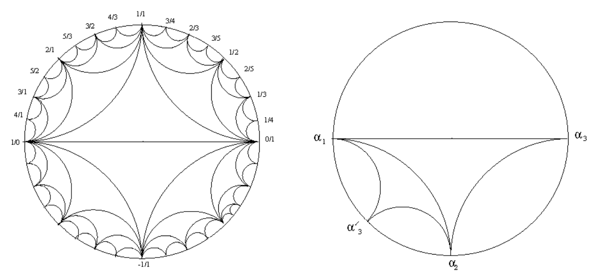


Figure 1

FIGURE 1

$\mathcal{S}(\Sigma_{1,r}^s)$ $(r+s=1)$ and $\mathcal{S}(\Sigma_{0,r}^s)$ $(r+s=4)$ are characterized by two polynomial equations (in $\cosh(f/2)$) in the $(\mathbf{Q}P^1, PSL(2,\mathbf{Z}))$ structure on $\mathcal{S}$.

Recall that a subsurface $\Sigma' \subset \Sigma$ is *incompressible* if each essential loop in Σ' is still essential in Σ. Given two isotopy classes α and β in $\mathcal{S}(\Sigma)$, the *geometric intersection number* between α, β, denoted by $\mathrm{I}(\alpha,\beta)$ is $\min\{|a \cap b| \mid a \in \alpha \text{ and } b \in \beta\}$ where $|a \cap b|$ is the number of points in $a \cap b$.

Theorem 1 also holds for surfaces of infinite types.

Given a surface Σ, let $\mathcal{S}'(\Sigma)$ be the set of isotopy classes of essential, non-boundary parallel nonparabolic simple loops in Σ. For surfaces $\Sigma = \Sigma_{1,r}^s$ $(r+s=1)$ and $\Sigma_{0,r}^s$ $(r+s=4)$, it is well known that there exists a bijection $\pi : \mathcal{S}'(\Sigma) \to \mathbf{Q}P^1(=\hat{\mathbf{Q}})$ so that $p'q - pq' = \pm 1$ if and only if $I(\pi^{-1}(p/q), \pi^{-1}(p'/q')) = 1$ (for $\Sigma_{1,r}^s$) and 2 (for $\Sigma_{0,r}^s$). See Figure 1. We say that three classes $\alpha_1, \alpha_2, \alpha_3$ in $\mathcal{S}'(\Sigma)$ form an *ideal triangle* if they correspond to the vertices of an ideal triangle in the modular relation under the map π.

For the rest of the paper, we introduce the *trace function* $t_m(\alpha) = 2\cosh l_m(\alpha)/2$ from $\mathcal{S}(\Sigma)$ to $\mathbf{R}_{\geq 2}$. We will deal with the trace function t_m instead of l_m.

Theorem 2. *(a) For surface $\Sigma_{1,r}^s$, $r+s=1$ with b as the isotopy class of the boundary loop or the parabolic loop, a function $t : \mathcal{S} \to \mathbf{R}_{\geq 2}$*

is a trace function if and only if the following hold:

$$\prod_{i=1}^{3} t(\alpha_i) = \sum_{i=1}^{3} t^2(\alpha_i) + t(b) - 2 \quad \textit{and} \tag{1}$$

$$t(\alpha_3)t(\alpha_3') = \sum_{i=1}^{2} t^2(\alpha_i) + t(b) - 2,$$

where $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1, \alpha_2, \alpha_3')$ are distinct ideal triangles in $\mathcal{S}'$.

(b) For surface $\Sigma_{0,r}^s$, $r + s = 4$, let b_1, b_2, b_3, b_4 be four isotopy classes of simple loops represented by the boundary components and the parabolic loops, a function $t : \mathcal{S} \to \mathbf{R}_{\geq 2}$ is a trace function if and only if for each ideal triangle $(\alpha_1, \alpha_2, \alpha_3)$ so that (α_i, b_j, b_k) bounds a $\Sigma_{0,3}^0$ in $\Sigma_{0,r}^s$ the following hold:

$$\begin{aligned}\prod_{i=1}^{3} t(\alpha_i) = \sum_{i=1}^{3} t^2(\alpha_i) &+ \sum_{j=1}^{4} t^2(b_j) + \prod_{j=1}^{4} t(b_j) \\ &+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{4} f(\alpha_i)f(b_j)f(b_k) - 4\end{aligned} \tag{2}$$

and

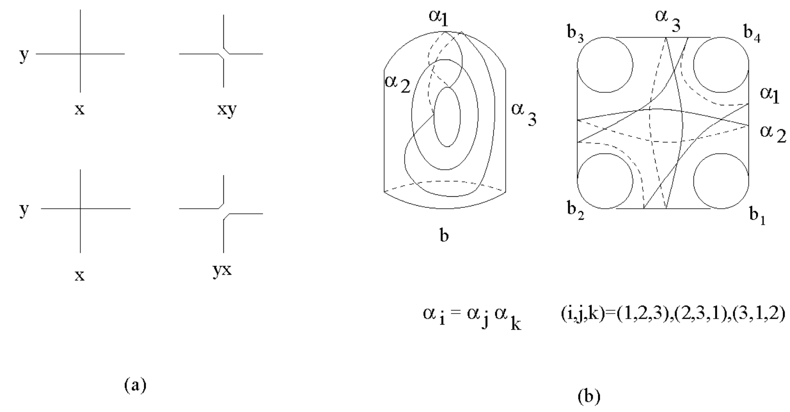
$$\begin{aligned}f(\alpha_3)f(\alpha_3') = \sum_{i=1}^{2} t^2(\alpha_i) &+ \sum_{j=1}^{4} t^2(b_j) + \prod_{j=1}^{4} t(b_j) \\ &+ \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{4} f(\alpha_i)f(b_j)f(b_k) - 4,\end{aligned}$$

where $(\alpha_1,\ \alpha_2,\ \alpha_3')$ and $(\alpha_1, \alpha_2, \alpha_3)$ are two distinct ideal triangles in $\mathcal{S}'$.

Part (a) of Theorem 2 was a result of Fricke-Klein [9] and Keen [21].

Thurston's compactification of the Teichmüller space $T(\Sigma)$ (see [3], [10], [35]) uses the embedding $\tau : T(\Sigma) \to \mathbf{R}^{\mathcal{S}(\Sigma)}$ sending m to l_m. Theorems 1 and 2 give a complete description of the image of the embedding.

The modular relation on $\mathcal{S}$ is derived from an intrinsic combinatorial structure on $\mathcal{S}$ as. If two simple closed curves a and b intersect at one point transversely (resp. $\alpha, \beta \in \mathcal{S}(\Sigma)$ with $\mathrm{I}(\alpha, \beta) = 1$), we denote it by $a \perp b$ (resp. $\alpha \perp \beta$); if two simple closed curves a and b intersect at two points of different signs transversely and $\mathrm{I}([a], [b]) = 2$, we denote it by



Right-hand orientation on the front face

Figure 2

FIGURE 2. Right-hand orientation on the front face

$a \perp_0 b$. In this case, we denote the relation between their isotopy classes by $[a] \perp_0 [b]$. Suppose x and y are two arcs in Σ so that x intersects y transversely at one point. Then *the resolution of $x \cap y$ from x to y* is defined as follows. Take any orientation on x and use the orientation on Σ to determine an orientation on y. Now resolve the intersection point $x \cap y$ according to the orientations (see Figure 2(a)). If $a \perp b$ or $a \perp_0 b$, we define ab to be the curve obtain by resolving intersection points in $a \cap b$ from a to b. We define $\alpha\beta = [ab]$ where $a \in \alpha, b \in \beta$ with $|a \cap b| = I(\alpha, \beta)$. It follows from the definition that $\alpha\beta \perp$ (resp. $\perp_0$) α, β if $\alpha \perp \beta$ (resp. $\alpha \perp_0 \beta$). Furthermore, $\alpha(\beta\alpha) = (\alpha\beta)\alpha = \beta$. For surface $\Sigma = \Sigma_{1,r}^s$ $(r + s = 1)$ and $\Sigma_{0,r}^s$ $(r + s = 4)$, three elements $\alpha_1, \alpha_2, \alpha_3$ in $\mathcal{S}'(\Sigma)$ form an ideal triangle if and only if $\alpha_1 \perp \alpha_2$ or $\alpha_1 \perp_0 \alpha_2$ and $\alpha_3 = \alpha_1\alpha_2$ or $\alpha_2\alpha_1$. In particular the two distinct ideal triangles in Theorem 2 are $(\alpha_1, \alpha_2, \alpha_1\alpha_2)$ and $(\alpha_1, \alpha_2, \alpha_2\alpha_1)$.

The relations (1) and (2) come from trace identities for SL(2,**R**) matrices. Note that the second part of relations (1), (2) shows that $t(\alpha_3)$ and $t(\alpha_3')$ are the two roots of the quadratic equation (in $t(\alpha_3)$) in the first part of the relations. Thus we obtain two more relations as follows.

$$t(\alpha_1\alpha_2) + t(\alpha_2\alpha_1) = t(\alpha_1)t(\alpha_2), \quad \text{where } \alpha_1 \perp \alpha_2 \text{ and}$$

$$t(\alpha_1\alpha_2) + t(\alpha_2\alpha_1) = t(\alpha_1)t(\alpha_2) - t(b_i)t(b_j) - t(b_k)t(b_l),$$
$$\{i,j,k,l\} = \{1,2,3,4\},$$

where $\alpha_1 \perp_0 \alpha_2$ and $(\alpha_1\alpha_2, b_i, b_j)$ bounds a $\Sigma^0_{0,3}$.

The main part of the proof of theorems is to show that these relations are sufficient. To prove this, we use induction on $|\Sigma^s_{g,r}| = 3g + r + s$. There are two key ingredients involved in the proof: a gluing lemma and an iteration process.

For simplicity, we describe the gluing lemma for a compact surface Σ. Decompose $\Sigma = X \cup Y$ where X and Y are compact incompressible subsurfaces so that $X \cap Y \cong \Sigma^0_{0,3}$ (see Figure 3 (b), (c)). Let the three boundary components of $X \cap Y$ be a_1, a_2 and a_3. Then the gluing lemma states that for each hyperbolic metric m_X and m_Y on X and Y respectively so that a_i are geodesics in both metrics with $l_{m_X}(a_i) = l_{m_Y}(a_i)$ (i=1,2,3), there is a hyperbolic metric m in Σ unique up to isotopy so that the restriction of m to X is isotopic to m_X and the restriction of m to Y is isotopic to m_Y.

The iteration process is derived as follows. Given a function t on $\mathcal{S}(\Sigma)$ satisfying the relations (1) and (2), using the gluing lemma and the induction hypothesis, one constructs a hyperbolic metric on the surface so that t and the trace of the metric coincide on $\mathcal{S}(X) \cup \mathcal{S}(Y)$. To show that these two functions are the same on all simple closed curves, we observe that the second part of the relations (1) and (2) indicates that the value of t at $\beta\alpha$ is determined by the values of t on α, β, $\alpha\beta$ and $b_i's$. By iterated use of the relations together with the multiplicative structure on $\mathcal{S}$, we show that these two functions are the same.

By the work of Thurston, the degenerations of hyperbolic metrics become measured laminations, and the corresponding projective limits of geodesic length functions become geometric intersection numbers. Thus, relations (1) and (2) degenerate to universal relations for the geometric intersection numbers. It is shown in [24] that these degenerated equations determine Thurston's measured lamination spaces and Thurston's compactification of the Teichmüller spaces.

As another consequence of Theorem 1, we consider finite dimensional embeddings of the Teichmüller spaces. Given a subset F of $\mathcal{S}(\Sigma^s_{g,r})$, let $\pi_F : T(\Sigma^s_{g,r}) \to \mathbf{R}^F$ be the map $\pi_F(m) = t_m|_F$. It is well known from the work of Fricke-Klein [9] that there exists a finite set F so that π_F is an embedding. The work of Okumura [30], Schmutz [32], Seppälä-Sorvali [33], Sorvali [34] show that there exists a set F consisting of N (N= $6g+3r+2s-6$ if $r > 0$ and N=$6g+2s-5$ if $r = 0$) elements so that

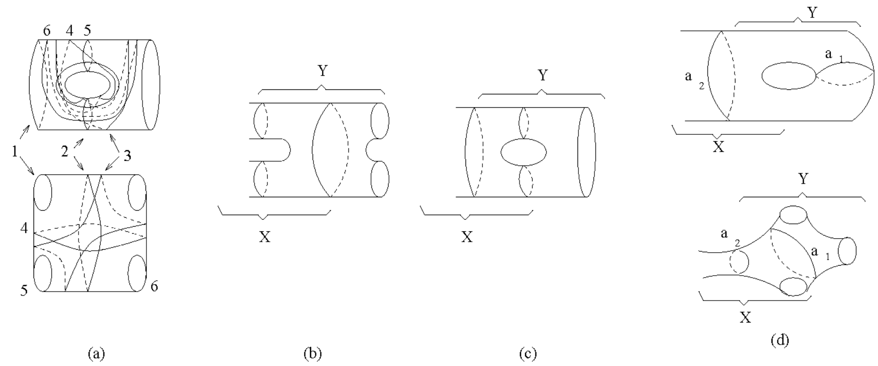


Figure 3

FIGURE 3

π_F is an embedding. This number N is necessarily the minimal number by a result of Wolpert [36] in case $r = 0$. We shall indicate a proof of the existence of such set F for compact surface with boundary below. By Theorem 2 and the gluing lemma, it is easy to show that hyperbolic metrics on $\Sigma^0_{0,4}$ and $\Sigma^0_{1,2}$ are determined by the geodesic lengths of six curves as shown in Figure 3(a). Now each compact oriented surface with boundary and Euler number smaller than -2 is obtained from $\Sigma^0_{0,4}$ and $\Sigma^0_{1,2}$ by repeated use of gluing along 3-holed spheres (see Figure 3(b), (c)). Furthermore, one of the subsurface used in the gluing (surface Y) is either $\Sigma^0_{0,4}$ or $\Sigma^0_{1,2}$. Thus, each time the Euler number of the resulting surface changes by -1 and the number of curves needed to determine the hyperbolic metric increases by 3 (the curves $3, 4, 6$ in Figure 3(a) are the needed ones and the curves $1, 2, 5$ are in the subsurface X).

The corollary below strength their result to conclude that the image of the embedding is an explicit semi-analytic set. Okumura [31] has also obtained the result for $s = r = 0$ using a different method. The semi-analytic property in the corollary also follows from the work of Brumfiel [4], Morgan-Shalen [28], and Helling [16].

Corollary. *(a) For surface $\Sigma^s_{g,r}$ of negative Euler number and $r > 0$, there exists a finite subset F in $\mathcal{S}(\Sigma^s_{g,r})$ consisting of $6g + 3r + 2s - 6$ elements so that the map $\pi_F : T(\Sigma^s_{g,r}) \to \mathbf{R}^F$ is a real analytic embedding onto an open subset which is defined by a finite set of explicit real analytic inequalities in the coordinates of π_F.*

(b) For surface $\Sigma^s_{g,0}$ of negative Euler number, there exists a finite subset F of $\mathcal{S}(\Sigma^s_{g,0})$ consisting of $6g + 2s - 5$ elements so that π_F :

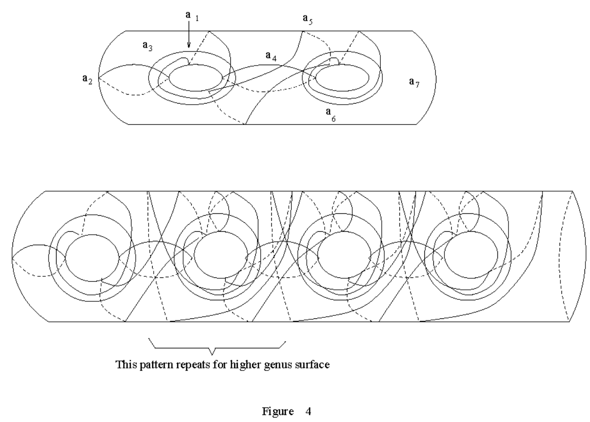


FIGURE 4

$T(\Sigma_{g,0}^s) \to \mathbf{R}^F$ is an embedding whose image in $\mathbf{R}^F$ is defined by one real analytic equation and finitely many explicit real analytic inequalities in the coordinates of π_F.

