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RIGIDITY OF HYPERBOLIC CONE-MANIFOLDS AND HYPERBOLIC DEHN SURGERY

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The local rigidity theorem of Weil [28] and Garland [12] for complete, finite volume hyperbolic manifolds states that there is no non-trivial deformation of such a structure through *complete* hyperbolic structures if the manifold has dimension at least 3. If the manifold is closed, the condition that the structures be complete is automatically satisfied. However, if the manifold is non-compact, there may be deformations through incomplete structures. This cannot happen in dimensions greater than 3 (Garland-Raghunathan [13]); but there are always non-trivial deformations in dimension 3 (Thurston [24]) in the non-compact case.

In this paper, we extend this rigidity and deformation theory to a class of finite volume, orientable 3-dimensional hyperbolic cone-manifolds, *i.e.*, hyperbolic structures on 3-manifolds with cone-like singularities along a knot or link. Our main result is that such structures are locally rigid if the cone angles are fixed, under the extra hypothesis that all cone angles are at most 2π. We can view the singular structure as an incomplete, smooth structure on the complement of the singular locus whose metric completion is the singular cone structure. The space of deformations of structures on this open manifold has non-zero dimension, so there will be deformations without the condition that the cone angles remain fixed. We show that it is possible to deform the structure so that the metric completion is still a cone-manifold, and that one

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can always deform a 3-dimensional hyperbolic cone-manifold to make arbitrary (small) changes in the cone angles. In fact, we show that the collection of cone angles locally parametrizes the set of cone-manifold structures.

The condition in the classical local rigidity theorem that the deformation be through *complete* hyperbolic structures can be interpreted as a boundary condition "at infinity" on the family of structures in the deformation. Specifically, it implies that the group elements corresponding to the end of the manifold must remain parabolic. In our case, too, the condition that the cone angles remain fixed can be viewed as the boundary condition that the traces of the group elements of meridians remain constant. In both cases, these algebraic boundary conditions lead to analytic growth conditions. Any infinitesimal deformation of a hyperbolic structure can be represented by an E-valued 1-form, where E denotes the flat vector bundle of local Killing vector fields (i.e., local infinitesimal hyperbolic isometries). The hypothesis that the cone angles remain fixed implies that this 1-form can be chosen to be in L^2. Our main result is really an L^2 rigidity theorem.

Our proof also gives information about hyperbolic 3-manifolds with "Dehn surgery" type singularities and gives new information on the structure of the "hyperbolic Dehn surgery spaces" introduced by Thurston in [24]. For example, the results described above imply that the hyperbolic Dehn surgery space is always a smooth manifold parametrized locally by Thurston's "Dehn surgery coordinates", near points corresponding to cone-manifolds with cone angles at most 2π. Equivalently, the space of representations of the fundamental group into $PSL_2(\mathbf{C})$ is locally a smooth manifold near the holonomy representation of a hyperbolic structure with these Dehn surgery type singularities. Previously these results were only known to be true in a neighborhood of the complete hyperbolic structure.

We conjecture that the current restriction on cone angles is not essential, and we hope to remove this restriction in a future paper. The techniques we develop also give specific models for the infinitesimal deformations which change the cone angle. These should lead to a better understanding of the changes in geometry as cone angles (and Dehn surgery coordinates) are changed. This, in turn, should imply new global results on the structure of hyperbolic Dehn surgery space.

It is conjectured that, as in the complete, finite volume case, a global (Mostow) rigidity should hold for hyperbolic cone-manifolds: Two hyperbolic cone-manifolds with the same underlying manifold and singular locus with the same cone angles are isometric. However, the known proofs for Mostow-Prasad rigidity all use the sphere at infinity to analyze the problem through ergodic theory or some other device. For cone-manifolds there is no known, reasonable analytic object to take the place of the sphere at infinity so that it is not so clear how to proceed along these lines.

However, our methods do provide controlled 1-parameter families of hyperbolic metrics with varying cone angles. Together with an understanding of the global analysis of how a sequence of hyperbolic metrics can degenerate, this should lead to some global results. The analysis of Hodge representatives in this paper suggests the following global conjecture: If some collection of cone angles is realizable, then all smaller cone angles are realizable. In particular, it should be possible to decrease all of the cone angles to zero which corresponds to a finite volume, complete structure on the complement of the singular locus. This would, in turn, prove the global rigidity conjecture for hyperbolic cone-manifolds. The argument is simply to deform any two, possibly distinct, structures with the same singular data back to the complete structure. There, they must be isometric by the standard Mostow-Prasad global rigidity. By the results of this paper, there are no branch points in the varieties along the way, so the two families of structures must have been the same throughout.

More generally, it is conjectured that hyperbolic Dehn surgery space is star-like with respect to the rays from the origin to infinity. The above conjecture on cone-manifolds is equivalent to this conjecture for lines of rational slope.

Analyzing what happens as the cone angles are increased seems much more difficult. Of central interest is the problem of finding sufficient conditions to guarantee that angle 2π can be reached for all components. If all cone angles equal 2π, the metric is smooth; reaching these angles would imply that the underlying closed manifold has a hyperbolic structure with geodesics in the given isotopy classes.

Here is an outline of the rest of the paper. Section 1 gives the statement and outlines the proof of our L^2 rigidity result (Theorem 1.1). We present preliminary material and notation and outline the proof of infinitesimal rigidity in the compact and in the complete, finite volume cases. These results are due to Calabi [5], Weil [28], and Garland [12]. Our presentation follows that in Matsushima-Murakami [21] and makes use of Hodge theory and a Weitzenböck formula to prove a cohomology vanishing theorem. In Section 2, we describe the general structure of the closed and co-closed forms which arise in the deformation theory of hyperbolic 3-manifolds. We then adapt this to derive the necessary Hodge theory for the cone-manifolds studied in this paper. In section 3 we study the asymptotic behavior of our harmonic forms near the singular locus. This allows us to complete the proof of Theorem 1.1. Finally, in section 4, we give some general results relating the topology of representation spaces to cohomology groups. We also explain some of the consequences of Poincaré duality for the study of representation spaces of 3-manifold groups. From these results and Theorem 1.1, we obtain proofs of the results on deformation spaces of hyperbolic cone-manifolds and hyperbolic Dehn surgery spaces mentioned above. In the appendix, we give a proof that the Laplacian on real-valued forms on a 3-dimensional cone-manifold is self-adjoint, with the domain used in section 2 in the proof of our Hodge theorem.

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1. An L^2 rigidity theorem

In this section, we state our main results and outline the methods of Calabi [5], Weil [28] and Matsushima-Murakami [21] used to prove infinitesimal rigidity in the closed manifold case. (See Raghunathan [26, Chap. 7] for a detailed exposition of these methods.)

We first consider the following general situation. Let M be an orientable n-dimensional smooth manifold, possibly with boundary. Assume that M has a geometric structure modelled on (X, G) where X is a simply connected, analytic Riemannian manifold, and G is a Lie group of isometries acting transitively on X. Then $X = G/K$, where K is the (compact) stabilizer of a point in X under the G action. Let $\mathcal{G}$ denote the Lie algebra of G, and Ad the adjoint representation of G on $\mathcal{G}$. Associated to a (X, G) structure on M is the holonomy representation $\rho : \pi_1(M) \to G$, which is defined in terms of the developing map $\tilde{M} \to X$ for the structure (see [24] or [27]).

Then there is a flat $\mathcal{G}$ vector bundle E over M associated with $Ad \circ \rho$; this can be interpreted geometrically as the bundle of (germs of) Killing vector fields on M. Using the flat connection on E gives an exterior derivative d on forms with values in E, which we denote by $\Omega^*(M; E)$. The kth (de Rham) cohomology group of M with coefficients in E, denoted $H^k(M; E)$, is defined to be the closed forms in $\Omega^k(M; E)$

modulo the image of $\Omega^{k-1}(M;E)$ under d. There is also a natural metric on E. Together with the metric on M, this allows us to define an inner product on $\Omega^k(M;E)$ and, in particular, to define the notion of an element of $\Omega^k(M;E)$ being in L^2. We will give details below.

We will be concerned with closed orientable 3-manifolds which have a singular hyperbolic structure, with "cone-like singularities" along a link. In particular, on the complement of the link there will be a smooth hyperbolic structure, i.e., a geometric structure modelled on (X,G), where $X = \mathbf{H}^3$ is hyperbolic 3-space, and $G = PSL_2\mathbf{C}$ is the group of orientation preserving isometries of $\mathbf{H}^3$. We let $\overline{M}$ denote the closed manifold with its singular structure, and let M denote the complement of the singular locus. Then M has a smooth, but *incomplete* hyperbolic structure, and the holonomy representation $\rho : \pi_1(M) \to G$ as above is still defined.

We will call the singular manifold $\overline{M}$ a *hyperbolic cone-manifold.* The precise definition of such a structure is that there is an incomplete hyperbolic structure on the complement of a link whose metric completion determines a singular metric with singularities along the link. The link is totally geodesic, and in cylindrical coordinates around a component of the singular locus, the metric has the form

$$dr^2 + \sinh^2 r\, d\theta^2 + \cosh^2 r\, dz^2,$$

where r is the distance from the singular locus, z is the distance along the singular locus, θ is the angular measure around the singular locus defined *modulo* α for some $\alpha > 0$. Then α is called the *cone angle* at that component. Note that the metric in a disk in the (r,θ)-plane is induced from that of a wedge of angle α in the hyperbolic plane $\mathbf{H}^2$, with sides identified. This is the cone from which the name arises.

Our main goal in this paper is to prove the following:

Theorem 1.1. *Let $\overline{M}$ be a finite volume, 3-dimensional hyperbolic cone-manifold, whose singular locus Σ is a knot or link. Let M denote the open, incomplete hyperbolic manifold $\overline{M}-\Sigma$. If all cone angles along Σ are at most 2π, then every closed L^2 form in $\Omega^1(M;E)$ represents the trivial cohomology class in $H^1(M;E)$.*

Remark. Although we don't prove it here, it is not hard to show that the L^2 forms in the theorem above are the images of sections which are also in L^2. Hence, the L^2 cohomology group $H^1_{L^2}(M;E)$ is trivial.

If $N = \overline{M} - U(\Sigma)$, where $U(\Sigma)$ is an (embedded) open tubular neighborhood of Σ, then every class in the image of the natural map

$(H^1(N, \partial N; E) \to H^1(N; E)$ can be represented by a form with compact support in $M = \overline{M} - \Sigma$. More generally, we shall see (in Lemma 3.3) that any infinitesimal deformation preserving cone angles can be represented by an L^2 form. This gives the following result:

Corollary 1.2. *For $\overline{M}$, M as in Theorem 1.1 and $N = \overline{M} - U(\Sigma)$, the following hold:*

(a) the natural map $H^1(N, \partial N; E) \to H^1(N; E)$ is zero,

(b) there are no infinitesimal deformations of the hyperbolic structure keeping the cone angles fixed. More precisely, if γ_i denotes a meridian curve about the ith component of Σ and $\gamma = \cup_i \gamma_i$, then the natural map $H^1(M, \gamma; E) \to H^1(M; E)$ is zero.

Our proof of Theorem 1.1 involves the use of Hodge theory and a Weitzenböck type formula (as in [28], [21]) to prove the desired cohomology vanishing theorem. We first outline the background material needed in the classical situation before adapting it to our particular situation in Section 2.

We begin with the general situation of a (X, G) structure on a manifold M as above. The model space X is of the form G/K, where K is the compact stabilizer of a point in X under the G action. We identify the Lie algebra $\mathcal{G}$ of G with the *right* invariant vector fields on G. Because K acts on the right there is an induced $\mathcal{G}$ bundle over X which is a product. We let G act on X on the left as a group of isometries; the induced action on the fibers of the bundle, identified with $\mathcal{G}$, is the adjoint action. A (X, G) structure on M is determined by charts on M, mapped diffeomorphically to open subsets of X. Each transition map is required to be the restriction of an element of G acting on X. Since the transition maps are locally constant, the bundle over X pulls back to a *flat* $\mathcal{G}$ bundle over M, which we denote by E. We can define cohomology groups on M with coefficients in this bundle in the usual way since it is a flat bundle.

Although the flat structure on the bundle E allows one to define cohomology groups, it is not well-adapted to the local geometry on X, or equivalently, on M. A single element in $\mathcal{G}$, acting globally on X can behave geometrically in very different ways at different points of X. For example, an element of the Lie algebra of K, viewed as an infinitesimal isometry of X fixes some points but has a huge effect on points far away from the fixed points. On the fibers of the bundle over

X there is a natural positive definite metric which does not respect the flat structure but reflects the way in which an element of the fiber over a point $x \in X$ acts on X near x. It can be described geometrically as follows:

At each point of X, the fiber, viewed as the germs of isometries of X, decomposes as a direct sum $\mathcal{P} \oplus \mathcal{K}$ where $\mathcal{K}$ consists of the infinitesimal rotations about the point, and $\mathcal{P}$ consists of the infinitesimal "pure translations" at the point. This decomposition is defined to be an *orthogonal* sum; to define a metric on the fiber it suffices to define one on $\mathcal{P}$ and one on $\mathcal{K}$. But $\mathcal{P}$ is naturally identified with the tangent space of X at the base of the fiber and we give it the metric induced from the Riemannian metric on X. Similarly, since an element of $\mathcal{K}$ operates linearly and isometrically on the tangent space, the metric on $\mathcal{K}$ comes from identifying it with a subspace of $o(n)$ with its usual metric. (In our case of constant curvature, G acts transitively on the frames based at any point so $\mathcal{K} \cong o(n)$.) Again, we emphasize that this metric is not compatible with the product structure of the bundle over X; from the point of view of the product structure the metric varies with the point of X. On the other hand, the metric is, by definition, invariant under isometries of X. In particular, it pulls back to a metric on the bundle E over M.

Remark. This metric on the bundle E and its associated orthogonal decomposition are often derived more algebraically in the Lie groups literature. The bundle over X is lifted to the Lie group G where, as before, it is identified with the product bundle of *right* invariant vector fields with fiber the Lie algebra $\mathcal{G}$. However, the bundle on G is also isomorphic to the bundle of *left* invariant vector fields (via the adjoint action); it is again realized as a product bundle with $\mathcal{G}$ as fiber. The Lie algebra of G has a standard splitting (the Cartan splitting) which is invariant under the adjoint action of K. There is also a standard inner product, derived from the Killing form on $\mathcal{G}$ which is invariant under this action and in which this splitting is orthogonal. Using the identification of the fibers with the Lie algebra, this induces a metric on the lifted bundle. By construction it is invariant under the action of G on itself by *left* multiplication. The product structure of the bundle of left invariant vector fields doesn't descend to a product structure on X since K acts on the right; the bundle over X has structure group K, with the adjoint action, from this point of view. However, since the metric and the splitting are invariant under the adjoint action of K, they descend to the bundle over X. Since G acts on the left as a group of isometries of X, the metric and the splitting are again seen to be invariant under isometry. Hence they are well-defined on the bundle E over M.

The metric on E gives an inner product $\langle\ ,\ \rangle$ on $\Lambda^* TM^* \otimes E$ and an L^2 inner product $(\ ,\ )$ on E-valued forms defined by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle,$$

when the integral exists. Using this we obtain an adjoint δ of d such that $(\delta\alpha, \beta) = (\alpha, d\beta)$ whenever α or β has compact support. Then the associated Laplacian is given by $\Delta = d\delta + \delta d$.

If M is a closed manifold, then the classical Hodge theorem shows that each cohomology class in $H^*(M; E)$ can be represented by a form ω which is harmonic, i.e., $\Delta\omega = 0$. Further, such a harmonic form is automatically closed and co-closed, i.e., $d\omega = 0$ and $\delta\omega = 0$. To study these cohomology groups, we need to analyze the Laplacian in more detail.

Above we have described two ways of viewing the bundle E. The first, considering it as a flat bundle, leads to the definition of cohomology groups which are connected with variation of the holonomy representation of hyperbolic structures. The second, endowing it with a metric structure and splitting derived directly from the local differential geometry on the manifold, will lead to a description of the variation of the metric itself. (See Section 2 for some further discussion of these approaches.) The key to the vanishing results for $H^1(M; E)$, which lead to local rigidity for hyperbolic manifolds, is the relationship between the operators d and δ which are connected with the flat structure on E and the covariant exterior derivative (and its adjoint) associated with the metrics on M and E.