The inequalities and the equation in the corollary are given by functions which are obtained from the coordinates of π_F by a finite number of algebraic operations (summation, multiplication, and division over the rationals) and the square root operation.

Some examples of the collection F and the images of the Teichmüller spaces are as follows. For $\Sigma_{2,0}^0$, take

$$F = \{[a_1], [a_2], [a_3], [a_4], [a_5], [a_6], [a_7]\}$$

as in Figure 4. Then the map π_F is an embedding with image $\pi_F(T_{2,0}^0) = \{(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \in \mathbf{R}_{>2}^7 \mid t_8 > 2,\ t_9 > 2,\ t_8 = t_6t_7t_9 - t_6^2 - t_7^2 - t_9^2 + 2$, where $t_8 = t_1t_2t_3 - t_1^2 - t_2^2 - t_3^2 + 2$, and

$$(2 + t_2^2 + t_8)t_9^2 + 2t_2(t_4 + t_5)t_9 + 2t_2^2 + t_4^2 + t_5^2 + t_8^2 + t_2^2t_8 - t_4t_5t_8 - 4 = 0\}.$$

The explicit equations and inequalities in the corollary for the surface $\Sigma_{1,r}^s$ ($r + s = 1$) are as follows. For $\Sigma_{1,1}^0$ (resp. $\Sigma_{1,0}^1$), Keen [21] proved that one takes $F = \{[a_1], [a_2], [a_3]\}$ to be an ideal triangle and the image $\pi_F(T(\Sigma_{1,1}^0))$ is $\{(t_1, t_2, t_3) \in \mathbf{R}_{>2}^3 \mid$ formula (3) holds$\}$:

$$t_1t_2t_3 > t_1^2 + t_2^2 + t_3^3. \tag{3}$$

$(\pi_F(T(\Sigma_{1,0}^1)) = \{(t_1, t_2, t_3) \in \mathbf{R}_{>2}^3 \mid t_1t_2t_3 = t_1^2 + t_2^2 + t_3^2\})$. For $\Sigma_{0,r}^s$ with $r + s = 4$, we take the collection F to be the isotopy classes of six curves b_1, b_2, b_3, a_{12}, a_{23}, and a_{31} where $[a_{ij}]$ forms an ideal triangle and (a_{ij}, b_i, b_j) bounds a $\Sigma_{0,3}$. Then π_F is an embedding whose image $\pi_F(T(\Sigma^0_{0,4}))$ is given by $\{(t_1, t_2, t_3, t_{12}, t_{23}, t_{31}) \in \mathbf{R}^6_{>2} |$ so that formula (4) holds$\}$:

$$
\begin{aligned}
t_{12}t_{23}t_{31} >\, & t_{12}^2 + t_{23}^2 + t_{31}^2 + t_1^2 + t_2^2 + t_3^2 + t_{12}t_1t_2 + t_{23}t_2t_3 \\
& + t_{31}t_3t_1 + 2t_1t_{23} + 2t_2t_{31} + 2t_3t_{12} + 2t_1t_2t_3.
\end{aligned} \tag{4}
$$

The organization of the paper is as follows. In §1, we prove a gluing lemma and recall basic facts on discrete subgroups of $SL(2, \mathbf{R})$ and the spin structures on surfaces. We prove Theorem 2 in §2. In §3, we establish a proposition on the multiplicative structure on $\mathcal{S}$. Theorem 1 is proved in §4. In §5, we discuss applications. In the main body of the paper (§2, §3, and §4) we shall treat hyperbolic metrics without cusp ends in order to reduce the length of the paper. No new ideas are needed for metrics with cusps. The proofs of the Theorems 1 and 2 for metrics with cups ends will be discussed briefly in §5.3.

Acknowledgment

I would like to thank F. Bonahon for calling my attention to several literature. This work is supported in part by the NSF.

1. Preliminaries on discrete subgroups of SL(2,R)

We prove a gluing lemma in §1.1. Basic facts about discrete representations of surface groups into SL(2,$\mathbf{R}$) and spin structures on surfaces will be recalled in §1.2 - 1.4.

We shall use the following notation throughout the paper. Let $\Sigma_{g,r} = \Sigma^0_{g,r}$, $\Sigma_g = \Sigma^0_{g,0}$, and $T_{g,r} = T^0_{g,r}$. We use $cl(X)$ and $int(X)$ to denote the closure and the interior of a submanifold X. The isotopy class of a simple loop a is denoted by $[a]$, and the isotopy class of a hyperbolic metric d is denoted by $[d]$. If $f : \mathcal{S} \to \mathbf{R}$ is a function and a is a simple loop, we define $f(a)$ to be $f([a])$. In particular, $I(a, b) = I([a], [b]) = I(a, [b])$. A regular neighborhood of a submanifold X is denoted by $N(X)$. Regular neighborhoods are always assumed to be small. All intersections of curves are assumed to be transverse.

An RC-function (compass and ruler constructible function) in variables $x_1, ..., x_n$ is a function obtained from $1, x_1, ..., x_n$ by a finite number of algebraic operations and the square root operation. The set of RC-functions is closed under algebraic operations and compositions. Note that $|x| = \sqrt{x^2}$ is an RC-function. An RC-function is continuous in its natural domain and is analytic away from its singular set.

1.1. A gluing lemma.

First some definitions and conventions. A surface Σ is oriented and connected which is either $\Sigma^s_{g,r}$ or obtained from $\bar{\Sigma} = \Sigma^s_{g,r}$ by removing some boundary components. Each boundary component of $\bar{\Sigma}$ is called a boundary component of Σ. A hyperbolic metric with geodesic boundary and cusp ends on Σ is a hyperbolic metric whose completion is a hyperbolic metric on $\bar{\Sigma}$ with geodesic boundary and cusp ends. Two hyperbolic metrics are *isotopic* if between them there is an isometry which is isotopic to the identity. The Teichmüller space of hyperbolic metrics with geodesic boundary and cusp ends on Σ is denoted by $T(\Sigma)$. It is canonically isomorphic to $T(\bar{\Sigma})$.

A subsurface X of Σ is *incompressible* if the inclusion map induces a monomorphism in fundamental groups. If the subsurface is compact, then it is incompressible if and only if each boundary component of X is essential in Σ. A *good incompressible* subsurface is an incompressible subsurface whose interior is a component of the complement of a finite union of disjoint, pairwise non-parallel, non-boundary parallel, non-parabolic simple closed curves in Σ. For instance, if s is a non-separating simple closed curve in Σ, then $\Sigma - s$ is a good incompressible subsurface but $\Sigma - N(s)$ is not. If X is an incompressible subsurface of negative Euler number, then $int(X)$ is isotopic to a good incompressible subsurface. For a good incompressible subsurface X of Σ, we define the *restriction map* $R_X = R^{\Sigma}_X : T(\Sigma) \to T(X)$ as follows. Given $[d] \in T(\Sigma)$, there is a homeomorphism h of Σ isotopic to the identity so that the frontier of X, $cl(X) - int(X)$, is a union of geodesics in the pull back metric $h^*(d)$. We define $R_X([d])$ to be $[h^*(d)|_X]$. It follows from elementary hyperbolic geometry and topology of surfaces that R_X is well defined (see [6], or [5]). Furthermore, it follows from the definition that if X is good incompressible in Y and Y is good incompressible in Z, then $R^Z_X = R^Y_X R^Z_Y$. The restriction map is in general not onto. For instance, if we take X to be the complement of a non-separating simple closed curve in a surface Σ with negative Euler number, then R_X is not onto.

Lemma 1 (Gluing along a 3-holed sphere). *Suppose X and Y are two good incompressible subsurfaces of Σ whose union is Σ so that either (1) $X \cap Y \cong \Sigma_{0,3}$, or (2) $Y \cong \Sigma_{1,1}$ and $X \cap Y \cong \Sigma_{1,1} - s$ where s is a* *non-separating simple closed curve in int*(Y) *(see Figure 3(b), (c), (d)), or (3)* $X \cap Y \cong \Sigma_{0,2}^1$ *with the punctured end in* $\Sigma_{0,2}^1$ *being a punctured end of* Σ. *Then for any two elements* $m_X \in T(X)$ *and* $m_Y \in T(Y)$ *with* $R_{X\cap Y}(m_X) = R_{X\cap Y}(m_Y)$, *there exists a unique element* $m \in T(\Sigma)$ *so that* $R_X(m) = m_X$ *and* $R_Y(m) = m_Y$.

Proof. To show the existence, let $d_X \in m_X$ (resp. $d_Y \in m_Y$) be a representative so that $d_X|_{X\cap Y}$ (resp. $d_Y|_{X\cap Y}$) has geodesic boundary and cusp ends, i.e., $R_{X\cap Y}([d_X]) = [d_X|_{X\cap Y}]$ (resp. $R_{X\cap Y}([d_Y]) = [d_Y|_{X\cap Y}]$). Let $h : X\cap Y \to X \cap Y$ be an isometry from $d_X|_{X\cap Y}$ to $d_Y|_{X\cap Y}$, which is isotopic to the identity map. By the assumption on X and Y, we can extend h to a homeomorphism g of X, which is isotopic to the identity. Define a hyperbolic metric d on Σ with geodesic boundary and cusp ends as follows: $d|_X = g^*(X)$ and $d|_Y = Y$. It follows from the definition that $R_X([d]) = [d_X]$ and $R_Y([d]) = [d_Y]$. The uniqueness follows from the fact that an analytic automorphism of a complex structure on $\mathrm{int}(\Sigma_{0,3})$ which preserves each end is the identity map. q.e.d.

1.2. Monodromy representations and spin structures.

Given a hyperbolic metric d with geodesic boundary and cusp ends on Σ, its monodromy is a discrete faithful representation

$$\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbf{R})$$

unique up to PGL(2,$\mathbf{R}$) =GL(2,$\mathbf{R}$)$/\{\pm I\}$ conjugation so that there is an isometric embedding h from the universal cover $\tilde{\Sigma}$ with the pull back metric into the hyperbolic plane $\mathbf{H}$ satisfying $h(\gamma(x)) = \rho(\gamma)(h(x))$ for all $x \in \tilde{\Sigma}$ and $\gamma \in \pi_1(\Sigma)$. Isotopic metrics have the same PGL(2,$\mathbf{R}$) conjugacy class of monodromies. If the isometric embedding h is orientation preserving (resp. reversing), we say the monodromy ρ is orientation preserving (resp. reversing). Thus each $m \in T(\Sigma)$ gives rise to two PSL(2,$\mathbf{R}$) conjugacy classes of monodromy representations: one preserving the orientation and the other reversing the orientation. Let $R(\Sigma)$ be the set of all such monodromy representations with the topology induced by algebraic convergence of representations. Then $R(\Sigma)$ has two connected components corresponding to the two orientations. Each component is a trivial principal PSL(2,$\mathbf{R}$) bundle over $T(\Sigma)$ (see [12], [14], [29] for details). Each representation $\rho \in R(\Sigma)$ can be lifted to a representation $\tilde{\rho} : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbf{R})$ (see [1]), and there are exactly 2^N such liftings where $N = 2g$ if Σ has signature (g,0,0) and $N = 2g + r + s - 1$ if Σ has signature (g, r, s) $(r + s > 0)$. Given a lifting $\tilde{\rho}$ of ρ, all other liftings are obtained as follows. Let $\{\gamma_1, ..., \gamma_N\}$ be a set of generators for $\pi_1(\Sigma)$ and I a subset of $\{1, ..., N\}$. Then all other liftings are $\tilde{\rho}_I$ where $\tilde{\rho}_I(\gamma_i) = \tilde{\rho}(\gamma_i)$ if $i \in I$ and $\tilde{\rho}_I(\gamma_i) = -\tilde{\rho}(\gamma_i)$ if $i \notin I$. Let $\tilde{R}(\Sigma)$ be the set of all liftings of the monodromies with the algebraic convergent topology. The representation space $\tilde{R}(\Sigma)$ has 2^{N+1} many connected components. These components are classified into two types according to the orientation of the representations in $R(\Sigma)$. Each component corresponds to a spin structure on the surface. We shall recall briefly spin structures. Let $U\Sigma$ be the unit tangent bundle over the surface Σ with S^1 as a fiber. A *spin structure* on Σ is a two-fold covering space of $U\Sigma$ so that the S^1-fiber does not lift. Since two-fold covering spaces correspond to index-two subgroups of the fundamental groups, a spin structure is the same as an epimorphism $\eta : \pi_1(U\Sigma) \to \mathbf{Z}_2 = \{\pm 1\}$ (as a multiplicative group) so that $\eta(S^1) = -1$. Since $\mathbf{Z}_2$ is abelian, the epimorphism η is induced by an epimorphism (still denoted by) $\eta : H_1(U\Sigma, \mathbf{Z}_2) \to \mathbf{Z}_2$ with $\eta(S^1) = -1$. Given a smooth immersed curve c in Σ, let $\bar{c}$ be the unit tangent vectors of c in $U\Sigma$. We define $\eta(c)$ to be $\eta([\bar{c}])$. For instance, if c bounds a disc, then $\eta(c) = -1$ and if c is null homotopic with exactly one self intersection (a figure eight), then $\eta(c) = 1$.

Johnson in [19] provides an algorithm to calculate $\eta(c)$ which we summarize as follows.

Lemma 2 (Johnson). *(a) Suppose $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_m\}$ are two collections of disjoint simple closed curves in Σ so that $\Sigma_{i=1}^n [a_i] = \Sigma_{j=1}^m [b_j]$ in $H_1(\Sigma, \mathbf{Z}_2)$. Then $\Sigma_{i=1}^n [\bar{a}_i] + n[S^1] = \Sigma_{j=1}^m [\bar{b}_j] + m[S^1]$ in $H_1(U\Sigma, \mathbf{Z}_2)$.*

(b) Given $\alpha \in H_1(\Sigma, \mathbf{Z}_2)$, represent α as $\Sigma_{i=1}^n [a_i]$ in $H_1(\Sigma, \mathbf{Z}_2)$ where $\{a_1, ..., a_n\}$ is a collection of disjoint simple closed curves in Σ. Then $\eta^*(\alpha) = (-1)^n \Pi_{i=1}^n \eta(a_i)$ is a $\mathbf{Z}_2$-quadratic map from $H_1(\Sigma, \mathbf{Z}_2)$ to $\mathbf{Z}_2$, i.e., $\eta^*(\alpha + \beta) = (-1)^{<\alpha, \beta>} \eta^*(\alpha) \eta^*(\beta)$ where $< \alpha, \beta >$ is the $\mathbf{Z}_2$-intersection number.

As a simple consequence, if $\{a_1, a_2, a_3\}$ bounds a 3-holed sphere in Σ, then $\eta(a_1)\eta(a_2)\eta(a_3) = -1$; if b is the boundary of a subsurface of signature (g,1,0), then $\eta(b) = -1$; and if $a_1 \perp a_2$, then $\eta(a_1)\eta(a_2)\eta(a_1 a_2) = 1$.