Explicit expressions for d and δ which relate the two ways of viewing the bundle were calculated by Matsushima-Murakami in [21]. Using the formalism of Wu's paper [30, Chap. 6], these can be written as follows:

$$d = \sum_i \omega^i \wedge (\nabla_{e_i} + \mathrm{ad}(E_i))$$

and

$$\delta = -\sum_j i(e_j)(\nabla_{e_j} - \mathrm{ad}(E_j)).$$

Here $\{e_i\}$ can be any orthonormal frame field for TM, $\{\omega^i\}$ is the dual coframe field, and $i()$ denotes the interior product on forms. ∇ denotes the covariant differentiation in X induced by a natural metric connection on E, as described below. Further, in the expressions $\mathrm{ad}(E_j)$, E_j is the element in the fiber over any point in M which is the infinitesimal translation in the direction e_j at that point. The operator $\mathrm{ad}(E_j)$ sends an element Y in the fiber to $[E_j, Y]$ where the bracket is the usual one on Killing fields.

Then $d = D + T$ and $\delta = D^* + T^*$, where

$$D = \sum_i \omega^i \wedge \nabla_{e_i},$$

$$T = \sum_i \omega^i \wedge \mathrm{ad}(E_i),$$

$$D^* = -\sum_j i(e_j)\nabla_{e_j}, \tag{1}$$

$$T^* = \sum_j i(e_j)\mathrm{ad}(E_j).$$

Note that these expressions reduce to the formulas given in [21] or in Raghunathan's book [26] for a suitable choice of frame field.

There is a natural decomposition:

$$\Omega^*(M; E) = \Omega^*(M; \mathcal{P}) \oplus \Omega^*(M; \mathcal{K}). \tag{2}$$

Here, we can identify $\Omega^i(M; \mathcal{P})$ with the TM valued i-forms $\Omega^i(M; TM)$, by identifying a tangent vector to M at a point with the corresponding infinitesimal pure translation at that point. Then D and D^* preserve the decomposition (2), while T and T^* map $\Omega^*(M; \mathcal{P})$ to $\Omega^*(M; \mathcal{K})$ and vice versa. The metric connection which determines D restricted to TM-valued forms is the standard metric (Levi-Civita) connection on M. On $\mathcal{K} \subset so(n) \subset Hom(TM, TM)$, it is again the standard metric connection, induced by the one on TM. In other words, a section is flat if it is represented by a constant matrix with respect to a parallel orthonormal frame for TM. Expanding out the expressions for d and δ, we get

$$\triangle = d\delta + \delta d = (DD^* + D^*D + TT^* + T^*T) + (DT^* + TD^* + D^*T + T^*D),$$

but the last term vanishes: *i.e.*, $DT^* + TD^* + D^*T + T^*D = 0$. (See [21].) In particular, the Laplacian Δ preserves the decomposition (2). This gives us a Weitzenböck formula of the form

$$\Delta = \Delta_D + H, \tag{3}$$

where $\Delta_D = DD^* + D^*D$ and $H = TT^* + T^*T$. Using the above expressions for T, T^* we can also write

$$H = \sum_i \operatorname{ad}(E_i)^2 - \sum_{i,j} \omega^i \wedge i(e_j)\operatorname{ad}([E_j, E_i]). \tag{4}$$

Applying $\Delta = d\delta + \delta d$ to ω and integrating by parts gives the following formula which will be crucial for our rigidity results:

Proposition 1.3. *Let M be a compact, oriented manifold with boundary and let $\omega \in \Omega^1(M; E)$ be an E-valued 1-form. Then*

$$(d\omega, d\omega) + (\delta\omega, \delta\omega) = (D\omega, D\omega) + (D^*\omega, D^*\omega) + (H\omega, \omega) + B, \tag{5}$$

where B denotes the boundary term

$$B = -\int_{\partial M} (*T\omega \wedge \omega + T^*\omega \wedge *\omega). \tag{6}$$

Here, $\alpha \wedge *\beta$ denotes the *real-valued* form obtained by using the Hodge star operator $*$ on M and the inner product on E.

Proof. This formula is derived from $\langle \delta d\omega + d\delta\omega, \omega\rangle = \langle \Delta_D\omega, \omega\rangle + \langle H\omega, \omega\rangle$ by integrating both sides by parts and subtracting the boundary term on the left from that on the right.

To derive the boundary terms we use the identity

$$\hat{d}(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^p \alpha \wedge D\beta,$$

where α and β are E-valued forms, $p = \deg \beta$, $\wedge$ denotes the exterior product using the inner product on E, and $\hat{d}$ is the exterior derivative on real-valued forms. (Compare [16, Chap. 3] and [30].)

In particular, this gives

$$\hat{d}(\alpha \wedge *\beta) = (\langle D\alpha, \beta\rangle - \langle \alpha, D^*\beta\rangle)\ \mathrm{dvol},$$

where $D^* = (-1)^{m(p+1)+1} * D*$, $p = \deg \beta = \deg \alpha + 1$ and $m = \dim M$.

We also have:

$$0 = T\alpha \wedge *\beta - \alpha \wedge *T^*\beta,$$

and since $d = D + T$, $\delta = D^* + T^*$ this gives:

$$\hat{d}(\alpha \wedge *\beta) = (\langle d\alpha, \beta\rangle - \langle \alpha, \delta\beta\rangle) \text{ dvol},$$

when $\deg\beta = \deg\alpha + 1$.

The boundary terms now follow immediately. For example the integrand in the boundary term for $(\delta d\omega, \omega) = (d\omega, d\omega) + B'$ is $*d\omega \wedge \omega$ and for $(D^*D\omega, \omega) = (D\omega, D\omega) + B''$ the integrand is $*D\omega \wedge \omega$. Thus the integrand for $B'' - B'$ is $- * T\omega \wedge \omega$ since $d = D + T$. The other summand in the integrand of B in (4) is derived similarly. q.e.d.

Now Weil showed that the algebraic term $\langle H\omega, \omega\rangle$ is (strictly) positive definite when G satisfies the hypotheses of the Mostow rigidity theorem. Then there is a constant $c > 0$ such that

$$(H\omega, \omega) \geq c(\omega, \omega) \tag{7}$$

for all 1-forms $\omega \in \Omega^1(M; E)$. In particular, this holds for a hyperbolic manifold M with $G = \text{isom}\,\mathbf{H}^n$ and $X = \mathbf{H}^n$, when $n \geq 3$.

If M is closed, every class in $H^1(M; E)$ is represented by a (unique) harmonic form ω such that $d\omega = \delta\omega = 0$ and the boundary term $B = 0$. Then (5) and (7) show that $0 \geq c(\omega, \omega)$, so $\omega = 0$ and $H^1(M; E) = 0$. This is the essence of the Calabi-Weil argument. In the complete, finite volume case, Garland [12] works with cut-off functions to deduce that any L^2 harmonic form is a coboundary. From this, he deduces that there are no non-trivial deformations of the hyperbolic structure through complete structures.

To prove Theorem 1.1, we apply a similar argument in the situation where $\overline{M}$ is a finite volume 3-dimensional hyperbolic cone-manifold and $M = \overline{M} - \Sigma$. Since M is both non-compact and non-complete, both the Hodge theory and the control of the boundary terms in (5) are much subtler.

We begin this analysis in the next section, and prove a Hodge theorem for hyperbolic cone-manifolds. If $\overline{M}$ is a compact 3-dimensional hyperbolic cone-manifold and $M = \overline{M} - \Sigma$, then we will show that any cohomology class in $H^1(M; E)$ can be represented by a form in $\Omega^1(M; E)$ which is *closed* and *co-closed* with controlled behavior near Σ. Note that in this situation, harmonic forms need not be closed and co-closed in general.

To prove our rigidity theorem we also need to study the boundary terms in (5) more carefully. The basic idea is to study the asymptotic behavior of the closed and co-closed forms given by our Hodge theorem in an open tubular neighborhood U_r of radius r about the singular locus, using separation of variables to obtain a Fourier series type expansion. If $\overline{M}$ is a 3-dimensional hyperbolic cone-manifold we apply proposition 1.3 to the compact manifold $\overline{M} - U_r$, and show the boundary terms $B = B_r$ approach zero as $r \to 0$ whenever the cone angles are at most 2π. Then (5) and (7) again imply that each of our closed and co-closed L^2 forms is zero. The details of this asymptotic analysis will be given in Section 3 below.

2. The structure of harmonic forms

Throughout this section $\overline{M}$ will denote a closed, orientable 3-dimensional hyperbolic cone-manifold with singular locus Σ, and M will denote $\overline{M} - \Sigma$, endowed with its smooth, but incomplete hyperbolic metric. Our ultimate goal is to compute the cohomology group $H^1(M; E)$ where E denotes the bundle of local Killing vector fields on M. In particular, we want to show that its L^2 part is trivial. We will use the de Rham point of view in this section and consider cohomology classes as equivalence classes of E-valued 1-forms. In Section 4 we switch to a simplicial or group cohomology point of view and use our results to analyze representation spaces.

In this section we first discuss the structure of closed and co-closed E-valued 1-forms and prove some identities for such forms. We will then prove a Hodge theorem which will give us a unique closed and co-closed representative in each cohomology class with specific control on the behavior of the form as it approaches the singular locus. Since the point of a Hodge theorem is to pick out a particularly nice representative within a cohomology class, we first discuss the data provided by a representative and how this changes when we alter it by a coboundary; i.e., by d of a section of E.

Consider an element in $H^1(M; E)$, regarded as an equivalence class of closed E-valued 1-forms. Let ω be any closed representative. Choosing a point $x \in M$, we can *locally* define a section s of the bundle E by integrating ω along paths beginning at x. (Note that we are using the flat connection on E to identify the fibers at different points along the path in order to do the integration.) Because ω is closed, the value of the integral depends only on the homotopy class of the path; a well-defined section is determined on any simply connected subset of M. Then $ds = \omega$ on such a subset.

In general, the section will not extend to a global section. Integration of ω around a closed loop based at x will give a non-zero element of the fiber of E at x, depending only on the homotopy class of the loop. We choose an isomorphism of the fiber at x with the Lie algebra $\mathcal{G}$; using the flat connection, this determines an isomorphism of the fibers to $\mathcal{G}$ along any path from x. This defines a map $z : \pi_1(M, x) \to \mathcal{G}$ which satisfies the cocycle condition,

$$z(\alpha\beta) = z(\alpha) + Ad(\rho(\alpha))z(\beta).$$

Thus, integration determines an element of the group cohomology, $H^1(\pi_1(M); Ad\rho)$. This element is trivial precisely when the 1-form ω is exact, or equivalently when the locally defined section, possibly plus a constant, extends to a global section.

The closed E-valued 1-form ω also determines an infinitesimal deformation of the hyperbolic structure on M (see below), and the cocycle z determines the corresponding infinitesimal deformation of its holonomy representation. Specifically, if $\rho_t : \pi_1(M) \to G$, $t \in \mathbf{R}$ is a smooth 1-parameter family of holonomy representations such that $\rho_0 = \rho$ is the holonomy for the initial hyperbolic structure on M, then

$$z(\gamma) = \left.\frac{d}{dt}\rho_t(\gamma)\rho(\gamma)^{-1}\right|_{t=0}.$$

We discuss group cohomology and the connection with representations $\pi_1(M) \to G$ in more detail in Section 4.

An alternative measurement of the non-triviality of the de Rham cohomology class determined by the closed form ω is given by looking at the universal cover. If we lift the form and the bundle to the universal cover $\tilde{M}$ of M, integration defines a global section of the lifted bundle $\tilde{E} \to \tilde{M}$. This section is a lift of the local section defined on M; it will still be denoted by s. The fundamental group, $\pi_1(M, x)$, acts on the universal cover $\tilde{M}$ by covering translations and on the fibers of $\tilde{E}$ as described above, so it acts on sections of $\tilde{E}$. (The result of $\gamma \in \pi_1(M, x)$ acting on a section s will be denoted by $\gamma_* s$.) In general, the section is not equivariant with respect to this action: it is equivariant exactly when the local section defined on M, possibly plus a constant, can be extended globally, hence when the form ω is exact. However, the lifted section does always satisfy an *automorphic* property:

Lemma 2.1. *Let ω be a closed E-valued 1-form on M and $s : \tilde{M} \to \tilde{E}$ be a global section defined by integrating ω. Then for any element γ of $\pi_1(M, x)$, the new section defined by $y \mapsto (\gamma_*^{-1} s)(\gamma(y)) - s(y)$ is d-closed.*

Remark. If the fibers of $\tilde{E}$ are identified with the Lie algebra $\mathcal{G}$ using the flat connection, $y \mapsto (\gamma_*^{-1} s)(\gamma(y)) - s(y)$ becomes a map from $\tilde{M}$ to $\mathcal{G}$. Lemma 2.1 says that the map is *constant*; i.e., independent of y. For any $\gamma \in \pi_1(M, x)$, this element of $\mathcal{G}$ is the value $z(\gamma)$ at γ of the cocycle z defined above.

Proof. The lemma follows from the fact that the 1-form ω is globally defined on M. Since ds is the lift of ω to $\tilde{M}$, it is equivariant; i.e., $\gamma_*^{-1}(ds) = ds$. Since the action of the fundamental group on sections commutes with d, the derivative of the section $y \mapsto (\gamma_*^{-1} s)(\gamma(y)) - s(y)$ is zero. q.e.d.

We will call a section satisfying the conclusion of Lemma 2.1 *automorphic.*

Although the flat connection on E is important in defining and understanding the cohomology theory related to deformations of hyperbolic structures and in relating it to deformations of representations, we will find an alternative structure on E more useful. It will help us understand the cohomology theory in terms of purely differential geometric concepts. It is based on the decomposition, as described in Section 1, of the fiber of E at each point in M as $\mathcal{P} \oplus \mathcal{K}$. This decomposes E into an orthogonal direct sum of two sub-bundles (which are *not* flat) which we also denote as $\mathcal{P}$ and $\mathcal{K}$, respectively. We emphasize that this splitting is *not* compatible with the flat connection; i.e., it is not preserved under parallel translation using the flat connection. In particular, on the lifted bundle, $\tilde{E}$, over the universal cover of M, where the flat connection allows a global identification of the fibers with the Lie algebra, this splitting does *not* come from a single splitting of the Lie algebra. Similarly, the flat exterior derivative d does not take (local) sections of the sub-bundles to (local) 1-forms with values in the sub-bundles. On the other hand, the covariant exterior derivative D does take (local) sections of the sub-bundles to (local) 1-forms with values in the sub-bundles. The relations between the two exterior derivatives and between their adjoints (as in equation (1)) are the key to the analysis in this section. We have found that these relations are best expressed in terms of this decomposition of the bundle.

The two sub-bundles of E lift to sub-bundles of $\tilde{E}$, denoted $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{K}}$, respectively. A section of E (resp., $\tilde{E}$) is equivalent to the pair of sections of $\mathcal{P}$ and $\mathcal{K}$ (resp., $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{K}}$) obtained by orthogonal projection. The bundle $\mathcal{P}$ is naturally isomorphic to the tangent bundle TM of M; its lift is isomorphic to the tangent bundle $T\tilde{M}$ of $\tilde{M}$. The action of the fundamental group respects the isomorphism between $\tilde{\mathcal{P}}$ and $T\tilde{M}$. Thus a section of the lifted sub-bundle can be interpreted as a vector field on $\tilde{M}$. It is equivariant under the action of the fundamental group precisely when it corresponds to a vector field defined *on* M. From now on we will pass freely between the notion of sections of $\mathcal{P}$ (resp., $\tilde{\mathcal{P}}$) and vector fields on M (resp., $\tilde{M}$) without distinction.

The section of $\tilde{E}$ determined by integrating a closed E-valued 1-form was seen to be automorphic in Lemma 2.1. Similarly, we will say that a vector field v on $\tilde{M}$ is *automorphic* if for any $\gamma \in \pi_1(M, x)$, the difference $(\gamma_*^{-1})v(\gamma(y)) - v(y)$ is a vector field which is the projection to $\tilde{\mathcal{P}}$ of a d-closed section of $\tilde{E}$. This definition is clearly made so that the projection of an automorphic section to $\tilde{\mathcal{P}}$ will be automorphic. In Lemma 2.3 below we will see how to lift an automorphic vector field to an automorphic section.

We now comment on the geometric interpretation of the automorphic vector field determined by integrating a closed E-valued 1-form and then taking the vector field part of the resulting automorphic section. A hyperbolic structure on M is determined by local charts modelled on $X = \mathbf{H}^n$. These determine, via an analytic continuation argument, a map $\Phi : \tilde{M} \to X$, called the *developing map*, which is determined uniquely up to post-multiplication by an element of $G = \mathrm{isom}(X)$. The developing map satisfies the equivariance property $\Phi(\gamma m) = \rho(\gamma)\Phi(m)$, for all $m \in \tilde{M}$, $\gamma \in \pi_1(M)$, where $\pi_1(M)$ acts on $\tilde{M}$ by covering transformations, and $\rho : \pi_1(M) \to G$ is the *holonomy representation* of the structure. (See [24] and [27] for a complete discussion of these ideas.)