The relationship between a lifting $\tilde{\rho} \in \tilde{R}(\Sigma)$ of $\rho \in R(\Sigma)$ and a spin structure is as follows. We first identify PSL(2,$\mathbf{R}$) with $U\mathbf{H}$ by sending an isometry g to $g(v_0)$ where v_0 is a specified element in $U\mathbf{H}$. Under this identification, given a hyperbolic metric with geodesic boundary and cusp ends on Σ whose monodromy is ρ, $U\Sigma$ is canonically identified with a deformation retractor (U(Nielsen core)) of PSL(2,$\mathbf{R}$)$/\rho(\pi_1(\Sigma))$. Let P : SL(2,$\mathbf{R}$) $\to$ PSL(2,$\mathbf{R}$) be the canonical projection. It is a two-fold covering map so that the S^1 fiber (corresponding to PSO(2) in PSL(2,$\mathbf{R}$)) does not lift. Then P induces a two-fold covering map from SL(2,$\mathbf{R}$)$/\tilde{\rho}(\pi_1(\Sigma))$ to PSL(2,$\mathbf{R}$)$/\rho(\pi_1(\Sigma))$ so that the S^1 fiber does not lift. Thus we have a spin structure η on Σ associated to the lifting $\tilde{\rho}$ of ρ. A simple calculation shows that

$$\eta(\gamma_*) = sign(tr(\tilde{\rho}(\gamma))), \qquad \gamma \in \pi_1(\Sigma), \tag{5}$$

where γ_* is the geodesic representative or a multiple of a parabolic simple closed curve in the conjugacy class of γ.

1.3. Trace identities and representations of surface groups into SL(2,R). Given three matrices A_1, A_2, A_3 in SL(2,$\mathbf{C}$), we have the following identities on the traces of their products (see [9], [13], [17], or [26]). The basic trace identity is $trA_1A_2 + trA_1^{-1}A_2 = trA_1trA_2$. By iterated use of it, one obtains the following relations:

$$trA_1A_2trA_1^{-1}A_2 = tr^2A_1 + tr^2A_2 - tr[A_1, A_2] - 2. \tag{6}$$

$$\begin{aligned} tr[A_1, A_2] + 2 =& tr^2A_1 + tr^2A_2 + tr^2A_1A_2 \\ &- trA_1trA_2trA_1A_2. \end{aligned} \tag{7}$$

$$\begin{aligned} trA_1A_2A_3 + trA_1A_3A_2 =& trA_1trA_2A_3 + trA_2trA_3A_1 \\ &+ trA_3trA_1A_2 - trA_1trA_2trA_3. \end{aligned} \tag{8}$$

$$\begin{aligned} trA_1A_2A_3trA_1A_3A_2 =& tr^2A_1 + tr^2A_2 + tr^2A_3 \\ &+ tr^2A_1A_2 + tr^2A_2A_3 \\ &+ tr^2A_3A_1 + trA_1A_2trA_2A_3trA_3A_1 \\ &- trA_1trA_2trA_1A_2 - trA_2trA_3trA_2A_3 \\ &- trA_3trA_1trA_3A_1 - 4. \end{aligned} \tag{9}$$

Combining formulas (8) and (9), we see that $trA_1A_2A_3$ and $trA_1A_3A_2$ are the two roots of the quadratic equation (10) below where P and Q stand for the right-hand sides of formulas (8) and (9) respectively.

$$x^2 - Px + Q = 0. \tag{10}$$

Using the basic trace relation, one obtains the following (see [17],[7]).

Lemma 3 (**Fricke-Klein**). *Suppose F_n is the free group on n generators $\gamma_1, ..., \gamma_n$. Then for each element w in F_n, there is a polynomial P_w with integer coefficient in $2^n - 1$ variables $x_{i_1...i_k}$ with*

$$1 \leq i_1 < ... < i_k \leq n$$

so that for any representation $\rho : F_n \to SL(2,\mathbf{R})$

$$tr\rho(w) = P_w(x_1, x_2, ..., x_{i_1...i_k}, ..., x_{12...n}),$$

where $x_{i_1...i_k} = tr\rho(\gamma_{i_1}...\gamma_{i_k})$. Furthermore, if ρ_1 and ρ_2 are two representations with the same character, and $\rho_1(F_n)$ is not a solvable group, then ρ_1 is conjugated to ρ_2 by a $GL(2,\mathbf{R})$ matrix.

In particular, if $n = 2$, then the three variables are $tr\rho(\gamma_1)$, $tr\rho(\gamma_2)$ and $tr\rho(\gamma_1\gamma_2)$; if $n = 3$, the seven variables are $tr\rho(\gamma_i)$, and $tr\rho(\gamma_i\gamma_j)$ and $tr\rho(\gamma_1\gamma_2\gamma_3)$ where $i, j = 1, 2, 3$ and $i < j$.

The discrete faithful representations of $\pi_1(\Sigma_{0,r}^s)$ $(r + s = 3)$ and $\pi_1(\Sigma_{1,r}^s)$ $(r + s = 1)$ which uniformize hyperbolic structures on $\Sigma_{0,r}^s$ $(r + s = 3)$ and $\Sigma_{1,r}^s$ $(r + s = 1)$ are as follows. See [11], [21] for details.

For surface $\Sigma_{0,r}^s$, $r + s = 3$, we choose a set of geometric generators γ_1 and γ_2 in $\pi_1(\Sigma_{0,r}^s)$ so that γ_1, γ_2 and $\gamma_3 = \gamma_1\gamma_2$ are represented by simple closed curves homotopic into the three ends of $int(\Sigma_{0,r}^s)$. $\Sigma_{0,2}^1$ has the puncture at the end corresponding to γ_3 and $\Sigma_{0,1}^2$ has the punctures at the ends corresponding to γ_2 and γ_3.

Lemma 4 (**Fricke-Klein**). *(a) If $\rho \in \tilde{R}(\Sigma_{0,r}^s)$ with $r + s = 3$, then $tr\rho(\gamma_1)tr\rho(\gamma_2)tr\rho(\gamma_3) < 0$ and $|tr\rho(\gamma_i)| \geq 2$ for $i = 1, 2, 3$ so that the equality holds if and only if the corresponding end is a cusp.*

(b) Given three real numbers t_1, t_2 and t_3 with $t_1t_2t_3 < 0$ and $|t_i| > 2$ $(i = 1, 2, 3)$, there exist two elements ρ_1 and ρ_2 in $\tilde{R}(\Sigma_{0,3})$ unique up to $SL(2,\mathbf{R})$ conjugation so that $tr\rho_i(\gamma_j) = t_j$ $(i = 1, 2; j = 1, 2, 3)$. These two representations are $GL(2,\mathbf{R})$ conjugated and are related by $\rho_1(\gamma_i) = \rho_2(\gamma_i)^{-1}$. Furthermore, if $\rho(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, and $\rho(\gamma_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c = 1$, then a, b, d and λ are real analytic RC-functions of t_1, t_2 and t_3 in the domain defined by $t_1t_2t_3 < 0$ and $|t_i| > 2$ $(i = 1, 2, 3)$.

(c) Given three numbers t_1, t_2 and t_3 with $t_1t_2t_3 < 0$ and $|t_1| > 2, |t_2| > 2$ and $|t_3| = 2$ (resp. $|t_1| > 2, |t_2| = |t_3| = 2$), there

exist two elements ρ_1 and ρ_2 in $\tilde{R}(\Sigma^1_{0,2})$ (resp. $\tilde{R}(\Sigma^2_{0,1})$) unique up to SL(2,$\mathbf{R}$) conjugation so that $tr\rho_i(\gamma_j) = t_j$ ($i = 1,2; j = 1,2,3$). These two representations are GL(2,$\mathbf{R}$) conjugated and are related by $\rho_1(\gamma_i) = \rho_2(\gamma_i)^{-1}$. Furthermore, if $\rho(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, and $\rho(\gamma_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c = 1$, then a, b, d and λ are real analytic RC-functions of t_1, t_2 and t_3 in the domain defined by $t_1t_2t_3 < 0$, $|t_1| > 2$ and $|t_2| > 2$.

(d) $T(\Sigma^3_{0,0})$ consists of one point.

Note that part (a) is a consequence of Lemma 2 and formula (5). To find the explicit expression of a, b, d and λ in terms of $t_i's$, see [14, p.305].

For surface $\Sigma^s_{1,r}$ ($r + s = 1$), we take a set of geometric generators $\{\gamma_1, \gamma_2\}$ in $\pi_1(\Sigma^s_{1,r})$ so that they are represented by two simple closed curves a_1 and a_2 with $a_1 \perp a_2$. The multiplication $\gamma_3 = \gamma_1\gamma_2$ is represented (in the free homotopy class) by either a_1a_2 or a_2a_1 depending on the orientation of the surface. The commutator $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ is represented by the simple closed curve $\partial N(a_1 \cup a_2)$ homotopic into the end of $int(\Sigma^s_{1,r})$.

Lemma 5. **(Fricke-Klein, Keen)** *(a) If $\rho \in \tilde{R}(\Sigma^s_{1,r})$ with $r+s = 1$, then $tr\rho(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) \leq -2$ so that equality holds if and only if $s = 1$. In particular, $tr^2\rho(\gamma_1) + tr^2\rho(\gamma_2) + tr^2\rho(\gamma_3) - tr\rho(\gamma_1)tr\rho(\gamma_2)tr\rho(\gamma_3) \leq 0$ so that equality holds if and only if $s = 1$.*

(b) Give three numbers t_i, $i = 1, 2, 3$ with $|t_i| > 2$ and $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 < 0$ (resp. $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 = 0$), there exist two representations ρ_1 and ρ_2 in $\tilde{R}(\Sigma^0_{1,1})$ (resp. $\tilde{R}(\Sigma^1_{1,0})$) unique up to SL(2,$\mathbf{R}$) conjugation so that $tr\rho_i(\gamma_j) = t_j$ ($i = 1,2; j = 1,2,3$). These two representations are GL(2,$\mathbf{R}$) conjugated and are related by $\rho_1(\gamma_i) = \rho_2(\gamma_i)^{-1}$. Furthermore, if $\rho(\gamma_1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, and $\rho(\gamma_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c = 1$, then a, b, c, d and λ are real analytic RC-functions of t_1, t_2, t_3.

The first part of the lemma also follows from Lemma 2 and formula (5). Below is a proof of part (b) (known to J. Gilman). By Lemma 3, it suffices to show the existence of $\rho \in \tilde{R}(\Sigma^s_{1,r})$ with $tr(\rho(\gamma_j)) = t_j$, $j = 1, 2, 3$. We first construct three points A_1, A_2 and A_3 in $\mathbf{H}$ so that their pairwise hyperbolic distance $d(A_i, A_j)$ is determined by $2\cosh d(A_i, A_j)/4 = |t_k|$ where $i \neq j \neq k \neq i$. That the pairwise distances satisfy the triangular inequalities follows from the given condition on t_i's. Let h_{A_i} be the hyperbolic isometry which rotates by degree π at the point A_i (a half-turn). Then $h_{A_i}h_{A_j}$ $(i \neq j)$ is a hyperbolic isometry so that the absolute value of its trace is $|t_k|$ by the construction $(k \neq i, j)$. Furthermore, $tr(h_{A_1}h_{A_2}h_{A_3})^2 = tr[h_{A_1}h_{A_2}, h_{A_3}h_{A_1}]$ $= t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 - 2$ which is at most -2. Thus the isometry $h_{A_1}h_{A_2}h_{A_3}$ has a fixed point p at the circle at the infinity of $\mathbf{H}$. By the construction, the three vertices of the triangle A_1, A_2 and A_3 are on the three sides of the ideal hyperbolic triangle Δ with vertices p, $h_{A_3}(p)$ and $h_{A_2}h_{A_3}(p)$. The four components of the complement of the ideal quadrilateral $\Delta \cup h_{A_3}(\Delta)$ give rise to a Schottky condition for the group $< h_{A_1}h_{A_3}, h_{A_3}h_{A_2} >$. Thus by Poincaré polyhedron theorem, the group $< h_{A_1}h_{A_3}, h_{A_3}h_{A_2} >$ uniformizes either $\Sigma_{1,1}^0$ or $\Sigma_{1,0}^1$ so that the geodesics of $h_{A_1}h_{A_3}$ and $h_{A_3}h_{A_2}$ are simple closed curves intersecting at one point. Let Y be the lifting of $h_{A_1}h_{A_3}$ to SL(2,$\mathbf{R}$) with $t_2 trY > 0$, and X be the lifting of $h_{A_3}h_{A_2}$ to SL(2,$\mathbf{R}$) with $t_1 trX > 0$. Then $trX = t_1$ and $trY = t_2$ and $tr(XY) = t_3$ due to the spin structure. This finishes the proof.

2. Proof of Theorem 2

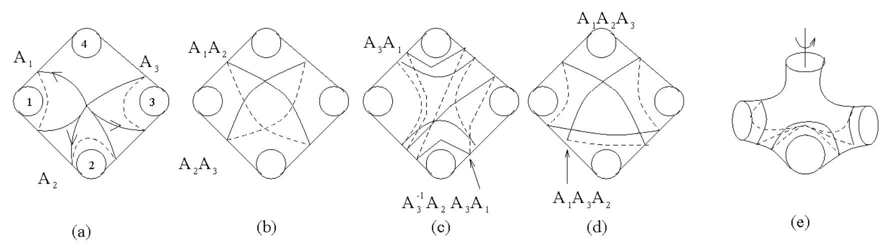
Given a hyperbolic metric m on Σ and a monodromy $\rho \in \tilde{R}(\Sigma)$ of the metric m, we have $t_m(x) = |tr(\rho(x))|$ where x is the homotopy class of a loop.

2.1. Proof of Theorem 2 for $\Sigma_{1,1}$.

To show that condition (1) in part (a) is necessary, take three classes $\alpha_1, \alpha_2, \alpha_3$ forming an ideal triangle in $\mathcal{S}$. Choose $\gamma_1, \gamma_2 \in \pi_1(\Sigma)$ so that the homotopy classes $\gamma_1, \gamma_2, \gamma_1\gamma_2$ and $\gamma_1^{-1}\gamma_2$ represent $\alpha_1, \alpha_2, \alpha_3$ and α_3' respectively. If t_m is a trace function corresponding to a monodromy $\rho \in \tilde{R}(\Sigma_{0,4})$, then condition (1) follows from the trace identities (6), (7) and Lemma 5 where $A_i = \rho(\gamma_i)$.

To show that condition (1) is also sufficient, we note that the modular relation implies that the value of t is determined by t on $\{\alpha_1, \alpha_2, \alpha_3\}$ where α_i's form an ideal triangle. Now since $t(b) \geq 2$, by condition (1), $t_i = t(\alpha_i)$ satisfies the inequalities in Lemma 5. By Lemma 5, we construct a hyperbolic metric m so that $t_m(\alpha_i) = t_i$. Thus, $t = t_m$ on $\mathcal{S}$ by the modular relation.

The proof of Theorem 2 for $\Sigma_{0,4}$ is in the same spirit, but technically is more complicated.



Left-hand orientation on the front face

Figure 5

FIGURE 5. Left-hand orientation on the front face

2.2. Necessity of condition (2) in Theorem 2.

Given three classes $\alpha_1, \alpha_2, \alpha_3$ forming an ideal triangle in $\mathcal{S}$, we take $a_{ij} \in \alpha_k$, $(i,j,k) = (1,2,3),(2,3,1),(3,1,2)$ so that $|a_{ij} \cap a_{jk}| = 2$. Without loss of generality, we may assume that (a_{ij}, b_i, b_j) bounds $\Sigma_{0,3}$ in $\Sigma_{0,4}$. Choose in $\Sigma_{0,4}$ a set of generators $\{A_1, A_2, A_3\}$ for $\pi_1(\Sigma_{0,4})$ as in Figure 5 (multiplication of loops in π_1 starts from left to right) so that (1) the boundary components b_1, b_2, b_3, and b_4 of $\Sigma_{0,4}$ are homotopic to representatives in A_1, A_2, A_3 and $A_1A_2A_3$ respectively; (2) the curves a_{12}, a_{23}, and a_{31} are homotopic to representatives in A_1A_2, A_2A_3, and A_3A_1 respectively; and (3) the generators are symmetric with respect to a $\mathbf{Z}_3$ action on $\Sigma_{0,4}$ preserving b_4 (Figure 5(e)).

Given $\rho \in \tilde{R}(\Sigma_{0,4})$ representing the monodromy of a hyperbolic metric m, we shall identify $\rho(A)$ with A for $A \in \pi_1(\Sigma_{0,4})$ for simplicity in this section. Thus $A_i's$ are SL(2,$\mathbf{R}$) matrices. By choosing a different lifting if necessary, we may assume that $trA_i < 0$ $(i = 1,2,3)$. By Lemma 4, $trA_iA_j < 0$ $(i \neq j)$, and $trA_1A_2A_3 < 0$. Then the first equation in condition (2) is given by trace identity (10). To see the second equation (which is the statement that $f(\alpha_3), f(\alpha_3')$ are the two roots in the first equation), we shall derive the equivalent equation

$$t_m(\alpha_1\alpha_2) + t_m(\alpha_2\alpha_1) = t_m(\alpha_1)t_m(\alpha_2) - t(b_1)t_m(b_2) - t_m(b_3)t_m(b_4).$$

To see this, we note that $\alpha_1\alpha_2 = \alpha_3$ and $\alpha_2\alpha_1$ are represented by A_1A_2 and $A_3^{-1}A_2A_3A_1$ respectively. Furthermore, by Lemma 4,

$$tr(A_3^{-1}A_2A_3A_1) < 0.$$

Thus the above formula is a consequence of the trace identity:

$$tr(A_3^{-1}A_2A_3A_1) + tr(A_1A_2) = tr(A_1)tr(A_2) + tr(A_3)tr(A_1A_2A_3) - tr(A_2A_3)tr(A_3A_1).$$

We shall write the first equation in condition (2) (i.e., equation (10)) explicitly as follows. Let $t_i = t_m(b_i)$ and $t_{ij} = t_m(\alpha_k)$. Then formulas (8) and (9) become:

$$-t_4 + tr(A_1A_3A_2) = t_1t_{23} + t_2t_{31} + t_3t_{12} + t_1t_2t_3. \tag{11}$$

$$\begin{aligned} -t_4 tr(A_1A_3A_2) = & t_1^2 + t_2^2 + t_3^2 + t_{12}^2 + t_{23}^2 + t_{31}^2 \\ & + t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{31} \\ & - t_{12}t_{23}t_{31} - 4. \end{aligned} \tag{12}$$

Thus equation (10) becomes

$$\begin{aligned} & t_4^2 + t_4(t_1t_{23} + t_2t_{31} + t_3t_{12} + t_1t_2t_3) + \sum_{i=1}^{3} t_i^2 \\ & + \sum_{(i,j)\in I} t_{ij}^2 + t_it_jt_{ij} - 4 - t_{12}t_{23}t_{31} = 0, \end{aligned} \tag{13}$$

where I = {(1,2), (2,3), (3,1)}. As a quadratic equation in $-t_4 = tr(A_1A_2A_3)$, it becomes $x^2 - Px + Q = 0$ where $P > 0$ and (thus) $Q < 0$. This implies that the equation has two real roots of different signs and $-t_4$ is the negative root, i.e.,

$$t_4 = (-P + \sqrt{P^2 - 4Q})/2. \tag{14}$$

In particular, the number t_4 is determined by the rest of the six numbers. Since $t_4 > 2$, we obtain the (equivalent) condition that $-Q > 2P + 4$ which is exactly condition (4). Conversely, if $-Q > 2P + 4$ and $P > 0$, then $t_4 > 2$.

Remark 2.1. We have shown that each hyperbolic metric m on $\Sigma_{0,4}$ is determined by its lengths on six curves $\{a_{ij}, b_1, b_2, b_3\}$. This was first observed by Schmutz ([32, Lemma 2]).

2.3. Sufficiency of condition (2) in Theorem 2.

We use the same notation as in §2.2. Given a function $t : \mathcal{S}(\Sigma_{0,4}) \to \mathbf{R}_{>2}$ satisfying condition (2), we note that the modular relation implies that the values of t is determined by t on $\{\alpha_1, \alpha_2, \alpha_3, b_1, b_2, b_3, b_4\}$ where α_i's form an ideal triangle. Thus, it suffices to find $\rho \in \tilde{R}(\Sigma_{0,4})$ so that $tr\rho(A_i) = -t(b_i)$, $tr\rho(A_iA_j) = -t(\alpha_k)$ and $tr\rho(A_1A_2A_3) = -t(b_4)$.

Let $t_i = t(b_i)$ $(i = 1, 2, 3, 4)$ and $t_{ij} = t(\alpha_k)$. Then $t_i, t_{ij} \in \mathbf{R}_{>2}$ and equation (13) holds. By the remark in the last paragraph, this is the same as assuming condition (4) holds for t_1, t_2, t_3 and t_{ij}. We shall first construct three matrices A_i $(i = 1, 2, 3)$ in $\mathrm{SL}(2,\mathbf{R})$ so that $trA_i = -t_i$, $tr(A_iA_j) = -t_{ij}$ and furthermore $trA_1A_2A_3 < -2$. Then we show that $tr(A_1A_2A_3) = -t_4$ and the corresponding representation ρ is in $\tilde{R}(\Sigma_{0,4})$.

Since conditions (13) and (4) are symmetric in t_{12}, t_{23} and t_{31} and the set of generators A_1, A_2, and A_3 are also symmetric, we may assume without loss of generality that $t_{23} = \max(t_{12}, t_{23}, t_{31})$.

To solve $trA_i = -t_i$ and $tr(A_iA_j) = -t_{ij}$, let $A_1 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A_3 = \begin{pmatrix} -\lambda d & \lambda^{-1} b \\ \lambda c & -\lambda^{-1} a \end{pmatrix}$ be $\mathrm{SL}(2,\mathbf{R})$ matrices. We have,

$$A_2A_3 = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix},$$

$$A_1A_2 = \begin{pmatrix} ax + cy & * \\ * & bz + dw \end{pmatrix},$$

$$A_3A_1 = \begin{pmatrix} -\lambda dx + \lambda^{-1} bz & * \\ * & \lambda cy - \lambda^{-1} aw \end{pmatrix},$$

$$A_1A_2A_3 = \begin{pmatrix} -\lambda x & * \\ * & -\lambda^{-1} w \end{pmatrix}.$$

By the condition $trA_i = -t_i$ and $tr(A_iA_j) = -t_{ij}$, we obtain a system of quadratic and linear equations in a, b, c, d, x, y, z, w and λ as follows.

(E1) $$a + d = -t_2.$$

(E2) $$\lambda^{-1} a + \lambda d = t_3.$$

(E3) $$\lambda + \lambda^{-1} = t_{23}.$$

(E4) $$ad - bc = 1.$$

(E5) $$x + w = -t_1.$$

(E6) $$ax + cy + bz + dw = -t_{12}.$$

(E7) $$-\lambda dx + \lambda cy + \lambda^{-1} bz - \lambda^{-1} aw = -t_{31}.$$

(E8) $$xw - yz = 1.$$

By (E3), λ is a positive real number not equal to 1 and is determined up to reciprocal. Let us fix $\lambda > 1$. By (E1) and (E2), we have $a = -(\lambda t_2 + t_3)/(\lambda - \lambda^{-1})$ and $d = (\lambda^{-1} t_2 + t_3)/(\lambda - \lambda^{-1})$. Thus $ad < 0$ and $bc = ad - 1 < 0$. Fix $c = 1$. We obtain a set of solutions in a, b, c, d and λ

which are real analytic RC-functions in t_i's and t_{ij}'s. We now claim that there are solutions for x, y, z, and w satisfying (E5)-(E8) in the complex number field **C**. Indeed, by (E6) and (E7), we express y and z in terms of x and w as follows. $y = (t_3 x - \lambda^{-1} t_2 w + \lambda^{-1} t_{12} - t_{31})/(c(\lambda - \lambda^{-1}))$ and

$$(E9) \qquad z = (t_2 \lambda x - t_3 w - \lambda t_{12} + t_{31})/(b(\lambda - \lambda^{-1})).$$

Using (E5), we have $w = -x - t_1$. Thus, $y = (\lambda^{-1} t_2 + t_3)x/(c(\lambda - \lambda^{-1})) + const$ and $z = (\lambda t_2 + t_3)x/(b(\lambda - \lambda^{-1})) + const$. Substitute these new equations and $w = -x - t_1$ into (E8). We obtain a quadratic equation in x whose leading coefficient (after a simple calculation) is $1/(bc) \neq 0$. Thus there is a solution for x in **C**. This implies the existence of solutions for y, z and w in **C**.

We next claim that x, y, z, and w are real numbers, i.e., A_1 is in SL(2,**R**). Indeed, the quadratic equation (in $-t_4$) (13) $x^2 - Px + Q = 0$ has two real roots of different signs. By (13), both $trA_1A_2A_3$ and $trA_1A_3A_2$ are solutions of the equation. Thus $trA_1A_2A_3$ is a real number. But $trA_1A_2A_3 = -\lambda x - \lambda^{-1} w$. This together with equation (E5) shows that both x and w are real numbers. Hence y and z are real numbers as well.

Now by choosing a different set of solution if necessary, we may assume that $trA_1A_2A_3$ is the negative root $-t_4$ of the equation $t^2 - Pt + Q = 0$, i.e.,

$$(E10) \qquad \lambda x + \lambda^{-1} w = t_4.$$

Indeed, if $trA_1A_2A_3$ is the positive root, we use the new set of solution $(A_1^{-1}, A_2^{-1}, A_3^{-1})$ to the equations $trX_i = -t_i$ and $trX_iX_j = -t_{ij}$ instead of (A_1, A_2, A_3) and use the fact that $trA_1^{-1}A_2^{-1}A_3^{-1} = trA_1A_3A_2$.

By the proof of above, we see that the solution a, b, c, d, x, y, z, w and λ are real analytic RC-functions in t_i and t_{ij} (i,j=1,2,3).

By condition (4), the negative root $trA_1A_2A_3$ is less than -2, i.e., $t_4 > 2$. Thus both representations of $\pi_1(\Sigma_{0,3})$ (in term of the pair of matrices) given by $< A_1^{-1}, A_1A_2A_3 >$ and $< A_2, A_3 >$ are in $\tilde{R}(\Sigma_{0,3})$ by Lemma 4. Furthermore, these two groups share a common generator $A_1^{-1}(A_1A_2A_3) = A_2A_3$. To apply the Maskit combination theorem [27] to amalgamate these two groups, we need to verify that the Nielsen convex cores for the two groups $< A_1^{-1}, A_1A_2A_3 >$ and $< A_2, A_3 >$ in **H** lie on the different sides of the axis of A_2A_3. The following lemma characterizes the side of the axis which contains the Nielsen core.

Lemma 6. *Suppose* $X = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}$ *and* $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *are* $SL(2,\mathbf{R})$ *matrices so that* $trX < -2$, $trY \leq -2$, $trXY \leq -2$. *Then the side of the axis of* X *which contains the Nielsen convex core for the discrete group* $< X, Y >$ *is* $\{(x,y)|x > 0, y > 0\}$ *if and only if* $c(\lambda - \lambda^{-1}) > 0$.

Proof. Let $trX = -t_1$, $trY = -t_2$, $trXY = -t_3$ with $t_1 > 2$ and $t_2, t_3 \geq 2$. Then we have $\lambda + \lambda^{-1} = t_1$, $a + d = -t_2$, $\lambda a + \lambda^{-1} d = t_3$, and $ad - bc = 1$. We solve for a and d and obtain: $a = (\lambda^{-1}t_2 + t_3)/(\lambda - \lambda^{-1})$ and $d = -(\lambda t_2 + t_3)/(\lambda - \lambda^{-1})$. The fixed points r_1 and r_2 of Y at the circle at the infinite of $\mathbf{H}$ are the roots of the equation

$$ct^2 + (d-a)t - b = 0.$$

In particular,

$$r_1 + r_2 = -(d-a)/c,$$

which is

$$(t_1 t_2 + 2t_3)/(c(\lambda - \lambda^{-1})).$$

Since the fixed points r_1 and r_2 are in the Nielsen core, the result follows. q.e.d.

Now to finish the proof, we verify the side condition by taking $X = A_2A_3$, and $Y = A_1$ for the group $< A_1^{-1}, A_1A_2A_3 >$, and taking $X = A_2A_3$, $Y = A_2^{-1}$ for $< A_2, A_3 >$. Thus it suffices to show $-zc < 0$, or the same $zb < 0$.

By (E5) and (E10), we have $x = (\lambda^{-1}t_1 + t_4)/(\lambda - \lambda^{-1})$ and $w = -(\lambda t_1 + t_4)/(\lambda - \lambda^{-1})$. Substitute them into (E9) and simplify it, we have,

$$bz(\lambda - \lambda^{-1})^2 = t_2\lambda(\lambda^{-1}t_1 + t_4) + t_3(\lambda t_1 + t_4) - (\lambda - \lambda^{-1})\lambda t_{12} + (\lambda - \lambda^{-1})t_{31}.$$

By (E3), we replace λ^2 by $\lambda t_{23} - 1$ and λ^{-1} by $t_{23} - \lambda$ in the above equation and obtain,

$$bz(\lambda - \lambda^{-1})^2 = \lambda(t_1t_3 + 2t_{31} + t_2t_4 - t_{23}t_{12}) + (t_1t_2 + 2t_{12} + t_3t_4 - t_{23}t_{31}).$$

We claim that under the condition $t_{23} = \max(t_{12}, t_{23}, t_{31})$ and equation (13) both $t_1t_3 + 2t_{31} + t_2t_4 - t_{23}t_{12}$ and $t_1t_2 + 2t_{12} + t_3t_4 - t_{23}t_{31}$ are negative. Indeed, since $t_{23} = \max(t_{12}, t_{23}, t_{31})$, and t_i, t_{ij} are at least 2, by equation (13), we have,

$$\begin{aligned} t_{12}t_{23}t_{31} >& t_1t_3t_{31} + t_{31}^2 + t_{23}^2 + t_2t_{31}t_4 \\ \geq& t_1t_3t_{31} + 2t_{31}^2 + t_2t_{31}t_4 \\ =& t_{31}(t_1t_3 + 2t_{31} + t_2t_4). \end{aligned}$$

This shows $t_1t_3 + 2t_{31} + t_2t_4 - t_{23}t_{12} < 0$. The other inequality follows by the same argument since the inequality is obtained from the previous one by interchanging the indices 2 and 3. q.e.d.

The proof shows that all the entries of the matrices A_1, A_2, A_3 are RC functions in t_i, t_{ij} where $i = 1, 2, 3, (i, j) = (1, 2), (2, 3), (3, 1)$.

Corollary 2.1. *For surface $\Sigma_{0,4}$ with $\partial\Sigma_{0,4} = b_1 \cup b_2 \cup b_3 \cup b_4$, let*

$$F = \{[a_{12}], [a_{23}], [a_{31}], b_1, b_2, b_3\}$$

so that $[a_{ij}]$ forms an ideal triangle and (a_{ij}, b_i, b_j) bounds a $\Sigma_{0,3}$. Then the map π_F: $T(\Sigma_{0,4}) \to \mathbf{R}^6$ is an embedding so that its image is given by $\{(t_1, t_2, t_3, t_{12}, t_{23}, t_{31}) \in \mathbf{R}^6_{>2} |$ formula (4) holds$\}$. Furthermore, there exits a continuous function f: $T(\Sigma_{0,4}) \to \tilde{R}(\Sigma_{0,4})$ sending $m \in T(\Sigma_{0,4})$ a representation $f(m)$ which is a lifting of a monodromy of m so that the entries of the matrices $f(m)(\gamma)$ are real analytic RC-functions of the coordinates of $\pi_F(m)$, for each $\gamma \in \mathcal{S}(\Sigma_{0,4})$.