A 1-parameter family of hyperbolic structures defines a 1-parameter family of developing maps Φ_t. The derivative at $t = 0$ of this family defines a map $\dot{\Phi} : \tilde{M} \to TX$. For any point $m \in \tilde{M}$, $\Phi_t(m)$ is a curve in X describing how the image of m is moving under the developing maps; $\dot{\Phi}(m)$ is the initial tangent vector to the curve. The tangent bundle TX of X is isomorphic to the bundle $\tilde{\mathcal{P}}$ over X, and the analogous bundle $\tilde{\mathcal{P}}$ over $\tilde{M}$ is the pull-back of the bundle over X via the initial developing map Φ_0. Thus the automorphic vector field is just describing the infinitesimal motion of the points of $\tilde{M}$ under the variation of hyperbolic structures.

In this way, we see how a choice of representative for a cohomology class in $H^1(M;E)$ determines an infinitesimal variation of the developing map. Taking a different representative for the cohomology class amounts to changing it by ds for a globally defined section s of E. The corresponding new infinitesimal variation of the developing map has been altered by the lift of a *globally defined* vector field on M. This is the infinitesimal result of changing the developing map by an isotopy on M; it determines an equivalent hyperbolic structure. This suggests that we should be able to study the cohomology theory in terms of automorphic vector fields modulo equivariant vector fields.

To do this, we need to analyze the operator d on local sections of E. There are some special features of the 3-dimensional situation, which simplify this analysis. The Lie algebra $\mathcal{G} = sl_2\mathbf{C}$ has a natural complex structure which is related to the decomposition $E = \mathcal{P} \oplus \mathcal{K}$ by $\mathcal{K} = i\,\mathcal{P}$. To see that multiplication by i in the Lie algebra induces a bundle isomorphism from $\mathcal{P}$ to $\mathcal{K}$, we recall from the previous section that this splitting of E comes from viewing the fibers as *left* invariant vector fields on G. This identification is only well-defined up to the adjoint action of K, but the adjoint action commutes with multiplication by i. More geometrically, if t denotes an infinitesimal translation, then it is an infinitesimal rotation whose axis is the axis of t. Further, t and it are orthogonal, and $\|t\| = \|it\|$ is the length of the tangent vector in M corresponding to t. We will now think of $\Omega^*(M;\mathcal{P})$ and $\Omega^*(M;\mathcal{K})$ as the *real* and *imaginary* parts of $\Omega^*(M;E)$. Recall the decompositions $d = D+T$, $\delta = D^* + T^*$ from equation (1) of the previous section. Then D and T (respectively D^* and T^*) are precisely the real and imaginary parts of d (respectively δ).

Using the complex structure on the fibers, a local section of E determined by integrating an E-valued 1-form is just a pair of vector fields, defined locally on M, called the real and imaginary vector fields. An E-valued 1-form is exact precisely when it is the derivative of a global section; i.e., when these vector fields are globally defined. We are free to alter the local vector fields by adding globally defined ones without changing the cohomology class. The image of d lies in the space of E-valued 1-forms, and can be decomposed into real and imaginary vector valued 1-forms. If v is a real vector field on M, then the real part of dv is Dv; by (1) this is the vector valued 1-form ∇v. The imaginary part of dv is another vector valued 1-form Tv, where T is the purely algebraic operator from (1).

A vector valued 1-form is an element of $Hom(TM, TM)$; using the metric on TM, it decomposes into its symmetric and skew-symmetric parts. Furthermore, in dimension 3, there is a canonical isomorphism from vector fields to *skew-symmetric* elements of $Hom(TM, TM)$, which, for example, sends e_k to $e_l \otimes \omega_j - e_j \otimes \omega_l$, when $\{e_k, e_l, e_j\}$ is a positively oriented frame and $\{\omega_i\}$ is the dual co-frame.

Lemma 2.2. *Let v be a locally defined, real vector field on M. Then the element $Tv \in iHom(TM, TM)$ is skew-symmetric and corresponds to the vector field $-iv$ under the above canonical isomorphism.*

Proof. This follows from the formula (1) for T and the equations for the bracket in the Lie algebra $\mathcal{G} = sl_2\mathbf{C}$. Choose a local frame $\{e_i\}$ on M, and denote by $\{\omega_i\}$ the dual co-frame and by $\{E_i\} \in \mathcal{P}$ the corresponding infinitesimal translations. Then the formula for T gives that $TE_k = \Sigma_j[E_j, E_k]\omega_j$. From $[E_j, E_k] = -iE_l$ when $\{e_j, e_k, e_l\}$ form an oriented basis, we see that the sum has two terms, which are of opposite signs. Since T is linear and purely algebraic, this implies that Tv is skew-symmetric. Further Tv is the image of $-iv$ under the above isomorphism. q.e.d.

The decomposition of Dv as an element of $Hom(TM, TM)$ has a useful geometric interpretation. The skew-symmetric part is called the *curl* of v. The trace of the linear transformation is the *divergence* of v. The traceless, symmetric part is often called the *strain* of v. The symmetric part can be considered as a symmetric 2-tensor (using the metric), which describes the infinitesimal change in the metric under the deformation. The divergence measures the infinitesimal change in the volume and is the conformal part of the change in metric. The strain measures the non-conformal change in the metric. Under the identification of skew-symmetric transformations with vectors, *curl* v, becomes a vector field. (This vector field is *half* the usual *curl* considered in elementary vector calculus.)

The following result allows us to work exclusively with vector fields, rather than E-valued 1-forms. It also shows that a cohomology class is completely determined by the infinitesimal change in the metric induced by the real vector field. This is to be expected since the cohomology class represents an infinitesimal variation of the hyperbolic structure, but it is useful to see explicitly how this fits in with the cohomology theory.

Consider the local section, $s_v = v - i\ curl\ v$, which we call the *canonical lift of v*. Clearly, it is determined by its real part.

Proposition 2.3. *Let v be an automorphic vector field on $\tilde{M}$. Then the following hold:*

(a) The canonical lift $s_v = v - \ i\ curl\ v$ is an automorphic section of $\tilde{E}$.

(b) The real part of ds_v is symmetric. The imaginary, skew-symmetric part of ds_v is $-(curl\ curl\ v\ + v)$.

(c) If s is any automorphic section of $\tilde{E}$, then the cohomology class of ds on M is determined by its real, symmetric part. Furthermore, any cohomology class may be represented by a form whose associated local section is the canonical lift of its associated local vector field.

Proof. We first show that a d-closed section of $\tilde{E}$ is determined by its projection to $\tilde{\mathcal{P}}$: if v is the projection of a d-closed section s, then $s = v \ - \ i\ curl\ v$.

To see this, note that if $s = v + \ i\ w$ and $ds = 0$, then the real part of ds, which equals $Dv + i\ Tw$, is zero. By Lemma 2.2, $i\ Tw$ is skew-symmetric, corresponding to w. Thus Dv is skew-symmetric equal to $-w$. By definition, the skew-symmetric part of Dv equals $curl\ v$.

(a) A vector field v is automorphic if, for any $\gamma \in \pi_1(M, x)$, the vector field $y \mapsto (\gamma_*^{-1})v(\gamma(y)) - v(y)$ is the projection of a d-closed section of $\tilde{E}$. From the observation above, this d-closed section equals

$$y \mapsto (\gamma_*^{-1})v(\gamma(y)) - v(y) - \ i\ curl((\gamma_*^{-1})v(\gamma(y)) - v(y)).$$

Since the action of γ is by isometries, it commutes with taking the *curl*. Thus the section is to equal $y \mapsto (\gamma_*^{-1})s(\gamma(y)) - s(y)$ for the canonical lift $s = v \ - \ i\,curl\ v$. This shows that the canonical lift s is automorphic.

(c) Let $s = v + iw$ be any automorphic section with vector fields v and w as its real and imaginary parts. Then ds is equivariant, and, in particular, its real skew-symmetric part $ds_{real,skew}$ is equivariant. Thus, it determines a vector field *on* M. On the other hand, the decomposition of d shows that

$$\begin{aligned} ds =& dv + idw = (Dv + Tv) + i(Dw + Tw) \\ =& (Dv + iTw) + (Tv + iDw), \end{aligned} \tag{8}$$

where we have decomposed ds into its real and imaginary parts. Further, by Lemma 2.2, Tv and Tw are skew-symmetric tensors identified with the vector fields $-iv$ and $-iw$. So the real, skew-symmetric part of ds, interpreted as a vector field, is

$$ds_{real,skew} = curl\ v + i\ Tw = curl\ v + w.$$

It follows that w differs from $-curl\ v$ by a globally defined vector field on M; hence, s differs from the canonical lift of v by a global section. In particular, the cohomology class is determined by the canonical lift of v, hence by the real part of s. Furthermore, the canonical lift of v may be chosen to represent this cohomology class.

Note also that the real, symmetric part of ds is not affected by the imaginary part of a local section s. Thus, the cohomology class of ds is completely determined by the real, symmetric part.

(b) If $s = v+iw$ is the canonical lift of v, then $w = -curl\ v$, so the real skew-symmetric part of its derivative $(ds)_{real,skew} = curl\ v + w = 0$, and the imaginary, skew-symmetric part of its derivative is

$$(ds)_{imag,skew} = Tv + curl\ w = -v - curl\ curl\ v.$$

q.e.d.

We now consider the structure of a closed and co-closed form in $\Omega^1(M; E)$. Note that since M is not complete this is a stronger condition than saying that the form is harmonic. Later we will prove a Hodge theorem which shows that there is a unique such representative in each cohomology class with certain controlled behavior at the singular locus. First we give a local description of such forms.

A closed and co-closed form ω in $\Omega^1(M; E)$ determines a locally defined harmonic section s by integration, since s satisfies $\Delta s = \delta ds = \delta\omega = 0$. By equation (3), the Laplacian on sections preserves the real and imaginary parts of the section: if $s = v + iw$, then $\Delta s = \Delta v + i\Delta w$. So, to understand locally defined harmonic sections, it suffices to understand the structure of locally defined harmonic vector fields. The Laplacian on vector fields has a Weitzenböck formula which decomposes into a second order operator plus a order zero operator. Specifically, we have

$$\Delta v = \nabla^* \nabla\, v + 2v, \tag{9}$$

where the operator ∇^* is the adjoint of ∇. (Sometimes $\nabla^*\nabla$ is called the "rough Laplacian".) The factor of 2 is a reflection of the fact that the Ricci curvature of M is -2. This formula can be easily derived from

(3) and (4) of the previous section since $\nabla^*\nabla = D^*D$. In this case, $H = T^*T = \Sigma_i \operatorname{ad}(E_i)^2$ is multiplication by 2 since $[E_i, [E_i, E_j]] = E_j$ for $i \neq j$. Note that if v is automorphic, then Δv is equivariant, since for any automorphic lift s of v, ds is equivariant and the real part of δds is Δv.

Most of the identities that we need are derived most easily by considering the local, real-valued 1-forms on M, which are *dual* to the local vector fields with respect to the metric. We will denote by $\hat{v}$ the 1-form dual to v and by $\hat{\Delta}$ the usual Laplacian on 1-forms. Then the relationship between the various Laplacians is determined by

$$\hat{\Delta}\hat{v} = (\nabla^*\nabla\, v)\hat{} - 2\hat{v}. \tag{10}$$

This follows from the general relation between the Laplacians (see [30] or [31]), using the fact that the Ricci curvature of M is -2. From this and (9) above, we see that

$$\widehat{\Delta v} - 4\hat{v} = \hat{\Delta}\hat{v}. \tag{11}$$

If we denote by $\hat{d}, \hat{\delta}$ the usual exterior derivatives of real-valued forms on M and their adjoints, then $\hat{\Delta} = \hat{d}\hat{\delta} + \hat{\delta}\hat{d}$. It is well-known that the function $\hat{\delta}\hat{v}$ is the divergence of v, that the 1-form $*\hat{d}\hat{v}$ is the dual of the vector field 2 *curl* v, and that $\hat{d}f$ is the dual of the gradient vector field *grad* f. Thus by taking the dual of formula (11) we obtain

$$\Delta v - 4v = \mathit{grad}\ \mathit{div}\ v + 4\ \mathit{curl}\ \mathit{curl}\ v. \tag{12}$$

The last equation immediately yields several useful properties of harmonic vector fields.

Lemma 2.4. *Suppose that v is a harmonic vector field, i.e., v satisfies $\Delta v = 0$. Then the following hold:*

(a) div v satisfies

$$\hat{\Delta}(\mathit{div}\ v) = -4\ \mathit{div}\ v, \tag{13}$$

(b) $w = $ curl v is harmonic and satisfies

$$\mathit{curl}\ \mathit{curl}\ w = -w, \tag{14}$$

(c) if v also satisfies div $v = 0$ (i.e., it is volume preserving) then it satisfies

$$(15) \qquad \textit{curl curl } v = -v.$$

Proof. By taking *div* of (12) and noting that $\hat{d}$ and $\hat{\delta}$ commute with $\hat{\Delta}$, we see that *div v* satisfies

$$\hat{\Delta}(\textit{div } v) = \textit{div grad } (\textit{div } v) = -4 \textit{ div } v.$$

Similarly, by taking *curl* of (12) we obtain equation (14). Since *div curl v* = 0, it follows from (12) and (14) that $\Delta(\textit{curl } v) = 0$. If v satisfies $\textit{div } v = 0$, then, by equation (12), it satisfies $\textit{curl curl } v = -v$ itself. q.e.d.

Remark. In particular, part (c) shows that the identity *curl curl w* $= -w$ holds for the vector field induced by a d-closed section (i.e., a Killing vector field). This also follows from the proof of Lemma 2.3 (b).

We will use these observations to study the canonical lift of a harmonic vector field. But first we need to establish the formulae for T and T^* on real, E-valued 1-forms.

Lemma 2.5. *The algebraic operator T^*, acting on sections of $Hom(TM, TM)$ sends each skew-symmetric element to $2i$ times the corresponding vector field and sends each symmetric element to 0. The operator $*T$ sends a traceless element of $Hom(TM, TM)$ to i times its transpose and multiplies the identity by $-2i$, i.e.,*

$$*T(\eta) = i(\eta^T - tr(\eta)I).$$

Proof. This is proved in the same manner as Lemma 2.2. Choose a local frame $\{e_i\}$ on M, and denote by $\{\omega_i\}$ the dual co-frame and by $\{E_i\} \in \mathcal{P}$ the corresponding infinitesimal translations. Then the formula for T^* gives that $T^*(E_j \otimes \omega_k) = [E_k, E_j]$ which equals iE_l when $\{e_j, e_k, e_l\}$ form an oriented basis. Thus $T^*(E_j \otimes \omega_k + E_k \otimes \omega_j) = 0$ (including the case $k = j$) and $T^*(E_j \otimes \omega_k - E_k \otimes \omega_j) = 2iE_l$ as claimed. Similarly, $T(E_j \otimes \omega_k) = \Sigma_l [E_l, E_j] \otimes \omega_l \wedge \omega_k$. If $j \neq k$, the only non-zero term is when $l \neq j, k$. Taking the $*$ of this gives $iE_k \otimes \omega_j$. If $j = k$ we get $*T(E_j \otimes \omega_j) = -i(E_k \otimes \omega_k + E_l \otimes \omega_l)$. Thus $*T$ sends the difference of any two such diagonal elements to i times itself and multiplies the identity by $-2i$. q.e.d.

Proposition 2.6. *Let s be the canonical lift of a divergence-free, harmonic vector field (locally defined). The E-valued 1-form ds is closed and co-closed, and $ds = \eta + i * D\eta$, where η and $*D\eta$ are symmetric and traceless. Futhermore, $D^*\eta = 0$, $D^*D\eta = -\eta$.*

Proof. Given a harmonic vector field v, its canonical lift, $s = v - i\ curl\ v$, is a harmonic section since $curl\ v$ is harmonic by Lemma 2.4(b). Thus ds is a closed and co-closed E-valued 1-form. The real part of ds is symmetric and the imaginary, skew-symmetric part is $-(curl\ curl\ v + v)$; this is the case for any canonical lift by Lemma 2.3(b).