Remark 2.2. The above proof works for hyperbolic metrics with cusp ends as well since Lemmas 3, 6 and Maskit combination theorem still hold. In particular, we obtain the following parametrization of the Teichmüller space of $T^4_{0,0}$ by the geodesic lengths t_{12}, t_{23} and t_{31} (other variables t_1, t_2, t_3 and t_4 are 2). Take $F = \{[a_{12}], [a_{23}], [a_{31}]\}$. Then the image of the embedding π_F of $T^4_{0,0}$ is

$$\{(t_{12}, t_{23}, t_{31}) \in \mathbf{R}^3_{>2} | t_{12}t_{23}t_{31} = t_{12}^2 + t_{23}^2 + t_{31}^2 + 8t_{12} + 8t_{23} + 8t_{31} + 28\}.$$

3. A combinatorical structure on the set of isotopy classes of simple closed curves

We introduce the following notation for convenience. If $\alpha \perp_0 \beta$ (resp. $\alpha \perp \beta$), then $\partial N(\alpha \cup \beta)$ denotes the union of the isotopy classes of four boundary components of $N(a \cup b)$ where $a \in \alpha$, $b \in \beta$ with $|a \cap b| = I(a, b)$ (resp. $\partial N(\alpha \cup \beta) = [\partial N(a \cup b)]$).

The goal of this section is to prove the following proposition.

Proposition 1. *(a) Given a set of disjoint simple closed curves and proper arcs $\{c_1, ..., c_n\}$ in a compact oriented surface Σ, let $G_0 = \{\alpha \in \mathcal{S}(\Sigma)|\ I(\alpha, [c_i]) \leq 2$ so that for each index i, if equality holds then the two points of intersection have different signs$\}$. Then $\mathcal{S}(\Sigma) = \cup_{i=0}^{\infty} G_i$ where $G_{i+1} = G_i \cup \{\alpha|\ \alpha = \beta\gamma$ where either (1) $\beta \perp \gamma$, and β, γ, $\gamma\beta$ are in G_i, or (2) $\beta \perp_0 \gamma$, and β, γ, $\gamma\beta$, and each component of $\partial N(\beta \cup \gamma)$ are in $G_i\}$.*

(b) Under the same assumption as in (a), if f is a function defined on $\mathcal{S}(\Sigma)$ so that (1) $f(\alpha\beta)$ is determined by $f(\alpha)$, $f(\beta)$, and $f(\beta\alpha)$ whenever $\alpha \perp \beta$, and (2) $f(\alpha\beta)$ is determined by $f(\alpha)$, $f(\beta)$, $f(\beta\alpha)$, and $f(\gamma_i)$ $(i = 1, 2, 3, 4)$ whenever $\alpha \perp_0 \beta$ with $\partial N(\alpha \cup \beta) = \cup_{i=1}^{4} \gamma_i$, then f is determined by $f|_{G_0}$.

Part (b) of the proposition follows from part (a). The proof of part (a) of the proposition is a simple application of the lemma below by induction on the number $\max\{I(\alpha, [c_i])|i = 1, ..., n\}$ for $\alpha \in \mathcal{S}(\Sigma)$. This lemma is inspired by Lemma 2 in [22].

Lemma 7. *Suppose a is a simple closed curve, and b is either a simple closed curve or an arc so that either $I(a, b) = |a \cap b| \geq 3$ or a intersects b at two points of the same intersection signs. Let $\{c_1, .., c_n\}$ be a collection of disjoint simple closed curves or arcs so that $int(b) \cap c_i = \emptyset$ for all $i = 1, ..., n$. Then there exist two simple closed curves p_1 and p_2 in $N(a \cup b)$ so that*

(1) $a = p_1 p_2$ where either $p_1 \perp p_2$ or $p_1 \perp_0 p_2$,

(2) $|p_i \cap b| < |a \cap b|$, $|p_2 p_1 \cap b| < |a \cap b|$, $|p_i \cap c_j| \leq |a \cap c_j|$ and $|p_2 p_1 \cap c_j| \leq |a \cap c_j|$ for $i = 1, 2$ and $j = 1, 2, ..., n$, and,

(3) if $p_1 \perp_0 p_2$, there are four simple closed curves d_1, d_2, d_3, and d_4 isotopic to four boundary components of $N(p_1 \cup p_2)$ so that $|d_i \cap b| < |a \cap b|$ and $|d_i \cap c_j| \leq |a \cap c_j|$ for $i = 1, 2, 3, 4$, and $j = 1, ..., n$.

Proof. We need to consider two cases.

Case 1. There exist two adjacent intersection points x and y in b which have the same intersection signs (see Figure 6). Let c be an arc in b joining x and y so that $int(c) \cap a = \emptyset$. Then the curves p_1 and p_2 as shown in Figure 6 (with the right-hand orientation on the surface) satisfy $p_1 \perp p_2$ and all conditions in the lemma.

Case 2. Suppose any pair of adjacent intersection points in b has different intersection signs. Then $|a \cap b| \geq 3$. Take three intersection

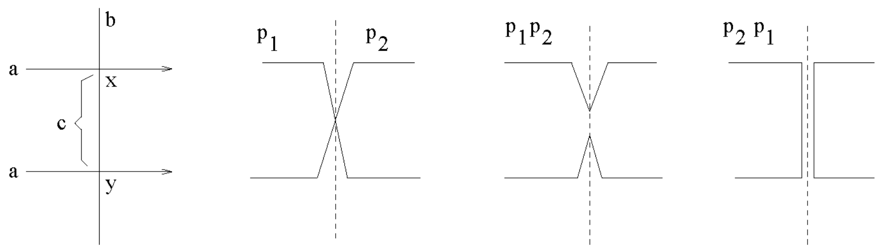


Figure 6

FIGURE 6

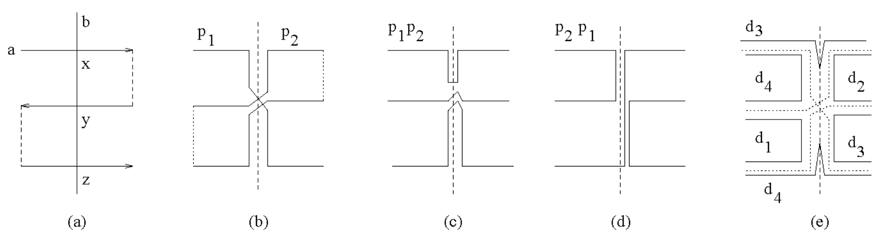


Figure 7

FIGURE 7

points x, y, z in b so that x, y and y, z are adjacent. Their intersection signs alternate. Fix an orientation on a so that the arc from x to y in a does not contain z as shown in Figure 7(a). If the surface Σ is right-hand oriented as in Figure 7(a), take p_1 and p_2 as in Figure 7(b). Then $p_1 \perp_0 p_2$ in $N(p_1 \cup p_2)$. We claim that $p_1 \perp_0 p_2$ in Σ. To see this, it suffices to show that $N(p_1 \cup p_2)$ is incompressible in Σ. Indeed, each boundary components of $N(p_1 \cup p_2)$ is isotopic to a simple loop d_i made by the arcs with ends x, y, z along a and d. Since $|a \cap d| = I(a, d)$, these loops d_i are essential. Thus the claim follows. By the construction conditions (1), (2) and (3) follow from Figure 7(c), (d) and (e). If Σ is left-hand oriented, we simply interchange p_1 and p_2.

As an application of the proposition, we show that the mapping class group is finitely generated by Dehn twists. Take f in the proposition to be the map sending $\alpha \in \mathcal{S}(\Sigma)$ to the isotopy class of positive Dehn twist along α. First of all, there are two basic relations on the Dehn twists: (1) (braid relation) $D_{\alpha\beta} = D_\alpha D_\beta D_\alpha^{-1}$ for $\alpha \perp \beta$ and (2) (lantern relations) $D_\alpha D_\beta D_{\alpha\beta} = D_{\partial N(\alpha \cup \beta)}$ for $\alpha \perp_0 \beta$. Thus by the proposition, the mapping class group is generated by elements in G_0. For all surfaces, it is easy to construct a finite set G_0 so that $G_\infty = \mathcal{S}$. For instance, if surface $\Sigma_{g,r}$ has $r > 0$, let $\{c_1, ..., c_n\}$ $(n = 6g + 3r - 6)$ be an ideal triangulation of it, i.e., a maximal collection of disjoint pairwise non-isotopic, essential arcs in $\Sigma_{g,r}$. Then the corresponding collection G_0 in the corollary is a finite set, indeed $|G_0| \leq 3^n$ since each $\alpha \in CS(\Sigma)$ is determined by the n-tuple $(I(\alpha, [c_1]), ..., I(\alpha, [c_n]))$.

Remark. The lantern relation was discovered and used by M. Dehn ([8, p.333]) and rediscovered independently by Johnson in 1979; (see for instance [2, p.19]. Also the braid relation (1) implies the Artin's relation $D_\alpha D_\beta D_\alpha = D_\alpha D_\beta D_\alpha$.

4. Proof of Theorem 1

We prove Theorem 1 for compact surface $\Sigma_{g,r}$ with or without boundary in §4.1-4.3. In §4.4, we indicate the modification needed for non-compact surfaces. By the proof of Theorem 2, it suffices to show that conditions (1) and (2) are sufficient.

4.1. Reduction to the surfaces $\Sigma_{0,5}$ and $\Sigma_{1,2}$.

We shall prove Theorem 1 by induction on the norm $|\Sigma_{g,r}| = 3g + r$ of a compact surface. The goal of this section is to show that Theorem

1 for all surfaces follows from Theorem 1 for $\Sigma_{0,5}$ and $\Sigma_{1,2}$.

Given $\Sigma = \Sigma_{g,r}$ with $|\Sigma| \geq 5$, and a function $f : \mathcal{S}(\Sigma) \to \mathbf{R}$ which is a trace function on each incompressible subsurface Σ' of norm 4, we decompose $\Sigma = X \cup Y$ so that X, Y are incompressible of smaller norms with $int(X \cap Y) \cong int(\Sigma_{0,3})$ as Figure 3(d). To be more precise, we take $X = \Sigma_{0,r-1}$, $Y = \Sigma_{0,4}$ if $g = 0$, and take $X = \Sigma_{g-1,r+2}$, $Y = \Sigma_{1,1}$ if $g \geq 1$. Consider the restrictions $f|_{\mathcal{S}(X)}$ and $f|_{\mathcal{S}(Y)}$. By the induction hypothesis we find hyperbolic metrics m_X and m_Y on X and Y respectively realizing the restrictions as the trace functions. By the gluing lemma, we construct a hyperbolic metric m on Σ whose restriction to X and Y are isotopic to m_X and m_Y. Thus the trace function t_m and f have the same values on $\mathcal{S}(X) \cup \mathcal{S}(Y)$.

The goal is to show that the above condition

$$f|_{\mathcal{S}(X)\cup\mathcal{S}(Y)} = t_m|_{\mathcal{S}(X)\cup\mathcal{S}(Y)}$$

implies $f = t_m$. To achieve this, let us rewrite the conditions (1), (2) satisfied by f and t_m as follows:

(1′) $f^2(\alpha) + f^2(\beta) + f^2(\alpha\beta) - f(\alpha)f(\beta)f(\alpha\beta) - 2 + f(\partial N(\alpha \cup \beta)) = 0$, if $\alpha \perp \beta$,

(2′) $f^2(\alpha) + f^2(\beta) + f^2(\alpha\beta) - f(\alpha)f(\beta)\, f(\alpha\beta) + f(\alpha)(f(\gamma_1)f(\gamma_2) + f(\gamma_3)f(\gamma_4)) + f(\beta)\ (f(\gamma_2)\ f(\gamma_3) + f(\gamma_1)f(\gamma_4)) + f(\alpha\beta)(f(\gamma_2)f(\gamma_4) + f(\gamma_1)f(\gamma_3)) + f^2(\gamma_1) + f^2(\gamma_2) + f^2(\gamma_3) + f^2(\gamma_4) + f(\gamma_1)\ f(\gamma_2)\ f(\gamma_3)\ f(\gamma_4) - 4 = 0$, if $\alpha \perp_0 \beta$,

(3′) $f(\alpha\beta) + f(\beta\alpha) = f(\alpha)f(\beta)$, if $\alpha \perp \beta$, and

(4′) $f(\alpha\beta) + f(\beta\alpha) = f(\alpha)f(\beta) - f(\gamma_1)f(\gamma_3) - f(\gamma_2)f(\gamma_4)$, if $\alpha \perp_0 \beta$, where $\gamma_i' s$ are the four components of $\partial N(\alpha \cup \beta))$ so that α separates $\{\gamma_1, \gamma_2\}$ from $\{\gamma_3, \gamma_4\}$ and β separates $\{\gamma_2, \gamma_3\}$ and $\{\gamma_1, \gamma_4\}$.

Note that relations (3′) and (4′) give rise to an iteration process. Namely, the value $f(\alpha\beta)$ is determined by the values of f at α, β, and $\beta\alpha$ if $\alpha \perp \beta$, and is determined by the values of f at α, β, $\beta\alpha$ and the four components of $\partial N(\alpha \cup \beta)$ if $\alpha \perp_0 \beta$.

Let a_1, a_2 be the simple loops in $\partial(X \cap Y)$ which is nonboundary parallel in Σ as in Figure 3(d). Applying Proposition 1 to f and to t_m with respect to the set $\{a_2\}$, we conclude that $f = t_m$ follows from $f(\alpha) = t_m(\alpha)$ where $\alpha \perp_0 [a_2]$. Assume that Theorem 1 holds for $\Sigma_{0,5}, \Sigma_{1,2}$. We show $f(\alpha) = t_m(\alpha)$ with $\alpha \perp_0 [a_2]$ as follows. Take $s \in \alpha$ so that $|s \cap a_2| = 2$. Then $Z = Y \cup N(s)$ is an incompressible subsurface homeomorphic either to $\Sigma_{1,2}$ or $\Sigma_{0,5}$. Let $X' = X \cap Z$, $Y' = Y \cap Z$. Then $Z = X' \cup Y'$ so that $X' \cap Y' = X \cap Y$. Consider $f|_{\mathcal{S}(Z)}$ and $t_m|_{\mathcal{S}(Z)}$.

By Theorem 1 for Z and the fact that f and t_m coincide on the subset $\mathcal{S}(X') \cup \mathcal{S}(Y')$, we conclude that $f = t_m$ on $\mathcal{S}(Z)$ by the gluing lemma. In particular, $f(\alpha) = t_m(\alpha)$.

It remains to show Theorem 1 for $\Sigma_{0,5}$ and $\Sigma_{0,5}$. By the same decomposition $\Sigma = X \cup Y$ as above, it suffices to show the following two lemmas.

For simplicity, we let $Im(\Sigma)$ be the set of all functions from $\mathcal{S}(\Sigma)$ to $\mathbf{R}_{>2}$ satisfying conditions $(1')$, $(2')$, $(3')$, and $(4')$. Two classes α and β are *disjoint* if they are distinct and have disjoint representatives.

Lemma 8. *Suppose α_1 and α_2 are two disjoint elements in $\mathcal{S}'(\Sigma_{0,5})$. If two elements f and g in $Im(\Sigma_{0,5})$ satisfy $f(\alpha) = g(\alpha)$ for all $\alpha \in \mathcal{S}(\Sigma_{0,5})$ with $I(\alpha, \alpha_1)I(\alpha, \alpha_2) = 0$, then $f = g$.*

Lemma 9. *Suppose α_1 and α_2 are two disjoint elements in $\mathcal{S}'(\Sigma_{1,2})$ so that α_1 is non-separating and α_2 is separating. If f and g are two elements in $Im(\Sigma_{1,2})$ so that $f(\alpha) = g(\alpha)$ for all $\alpha \in \mathcal{S}(\Sigma_{1,2})$ with $I(\alpha, \alpha_1)I(\alpha, \alpha_2) = 0$, then $f = g$.*

4.2. Proof of Lemma 8.

To prove Lemma 8, by Proposition 1, it suffices to show that $f(\alpha) = g(\alpha)$ for $\alpha \perp_0 \alpha_i$ for $i = 1, 2$. Let $a_i \in \alpha_i$ be a representative so that $|a_1 \cap a_2| = 0$, and let $x \in \alpha$ so that $x \perp_0 a_i$ for $i = 1, 2$. Note that if $x' \perp_0 a_i$ for $i = 1, 2$, there is an orientation preserving homeomorphism h of $\Sigma_{0,5}$ sending x to x' and preserving each a_i (since both $N(a_1 \cup a_2 \cup x)$ and $N(a_1 \cup a_2 \cup x')$ are strong deformation retractors for $\Sigma_{0,5}$). Thus we may draw x as in Figure 8(a). Let a, b, c, d, e and b_1, b_2, c_1, c_2, d_1, and d_2 be curves as in Figures 8(a), (b) and (c) so that each of them is either disjoint from a_1 or from a_2.

Claim. There is a rational function R so that for each $h \in Im(\Sigma_{0,5})$, $h(x) = R(h(a), h(b), \ldots, h(e), h(a_1), h(a_2), h(b_1), h(b_2), h(c_1), h(c_2), h(d_1), h(d_2))$.

It follows from the claim that $f(\alpha) = g(\alpha)$. This finishes the proof of Lemma 8.

Before begin the proof of the claim, let us simplify the notation by making the following conventions:

(C1) The value of h at a curve s will be denoted by s.

(C2) The multiplication of two curves s_1 and s_2 will be denoted by $s_1 \circ s_2$.

(C3) Surfaces drawn in the figures have the right-hand orientation in the front face.

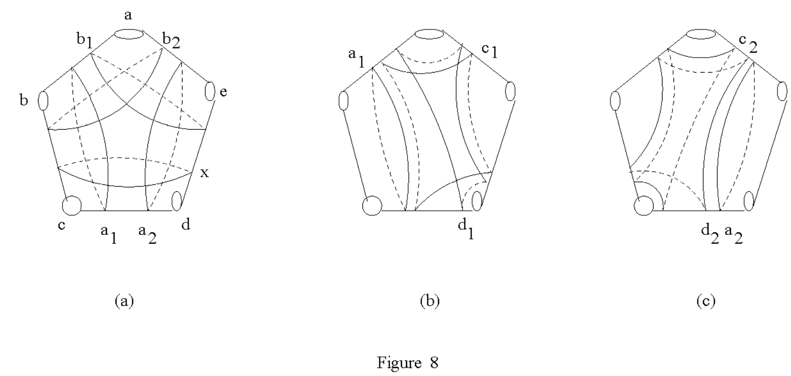


FIGURE 8

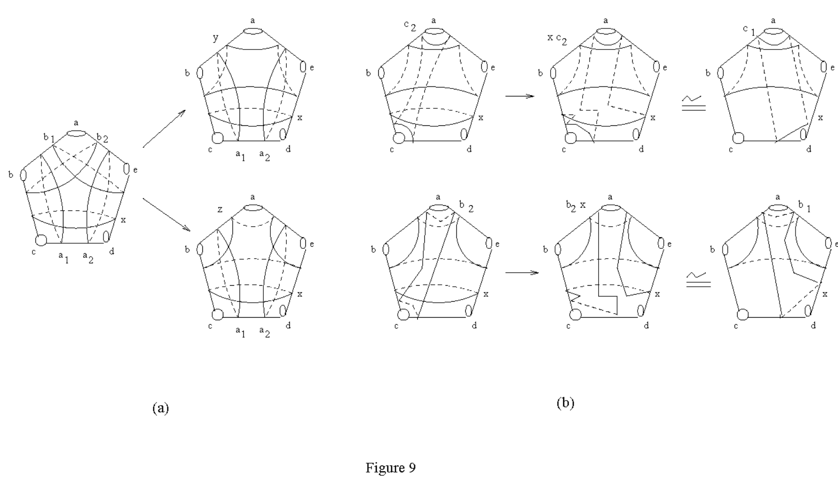


FIGURE 9

Let $y = b_1 \circ b_2$ and $z = b_2 \circ b_1$ as in Figure 9(a).

Since $b_1 \perp_0 b_2$ and $\partial N(b_1 \cup b_2) \cong a \cup b \cup e \cup x$, applying relation (2′) in $N(b_1 \cup b_2)$ with respect to b_1, b_2 and y, we obtain: $x^2 + a^2 + b^2 + e^2 + y^2 + b_1^2 + b_2^2 - b_1 b_2 y + abex + b_1(ae + bx) + b_2(ex + ab) + y(ax + be) - 4 = 0$. This can be written as:

$$x^2 + y^2 + axy + p_1 x - p_2 y + p_3 = 0, \tag{15}$$

where p_j are some polynomials in $a, b, c, d, e, a_i, b_i, c_i$, and d_i (the same notation apply below) and $p_j > 0$ for $j = 1, 2, 3$. Note that $p_2 = b_1 b_2 - be > 0$ due to equation (2).

Similarly,

$$x^2 + z^2 + axz + p_1 x - p_2 z + p_3 = 0. \tag{16}$$

Furthermore, by (4′) we have $y + z = b_1 b_2 - ax - be$, i.e.,

$$ax + y + z = p_4. \tag{17}$$

Now $c_2 \perp_0 x$ and $x \circ c_2 = c_1$ (see Figure 9(b)). Applying the relation (2′) to $N(c_2 \cup x)$ with respect to c_2, x, c_1 and using $\partial N(c_2 \cup x) \cong a \cup c \cup d \cup y$, we obtain $y^2 + a^2 + c^2 + d^2 + x^2 + c_1^2 + c_2^2 - c_1 c_2 x + acdy + x(ay + cd) + c_2(ac + dy) + c_1(ad + cy) - 4 = 0$, i.e.,

$$x^2 + y^2 + axy - p_5 x + p_6 y + p_7 = 0, \tag{18}$$

where p_5, p_6 and p_7 are positive.

Similarly, use of $d_2 \perp_0 x$ and $d_1 \circ x = d_2$ yields a relation:

$$x^2 + z^2 + axz - p_8 x + p_9 z + p_{10} = 0, \tag{19}$$

where p_8, p_9 and p_{10} are positive.

Consider the difference of (15) and (18). We obtain,

$$p_{11} x - p_{12} y = p_{13}, \tag{20}$$

where p_{11} and p_{12} are positive.

Similarly from (16) and (19) it follows that

$$p_{14} x - p_{15} z = p_{16}, \tag{21}$$

where p_{14} and p_{15} are positive.

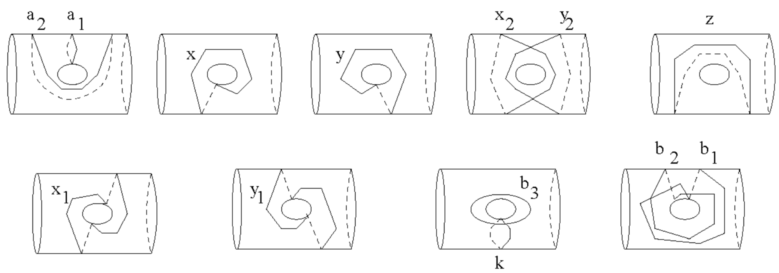


Figure 10

FIGURE 10

Now the system of linear equations (17), (20) and (21) in variables x, y and z has a unique solution since its determinant is positive. This ends the proof of the claim and thus finishes the proof of Lemma 8.

4.3. Proof of Lemma 9.

To prove Lemma 9, by Proposition 1, it suffices to show that $f(\alpha) = g(\alpha)$ for $\alpha \in \mathcal{S}(\Sigma_{1,2})$ with $\alpha \perp_0 \alpha_2$ and $\alpha \perp \alpha_1$ since there is no element $\beta \in \mathcal{S}(\Sigma_{1,2})$ such that $\beta \perp_0 \alpha_i$ for $i = 1, 2$. Fix such an α for the rest of the proof. Take $x \in \alpha$, $a_i \in \alpha_i$, $i = 1, 2$ so that $a_1 \cap a_2 = \emptyset$, $x \perp a_1$ and $x \perp_0 a_2$.

Let $Y = \Sigma_{1,2} - a_1$, and let X be the subsurface bounded by a_2 containing a_1. Then we have $f = g$ on the subset $\mathcal{S}(X) \cup \mathcal{S}(Y)$.

Claim. There exist a finit set of elements $\{\beta_1, ..., \beta_n\}$ in $\mathcal{S}(X) \cup \mathcal{S}(Y)$ and a function F such that for any element h in $Im(\Sigma_{1,2})$, $h(\alpha) = F(h(\beta_1), ..., h(\beta_n))$.

It follows from the claim that $f(\alpha) = g(\alpha)$. This finishes the proof of Lemma 9.

We shall adopt the same convention as in §4.2 by identifying $h(s)$ with the simple closed curve s for the rest of the proof.

Proof of the claim. Since any other simple closed curve x' with $x' \perp a_1$ and $x' \perp_0 a_2$ is an image of x under an orientation preserving self-homeomorphism preserving a_1 and a_2, we may draw x as in Figure 10. Introduce a few more curves y, z, x_1, y_1, x_2, y_2, b_1, b_2, b_3, k as in Figure 10. Note that the curves b_1, b_2, b_3, and k are either in X or in Y.

There are many relations among these curves as shown in Figure 11.

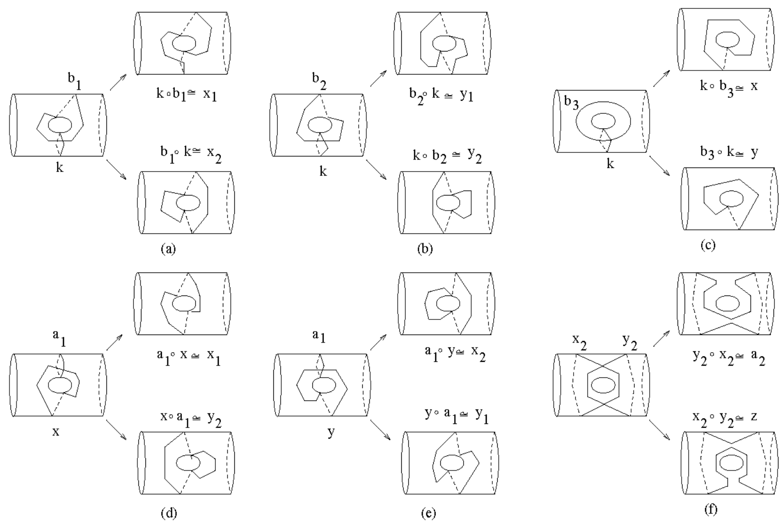


Figure 11

FIGURE 11

We obtain a system of equations in x, y, x_1, y_1, x_2, y_2 and z by applying formulas $(1')$, $(2')$, $(3')$ and $(4')$.

By Figure 11(a), we have,

$$x_1 + x_2 = kb_1. \tag{e1}$$

By Figure 11(b), we have,

$$y_1 + y_2 = kb_2. \tag{e2}$$

By Figure 11(c), we have,

$$x + y = kb_3. \tag{e3}$$

By Figure 11(d) and that $\partial N(x \cup a_1) \subset Y$, we have,

$$x_1 + y_2 = a_1 x, \tag{e4}$$

and,

$$y_2^2 + x^2 - a_1 x y_2 = p_1. \tag{e5}$$

By Figure 11(e) and that $\partial N(y \cup a_1) \subset Y$, we have,

$$y_1^2 + y^2 - a_1 y y_1 = p_2. \tag{e6}$$

By Figure 11(c) that $x = k \circ b_3$ and $\partial N(b_3 \cup k) \cong z$, we have,

$$(e7) \qquad x^2 - kb_3x = -z + p_3.$$

By Figure 11(f) and $\partial N(x_2 \cup y_2) \cong a \cup b \cup b_3 \cup b_3'$ where b_3' is a parallel copy of b_3 and $\partial\Sigma_{1,2} = a \cup b$, we have,

$$(e8) \qquad z = x_2y_2 + p_4.$$

Here and below, $p_i's$ denote some polynomials in some elements in $\mathcal{S}(X) \cup \mathcal{S}(Y)$.

Also from $a_1 \perp b_3$ with $b_3 \circ a_1 = b_2$ and $a_1 \circ b_3 = b_1$, we have,

$$(e9) \qquad b_1 + b_2 = a_1b_3.$$

The goal is to show that the system of equations (e1)-(e8) has a unique solution in x. Assuming this, we conclude that the claim holds.

To this end, we shall first eliminate x_1, y_1, y and z from the above system and show that x_2 and y_2 are linear functions in x.

Subtracting (e1) from (e4) gives:

$$(e10) \qquad x_2 = y_2 - a_1x + kb_1,$$

and subtracting (e7) from (e8) yields:

$$(e11) \qquad x_2y_2 + x^2 - kb_3x = p_5.$$

By (e3), $y = kb_3 - x$ and by (e2), $y_1 = kb_2 - y_2$. Substituting them into (e6) and subtracting the result from (e5), we obtain:

$$(e12) \qquad (a_1b_3 - 2b_2)ky_2 + (a_1b_2 - 2b_3)kx = p_6.$$

Note that the coefficients of y_2 and x in (e12) cannot both be zero since $a_1 > 2$. If $a_1b_3 - 2b_2 = 0$, then x is determined uniquely. Suppose otherwise, then we solve y_2 in terms of x and obtain,

$$(e13) \qquad y_2 = p_7x + p_8,$$

where $p_7 = (2b_3 - a_1b_2)/(a_1b_3 - 2b_2)$. From (e10) it follows that

$$(e14) \qquad x_2 = (p_7 - a_1)x + p_8 + kb_1.$$

Now substituting (e13) into (e5), we get a quadratic equation in x as follows:

$$(e15) \qquad (p_7^2 - a_1p_7 + 1)x^2 + (2p_7p_8 - a_1p_8)x + p_9 = 0.$$

Substituting $(e13)$ and $(e14)$ into $(e11)$ gives a quadratic equation in x as follows.

$$(e16)\ \ (p_7^2-a_1p_7+1)x^2+(-b_3k+p_8(p_7-a_1)+p_7p_8+kb_1p_7)x+p_{10}=0.$$

Subtracting (e16) from (e15) we obtain a linear equation in x whose leading term is $-kb_3+kb_1p_7$. Replacing p_7 by $(2b_3-a_1b_2)/(a_1b_3-2b_2)$ and using (e9) that $b_1=a_1b_3-b_2$, we simplify the leading coefficient to $a_1k(b_2^2+b_3^2-a_1b_2b_3)/(a_1b_3-2b_2)$. The number $b_2^2+b_3^2-a_1b_2b_3$ is negative by relation $(1')$ that $a_1^2+b_2^2+b_3^2<a_1b_2b_3$. Thus we obtain a unique solution of x. This finishs the proof of Lemma 9. q.e.d.