If v is also volume-preserving ($div\ v = 0$), then $curl\ curl\ v + v = 0$ by (15), so both the real and imaginary parts of ds are symmetric and traceless. In particular,

$$ds = str\ v - i\ str\ curl\ v. \tag{16}$$

Here $str\ v$ denotes the strain of v (the traceless, symmetric part of ∇v). Let $\eta = str\ v$. Then $Dv = curl\ v + \eta$. The real part of $\delta ds = D^*\eta - T^*(i\ str\ curl\ v)$ is zero since s is harmonic. But, by Lemma 2.5, since $str\ curl\ v$ is symmetric, T^* of it is zero. Thus, if $div\ v = 0$, we get that

$$D^*\eta = 0. \tag{17}$$

Next, consider the equation $d^2 s = 0$. By equation (16) and the decomposition of d in (8), the real part of $d^2 s$ is $D\eta - T(i\ str\ curl\ v)$. By Lemma 2.5 $*T$ is just multiplication by i on this symmetric, traceless element of $Hom(TM, TM)$. It follows that $*D\eta = -str\ curl\ v$; in other words,

$$ds = \eta + i\ * D\eta. \tag{18}$$

Applying $*D$ to $*D\eta$ gives $str\ curl\ curl\ v$ since $curl\ v$ is also harmonic and volume preserving. Since $curl\ curl\ v = -v$ by (15), we get $*D * D\eta = str\ (-v) = -\eta$ or, since $D^* = *D*$,

$$D^*D\eta = -\eta. \tag{19}$$

q.e.d.

Remark. These equations have geometric derivations as well. Equation (19) represents the condition, in the presence of the divergence-free hypothesis and equation (17) that the curvature is (infinitesimally) remaining constant. Under the divergence-free hypothesis, equation (17) is the condition, that the deformation is L^2-perpendicular to those induced by *global* vector fields on M. These equations also arise in the deformation theory of Einstein structures ([3], Chapter 12).

We will now use the structure that we have developed for closed and co-closed forms to study the cohomology group $H^1(M; E)$. We make some preliminary assumptions about the E-valued 1-form representing the class. First, we assume that the local section which it determines is the canonical lift of the real vector field associated to it; this is possible by Prop. 2.3 (c). This allows us to restrict our attention to the real part of the associated section. Secondly, we assume that it is equal to a standard form in an open tubular neighborhood U of the singular locus. This form will depend only on the value of the cohomology class when applied to cycles on the boundary tori of the manifold $M - U$. If the cohomolgy class is trivial on the boundary, the standard form will have compact support. More generally, we will give explicit representatives in U using cylindrical coordinates. (See (23) and (24) in Section 3). For now, it suffices to say that in U, the form will be of the kind described in Proposition 2.6; in particular, it will be closed, co-closed, and traceless. We will say that a form satisfying the above conditions is in *standard form.* Once we give the explicit local representatives, it will be immediate that each cohomology class has a representative in standard form. (See Lemma 3.3).

Our main analytic result is that in each cohomology class there is a unique representative which is closed, co-closed and traceless, and differs from any standard form by the derivative of an L^2 section.

Theorem 2.7 (Hodge Theorem for Cone-Manifolds). *Let $\overline{M}$ be a closed, orientable 3-dimensional hyperbolic cone-manifold and $M = \overline{M} - \Sigma$, where Σ is the singular locus. Given a cohomology class in $H^1(M; E)$, let $\tilde{\omega} \in \Omega^1(M; E)$ be a smooth, E-valued 1-form in standard form representing that class. There is a closed and co-closed representative ω in the same cohomology class, whose associated local section is the canonical lift of a divergence-free, harmonic vector field. There is a unique such representative satisfying the condition that $\tilde{\omega} - \omega = ds$ where s is a globally defined L^2 section of E.*

Remark 1. A 1-parameter family of complete hyperbolic structures determines a vector field on hyperbolic space which often (e.g. for closed surfaces) extends continuously to a vector field v_∞ on the sphere at infinity. On the interior of hyperbolic space there is a new vector field, called the *visual average* of v_∞, which determines an equivalent infinitesimal variation of structure. (See [24, Chap. 11], [22] and [1].) It is canonically defined, depending only on the vector field at infinity; hence, it is invariant under infinitesimal change of co-ordinates with compact support on the underlying manifold. The induced infinitesimal change of metric η coming from such a vector field has exactly the same local properties as those in Proposition 2.6. Our choice of Hodge representatives can be viewed as a substitute in the cone-manifold setting, where there is no nice sphere at infinity, for this canonical representative of the variation of structure.

Remark 2. It should be noted that the closed and co-closed 1-form guaranteed by the theorem above is not necessarily in L^2 itself, even if the standard form $\tilde{\omega}$ is in L^2. This will only be the case under the condition that the cone angles are at most 2π. A seemingly more natural Hodge theorem would be one in which the unique representative 1-form is in L^2. Such a theorem is also true; however, for cone angles greater than 2π, the Hodge representative will not necessarily be traceless; i.e., the deformation is not volume preserving. For a more detailed discussion of these issues and the role which they play in the deformation theory of surfaces, see [18].

Proof. Consider the globally defined section $\delta\tilde{\omega}$. We want to solve the equation $\Delta s = \delta\tilde{\omega}$ for a globally defined section s. Then we can define ω by $\omega = \tilde{\omega} - ds$, and ω will be closed and co-closed. By restricting the domain of the Laplacian carefully, we can insure that ω has the required properties.

Since, by hypothesis, $\tilde{\omega}$ is in standard form, $\delta\tilde{\omega}$ has compact support; in particular, it is in L^2. Being in standard form also means that the local section determined by $\tilde{\omega}$ is the canonical lift of its real part w, say, $\tilde{\omega} = d(w - i\ curl\ w)$. Then $\delta\tilde{\omega} = \delta d(w - i\ curl\ w) = \Delta(w - i\ curl\ w) = \Delta w - i\ curl\ \Delta w$, since *curl* and Δ (on local vector fields) commute. It follows that $\delta\tilde{\omega}$ is the canonical lift of its real part. Thus, it suffices to solve $\Delta v = \phi$, where ϕ is the real part of $\delta\tilde{\omega}$, for a globally defined vector field v.

It is easiest to describe the domain of the Laplacian on (globally defined) vector fields, by considering the dual 1-forms as before. Thus, if ζ is the dual to ϕ then, by equation (11), we want to find a 1-form τ satisfying

$$(\hat{\Delta} + 4)\tau = \zeta.$$

Note that ζ is C^∞ with compact support, so it is in L^2.

To describe the domain of $\hat{\Delta}$, first assume that τ is C^∞. Then the conditions are that $\tau, \hat{d}\tau, \hat{\delta}\tau, \hat{d}\hat{\delta}\tau, \hat{\delta}\hat{d}\tau$ are all in L^2. We then take the closure, in the L^2 sense, of these conditions. With this domain, the operator $(\hat{\Delta}+4)$ is a self-adjoint elliptic operator with trivial kernel (see Theorem 5.4). The domain is essentially the same as that considered by Cheeger [6] for manifolds with conical metrics and the proof of the properties of the operator are similar. Since our metrics are not conical in his sense, but rather asymptotically like a cone crossed with a circle, we will give the proof in an appendix.

Assuming this result, we can solve the above equation uniquely with τ in our domain. By the usual regularity theory for elliptic operators, τ will be C^∞ since ζ is C^∞. (See Gaffney [10], [11].)

We now let $\omega = \tilde{\omega} - ds$, where s is the canonical lift of the vector field v which is dual to τ. It is closed and co-closed by construction. Furthermore, since τ and $\hat{d}\tau$ are in L^2, v and $curl\ v$ (which is dual to $\frac{1}{2} * \hat{d}\tau$) are in L^2. Thus $s = v - i\ curl\ v$ is in L^2 as desired.

Finally, to see that ω is traceless, recall that the trace of $\tilde{\omega}$ has compact support by definition of "standard". Since $\hat{\delta}\tau$ and $\hat{d}\hat{\delta}\tau$ are in L^2, $div\ v$ and its derivative are in L^2. Thus, the trace of ω and its derivative are in L^2. However, since ω is closed and co-closed, its trace satisfies

$$\hat{\Delta} tr(\omega) = -4\ tr(\omega),$$

by equation (13). (Recall that the trace is just the globally defined function which is the divergence of the local vector fields determined by the E-valued 1-form.) As with 1-forms, $\hat{\Delta}$ on functions is self-adjoint with non-negative spectrum, if the domain is restricted to functions which together with their derivatives and Laplacians are in L^2. Thus, $tr(\omega) = 0$. q.e.d.

3. Asymptotic behavior of harmonic forms near singular locus

In this section we study the asymptotic behavior near the singular locus of the harmonic forms given by the Hodge theorem for hyperbolic cone-manifolds. Using this we will be able to analyze the boundary term in formula (5) of Section 1 and prove Theorem 1.1.

We assume that the form $\omega \in \Omega^1(M; E)$ representing the cohomology class is of the form guaranteed by the Hodge Theorem 2.7. This means that $\omega = \eta + i * D\eta$, where both η and $*D\eta$ are traceless and symmetric, representing the strains of a locally defined vector field and of minus the *curl* of the vector field, respectively. We want to study the behavior of the boundary term arising in (5) when we remove a tubular neighbourhood of radius r about the singular locus and let r approach 0. To do this, we first investigate the growth rate of the forms η and $*D\eta$ near the singular locus.

The behavior of the forms η and $*D\eta$ in a tubular neighborhood of the singular locus is determined by our choice of the domain of the Laplacian in the proof of the Hodge Theorem 2.7. We began with a form $\tilde{\omega}$ which equals a multiple of a standard closed and co-closed form (whose explicit description is given in (24) below) in a neighborhood of the singular locus. In particular, $\delta\tilde{\omega}$ has compact support. The form ω given by the Hodge theorem equals $\tilde{\omega} - ds$ where s is a globally defined L^2 section satisfying $\Delta s = \delta\tilde{\omega}$. Thus, in a neighborhood of the singular locus, ω equals ds plus a multiple of (24), where $\Delta s = 0$ and s is in our chosen domain.

The section s is the canonical lift $v - i\ curl\ v$ of a globally defined harmonic vector field v. The domain of the Laplacian acting on sections was given in terms of the 1-form τ dual to v. For smooth forms the condition is that $\tau, \hat{d}\tau, \hat{\delta}\tau, \hat{d}\hat{\delta}\tau, \hat{\delta}\hat{d}\tau$ are all in L^2. The equation $\Delta v = 0$ is equivalent (by (11)) to $\hat{\Delta}\tau = -4\,\tau$. Thus we will analyze the asymptotic behavior near the singular locus of forms τ satisfying this equation, under these L^2 constraints. We will do this by separation of variables, essentially expanding τ into a Fourier series.

A sufficiently small neighborhood of the singular locus will be mapped by the developing map into a neighborhood in $\mathbf{H}^3$ of a geodesic. If we use cylindrical coordinates, (r, θ, z), the hyperbolic metric is $dr^2 + \sinh^2 r\, d\theta^2 + \cosh^2 r\, dz^2$, where the angle θ is defined modulo the cone angle α. We denote the moving co-frame adapted to this coordinate system by $(\omega_1, \omega_2, \omega_3) = (dr, \sinh r\, d\theta, \cosh r\, dz)$.

Consider a real-valued 1-form, τ, defined in this neighborhood. If we express it in these coordinates as

$$\tau = f(r,\theta,z)\omega_1 + g(r,\theta,z)\omega_2 + h(r,\theta,z)\omega_3,$$

then, by explicit computation, we get the following expressions for the ω_i components of $(\hat{\Delta} + 2)\tau$, for $i = 1, 2, 3$ respectively:

$$\begin{aligned}-f_{rr} - (tanh + coth)f_r + (tanh^2 + coth^2)f\ - csch^2\ f_{\theta\theta} - sech^2\ f_{zz}\\ +2csch\ coth\ g_\theta + 2sech\ tanh\ h_z\end{aligned}$$

$$-g_{rr} - (tanh + coth)g_r + coth^2\ g - csch^2\ g_{\theta\theta} \\ -sech^2\ g_{zz} - 2csch\ coth\ f_\theta$$

$$-h_{rr} - (tanh + coth)h_r + tanh^2\ h - csch^2\ h_{\theta\theta} \\ -sech^2\ h_{zz} - 2sech\ tanh\ f_z.$$

(Subscripts denote derivatives with respect to a variable and "*tanh*" is short for "$\tanh r$", etc.)

Since τ comes from a neighborhood of the singular locus on the cone-manifold it will satisfy equivariance properties, depending on the shape of the neighborhood. If the cone angle equals α then $\tau(r, \theta + \alpha, z) = \tau(r, \theta, z)$, and if the singular locus has length ℓ, it will further satisfy $\tau(r, \theta, z + \ell) = \tau(r, \theta + t, z)$, where t measures the twist in the normal direction along the singular locus. (The complex number $\ell + it$ is called the *complex length* of the singular locus.)

Because of the decomposition of the Laplacian in a neighborhood of the singular locus, we can use separation of variables, assuming that $f(r, \theta, z)$ equals a function $f(r)$ times a function on the torus which is the boundary of the tubular neighborhood of the singular locus. Similarly for the other functions. It suffices further to decompose the functions on the torus into eigenfunctions of the Laplacian, which are of the forms $\cos(a\theta + bz)$ and $\sin(a\theta + bz)$, where a and b denote the quantities $\frac{2\pi n}{\alpha}$ and $\frac{(2\pi m + at)}{\ell}$, for $n, m \in \mathbf{Z}$. We now assume that such a 1-form τ is an *eigenform* of the Laplacian. Then, from the explicit expression for the Laplacian, we see that τ must be of the following type (or the same form with sin and cos interchanged):

$$(20)\quad \tau = f(r)\cos(a\theta + bz)\omega_1 + g(r)\sin(a\theta + bz)\omega_2 + h(r)\sin(a\theta + bz)\omega_3.$$

Then the ω_i coefficients of $(\hat{\Delta} + 2)\tau$ are given by the following (times $\cos(a\theta + bz)$ when $i = 1$ and times $\sin(a\theta + bz)$ when $i = 2, 3$):

$$-f'' - (tanh + coth)f' + (tanh^2 + coth^2 + a^2 csch^2 + b^2 sech^2)f \\ +(2a\ csch\ coth)g + (2b\ sech\ tanh)h$$

$$\begin{aligned}(21)\qquad &-g'' - (tanh + coth)g' + (coth^2 + a^2csch^2 + b^2sech^2)g \\ &\qquad + (2a\ csch\ coth)f \\ &-h'' - (tanh + coth)h' + (tanh^2 + a^2csch^2 + b^2sech^2)h \\ &\qquad + (2b\ sech\ tanh)f.\end{aligned}$$

Expanding the functions in r as Taylor series,

$$\begin{aligned}f(r) &= r^k(f_0 + f_1r + \cdots), \\ g(r) &= r^k(g_0 + g_1r + \cdots), \\ h(r) &= r^k(h_0 + h_1r + \cdots),\end{aligned}$$

we get linear equations for the coefficients. Solving the linear equations, we get the following lemma whose proof we leave to the reader.

Lemma 3.1. *If τ is an eigenform of the Laplacian of the form (20), then the indicial roots (leading powers) for τ are $k = 1 \pm a, \pm a, -1 \pm a$. The leading coefficients for $(f(r), g(r), h(r))$ are multiples of $(1, \pm 1, 0)$, $(0, 0, \pm 1), (1, \mp 1, 0)$ respectively. (Without loss of generality we will always assume that $a \geq 0$.)*

Note that these are independent of b and the eigenvalue of the eigenform τ; they depend only on the cone angle and the frequency n in the θ direction. A general solution will be a linear combination of these six solutions. As usual, it may be necessary to introduce a logarithmic factor when the indicial roots differ by a positive integer (which is always the case here), and when there are repeated roots. The analysis below involves only the leading coefficients, so it will only be affected when the logarithmic term is in the leading power; i.e., when there are repeated roots. This occurs only when $a = 0$, $a = \frac{1}{2}$, and $a = 1$. These cases will be analyzed separately below.

We will first show that our choice of domain for the Laplacian requires that only the "positive" roots, $k = 1 + a, a, -1 + a$, appear.

Proposition 3.2. *Let τ be an eigenform for the Laplacian on real-valued 1-forms of the type (20), defined on a neighborhood of a component of the singular locus of M. Then τ has the property that $\tau, \hat{d}\tau, \hat{\delta}\tau, \hat{d}\hat{\delta}\tau, \hat{\delta}\hat{d}\tau$ are all in L^2 in a neighborhood of the singular locus if and only if τ has leading powers r^k with no leading logarithmic term, where*

$k = 1 + a, a, -1 + a$ if $a \neq 0$, and $k = 0, 1$ if $a = 0$.

Proof. First assume that $a \neq 0, \frac{1}{2}, 1$. Then the leading terms will be powers of r with no leading logarithmic terms. Note that a form whose pointwise norm is $\sim r^k$ is in L^2 precisely when $k > -1$. Since we are assuming that $a \geq 0$, we see that $k = -1 - a$ is never allowed and $k = 1 + a$ is always allowed.

The divergence of the vector field dual to τ, is

$$\hat{\delta}\tau = (f' + (tanh + coth)f + a\ csch\ g + b\ sech\ h)\cos(a\theta + bz).$$

Its leading coefficient, of order $k - 1$, equals $(1 + k)f_0 + a\ g_0$ which is non-zero only when $k = 1 \pm a$. In this case it equals $(2 \pm 2a)f_0$ which is non-zero unless $a = 1$ and $k = 0$ (which is considered later). Similarly, the coefficients of r^{k-2} in $\hat{d}\hat{\delta}\tau$ are

$$((k-1)((1+k)f_0 + a\ g_0), -a((1+k)f_0 + a\ g_0), 0),$$

which is non-zero as long as $a \neq 0$. It follows that, if $\hat{d}\hat{\delta}\tau$ is in L^2, then $k \neq 1 - a$. Conversely, if $k = -1 + a$ or a, then $\hat{\delta}\tau$ is still in L^2 since the $k - 1$ coefficient vanishes. Since $\hat{d}\hat{\delta}\tau$ is again an eigenform for the Laplacian, its leading power must be at least $-1+a$ (rather than $-2+a$, which is not an indicial root). Thus, if $k = -1 + a$ or a, $\hat{d}\hat{\delta}\tau$ will still be in L^2.

We also compute that the ω_i coefficients of $*\hat{d}\tau$ are given by the following (times the respective trigonometric function):

$$\begin{aligned} &- b\ sech\ g + a\ csch\ h \\ &- (h' + tanh\ h + b\ sech\ f) \\ &g' + coth\ g + a\ csch\ f. \end{aligned} \tag{22}$$

The coefficients of r^{k-1} are $(ah_0, -kh_0, (k+1)g_0 + af_0)$. The first two are non-zero when $k = \pm a$ so, if we assume that $\hat{d}\tau$ is in L^2, then $k \neq -a$. On the other hand, when $k = -1 + a$, the coefficients are all zero, so, even in this case, $\hat{d}\tau$ will be in L^2. Since $*\hat{d}\tau$ is again an eigenform for the Laplacian, the same analysis holds for it and we conclude that $\hat{\delta}\hat{d}\tau = *\hat{d}(*\hat{d}\tau)$ will be in L^2. Thus, for $k = -1 + a$ or a, both $\hat{d}\tau$ and $\hat{\delta}\hat{d}\tau$ will be in L^2.

The case of repeated roots is only slightly more complicated.

The vectors of leading coefficients for $k = a$ and $k = 1-a$ are linearly independent, so no logarithmic term is necessary for the repeated root that occurs when $a = \frac{1}{2}$. The argument above goes through unchanged.

When $a = 1$, then solutions for $k = -1 + a$ and $k = 1 - a$ coalesce and it is necessary to introduce a logarithmic solution. For both solutions the leading coefficients are $(1, -1, 0)$. From the formula above for the divergence, it follows that the divergence of the solution with the logarithmic term is not in L^2, so this case is not allowed (and plays the role of the $1 - a$ solution). The argument that the other solution is in the domain is the same as that for the general $-1 + a$ case above.

The case $a = 0$ involves three repeated roots. If $k = \pm a = 0$, then it is necessary to introduce a logarithmic solution in order to a get a 2-dimensional solution space. However, from (22) above we see that the ω_2 coordinate of $*\hat{d}\tau$ will have leading power -1, and hence this case is not in our domain. This plays the role of the negative solution above. The positive solution, without the logarithmic term, is seen to be in our domain by the previous analysis. Since we will include this case in the analysis below, we omit the details. If $k = -1$ or $k = 1$, the solution space is already two-dimensional, spanned by $(1, 0, 0)$ and $(0, 1, 0)$ in both cases. Clearly the case $k = -1$ is not in our domain. To check that the entire solution space is in the domain for $k = 1$, it suffices to check that $\hat{d}\hat{\delta}\tau$ and $\hat{\delta}\hat{d}\tau$ are. Again, we omit the details. However, we note that, in a sense, the $1 - a$ solution has been allowed in this case. The significance of this fact is unclear. q.e.d.

Remark. At this stage, we can see why the size of the cone angle, α, might play a role in our analysis. It is related to the growth rates of the forms via the quantity, a, which equals $\frac{2\pi n}{\alpha}$. If $n \neq 0$, a will be ≥ 1 precisely when $\alpha \leq 2\pi$. For the eigenforms above which are in our domain, when either $a = 0$ or $a \geq 1$, the pointwise norms of τ and $d\tau$ (and hence of v and *curl* v) are bounded. This plays a key role in the proof of Theorem 1.1 later in this section. Even more significantly, this angle restriction is necessary to ensure that the Hodge representative $\omega = \eta + i \ * D\eta$ from Theorem 2.7 will be in L^2 when the cone angle is unchanged. The reason for this is suggested by the above computation because η and $*D\eta$ differ from the standard form in a neighborhood of the singular locus by the strains of vector fields v and *curl* v respectively. Since these are dual to τ and $d\tau$, their Fourier components will have growth rates of, at most, r^{-1+a} if $a > 0$. This means that the strains will generally have growth rates of, at most, r^{-2+a}. This is in L^2 only if $a > 1$. This illustrates the fact that, while our domain for the Laplacian ensures that some combinations of the covariant derivatives of the vector fields are in L^2, it doesn't control all of them. For large cone angles, no self-adjoint extension can do this. This is one of the main subtleties of this kind of singularity.

We will now go back to our analysis of the closed and co-closed E-valued 1-forms which, by Theorem 2.7, will represent the cohomology classes in $H^1(M;E)$. In a tubular neighborhood U of the singular locus, these will differ from ds, where s is a global section of E which is in the domain analyzed above, by certain standard forms. The infinitesimal change of metric induced by the standard forms is invariant under the entire stabilizer of the singular axis in isom$\mathbf{H}^3$ and preserves the orthogonality of the vectors in the cylindrical orthonormal frame in a neighborhood of the axis. Thus, when written in terms of cylindrical co-ordinates, the real part of the E-valued 1-form depends only on the radial distance from the axis. Considered as an element of $Hom(TM, TM)$ it will be diagonal in these co-ordinates. Explicit forms which induce all possible infinitesimal changes in the holonomy in a neighborhood of the singular axis are given below.

Since the tubular neighborhood U of a component of the singular locus is diffeomorphic to a torus T^2 cross an open interval, $\pi_1(U) \cong \pi_1(T^2) \cong \mathbf{Z} \oplus \mathbf{Z}$ and the cohomology class is determined by its value on any two generators of the fundamental group of the torus. We take as generators the meridian, which is the class which wraps around the singular locus once and bounds a singular disk with cone angle α, and a longitude which intersects the meridian once.

To define our standard forms, we use the cylindrical coordinates on U defined above, and we denote by e_1, e_2, e_3 the orthonormal frame in U dual to the co-frame $\omega_1, \omega_2, \omega_3$ defined above. In particular, e_2 is tangent to the meridian and e_3 is tangent to the singular locus, which is homotopic in the cone-manifold to the longitude. We can interpret an E-valued 1-form as a complex-valued section of $TM \otimes T^*M \cong Hom(TM, TM)$. Then an element of $TM \otimes T^*M$ can be described as a matrix whose (i,j) coordinate is the coefficient of $e_i \otimes \omega_j$.

The form in equation (23) below is a "standard" closed and co-closed (non-L^2) form which represents an infinitesimal deformation which decreases the cone angle but does not change the real part of the complex length of the meridian. It preserves the property that the meridian is elliptic and, hence, that there is a cone-manifold structure.

$$\begin{bmatrix} \frac{-1}{\cosh^2(r)\sinh^2(r)} & 0 & 0 \\ 0 & \frac{1}{\sinh^2(r)} & \frac{-i}{\cosh(r)\sinh(r)} \\ 0 & \frac{-i}{\cosh(r)\sinh(r)} & \frac{-1}{\cosh^2(r)} \end{bmatrix} \tag{23}$$

The form in equation (24) below is a "standard" closed and co-closed, L^2 form which stretches the singular locus, but leaves the holonomy of the meridian (hence the cone angle) unchanged.

$$\begin{bmatrix} \frac{-1}{\cosh^2(r)} & 0 & 0 \\ 0 & -1 & \frac{-i\sinh(r)}{\cosh(r)} \\ 0 & \frac{-i\sinh(r)}{\cosh(r)} & \frac{\cosh(r)^2+1}{\cosh(r)^2} \end{bmatrix} \tag{24}$$

We now explain how to find the infinitesimal change in holonomy representation produced by a closed E-valued 1-form ω on U. The hyperbolic structure on U determines a holonomy representation $\rho : \pi_1(U) \cong \mathbf{Z} \oplus \mathbf{Z} \to G = \text{isom}\mathbf{H}^3$, which is defined up to conjugacy. The image of ρ preserves a geodesic axis A in $\mathbf{H}^3$, namely the image of the singular locus under a developing map. Hence, ρ is determined up to conjugacy by the *complex lengths* of two generators. (The complex length of a group element is the translation distance along its axis plus i times its rotation angle around the axis. See Section 4, for further discussion.)

The form ω determines an infinitesimal deformation of the holonomy representation given by the cocycle $z : \pi_1(U) \to \mathcal{G}$ obtained by integrating ω around loops based at a point in U, after identifying each fiber of E along the loop with the Lie algebra $\mathcal{G}$. Let $\mathcal{A}$ denote the subspace of $\mathcal{G}$ consisting of infinitesimal translations and rotations about the singular axis A. Choose a point x on A and use the metric at that point to define the subspace $\mathcal{A}^\perp$ of $\mathcal{G}$ orthogonal to $\mathcal{A}$. $\mathcal{A}^\perp$ is generated by translations and rotations with axes, based at x, perpendicular to the axis A. So elements of $\mathcal{A}^\perp$ give infinitesimal deformations which move the invariant axis, but do not change any complex lengths. It follows that the infinitesimal change in the complex length of a group element γ is precisely the orthogonal projection of $z(\gamma)$ onto the subspace $\mathcal{A}$ of $\mathcal{G}$.

In order to compute $z(\gamma)$ by integration along a curve, $\mathcal{G}$ is identified with the fiber at a general point in U by parallel translation along an arc from x. Note that $\mathcal{A}$ is elementwise invariant and $\mathcal{A}^\perp$ is setwise invariant under translation and rotation along A, so orthogonal projection and the $\mathcal{A}$ component of an element of the fiber are independent of $x \in A$. Similarly, since the holonomy image, $\rho(\pi_1(U))$, is contained in the group of translations and rotations around A, it commutes with projection onto $\mathcal{A}$. Thus, the $\mathcal{A}$ component of an element of the fiber over a point in U is independent of the path from x. The infinitesimal change in the complex length of γ is obtained by integrating the $\mathcal{A}$ component of ω around any representative in U of γ.

In terms of the cylindrical coordinates used above, the subspace $\mathcal{A}$ of $\mathcal{G}$ consists of complex multiples of t_A, where t_A represents the infinitesimal pure translation in the e_3 direction, along the singular axis A where $r = 0$. The $\mathcal{A}$ components of the elements $\{e_1, e_2, e_3\}$ at a point with cylindrical coordinates (r, θ, z) are seen to be $\{0,\ -i\sinh r,\ \cosh r\}$ times t_A, respectively. The $\mathcal{A}$ component of ω is the **C**-valued 1-form $\omega_{\mathcal{A}} = v_A \cdot \omega$ obtained by taking a pointwise inner product with the complex-valued vector field $v_A = \cosh r\, e_3 - i \sinh r\, e_2$ using the inner product on M. (Note that this product satisfies $ia \cdot ib = -a \cdot b$.) So, explicitly, the derivative of complex length of γ is:

$$\int_\gamma (\cosh r\, e_3 \cdot \omega - i \sinh r\, e_2 \cdot \omega).$$

To see the effect of the form defined by (23) on the holonomy of the meridian, it suffices to integrate the $\mathcal{A}$-part of the form around a meridian at radius r. This integration is in the e_2 direction, so only involves the second column of (23). From the previous discussion, we see that the derivative of the complex length of the meridian is

$$\int_0^\alpha (\cosh r\, e_3 - i \sinh r\, e_2) \cdot \left(\frac{1}{\sinh^2 r} e_2 - \frac{i}{\cosh r \sinh r} e_3\right) \sinh r\, d\theta = -2i\alpha.$$

In particular, the derivative of the cone angle is -2α. Also, the complex length $\mathcal{L}$ of the longitude has derivative $-2\mathcal{L}$. In particular, the length $\ell = Re(\mathcal{L})$ of the singular locus has derivative

$$\int_0^\ell (\cosh r\, e_3 - i \sinh r\, e_2) \cdot \left(\frac{-i}{\sinh r \cosh r} e_2 - \frac{1}{\cosh^2 r} e_3\right) \cosh r\, dz = -2\ell.$$

Similar calculations show that for the 1-form given by (24), the derivative of the complex length of the longitude is 2ℓ, while the complex length of the meridian has derivative zero.

Lemma 3.3. *Given any closed $\phi \in \Omega^1(M; E)$ there is a cohomologous form ω which equals a complex linear combination of the forms above in a neighborhood of the singular locus. Thus ω will be in standard form. It will be in L^2 if and only if the corresponding infinitesimal deformation leaves the complex length (or trace) of the meridian unchanged. If it is in L^2, it will actually be bounded.*

Proof. The cohomology group $H^1(T^2; E)$ is easily seen to have complex dimension 2 (see the proof of Theorem 4.4, Section 4), parametrized by the infinitesimal changes in the complex lengths (or traces) of the longitude and the meridian. By taking a complex linear combination of the forms above, we can define a form which effects any given change in these generators in the neighborhood of the singular locus. Thus, a general form will equal this linear combination plus an exact form. We can subtract off this exact form in our neighborhood and then damp it off outside of the neighborhood to get a cohomologous form which is now standard in the neighborhood of the singular locus.

The linear combination of the above forms will include (23) if and only if the trace of the meridian is changed. Since (23) is not in L^2 and (24) is bounded, the remainder of the lemma follows. q.e.d.

Note that when the cone angle is unchanged, the standard form will be bounded near the singular locus so the growth rates of the closed and co-closed forms representing the cohomology class in $H^1(M; E)$ will be determined, up to a bounded amount, by the derivative ds of a section s which is harmonic and is the domain studied above.

We also will use the following observation.

Lemma 3.4. *Let ω be an E-valued form which is L^2 in a neighborhood of the singular locus. Then ω does not change the complex length of a meridian around the singular locus.*

Proof. If the change in holonomy of a meridian is non-zero, then $\int_{S_r} \omega_A = c \neq 0$, for each meridian circle S_r at radius r. Then, using the Schwarz inequality for vectors and for functions on S_r, we see that

$$\begin{aligned} 0 < c^2 = \left(\int_{S_r} t_A \cdot \omega(e_2)\right)^2 &\leq \left(\int_{S_r} |t_A|\,|\omega(e_2)|\right)^2 \\ &\leq \left(\int_{S_r} |t_A|^2\right)\left(\int_{S_r} |\omega|^2\right) \\ &\leq c_1 r\left(\int_{S_r} |\omega|^2\right), \end{aligned}$$

where $|\ |$ denotes the pointwise norm of a form. It follows that

$$\int_{T_r} |\omega|^2 \geq \frac{c_2}{r},$$

where T_r denotes the torus at radius r from the singular locus. Now choose $0 < r_0 < r_1$ such that the cylindrical region $r_0 \leq r \leq r_1$ is contained in the tubular neighborhood of the singular locus. Then the L^2 norm of ω in this region is

$$\int_{r_0}^{r_1} \left(\int_{T_r} |\omega|^2 \right) dr \geq \int_{r_0}^{r_1} \frac{c_2}{r} dr.$$

But the last integral diverges as $r_0 \to 0$, so the form ω is not in L^2 in a neighborhood of the singular locus. q.e.d.

We now have enough information to prove the main result of this paper, Theorem 1.1, which we now recall.

Theorem 1.1. *Let $\overline{M}$ be a finite-volume, 3-dimensional hyperbolic cone-manifold, whose singular locus Σ is a knot or link. Let M denote the open, incomplete manifold $\overline{M} - \Sigma$. If all cone angles along Σ are at most 2π, then every closed L^2 form in $\Omega^1(M; E)$ represents the trivial cohomology class in $H^1(M; E)$.*

Proof. Let $\phi \in \Omega^1(M; E)$ be a closed, E-valued 1-form which is in L^2. Then Lemma 3.4 shows that ϕ does not change the trace of the meridian. So, by Lemma 3.3, ϕ is cohomologous to a form which is in standard form and is bounded. We denote this new form by ϕ as well.