4.4. Proof of Theorem 1 for metrics with cups ends.

We first recall Theorem 2 for metrics with cups ends. Let $\Sigma=\Sigma_{0,r}^s$ with $r+s=4, s<4$, be given with three simple closed curves a_{12},a_{23}, and a_{31} on it satisfying $a_{31}=a_{12}a_{23}$ and $a_{12}\perp_0 a_{23}$. Let b_i be four essential simple closed curves in $int(\Sigma_{0,r}^s)$ which are homotopic into the four ends so that a_{ij}, b_i and b_j bound a 3-holed sphere in the surface $(i\neq j, i,j\leq 3)$. Assume the cusp ends correspond to b_i $(i=1,2,..,s)$. Take the collection $F\subset\mathcal{S}(\Sigma)$ to be the isotopy classes of a_{ij} and $b_i's$ where $i\neq j$ and $i,j\leq 3$. Then the same argument used in the proof of Theorem 2 shows,

Lemma 10. *The map $\pi_F: T(\Sigma_{0,r}^s)\to\mathbf{R}_{\geq 2}^6$ is an embedding whose image is given by $\{(t_1,t_2,t_3,t_{12},t_{23},t_{31})\in\mathbf{R}_{\geq 2}^6|\ t_1=...=t_s=2,\ t_{s+1}>2,...,t_3>2$, so that formula (4) holds$\}$. Furthermore, there exists a real analytic map $f: T(\Sigma_{0,r}^s)\to\tilde{R}(\Sigma_{0,r}^s)$ so that for each m in $T(\Sigma_{0,r}^s)$, $f(m)$ is a lifting of a monodromy of m and the entries of the matrix $f(m)(\alpha)$ are real analytic RC-function of $\pi_F(m)$ for $\alpha\in\mathcal{S}(\Sigma_{0,r}^s)$.*

Now to construct metrics on $\Sigma_{g,r}^s$ with $s>0$, we use the decomposition $\Sigma_{g,r}^s=X\cup Y$ as in Figure 12. The first case (1) is given by $r>0$. We need to consider subcases (1.1), (1.2) and (1.3) where (1.1) corresponds to $g>0$, (1.2) corresponds to $g=0$ and $r+s>5$, and (1.3) corresponds to $g=0$ and $r+s\leq 5$. In cases (1.1), or (1.2), we choose $X\cong\Sigma_{g,r}^{s-1}$, $Y\cong\Sigma_{0,3}^1$, and $X\cap Y\cong\Sigma_{0,3}$. Case (1.3) with $r+s\leq 4$ then follows from Theorem 2. In case (1.3) and $r+s=5$, we choose $X\cong\Sigma_{0,r-1}^{s-1}$, $Y\cong\Sigma_{0,u}^v$, where $u+v=4$, $2\geq v\geq 1$, and $X\cap Y\cong\Sigma_{0,4-v}^{v-1}$. In the second case (2) $r=0$, we need to consider subcases (2.1) $s\geq 2$ and (2.2) $s=1$. In case (2.1) that $s\geq 2$, if (2.1.1) $g>0$, or (2.1.2) $g=0$ and $s>5$, then $X\cong\Sigma_{g,1}^{s-2}$, $Y\cong\Sigma_{0,2}^2$, and $X\cap Y\cong\Sigma_{0,3}$. If (2.1.3)

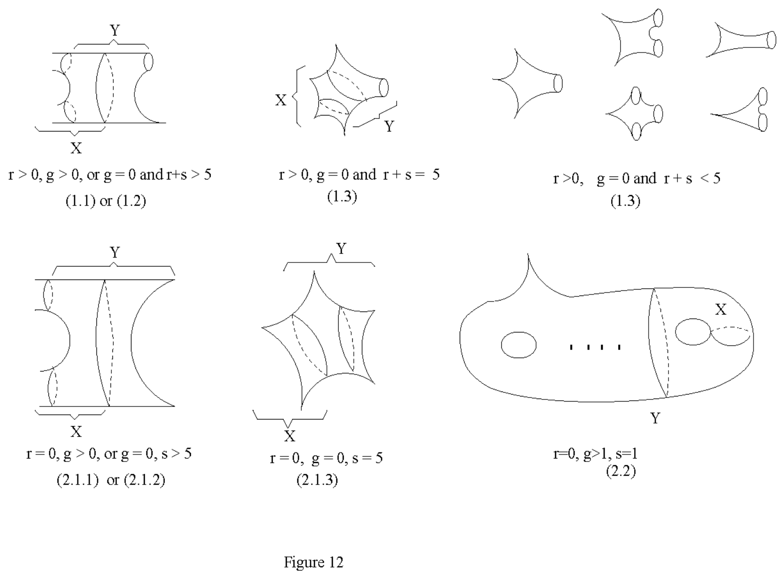


FIGURE 12

$5 \geq s \geq 2$ and $g = 0$, the theorem holds except for $s = 5$ where we decompose $\Sigma^5_{0,0}$ as a union of two $\Sigma^3_{0,1}$ with intersection $\Sigma^1_{0,2}$. Finally, in case (2.2) that $s = 1$, it suffices to consider $g \geq 2$. We take $X \cong \Sigma_{1,1}$, $Y \cong \Sigma^1_{g,1}$ and $X \cap Y \cong X - s$ where s is a non-separating simple closed curve in X.

These give the 3-holed sphere decomposition of the surface into two subsurfaces of smaller $|X|$ and $|Y|$ where $|\Sigma^s_{g,r}| = 3g + r + s$. Note that Lemmas 8 and 9 still hold for metrics with cups ends. Now by the gluing Lemma, Lemmas 8, 9, 10, Theorem 2, the same argument used in the previous sections applies. This gives a proof of Theorem 1 for metrics with cusp ends.

Remark. Teichmüller space is well known to be homeomorphic to a Euclidean space. This fact can also be derived from Theorem 2 and Lemma 1. Indeed, the gluing Lemma shows that the restriction map from $T(X \cup Y)$ to $T(X)$ is a fiber-bundle map. The fiber can be shown to be homeomorphic to a Euclidean space by solving a simple inequality (e.g. relations (3) or (4)).

5. Application to finite dimensional embeddings of Teichmüller spaces

We shall prove the following stronger version of the corollary for compact surfaces by induction on $|\Sigma_{g,r}| = 3g + r$ in this section. The proof for surfaces with cusp ends will be omitted.

Corollary. *(a) For surface $\Sigma_{g,r}$ of negative Euler number and $r > 0$, there exists a finite subset F in $\mathcal{S}(\Sigma_{g,r})$ consisting of $6g + 3r - 6$ elements so that the map $\pi_F : T(\Sigma_{g,r}) \to \mathbf{R}^F_{>2}$ is an embedding onto an open subset which is defined by a finite set of real analytic RC-inequalities in the coordinates of π_F. Furthermore, there exists a map $f : T(\Sigma_{g,r}) \to \tilde{R}(\Sigma_{g,r})$ so that for each m in $T(\Sigma_{g,r})$, $f(m)$ is a lifting of a monodromy of m and the entries of the matrix $f(m)(\alpha)$ are real analytic RC-functions of $\pi_F(m)$ for any $\alpha \in \mathcal{S}(\Sigma_{g,r})$.*

(b) For surface $\Sigma_{g,0}$ of negative Euler number, there exists a finite subset F of $\mathcal{S}(\Sigma_{g,0})$ consisting of $6g-5$ elements so that $\pi_F : T(\Sigma_{g,0}) \to \mathbf{R}^F_{>2}$ is an embedding whose image is defined by one real analytic RC-equation and finitely many real analytic RC-inequalities in the coordinates of τ_F. Furthermore, there exists a map $f : T(\Sigma_{g,0}) \to \tilde{R}(\Sigma_{g,0})$ so that for each m in $T(\Sigma_{g,0})$, $f(m)$ is a lifting of a monodromy of m and the entries of the matrix $f(m)(\alpha)$ are real analytic RC-functions of $\pi_F(m)$ for any $\alpha \in \mathcal{S}(\Sigma_{g,0})$

Note that the corollary without the statement about the lifting of monodromies follows immediately from the gluing Lemma, Theorems 1 and 2, and Lemmas 8 and 9. To prove the full statement, we need to strengthen the gluing lemma.

In §5.1, we prove an extended version of the gluing Lemma. In §5.2, we prove the corollary for $\Sigma_{1,2}$. The corollary for surfaces with non-empty boundary is proved in §5.3. In §5.4, we prove the corollary for closed surfaces.

5.1. Algebraic dependence in the gluing lemma.

We begin with a parametrized version of the Jordan canonical form theorem for SL(2,$\mathbf{R}$) matrices.

Lemma 11. *(a) If A =$[a_{ij}]$ in SL(2,$\mathbf{R}$) satisfies $|trA| > 2$ and $a_{12}a_{21} \neq 0$, then*

$$C^{-1}AC = \frac{1}{2}\begin{pmatrix} a_{11}+a_{22}+\sqrt{(a_{11}+a_{22})^2-4} & 0 \\ 0 & a_{11}+a_{22}-\sqrt{(a_{11}+a_{22})^2-4} \end{pmatrix},$$

where

$$C = \begin{pmatrix} \frac{2a_{12}}{a_{22} - a_{11} + \sqrt{(a_{11}+a_{22})^2 - 4}} & \frac{a_{11} - a_{22} - \sqrt{(a_{11}+a_{22})^2-4}}{2a_{21}} \end{pmatrix}.$$

(b) For $A = [a_{ij}]$ and $B = [b_{ij}]$ in $SL(2,\mathbf{R})$ with $trA > 2$ (resp. $trA < -2$), $a_{12}a_{21} \neq 0$ and $trABA^{-1}B^{-1} \neq 2$, there exist four real analytic RC-functions c_{ij} in eight variables so that $C^{-1}AC = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$ (resp. $\lambda < -1$) and $C^{-1}BC = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$, $|\delta| = 1$ where $C = [c_{ij}(A,B)] \in GL(2,\mathbf{R})$.

Proof. Part (a) follows by a direct calculation. Note that the matrix C is invertible since $a_{12}a_{21} \neq 0$. Part (b) follows from part (a). Indeed, by part (a), we may conjugate A to the required diagonal form A'. We also conjugate B by the same matrix to obtain B'. The trace of the commutator remains unchanged. Thus the new matrix $B' = [b'_{ij}]$ has non-zero (2,1)-entry. Now a further conjugation by the matrix $\begin{pmatrix} \sqrt{|b'_{21}|} & 0 \\ 0 & \sqrt{|b'_{21}|}^{-1} \end{pmatrix}$ will not change matrix A' but change B' into the required form. q.e.d.

We say a pair of matrices (A,B) is *normalized* if $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $|\lambda| > 1$ and the (2,1)-entry of B is 1. It follows from the normalized condition that if C is in GL(2,$\mathbf{R}$) so that both (A,B) and $(C^{-1}AC, C^{-1}BC)$ are normalized, then $(A,B) = (C^{-1}AC, C^{-1}BC)$, i.e., normalization is unique up to GL(2, $\mathbf{R}$) conjugation. Fix a pair of elements (γ_1, γ_2) in $\pi_1(\Sigma)$. A representation ρ in $\tilde{R}(\Sigma)$ is called *normalized* with respect to the pair if $(\rho(\gamma_1), \rho(\gamma_2))$ is normalized.

A *section* of the natural projection from $\tilde{R}(\Sigma)$ to $T(\Sigma)$ is a continuous map $f : T(\Sigma) \to \tilde{R}(\Sigma)$ so that $f(m)$ is a lifting of a monodromy of m. Given a section f, we may produce a new section whose image lies in any given component of $\tilde{R}(\Sigma)$ as follows. Conjugating representations in $f(T(\Sigma))$ by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives rise to a new section in a component of the opposite orientation type; and choosing a different lifting ρ_I associated to $\rho \in f(T(\Sigma))$ for a fixed index set I (see §1.2 for the definition) gives a section in a different component of the same orientation type. We call these new sections to be the ones obtained from f by different liftings and conjugations. An *RC-section* is a section so that (1) there exists an associated finite set $F \subset \mathcal{S}(\Sigma)$ such that the entries of the matrix $f(m)(\alpha)$ are real analytic RC-functions of the coordinates of $\pi_F(m)$ for all $\alpha \in \pi_1(\Sigma)$ and (2) each representation in the image of the section is normalized with respect to a fixed pair of elements in $\pi_1(\Sigma)$. By Lemmas 4, 5 and Theorem 2, the Teichmüller spaces $T_{0,3}$, $T_{1,1}$, and $T_{0,4}$ have RC-sections.

For simplicity, we shall identify curves, isotopy classes of curves, and homotopy classes of curves in incompressible subsurfaces with their images in the ambient spaces without mentioning the including maps.

Lemma 12. (Algebraic dependence) *Let X and Y be good incompressible subsurfaces of Σ such that $\Sigma = X \cup Y$ and either (1) $X \cap Y \cong \Sigma_{0,3}$, or (2) $Y \cong \Sigma_{1,1}$ and $X \cap Y = Y - s$ where s is a non-separating simple closed curve in int(Y), or (3) $X \cap Y \cong \Sigma^1_{0,2}$ so that the punctured end in $\Sigma^1_{0,2}$ is a punctured end of Σ. If $T(X)$ and $T(Y)$ both have RC-sections f_X and f_Y with associated sets F_X and F_Y respectively, then $T(\Sigma)$ has an RC-section with associated set $F_X \cup F_Y$.*

Proof. Let (α_1, α_2) (resp. (β_1, β_2)) be the pair in $\pi_1(X)$ (resp. $\pi_1(Y)$) such that each representation in the image of f_X (resp. f_Y) is normalized with respect to it. Choose two geometric generators γ_1 and γ_2 for $\pi_1(X \cap Y)$ so that $\gamma_1\gamma_2$ is represented by the third boundary component. Then one of the three elements γ_1, γ_2, $\gamma_1\gamma_2$, say γ_1, satisfies the condition that both subgroups $< \alpha_1, \gamma_1 >$ and $< \beta_1, \gamma_1 >$ are not solvable. Let γ_2 be one of the remaining element. Then $\pi_1(X \cap Y)$ is generated by γ_1 and γ_2. We extend $\{\gamma_1,\ \gamma_2\}$ to a minimal set of generators $\{\gamma_1,\ ...,\ \gamma_n\}$ for $\pi_1(\Sigma)$ so that each γ_i is either in $\pi_1(X)$ or in $\pi_1(Y)$.

By choosing a different lifting if necessary, we may assume that $f_X(m)(\alpha_1)$ and $f_Y(m)(\beta_1)$ are diagonal matrices with positive traces for $m \in T(X \cap Y)$ (f_X and f_Y are still sections but may not be normalized any more). Now by the choice of element γ_1, both matrices $f_X(m)(\gamma_1)$ and $f_Y(m)(\gamma_1)$ have non-zero off diagonal entries for all m, and the trace of the commutator of $f_X(m)(\gamma_1)$ and $f_X(m)(\gamma_2)$ (resp. $f_Y(m)(\gamma_1)$, and $f_Y(m)(\gamma_2)$) is not 2. Thus by Lemma 11, we may conjugate the pair $(f_X(m)(\gamma_1), f_X(m)(\gamma_2))$ (resp. $(f_Y(m)(\gamma_1), f_Y(m)(\gamma_2))$) to the form in Lemma 11(b) by a GL(2, $\mathbf{R}$) matrix whose entries are real analytic RC-functions in the coordinates of $\pi_{F_X}(m)$ (resp. in the coordinates of $\pi_{F_Y}(m)$). This produces two sections g_X and g_Y for $T(X)$ and $T(Y)$ respectively so that (1) for each $m \in T(X)$ (resp. $m \in T(Y)$), the entries of the matrices $g_X(m)(\gamma)$ (resp. $g_Y(m)(\gamma)$) are real analytic

RC-functions in the coordinates of $\pi_{F_X}(m)$ (resp. $\pi_{F_Y}(m)$), and (2) the matrix $g_X(m)(\gamma_1)$ (resp. $g_Y(m)(\gamma_1)$) is diagonal with (1,1)-entry bigger than one and the (2,1)-entry of $g_X(m)(\gamma_2)$ (resp. $g_Y(m)(\gamma_2)$) has absolute value one.