Using Proposition 2.6 and Theorem 2.7, the Hodge theorem for cone-manifolds, we can find a closed and co-closed representative in the cohomology class which is volume preserving and is of the form $\omega = \eta + i *D\eta$. Here η and $*D\eta$ are the strains of a locally defined vector field and the negative of its curl, respectively. In a neighborhood of the singular locus the local section determined by ω is the canonical lift of a globally defined vector field plus the local section determined by a complex multiple of (24). Since (24) is bounded, the growth of η and $*D\eta$ near the singular locus are determined by the strains of the globally defined vector field and its curl. We will denote by v this globally defined vector field; it and its curl are in L^2 by Theorem 2.7.

We now use formula (5), obtained by integrating by parts, to get

$$0 = (D\omega, D\omega) + (D^*\omega, D^*\omega) + (H\omega, \omega) + B,$$

where B is the boundary term

$$B = \int_{\partial N} -(*T\omega \wedge \omega + T^*\omega \wedge *\omega),$$

and N is the cone-manifold $\overline{M}$ minus a tubular neighbourhood of the singular locus. But, since ω is closed and co-closed, $D\omega = -T\omega$ and $D^*\omega = -T^*\omega$. Furthermore, since the real and imaginary parts of ω are traceless and symmetric, $T^*\omega = 0$, $*T\omega = i\ \omega$, and $H\omega = \omega$. (See Lemma 2.5.)

Thus, we get

$$0 = 2\ ||\omega||^2 + B,$$

where the integrand of B is $-*T\omega \wedge \omega = *D\omega \wedge \omega$. To prove that $\omega = 0$, it suffices to prove that the boundary term goes to zero as $r \to 0$ on the tori T_r which are the boundary of a tubular neighborhood of radius r around the singular locus. In fact, it suffices to find a sequence of radii tending to zero for which the boundary term goes to zero.

We first note that, since $\omega = \eta + i\ *D\eta$ and $*D*D\eta = -\eta$, the integrand for B, $*D\omega \wedge \omega$, equals $2\ *D\eta \wedge \eta$. (Here we use the facts that $a \wedge b = ia \wedge ib$ and $a \wedge ib = 0$ for $\mathcal{P}$-valued 1-forms a and b.) To get a more explicit expression for this integrand we interpret an element of $TM \otimes T^*M$ as a matrix whose (i,j) entry is the coefficient of $e_i \otimes \omega_j$. Then, explicitly, the integrand is $\eta(e_2) \cdot *D\eta(e_3) - \eta(e_3) \cdot *D\eta(e_2)$, where $\cdot$ denotes the inner product on M. Now

$$\left| \int_{\partial N} \eta(e_2) \cdot *D\eta(e_3) - \eta(e_3) \cdot *D\eta(e_2) \right| \le \int_{\partial N} |\eta(e_2) \cdot *D\eta(e_3)| + \int_{\partial N} |\eta(e_3) \cdot *D\eta(e_2)|.$$

We analyze the second term in this sum; the first is done similarly. By the Schwarz inequality for the inner product of vectors and the Schwarz inequality for the L^2 inner product, this is

$$\le \int_{\partial N} |\eta(e_3)|\ |*D\eta(e_2)| \le ||\eta(e_3)||_r\ ||*D\eta(e_2)||_r.$$

Here we have used $|\eta(e_3)|$ to denote the (pointwise) norm of the vector $\eta(e_3)$, and $||\eta(e_3)||_r$ to denote the L^2-norm of the function $|\eta(e_3)|$ on the torus T_r distance r from the singular locus with its induced metric.

At this point, we must invoke the hypothesis that the cone angle is at most 2π. As we noted in the Remark after the proof of Proposition

3.2, this is necessary to ensure that η and $*D\eta$ are in L^2. What we noted there was that the growth rates of the covariant derivatives of the Fourier components of the vector fields v and *curl* v are at most r^{-2+a} and that $a > 1$ when the cone angle is smaller than 2π. (The case $a = 1$ is handled separately.) We will see in Corollary 3.6 below that this also implies that η and $*D\eta$ are themselves in L^2. For now, we assume that this is true.

Since $*D\eta$ is in L^2, so is $*D\eta(e_2)$. An easy estimate (see (26) in the appendix) shows that, for a sequence of distances r going to zero,

$$|| * D\eta(e_2)||_r = o(r^{-1/2}|\log r|^{-1/2}).$$

If we compare the L^2-norm of a function f on T_1 with that of the same function measured in the metric on T_r we see that $||f||_r \sim r^{1/2}||f||_1$. Thus, to prove that the boundary term goes to zero it suffices to show that $||\eta(e_3)||_1$ is bounded near the singular locus. To do this, we will show that, up to bounded terms, it behaves asymptotically like $||v||_1$ and $||curl\ v||_1$. (Recall that v is a globally defined vector field and that $str(v + v_0) = \eta$, where v_0 is the local vector field determined by a multiple of (24).) This is a power of r slower than one would expect; indeed, it is slower than the overall growth of $||\eta||_1$.

By subtracting a complex multiple of (24) from ω, we can obtain a closed form whose local section in a neighborhood of the singular locus is the canonical lift of v. Since (24) is bounded, it suffices to consider the case where $\omega = ds$, $s = v - i\ curl\ v$. From the decomposition $d = \nabla + T$, we see that $\eta(e_3)$ differs from $\nabla_{e_3} v$ by terms from *curl* v. (Recall that T is a purely algebraic operation.) But, if $v = f(r,\theta,z)e_1 + g(r,\theta,z)e_2 + h(r,\theta,z)e_3$, then

$$\nabla_{e_3}\ v = (sech\ f_z - tanh\ h)e_1 + (sech\ g_z)e_2 + (sech\ h_z + tanh\ f)e_3.$$

This follows immediately from the formulae $\nabla_{e_3} e_1 = tanh\ e_3$, $\nabla_{e_3} e_3 = -tanh\ e_1$, and $e_3 = sech\ \frac{\partial}{\partial z}$. It suffices to show that $||v||_1$, $||curl\ v||_1$, and $||\nabla_{e_3} v||_1$ are bounded in a neighborhood of the singular locus.

We first will do the case of v; the case of *curl* v is identical and we will indicate the changes needed for the case $\nabla_{e_3} v$. In a neighborhood of the singular locus, the vector field v satisfies $\Delta v = 0$ and is globally defined there. Above we analyzed the asymptotic behavior of the dual one-forms by using separation of variables and a Fourier decomposition on each of the T_r. Thus v is a sum of the form

$$v = \sum_{m,n} v_{m,n},$$

where each $v_{m,n}$ is dual to a 1-form of the form (20), with a and b denoting the quantities $\frac{2\pi n}{\alpha}$ and $\frac{(2\pi m+at)}{\ell}$, respectively, as usual, and possibly sin and cos interchanged. The analysis there showed that, with our choice of domain, the leading power was at most r^{-1+a}, if $a > 0$. In particular, these vector fields vanish at the singular locus as long as $a > 1$. Since $a = \frac{2\pi n}{\alpha}$, where the cone angle is α, this occurs whenever the cone angle is less than 2π and $a \neq 0$. This is the second point at which we need to impose the cone angle restriction in our proof of Theorem 1.1. We will analyze separately the special cases $a = 0, 1$, where the individual terms are bounded near the singular locus but need not vanish there.

Unfortunately, the fact that the individual terms go to zero in norm is not enough to conclude that the vector field is bounded as one approaches the singular locus. For, if the convergence to zero is erratic enough, there could be terms further and further out which grew before going to zero. The fact that the vector fields are harmonic gives the needed control to rule this out. Specifically, we have the following lemma:

Lemma 3.5. *If v is a harmonic vector field, then it satisfies*

$$\hat{\Delta}||v||^2 = -2(2||v||^2 + ||Dv||^2).$$

In particular, if v is non-zero and $||v||^2$ vanishes along the singular locus, then $||v(r)||_1^2$ is an increasing function of r.

Proof. For any vector field one has the equality,

$$\hat{\Delta}||v||^2 = 2(\langle \Delta_D v, v\rangle - ||Dv||^2).$$

(See [30] or [31].) But, since v is harmonic, we have $\Delta_D v = -2v$, proving the above formula. Integrating over the torus (with the metric from $r = 1$), we get that

$$-(||v(r)||_1^2)'' - (tanh + coth)(||v(r)||_1^2)' \leq -4||v(r)||_1^2.$$

This implies that there are no positive local maxima for the function $||v(r)||_1^2$. Since $||v(r)||_1^2$ vanishes at the singular locus and is non-negative, it must be increasing in the neighborhood of the singular locus where it is globally defined (assuming it is non-zero). q.e.d.

We use this lemma to show that the results we obtained for Fourier components, by the earlier analysis of growth rates, are true more generally.

Corollary 3.6. *If the cone angles are at most 2π, then $||v(r)||_1$, $||curl\ v(r)||_1$, and $||\nabla_{e_3} v(r)||_1$ are all bounded in a neighborhood of the singular locus. Furthermore, Dv and $D(curl\ v)$, hence η and $*D\eta$, are in L^2 under this angle constraint.*

Proof. We apply Lemma 3.5 (when $a \neq 0, 1$) to each of the terms in the sum $v = \Sigma\ v_{m,n}$ to conclude that each of the $||v_{m,n}||_1^2$ are increasing in r. Since, by Parseval's theorem, $||v(r)||_1^2 = \Sigma\ ||v_{m,n}(r)||_1^2$, this implies that $||v(r)||_1$ is bounded near the singular locus as desired. To apply the lemma to the special cases where $a = 0, 1$, it suffices to note that the derivative $(||v(r)||_1^2)'$ vanishes at the singular locus, so the norm squared still is non-decreasing in r near the singular locus.

The same argument applies to *curl* v since it is also harmonic and each term in its Fourier decomposition satisfies the same growth conditions from our previous analysis. Similarly, the vector field $\nabla_z v = f_z e_1 + g_z e_2 + h_z e_3$ is also harmonic since ∇_z commutes with Δ. Hence, its norm is also bounded near the singular locus. From the explicit formula for $\nabla_{e_3}\ v$ given above, it follows that this is also bounded in norm.

Since η and $*D\eta$ are the symmetric parts of Dv and $D(curl\ v)$, they will be in L^2 if Dv and $D(curl\ v)$ are. We show that Dv is in L^2; the argument for $D(curl\ v)$ is the same.

From the formula in Lemma 3.5, Dv will be in L^2 if the integral of $\hat{\Delta}||v||^2$ over a neighborhood of the singular locus is finite. Integration by parts shows that this is finite precisely when the integral of the normal derivative of $||v||^2$ over the torus at distance r from the singular locus is bounded as $r \to 0$. This will fail to be bounded only if there is a sequence of such tori where the integral of the normal derivative goes to $-\infty$. (By Lemma 3.5, $\hat{\Delta}||v||^2$ and, hence, the integral of the normal derivative are non-positive.) Since the area of the torus at distance r is $\sinh r\ \cosh r$ times a constant, if the integral of the normal derivative goes to $-\infty$ as $r \to 0$, then $-(||v(r)||_1^2)'$ times r goes to ∞. But then there is a constant K such that $-(||v(r)||_1^2)' > \frac{K}{r}$, for all sufficiently small r. This implies, by integrating with respect to r, that $||v(r)||_1^2$ is not bounded near the singular locus, a contradiction. q.e.d.

Corollary 3.6 contains the estimates necessary to conclude that the boundary term B goes to zero on a sequence of T_r with $r \to 0$. This implies that $||\omega||^2 = 0$ as desired.

This completes the proof of Theorem 1.1. q.e.d.

4. Cohomology, Poincaré duality and representations of 3-manifold groups

In this section, we give some general results relating the topology of representation spaces to cohomology groups. We also explain some implications of Poincaré duality for the study of representation spaces of 3-manifold groups. From these results and Theorem 1.1, we obtain proofs of the results on deformation spaces of hyperbolic cone-manifolds and hyperbolic Dehn surgery spaces mentioned in the introduction.

First we introduce some notation. Let Γ be a finitely presented group, let G be a Lie group and let $\mathcal{G}$ denote the Lie algebra of G. Then $R(\Gamma, G)$ will denote the space of all representations (i.e., homomorphisms) from Γ into G. There is a natural action of G on $R(\Gamma, G)$ by conjugation: $(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$ for $g \in G$ and $\gamma \in \Gamma$. If Γ has a presentation with n generators and m relations, then $R(\Gamma, G)$ can be identified with the analytic subset of G^n consisting of n-tuples of elements of G satisfying m equations given by the relations of Γ. If G is an algebraic group, then $R(\Gamma, G)$ is an algebraic variety.

If $\rho : \Gamma \to G$ is a representation, we write $Ad\rho$ to denote the Γ-module $\mathcal{G}$ with the Γ action $Ad \circ \rho$, where Ad denotes the adjoint action of G on its Lie algebra $\mathcal{G}$. Let $Z^1(\Gamma; Ad\rho)$ denote the space of 1-cocycles of Γ with coefficients in the module $Ad\rho$, consisting of all maps $z : \Gamma \to \mathcal{G}$ satisfying the cocycle condition

$$z(ab) = z(a) + a \cdot z(b) = z(a) + Ad\rho(a)z(b),$$

for all $a, b \in \Gamma$, and let $B^1(\Gamma; Ad\rho)$ denote the space of 1-coboundaries, *i.e.*, all maps $z : \Gamma \to \mathcal{G}$ of the form

$$z(a) = v - a \cdot v = v - Ad\rho(a)v,$$

where $v \in \mathcal{G}$. Then the cohomology group $H^1(\Gamma; Ad\rho)$ is $Z^1(\Gamma; Ad\rho)/B^1(\Gamma; Ad\rho)$, and the cohomology group $H^0(\Gamma; Ad\rho)$ is $\{v \in \mathcal{G} : v = Ad\rho(\gamma)\, v \text{ for all } \gamma \in \Gamma\}$. The higher cohomology groups $H^i(\Gamma; Ad\rho)$ can be defined in a similar way.

It is useful to note that these cohomology groups can also be defined in terms of singular, simplicial, Čech or de Rham cohomology. In particular, given a manifold N and a representation $\rho : \pi_1(N) \to G$, we can define $H^*(N; Ad\rho)$ and $H^*(N, \partial N; Ad\rho)$. Then $H^*(\pi_1(N); Ad\rho) = H^*(N; Ad\rho)$ if N is a $K(\pi, 1)$ space, and in general $H^i(N; Ad\rho) = H^i(\pi_1(N); Ad\rho)$ for $i = 0, 1$. Note also that $H^*(N; Ad\rho) = H^*(N; E)$, where E denotes the flat $\mathcal{G}$-bundle over N associated with $Ad \circ \rho$ considered in the previous sections.

These cohomology groups are related to the local structure of the representation space $R(\Gamma, G)$ by the following fundamental results of Weil [29]. In order to state them properly, we need to define a scheme, $\mathcal{R}$, associated to $R(\Gamma, G)$, using a presentation of Γ. (See [17] and the proof of Theorem 4.4 below.)

Proposition 4.1. *If $\rho : \Gamma \to G$ is a representation, then:*

(a) the Zariski tangent space $T\mathcal{R}_\rho$ of $\mathcal{R}$ at ρ can be identified with $Z^1(\Gamma; Ad\rho)$,

(b) the Zariski tangent space to the scheme associated to the orbit of ρ under conjugation by G can be identified with $B^1(\Gamma; Ad\rho)$,

(c) $H^0(\Gamma; Ad\rho)$ is the infinitesimal centralizer of the representation ρ, i.e., the Lie algebra of

$$Z(\rho) = \{g \in G : g\rho(\gamma)g^{-1} = \rho(\gamma) \text{ for all } \gamma \in \Gamma\}.$$

q.e.d.

In particular, $H^1(\Gamma, Ad\rho)$ can be thought of as the "Zariski tangent space" to the scheme associated to the space $R(\Gamma, G)/G$ of representations $\Gamma \to G$ up to conjugacy.

Next we discuss how Poincaré duality can be applied to the calculation of these cohomology groups. Let N be a compact oriented n-manifold with boundary ∂N, and let $\rho : \pi_1(N) \to G$ be a representation into a Lie group G defined over a field $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. Given a symmetric bilinear pairing pairing $\langle\ ,\ \rangle : \mathcal{G} \times \mathcal{G} \to \mathbf{F}$ invariant under the $Ad\rho$-action on $\mathcal{G}$, there is an induced cup product map

$$\cup : H^i(N; Ad\rho) \times H^{n-i}(N, \partial N; Ad\rho) \to \mathbf{F}.$$

If the coefficient pairing is non-degenerate, then we have Poincaré duality: the cup product $\cup$ gives a non-degenerate (hence perfect) pairing and a natural isomorphism:

$$PD : H^i(N; Ad\rho) \cong H^{n-i}(N, \partial N; Ad\rho)^*,$$

defined by the map $\alpha \mapsto \alpha \cup \cdot$.

The most important example of such a non-degenerate pairing is the Killing form on a semi-simple Lie algebra $\mathcal{G}$. For example, for the matrix group $G = SL_m(\mathbf{C})$ the complex Killing form is essentially the trace map

$$sl_m(\mathbf{C}) \times sl_m(\mathbf{C}) \to \mathbf{C}, \quad (A, B) \mapsto tr(AB).$$

Now let N be a compact orientable 3-manifold with boundary, and $\rho : \pi_1(N) \to G$ a representation into a semi-simple Lie group G. From the exact sequence of the pair $(N, \partial N)$ and Poincaré duality one obtains the commutative diagram:

$$\begin{array}{ccccc} H^1(N; Ad\rho) & \xrightarrow{\alpha} & H^1(\partial N; Ad\rho) & \xrightarrow{\beta} & H^2(N, \partial N; Ad\rho) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(N, \partial N; Ad\rho)^* & \xrightarrow{\beta^*} & H^1(\partial N; Ad\rho)^* & \xrightarrow{\alpha^*} & H^1(N; Ad\rho)^*, \end{array}$$

where the vertical maps are the isomorphisms given by Poincaré duality. It follows that $\operatorname{im}\beta \cong \operatorname{im}\alpha^*$ but $\operatorname{im}\alpha \cong \operatorname{im}\alpha^*$ and $\operatorname{im}\alpha = \ker\beta$, so

$$\dim H^1(\partial N; Ad\rho) = \dim\ker\beta + \dim\operatorname{im}\beta = 2\dim\operatorname{im}\alpha.$$

Hence, $\dim\operatorname{im}\alpha = \frac{1}{2}\dim H^1(\partial N; Ad\rho)$. Putting $x = \dim\ker\alpha$, we obtain

$$\dim H^1(N; Ad\rho) = \dim\operatorname{im}\alpha + \dim\ker\alpha = \frac{1}{2}\dim H^1(\partial N; Ad\rho) + x.$$

So $\dim H^1(N; Ad\rho) \geq \frac{1}{2}\dim H^1(\partial N; Ad\rho)$ with equality if $x = 0$. Note that

$$x = \dim\operatorname{im}(H^1(N, \partial N; Ad\rho) \to H^1(N; Ad\rho))$$

from the exact sequence of the pair $(N, \partial N)$. We summarize this calculation for future reference.

Lemma 4.2. *Let N be a compact orientable 3-manifold with boundary, and let $\rho : \pi_1(N) \to G$ be a representation into a semi-simple Lie group G. If the natural map*

$$H^1(N, \partial N; Ad\rho) \to H^1(N; Ad\rho)$$

is zero, then

(a) the map $H^1(N; Ad\rho) \to H^1(\partial N; Ad\rho)$ is injective, and

(b) $\dim H^1(N; Ad\rho) = \frac{1}{2} \dim H^1(\partial N; Ad\rho)$. q.e.d.

We will be interested primarily in the case where $G = PSL_2(\mathbf{C})$ is the group of orientation preserving isometries of 3-dimensional hyperbolic space, and Γ is the fundamental group of a compact orientable 3-manifold N whose boundary ∂N is a union of tori. In this situation Thurston proved the following fundamental result (see [24], [8]).

Theorem 4.3. *Let N be a compact, connected 3-manifold with non-empty boundary consisting of t incompressible tori $T_1, \dots, T_t$. Let $\rho : \pi_1(N) \to PSL_2(\mathbf{C})$ be an irreducible representation such that $\rho(\pi_1(T_i)) \neq 1$ for each i. Then each irreducible component of $R = R(\pi_1(N), PSL_2(\mathbf{C}))$ containing ρ is a complex variety of complex dimension at least $t + 3$.*

We now refine this result by showing that under an extra cohomological condition, R is a smooth manifold near ρ of complex dimension equal to $t + 3$.

Theorem 4.4. *Let N be a compact, connected 3-manifold with non-empty boundary consisting of t incompressible tori $T_1, \dots, T_t$. Let $\rho : \pi_1(N) \to PSL_2(\mathbf{C})$ be an irreducible representation such that $\rho(\pi_1(T_i)) \neq 1$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ for each i. If the natural map*

$$H^1(N, \partial N; Ad\rho) \to H^1(N; Ad\rho)$$

is zero, then the representation space $R(\pi_1(N), PSL_2(\mathbf{C}))$ is a smooth manifold near ρ, of complex dimension equal to $t + 3$.

Proof. First we use Poincaré duality and Lemma 4.2 to calculate $H^1(N; Ad\rho)$ and $Z^1(\pi_1(N); Ad\rho)$.

For a torus T, a generic representation

$$\phi : \pi_1(T) \to G = PSL_2(\mathbf{C})$$

has centralizer $Z(\phi)$ of complex dimension 1. In fact, this is true unless the image of ϕ equals $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ or is trivial. Hence $\dim_{\mathbf{C}} H^0(T; Ad\phi) = 1$ since this is just the Lie algebra of $Z(\phi)$. Now $\sum_i (-1)^i \dim H^i(T; Ad\phi) = \chi(T) \dim \mathcal{G} = 0$, so it follows from Poincaré duality that

$$\dim_{\mathbf{C}} H^0(T; Ad\phi) = \dim_{\mathbf{C}} H^2(T; Ad\phi) = 1,$$

and $\dim_{\mathbf{C}} H^1(T; Ad\phi) = 2$.

Hence $\dim_{\mathbf{C}} H^1(\partial N; Ad\rho) = 2t$, and since $H^1(N, \partial N) \to H^1(N)$ is zero by our hypothesis, we can apply Lemma 4.2 to conclude that $\dim_{\mathbf{C}} H^1(N; Ad\rho) = t$. Since ρ has a trivial centralizer, we have $Z^0(\pi_1(N); Ad\rho) = 0$ and $\dim_{\mathbf{C}} B^1(\pi_1(N); Ad\rho) = 3$. Hence

$$\dim_{\mathbf{C}} Z^1(\pi_1(N); Ad\rho) = t + 3.$$

Suppose that $\Gamma = \pi_1(N)$ has a presentation with n generators and m relations. Then $R = R(\pi_1(N), PSL_2(\mathbf{C}))$ can be constructed as the preimage of $(1, 1, \ldots, 1)$ under the map $f : G^n \to G^m$ which evaluates an n-tuple of group elements on a set of defining relations for Γ. This defines a scheme $\mathcal{R}$ whose Zariski tangent space $T\mathcal{R}_\rho = \ker df_\rho$ can be identified with the 1-cocycles $Z^1(\Gamma; Ad\rho)$ (see Lemma 4.1). In general, $\dim T\mathcal{R} \geq \dim R$, but by the calculation above and Thurston's result (Theorem 4.3) we have at ρ:

$$t + 3 = \dim T\mathcal{R} \geq \dim R \geq t + 3.$$

Hence $\dim T\mathcal{R} = \dim R = t + 3$ and it follows that ρ is smooth, reduced point of $\mathcal{R}$ (see [17]). In particular, R is a smooth manifold of complex dimension $t + 3$ near ρ. q.e.d.

Our next result gives a local parametrization of representations by holonomies of certain peripheral elements. It is convenient to state the result in terms of complex lengths rather than traces. We recall that the *complex length* $\mathcal{L}$ of an element of $SL_2(\mathbf{C})$ is related to its trace by $2\cosh(\mathcal{L}/2) = \text{trace}$. The map $\mathcal{L} \mapsto \text{trace}$ is a local diffeomorphism, except when $\mathit{trace} = \pm 2$.

We will be interested in the complex lengths of the elements corresponding to the boundary tori of a compact 3-manifold N. Since the torus has abelian fundamental group, if the complex length of one of its elements has non-zero real part, then the whole group will consist of isometries which preserve a single axis. The real part of the complex length describes the amount of translation along the axis, and the imaginary part describes the amount of rotation around the axis. For example, when there is a cone-manifold structure, this applies to the manifold N obtained by removing a neighbourhood of the singular locus. In this case the imaginary part of the complex length can be defined as a real number, not just modulo 2π. There is also a choice of sign, which can be made locally in a well-defined way. See [24] for details.

Theorem 4.5. *Let N be a compact, connected 3-manifold with non-empty boundary consisting of t incompressible tori $T_1, \ldots, T_t$. Let*

$\rho : \pi_1(N) \to PSL_2(\mathbf{C})$ be a representation such that each $\rho(\pi_1(T_i))$ is non-trivial and preserves a geodesic in $\mathbf{H}^3$. Let $\gamma_1 \subset T_1, \dots, \gamma_t \subset T_t$ be simple closed curves in ∂N and let $\gamma = \cup_i \gamma_i$. If the natural map $H^1(N, \gamma; Ad\rho) \to H^1(N; Ad\rho)$ is zero, then the representations near ρ are parametrized up to conjugacy by the complex lengths of their values on the homotopy classes $g_i = [\gamma_i] \in \pi_1(N)$. More precisely, the map

$$\mathcal{L} : R(\pi_1(N), PSL_2(\mathbf{C})) \to \mathbf{C}^t$$

defined by $\mathcal{L}(\phi) = (\mathcal{L}\phi(g_1), \dots, \mathcal{L}\phi(g_t))$ is a submersion from a neighborhood of ρ onto a neighborhood of $\mathcal{L}(\rho)$, whose preimages are precisely the conjugacy classes of representations near ρ.

Proof. Since the composition

$$H^1(N, \partial N) \to H^1(N, \gamma) \to H^1(N)$$

is zero by our hypothesis, Theorem 4.4 shows that the representation space $R(\pi_1(N), PSL_2(\mathbf{C}))$ is a smooth manifold near ρ, of complex dimension equal to $t + 3$, and $\dim_{\mathbf{C}} H^1(N; Ad\rho) = t$. From the exact sequence of the pair (N, γ) we see that $H^1(N; Ad\rho) \to H^1(\gamma; Ad\rho)$ is injective since the map

$$H^1(N, \gamma; Ad\rho) \to H^1(N; Ad\rho)$$

is zero by our assumption.

First assume that each $\phi(g_i) \neq 1$. Then it is clear that

$$H^0(\gamma_i; Ad\rho) = H^1(\gamma_i; Ad\rho) = \mathbf{C}.$$

Hence $\dim_{\mathbf{C}} H^1(\gamma; Ad\rho) = t$, and the map $H^1(N; Ad\rho) \to H^1(\gamma; Ad\rho)$ is an isomorphism. Now the derivative of the complex lengths gives a natural isomorphism between $H^1(\gamma; Ad\rho) \equiv \bigoplus_i H^1(\gamma_i; Ad\rho)$ and $\mathbf{C}^t$. This follows since elements in $PSL_2(\mathbf{C})$ preserving an axis are parametrized up to conjugacy by their complex lengths. It follows that $\mathcal{L}$ is a submersion, whose preimages are conjugacy classes of representations.

If some $\rho(g_i) = 1$, then the previous argument breaks down since elements of $PSL_2(\mathbf{C})$ near the identity are not determined uniquely up to conjugacy by their complex lengths. (Nearby parabolics have the same complex length as the identity.) However, in this case we can find another element $h_i \in \pi_1(T_i)$ such that $\rho(h_i) \neq 1$ since $\rho(\pi_1(T_i)) \neq 1$.

Now g_i and h_i commute, and $\rho(h_i)$ is non-parabolic. Hence $\phi(g_i)$ and $\phi(h_i)$ are both non-parabolic, for each representation

$$\phi : \pi_1(N) \to PSL_2(\mathbf{C})$$

near ρ. It follows that the elements $\phi(g_i)$ are parametrized locally up to conjugacy by their complex lengths. Thus we obtain the result by the same arguments as before. q.e.d.

For the case of a 3-dimensional hyperbolic cone-manifold $\overline{M}$, we apply these results where N is the complement of an open tubular neighborhood of the singular locus, and therefore ∂N is a union of tori $T_1, \ldots, T_t$. First we collect some well-known facts about the holonomy representation of N.

Lemma 4.6. *Let $\overline{M}$ be a 3-dimensional hyperbolic cone-manifold of finite volume, whose singular locus Σ is a knot or link, and let N be the complement of an open tubular neighborhood of the singular locus. Let $T_1, \ldots, T_t$ be the tori making up ∂N, where T_i is the boundary of a tubular neighbourhood of the ith component Σ_i of Σ. Then the following hold:*

(a) The holonomy representation $\rho : \pi_1(N) \to PSL_2(\mathbf{C})$ is irreducible.

(b) The holonomy $\rho(\pi_1(T_i))$ is an infinite group preserving a geodesic axis in $\mathbf{H}^3$ for each boundary torus T_i. In fact, $\rho(\pi_1(T_i)) \cong \mathbf{Z} \oplus \mathbf{Z}$ (hence ∂T_i is incompressible) if the cone angle along Σ_i is not a multiple of 2π.

Proofs. (a) If ρ is reducible, then its image $\rho(\pi_1(M))$ fixes a point x_∞ on the sphere at infinity of hyperbolic space $\mathbf{H}^3$. Thus we can define a flow on $\mathbf{H}^3$ by moving each point at unit speed along the geodesic which passes through the point and ends at x_∞. This flow is volume decreasing and is invariant under the holonomy group $\rho(\pi_1(M))$, so it can be pulled back via the developing map $\Phi : \tilde{M} \to \mathbf{H}^3$ to give a volume decreasing flow on M. But this is impossible unless M has infinite volume.

Part (b) follows immediately from the local structure of a hyperbolic cone-manifold near its singular locus. In fact, the holonomy of any curve homotopic to a component Σ_i of Σ represents a translation plus rotation about a geodesic axis, where the translation length is equal to the length of the geodesic Σ_i. Since $\pi_1(T_i) = \mathbf{Z} \oplus \mathbf{Z}$ is abelian, $\rho(\pi_1(T_i))$ preserves the same axis. The last statement follows since the holonomy of a meridian curve around Σ_i is a rotation by the cone angle along Σ_i, and so is non-trivial unless this cone angle is a multiple of 2π. q.e.d.

Remark. In fact, N is a compact, irreducible, atoroidal 3-manifold (for any cone angles). One proof of this involves modifying the hyperbolic metric on $\overline{M}$ inside a tubular neighbourhood of Σ to produce a complete negatively curved Riemannian metric on the manifold $\overline{M} - \Sigma$, which is homeomorphic to the interior of N (compare [14], [4, §3].) The result then follows from standard results on the structure of negatively curved manifolds (e.g. [9], [2]). Topological arguments are given in [23] for the case where the cone angle is 2π. In [19], the local rigidity results of this paper are used to derive these further topological restrictions for all cone angles at most 2π.

Combining the results of Corollary 1.2, Theorem 4.5, and Lemma 4.6 gives the following:

Theorem 4.7. *Let $\overline{M}$ be a 3-dimensional hyperbolic cone-manifold, whose singular locus Σ is a knot or link, and let $M = \overline{M} - \Sigma$. Let $\rho : \pi_1(M) \to PSL_2(\mathbf{C})$ be the holonomy representation for M. If all cone angles along Σ are at most 2π, then the space*

$$R(\pi_1(M, PSL_2(\mathbf{C}))/PSL_2(\mathbf{C})$$

of representations up to conjugacy is a smooth manifold near ρ, of complex dimension equal to the number of components in the singular locus Σ. Further, if $g_1, \dots, g_t$ are homotopy classes of meridian curves for the components of Σ, then the complex length map

$$\mathcal{L} : R(\pi_1(M), PSL_2(\mathbf{C}))/PSL_2(\mathbf{C}) \to \mathbf{C}^t$$

defined by $\mathcal{L}(\phi) = (\mathcal{L}\phi(g_1), \dots, \mathcal{L}\phi(g_t))$ is a local diffeomorphism near ρ. q.e.d.

Now hyperbolic structures on M near a given structure are parametrized (locally) by the conjugacy classes of their holonomy representations $\pi_1(M) \to PSL_2(\mathbf{C})$ (see [24, Chap. 5] or [15, Chap. 1]). If the initial hyperbolic structure corresponds to a hyperbolic cone manifold, it is easy see that each conjugacy class of nearby representations corresponds to a unique hyperbolic cone-manifold structure provided that the holonomy of each meridian is elliptic. Other nearby representations correspond to nearby hyperbolic structures with Dehn surgery type singularities. We will see that from the results above it follows that these structures are locally parametrized by their Dehn surgery coordinates under the hypotheses given above. The reader unfamiliar with these concepts is referred to [24, Chap. 4], [15] or [27]. Below we will recall enough of the definitions to state our final theorem.