We may normalize the sections g_X and g_Y by choosing different lifting which changes the generator $\rho(\gamma_2)$ to $-\rho(\gamma_2)$ if necessary. Thus we may assume that both g_X and g_Y are normalized with respect to the pair (γ_1, γ_2).

We now define an RC-section for $T(\Sigma)$ as follows. By the gluing lemma, each $m \in T(\Sigma)$ corresponds to a pair $(m_X, m_Y) \in T(X) \times T(Y)$ so that $R_X(m) = m_X$, $R_Y(m) = m_Y$ and the restrictions of m_X and m_Y to $X \cap Y$ are the same. The restrictions of the two representations $g_X(m_X)$ and $g_Y(m_Y)$ to the subgroup $\pi_1(X \cap Y)$ uniformize the same element $R_{X\cap Y}(m)$. Since the pair (γ_1, γ_2) generates $\pi_1(X \cap Y)$, by the normalization condition for g_X and g_Y, we have $g_X(m_X)|_{\pi_1(X\cap Y)} = g_Y(m_Y)|_{\pi_1(X\cap Y)}$. By Maskit combination theorem (there is no need to verify the side condition since the gluing is along a 3-holed sphere), there exists a unique representation $\rho \in \tilde{R}(\Sigma)$ so that $\rho|_{\pi_1(X)} = g_X(m)$ and $\rho|_{\pi_1(Y)} = g_Y(m)$. The map from $T(\Sigma)$ to $\tilde{R}(\Sigma)$ sending m to ρ is a section normalized with respect to (γ_1, γ_2). To see the RC-dependence (which also shows the continuity of the map m to ρ), it suffices to check the condition for each generator γ_i. By the construction, $\rho(\gamma_i)$ is either $g_X(m)(\gamma_i)$ or $g_Y(m)(\gamma_i)$. Thus, each entry of the matrix $\rho(\gamma_i)$ is a real analytic RC-function in the coordinates of $\pi_{F_X \cup F_Y}(m)$. q.e.d.

5.2. Proof of the Corollary for $\Sigma_{1,2}$.

Let s_7 be an essential separating simple closed curve, and s_1 be a non-separating simple closed curve disjoint from s_7 in $\Sigma_{1,2}$ as in Figure 13(a). We decompose $\Sigma_{1,2}$ as a union $X \cup Y$ where X is the compact subsurface bounded by s_7 containing s_1, and Y is the complement of s_1. Then $X \cap Y$ is $X - s_1$. Let s_2, s_3 be simple closed curves in X so that $s_1 \perp s_2$ and $s_3 = s_1 s_2$; let s_4, s_5, s_6, s_1^+ and s_1^- be simple closed curves in Y so that s_1^+ and s_1^- are boundary components which are identified to be s_1 in $\Sigma_{1,2}$, $s_6 \subset \partial Y$ and $s_4 \perp_0 s_7$, $s_5 = s_4 s_7$. See Figures 13(b) and (c). By the gluing lemma and Lemma 4, the Teichmüller space $T(\Sigma_{1,2})$ can be identified with the subset $\{(m_X, m_Y) \in T(X) \times T(Y) | t_{m_X}(s_1) = t_{m_Y}(s_1^+) = t_{m_Y}(s_1^-)$ and $t_{m_X}(s_7) = t_{m_Y}(s_7)\}$. By Lemma 5, m_X is determined by $\pi_{F_X}(m_X) = (t_{m_X}(s_1), t_{m_X}(s_2), t_{m_X}(s_3))$ where $F_X = \{[s_1], [s_2], [s_3]\}$. By Theorem 2, m_Y is determined by $\pi_{F_Y}(m_Y) = (t_{m_Y}(s_1^+), t_{m_Y}(s_1^-), t_{m_Y}(s_4), t_{m_Y}(s_5), t_{m_Y}(s_6), t_{m_Y}(s_7))$. Fi-

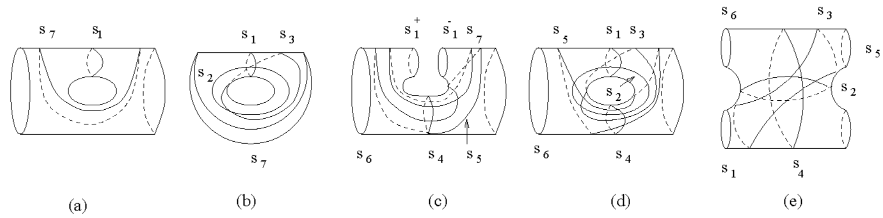


Figure 13

FIGURE 13

nally, formula (1) shows that $t_{m_X}(s_7) = t_{m_X}(s_1)t_{m_X}(s_2)t_{m_X}(s_3) + 2 - t^2_{m_X}(s_1) - t^2_{m_X}(s_2) - t^2_{m_X}(s_3)$. Combining these and Lemma 12, we obtain the following lemma.

Lemma 13. *For surface $\Sigma_{1,2}$, let F be the collection of isotopy classes of six curves s_1, s_2, s_3, s_4, s_5, s_6 as in Figure 13(d). Then $\pi_F : T(\Sigma_{1,2}) \to R^6$ is an embedding whose image is given by*

$$\begin{aligned}\{(t_1,t_2,t_3,t_4,t_5,t_6) \in \mathbf{R}^6_{>2} | t_1t_2t_3 > t_1^2 + t_2^2 + t_3^3,\\ t_4t_5t_7 > t_4^2 + t_5^2 + t_6^2 + t_7^2 + 2t_1^2 + 2t_1^2t_6 + t_1^2t_7\\ + t_1t_4t_6 + t_1t_5t_6 + 2t_1t_4 + 2t_1t_5 + 2t_6t_7,\\ \textit{where } t_7 = t_1t_2t_3 - t_1^2 - t_2^2 - t_3^2\}.\end{aligned}$$

Furthermore, there exists an RC-section for $T(\Sigma_{1,2})$ with associated set F.

5.3. Proof of the Corollary for $\Sigma_{g,r}$ with $r > 0$.

We prove the corollary by induction on $|\Sigma_{g,r}| = 3g + r$ with $r > 0$.

For surfaces $\Sigma_{0,3}$, $\Sigma_{0,4}$, $\Sigma_{1,1}$ and $\Sigma_{1,2}$, we have shown in the previous sections that the corollary holds. Given $\Sigma_{g,r}$ with either $3g + r = n > 5$ or $(g,r) = (0,5)$, if $r \geq 2$, we decompose $\Sigma_{g,r} = X \cup Y$ where $X \cong \Sigma_{g,r-1}$, $Y \cong \Sigma_{0,4}$ with $X \cap Y \cong \Sigma_{0,3}$ as in Figure 3(b); if $r = 1$, we decompose $\Sigma_{g,r} = X \cup Y$ where $X \cong \Sigma_{g-1,2}$, $Y \cong \Sigma_{1,2}$ and $X \cap Y \cong \Sigma_{0,3}$ as in Figure 3(c). Then $|X|$ and $|Y|$ are less than $|\Sigma_{g,r}|$. By the induction hypothesis, there exists a subset $F_X \subset \mathcal{S}(X)$ consisting of $6g + 3r - 9$ elements so that corollary holds. Let $F_Y \subset \mathcal{S}(Y)$ be the set $\{[s_1],[s_2],[s_3],[s_4],[s_5],[s_6]\}$ given by Theorem 2 as in Figure 13(e) if $Y \cong \Sigma_{0,4}$, and by Lemma 13 as in Figure 13(d) if $Y \cong \Sigma_{1,2}$. Let $F = F_X \cup \{[s_2],[s_3],[s_5]\}$ consisting of $6g + 3r - 6$ elements. We claim that the corollary holds for $\Sigma_{g,r}$ with respect to the set F. First to show that π_F is an embedding, we use the gluing lemma. It follows that $\pi_{F_X \cup F_Y}$ is an embedding. However, by the construction, s_1, s_4 and s_6 are in the subsurface X. Thus by the induction hypothesis, $t_m(s_1)$, $t_m(s_4)$ and $t_m(s_6)$ are real analytic RC-functions in the coordinates of $\pi_{F_X}(m)$. Hence, we may drop the three elements $[s_1], [s_4]$, and $[s_6]$ from the set $F_X \cup F_Y$ without effecting the embeddedness of $\pi_{F_X \cup F_Y}$. Applying Lemma 12 to F_X and F_Y and then dropping the three elements $[s_1], [s_4]$ and $[s_6]$, we see that $T(\Sigma_{g,r})$ has an RC-section with associated set F. Finally, we show that the image $\pi_F(T(\Sigma_{g,r}))$ is defined by a finite set of RC-inequalities in the coordinates of π_F. Indeed, by the induction hypothesis, $\pi_{F_X}(T(X))$ (resp. $\pi_{F_Y}(T(Y))$) is defined by a finite set of RC-inequalities. By the gluing Lemma 1, the image $\pi_{F_X \cup F_Y}(T(\Sigma_{g,r}))$ is given by the same set of RC-inequalities for $\pi_{F_X}(T(X))$, together with the RC-inequalities for $\pi_{F_Y}(T(Y))$, and three equations expressing that the lengths of the three simple closed curves in $\partial(X \cap Y)$ are the same in both metrics m_X and m_Y. Thus the result follows.

5.4. Proof of the Corollary for closed surface $\Sigma_{g,0}$ with $g \geq 2$. Given $\Sigma_g = \Sigma_{g,0}$, let Y be an incompressible subsurface of Σ_g homeomorphic to $\Sigma_{1,1}$ with boundary s_1, and let s_2 be a non-separating simple closed curve in $int(Y)$. Set $X = \Sigma_g - s_2$ as in Figure 3(d). Thus $\Sigma_g = X \cup Y$ and $X \cap Y = Y - s_2$. By the gluing Lemma 1, each metric $m \in T(\Sigma_g)$ is the same as a pair $(m_X, m_Y) \in T(X) \times T(Y)$ with $R_{X \cap Y}(m_X) = R_{X \cap Y}(m_Y)$. In particular the completion $\bar{X}$ of X under the metric m_X has the same geodesic lengths at the two boundary components. The following lemma describes hyperbolic metrics on $\Sigma_{0,4}$ which have the same lengths at two boundary curves.

Lemma 14. *Given $\Sigma_{0,4}$ with curves b_i ($i = 1,2,3,4$) as boundary components, let a_{ij} ($(i,j) = (1,2),(2,3),\ (3,1)$) be simple closed curves in $\Sigma_{0,4}$ so that $a_{12} \perp_0 a_{23}$ and $a_{31} = a_{12}a_{23}$ and b_i, b_j and a_{ij} bound a subsurface of signature $(0,3)$. Let $T'(\Sigma_{0,4})$ be the subspace of the Teichmüller space $T(\Sigma_{0,4})$ defined by $t_m(b_3) = t_m(b_4)$, and let $F' = \{[b_1],[b_2],[a_{12}],[a_{23}],[a_{31}]\}$. Then $\pi_{F'} : T'(\Sigma_{0,4}) \to \mathbf{R}^5_{>2}$ is an embedding whose image is defined by a real analytic RC-inequality in the coordinates of $\pi_{F'}$. Furthermore, there is an RC-section $f\colon T'(\Sigma_g) \to \tilde{R}'(\Sigma_g)$ where $\tilde{R}'(\Sigma_g)$ stands for the subset of $\tilde{R}(\Sigma_g)$ which projects onto $T'(\Sigma_g)$ so that the entries of $f(m)(\gamma)$ are real analytic RC-functions in the coordinates of $\pi_{F'}(m)$.*

Proof. Given a metric $m \in T'(\Sigma_g)$, let $t_i = t_m([b_i])$, $i = 1,2,3,4$, and let $t_{ij} = t_m([a_{ij}])$, $(i,j) = (1,2),(2,3),\ (3,1)$, where $t_3 = t_4$. Now these t_i and t_{ij} satisfy equation (13). Thus we obtain an equation in t ($= t_3 = t_4$) below,

$$(2 + t_1t_2 + t_{12})t^2 + (t_1t_{31} + t_1t_{23} + t_2t_{31} + t_2t_{23})t \\ + t_1^2 + t_2^2 + t_1t_2t_{12} + t_{12}^2 + t_{23}^2 + t_{31}^2 - t_{12}t_{23}t_{31} - 4 = 0.$$

The coefficient of t^2 is positive and the constant term is negative by (4). Thus the equation has two real roots of different signs and t_3 ($= t_4$) is the positive root of the equation. Hence $t_3(= t_4)$ is a real analytic RC-function of t_1, t_2, t_{12}, t_{23} and t_{31} which are the coordinates of $\pi_{F'}(m)$. This shows that $\pi_{F'}$ is an embedding. The rest of the lemma follows by the same argument used in the proof of Theorem 2. q.e.d.

Let $T'(X)$ be the subset of $T(X)$ so that $t_m(s_2^+) = t_m(s_2^-)$ where s_2^+ and s_2^- are the boundary components of $\bar{X}$. Then in the proof of the corollary for $\Sigma_{g-1,2}$ ($\cong X$) in §5.3, to construct $m \in T'(X)$, we decompose $X = X_1 \cup Y_1$ where $X_1 \cong \Sigma_{g-1,1}$, $Y_1 \cong \Sigma_{0,4}$ and $X_1 \cap Y_1 \cong \Sigma_{0,3}$. We use Lemma 14 instead of Theorem 2 for metrics on Y_1 in the gluing process. Thus, the same argument shows that there exists a subset $F_X \subset \mathcal{S}(\bar{X})$ consisting of $6g - 7$ elements so that

$$\pi_{F_X} : T'(X) \to \mathbf{R}^{6g-7}_{>2}$$

is an embedding whose image is an open set defined by a finite set of real analytic RC-inequalities in the coordinates of π_{F_X}.

Let s_3 and s_4 be two simple closed curves in $int(Y)$ so that $s_3 \perp s_2$ and $s_4 = s_2s_3$. Now by the gluing Lemma 1, each $m \in T(X \cup Y)$ is determined by a pair $(m_X, m_Y) \in T'(X) \times T(Y)$ so that the restrictions of m_X and m_Y to $X \cap Y$ are the same. The gluing condition on $X \cap Y$ is equivalent to that $t_{m_X}(s_2^+) = t_{m_Y}(s_2)$ and $t_{m_X}(s_1) = t_{m_Y}(s_1)$ by Lemma 4. Also Lemma 5 gives the complete description of $(t_{m_Y}(s_2), t_{m_Y}(s_3), t_{m_Y}(s_4))$. Let $F = F_X \cup \{[s_3], [s_4]\} \subset \mathcal{S}(\Sigma_g)$ consisting of $6g - 5$ elements. Combining the previous facts, we obtain (1) $\pi_F : T(\Sigma_g) \to \mathbf{R}^{6g-5}$ is an embedding, (2) the image $\pi_F(T(\Sigma_g))$ is defined by a finite set of RC-inequalities (from those of $\pi_{F_X}(T'(X))$ and of $\pi_{\{[s_2],[s_3],[s_4]\}}(T(\Sigma_{1,1}))$ where we replace $t_{m_Y}([s_2])$ by $t_{m_X}([s_2])$, and one real analytic RC-equation $t_{m_Y}(s_1) = t_{m_X}(s_1)$. Furthermore, by Lemmas 13 and 14, there is an RC-section for $T(\Sigma_g)$.

q.e.d.

Remark. The fact that $\pi_{F'}$ is an embedding in Lemma 10 was first proved by P. Schmutz ([32]).

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