If we choose two generators of the fundamental group of a cusp of M, and denote their complex lengths by μ, λ, then we can uniquely solve the equation

$$x\,\mu + y\,\lambda = 2\pi\, i \tag{25}$$

for real numbers x, y whenever μ and λ are linearly independent over the reals. The pair $(x, y) \in \mathbf{R}^2$ is called the (hyperbolic) *Dehn surgery coefficient* of the hyperbolic structure. This name refers to the first case studied by Thurston in [24], where the complement of the singular locus M was a knot complement, and the standard meridian and longitude generators were chosen. When $(x, y) = (p, q)$ are relatively prime integers, the hyperbolic structure on M completes to a *smooth* hyperbolic structure on the manifold $M_{(p,q)}$ obtained by doing (p, q) Dehn surgery on the knot. Similarly, if x, y have rational ratio, $(x, y) = \alpha\ (p, q)$, the completion of the hyperbolic structure on the complement defines a cone-manifold structure on $M_{(p,q)}$ with cone angle $\frac{2\pi}{\alpha}$. (In other words, the (p, q) curve bounds a singular, hyperbolic disk.)

In our case, we can choose generators so that the meridian bounds a singular disk; then the Dehn surgery coefficient is $(\alpha, 0)$ where the cone angle is $\frac{2\pi}{\alpha}$. Note that the complex length of the meridian is pure imaginary, and the real part of the complex length of the longitude is non-zero, so the two complex lengths are linearly independent over the reals as desired. An obvious problem is to describe how the Dehn surgery coefficient varies as the holonomy representation varies. We have seen that, near the holonomy representations of cone-manifolds (with our angle restriction), the representation variety modulo conjugacy is a smooth manifold of real dimension $2t$, where t is the number of components of the singular locus. This is the same as the dimension of the space of possible Dehn surgery coefficients if we define one for each component. The natural conjecture is that the map from the representation variety (modulo conjugation) to the space of Dehn surgery coefficients is a local diffeomorphism. Theorem 4.7 implies that this conjecture is true near the holonomy representations of cone-manifolds with cone angles at most 2π.

Theorem 4.8. *Let $\overline{M}$ be a 3-dimensional hyperbolic cone-manifold, whose singular locus Σ is a knot or link with t components, and let $M = \overline{M} - \Sigma$. Let $\rho : \pi_1(M) \to PSL_2(\mathbf{C})$ be the holonomy representation for M. If all cone angles along Σ are at most 2π, then the t Dehn surgery coefficients give a map from the space of representations up to conjugacy to the space $\mathbf{R}^{2t} = \{(x_i, y_i) | x_i, y_i \in \mathbf{R}, i = 1, \cdots, t\}$ which is a local diffeomorphism near ρ.*

Proof. We saw in the previous theorem that the map from the representation space to complex lengths of meridians is a local diffeomorphism. Thus, it suffices to prove that the map from complex lengths of the meridians to the Dehn surgery coefficients is a local diffeomorphism. It suffices to prove this for each component of the singular locus individually.

Let X denote the (column) vector representing the Dehn surgery coefficient (x, y), and let P be the 2×2 matrix whose columns represent the (real and imaginary parts of) complex lengths of the meridian and longitude respectively. Then the formula (25) for the Dehn surgery coefficient becomes the matrix equation $PX = C$ where C is the (column) vector $(0, 2\pi)$. The variational formula becomes $X' = -P^{-1} P' X$. Since $X = (\alpha, 0)$ at the cone-manifold representation, $P' X$ depends only on the variation of the first column of P, or, in other words, on the variation of the complex length of the meridian. Since P^{-1} is a isomorphism, this is exactly what we needed. q.e.d.

Remark. A hyperbolic cone-manifold structure with some cone angles of *zero* along Σ can be interpreted as a structure which is complete (*i.e.*, has cusps) along some components of Σ. Then the results of Theorems 4.7 and 4.8 extend to this case, provided $\mathcal{L}$ is regarded as a map into $(\mathbf{C}/\pm 1)^t$ in Theorem 4.7, and Dehn surgery coefficients are regarded as elements of $((\mathbf{R}^2 \cup \infty)/\pm 1)^t$ in Theorem 4.8. This follows from the arguments of this section together with the work of Garland [12].

5. Appendix: Self-adjointness of the Laplacian

In this appendix, we will sketch a proof of the self-adjointness of the Laplacian on real-valued forms on a 3-dimensional hyperbolic cone-manifold with the domain described in Section 2. This was used in the proof of Theorem 2.7, the Hodge Theorem for Cone-Manifolds. This proof follows closely Cheeger's proof in [6], but his proof is for singularities which are the cones on manifolds rather than for the $S^1\times$ cone case which we are considering. Some of the finer aspects of the functional analysis of such "edge singularities" are extremely delicate (see [20]), but, fortunately we can avail ourselves of the "softer" aspects of the theory which generalize fairly readily.

The main step is to prove a Stokes' theorem for L^2 forms on the non-complete manifold $\overline{M} - \Sigma$, where $\overline{M}$ is a cone-manifold with singular locus Σ. We will state and prove the theorem in the smooth case and then suggest afterwards how this extends to the case of L^2 distributional derivatives. Since there are no E-valued forms in this section, we will revert to using d (rather than $\hat{d}$) for the exterior derivative on real-valued forms, and we will use $\langle\ ,\ \rangle$ to denote the L^2 inner product of real-valued forms.

Theorem 5.1 (Stokes' Theorem). *Let $\overline{M}$ denote a closed, orientable 3-dimensional hyperbolic cone-manifold with singular locus Σ. Denote by M the non-complete manifold, $\overline{M} - \Sigma$. If α, β are smooth L^2-forms on M such that $d\alpha$ and $\delta\beta$ are in L^2, then the Stokes' theorem holds, i.e.,*

$$\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle.$$

Remark. We will give the proof in the case that α is a 1-form and β is a 2-form. For the remaining degrees, either α or $*\beta$ is a function, and the argument is similar, but easier.

Proof. The main goal is to control the behavior of the forms as they approach the singular locus so that the boundary term which one gets by applying the usual Stokes' theorem to the complement of a neighborhood of the singular locus goes to zero as the neighborhood shrinks down onto the singular locus. A smooth, L^2 form does not generally vanish as one approaches the boundary, but the extra condition that their derivatives are in L^2 implies that a crucial part of the forms does go to zero. This proof is patterned after that of Cheeger [6], who proves it in the case where the boundary is coned to a point. Due to the non-homogeneous form of our boundary (some directions coned to a point, some not), the proof has to be altered somewhat. Although the proof below is based on the explicit form of the neighborhood of the singular locus, it uses only the asymptotic behavior and only in a rough ("quasi-isometric") way.

The metric in a neighborhood of the singular locus is of the form $\sinh^2 r\, d\theta^2 + \cosh^2 r\, d\phi^2 + dr^2$, where r is the distance to the singular locus, and θ and ϕ are standard angular coordinates on the torus with ϕ defined modulo 2π and θ defined modulo the cone angle. For simplicity we assume instead that the metric is $r^2 d\theta^2 + d\phi^2 + dr^2$. The proof makes use only of the asymptotic growth of the metric so this will not affect the proof.

We remove a small tubular neighborhood N of the singular locus, consisting of all points in M whose distance to the singular locus is less than or equal to some small constant. The constant should be small enough so that N is topologically a solid torus minus the singular locus. Assume that the metric on N is the flat one given above. It suffices to prove that Stokes' theorem for manifolds with boundary holds for N. In other words the only boundary term comes from the outer torus boundary. Together with the usual Stokes' theorem with boundary, applied to M minus the interior of N, this implies Theorem 5.1.

The proof of Stokes' theorem with boundary for N will involve two main steps, both of which are essentially contained in [6]. However, we will outline how to put them together in our particular case, referring to [6] for some of the more technical points.

We first prove Stokes' theorem with boundary for a cross-section defined by fixing ϕ. This is topologically a disk minus a point, where the point is the cone point obtained by intersecting the closed disk with the singular locus. The closed disk is a 2-dimensional cone-manifold in the usual sense so this follows as a special case of Theorem 2.2 in [6]. However, it would seem to be useful to highlight the salient points of the argument in this special case.

Denote by D the (punctured) disk cross-section which is topologically a circle crossed with a half-open interval. We denote by S_r the circle $S^1 \times \{r\}$, at distance r from the singular locus, with the induced metric. If α is a form on the circle, we denote by $||\alpha||_r$ the L^2 norm of α on S_r (with its induced metric). If η is an L^2 form on D, then the norm of $\eta(r)$ grows slower than $r^{-1/2}$, where $\eta(r)$ is the restriction of η to S_r. In fact

$$||\eta(r)||_r = o(r^{-1/2}|\log r|^{-1/2}). \tag{26}$$

(See Cheeger [6, Lemma 1.2]. Strictly speaking, this estimate holds only on a sequence of radii, $r_i \to 0$, but this will not affect our argument.)

It is tempting to believe that if an L^2 form η also satisfies the condition that $d\eta$ is in L^2, then the above growth estimate could be improved.

Unfortunately, this is not the case. However, the growth estimate of a piece of the form can be improved. Except for a special "middle dimensional" case, one gets an improved estimate on the co-closed part of the portion of the form with no dr piece. ([6, Lemma 2.3]) This is enough to control the boundary terms in Stokes' theorem near the singular locus for a general space with cone-singularities, except for the case where the dimension of the link of the cone point is twice the degree of the form, where Stokes' theorem fails. In our case, one of the forms is actually a function, so the co-closed part is everything and the needed estimate is particularly easy.

Lemma 5.2. *Suppose g is an L^2 function on D such that dg is also in L^2. Then it satisfies the growth condition*

$$||g(r)||_r < Kr^{1/2}|\log r|^{1/2}.$$

Assuming this estimate we can prove an L^2 Stokes' theorem with boundary for the disk cross-section.

Proposition 5.3 (Stokes' Theorem for cross-sections). *Let D be a punctured disk which is a cross-section of the tubular neighborhood $N \cong D \times S^1$ of the singular locus. Let η be a smooth L^2 1-form, and τ a smooth L^2 2-form on D such that $d\eta$ and $\delta\tau$ are in L^2. Then Stokes' theorem with boundary holds for D, i.e.,*

$$\langle d\eta, \tau\rangle = \langle \eta, \delta\tau\rangle \ + \ \int_{\partial D} \eta \wedge *\tau.$$

Proof of Stokes' theorem for cross-sections. If we remove a small neighborhood of the puncture in D, we can apply the usual Stokes' theorem with boundary to the resulting annulus. We need to show that the boundary term coming from the boundary of the small neighborhood of the puncture goes to zero as the radius r of the neighborhood goes to zero. This term is of the form

$$\int_{S_r} \eta \wedge *\tau.$$

If we write

$$\eta = \rho + h dr$$

$$*\tau = g,$$

then this boundary term becomes

$$\int_{S_r} \rho g.$$

Using the estimate for g from the Lemma above and the general L^2 estimate (26) for ρ, it follows immediately from the Schwarz inequality that this inner boundary term goes to zero as r goes to zero.

This completes the proof of Stokes' theorem for cross-sections, assuming the estimate. q.e.d.

Proof of Lemma 5.2. To get the necessary estimate for $||g(r)||_r$, first we note the relationship between the L^2-norms on S_r for different values of r. If g is a fixed function on the circle S^1, then the norms satisfy

$$||g||_r = r^{1/2}||g||_1.$$

We choose a radius a so that S_a is contained in D, and then consider the integral of $\frac{\partial g}{\partial r}$ from R to a, where $R < a$. We see that

$$\left\|\int_R^a \frac{\partial g}{\partial r}\right\|_1 \leq \int_R^a \left\|\frac{\partial g}{\partial r}\right\|_1 = \int_R^a r^{-1/2}\left\|\frac{\partial g}{\partial r}\right\|_r.$$

By the Schwarz inequality, this is less than or equal to

$$\left(\int_R^a r^{-1}dr\right)^{1/2}\left(\int_R^a \left\|\frac{\partial g}{\partial r}\right\|_r^2\right)^{1/2} \leq K|\log R|^{1/2},$$

where the constant K depends on the L^2 norm of $\frac{\partial g}{\partial r}$ on the neighborhood of the singular locus. Note that $\frac{\partial g}{\partial r}$ is in L^2 because dg is by hypothesis. Thus

$$\left\|\int_r^a \frac{\partial g}{\partial r}\right\|_r = r^{1/2}\left\|\int_r^a \frac{\partial g}{\partial r}\right\|_1 \leq K|\log r|^{1/2}r^{1/2}.$$

But since $\int_r^a \frac{\partial g}{\partial r} = g(a) - g(r)$ and $||g(a)||_r = r^{1/2}||g(a)||_1$, we get that

$$||g(r)||_r \leq \tilde{K}|\log r|^{1/2}r^{1/2}$$

for some new constant $\tilde{K}$.

This completes the proof of the Lemma and hence of the L^2 Stokes' Theorem for cross-sections. q.e.d.

Finally, to prove the L^2 Stokes' Theorem for the tubular neighborhood N itself, we need to use a variant of Theorem 2.1 of [6] which shows that, if the the theorem is true for a manifold, then it is true for that manifold crossed with an open interval. The proof which we give is essentially the same as the one given there.

Proof of 5.1 (concluded). Let α be a 1-form and β a 2-form such that α, β, $d\alpha$ and $d*\beta$ are in $L^2(N)$. We write

$$\alpha = \omega + f d\phi, \qquad *\beta = \psi + g d\phi,$$

where ω and ψ are 1-forms with no $d\phi$ part, and f and g are functions. Let $\tilde{d}$ denote the exterior derivative restricted to a cross-section D, where ϕ is a constant. Then $d = \tilde{d} + d\phi \wedge \frac{\partial}{\partial\phi}$. Since $d\alpha$ and $d*\beta$ are in $L^2(N)$, the following are in $L^2(D)$ for almost every cross-section:

$$\omega,\ f,\ \tilde{d}\omega,\ (\frac{\partial\omega}{\partial\phi} - \tilde{d}f),$$

$$\psi,\ g,\ \tilde{d}\psi,\ (\frac{\partial\psi}{\partial\phi} - \tilde{d}g).$$

The only technical difficulty is that one can't immediately conclude that $\tilde{d}f$ and $\tilde{d}g$ are in $L^2(D)$ themselves. This is handled in [6] by considering the δ-regularization of the forms. For a form η this is defined to be

$$\frac{1}{\delta}\int_{\phi-\frac{\delta}{2}}^{\phi+\frac{\delta}{2}} \eta \, d\phi,$$

and denoted by η_δ. In particular, the δ-regularization of $(\frac{\partial\omega}{\partial\phi} - \tilde{d}f)$ is seen to be

$$\frac{\omega(\phi+\frac{\delta}{2}) - \omega(\phi-\frac{\delta}{2})}{\delta} - (\tilde{d}f)_\delta.$$

Since this and ω are in L^2, $(\tilde{d}f)_\delta$ is in L^2. The argument which we give below can then be done for the regularized forms for any δ. Letting δ go to zero, we get the desired result. We leave the details of this limiting argument to the reader (who can read it in [6]) and proceed under the assumption that $\tilde{d}f$ and $\tilde{d}g$ are actually in L^2.

We can now go through a standard integration by parts argument:

$$\begin{aligned}\int_{S^1\times D} d\alpha\wedge *\beta &= \int_{S^1\times D}\Big(\tilde{d}\omega + d\phi\wedge(\frac{\partial\omega}{\partial\phi} - \tilde{d}f)\Big) \wedge (\psi + g d\phi)\\ &= \int_{S^1}\Big(\int_D \tilde{d}\omega\, g + (\frac{\partial\omega}{\partial\phi} - \tilde{d}f)\wedge\psi\Big) d\phi.\end{aligned}$$

We can apply Stokes' theorem on D to the first and third term above to get

$$\int_{S^1}\left(\int_D \omega\wedge\tilde{d}g \;+\; f\tilde{d}\psi+\frac{\partial\omega}{\partial\phi}\wedge\psi\right)d\phi+\int_{S^1}\left(\int_{\partial D}\omega\, g \;-\; \psi\, f\right)d\phi.$$

If we interchange the order of integration and integrate the third term by parts we get

$$\int_{S^1\times D}\left(\omega\wedge\tilde{d}g \;+\; f\tilde{d}\psi-\omega\wedge\frac{\partial\psi}{\partial\phi}\right)\wedge d\phi+\int_{S^1}\left(\int_{\partial D}\omega\, g \;-\; \psi\, f\right)d\phi.$$

This is seen to be equal to

$$\int_{S^1\times D}\alpha\wedge d*\beta+\int_{S^1}\left(\int_{\partial D}\alpha\wedge *\beta\right).$$

Thus Stokes' theorem holds for N.

This completes the proof of the Theorem 5.1. q.e.d.

We want to apply this Stokes' Theorem to prove that the Laplacian on 1-forms with our desired domain is self-adjoint. The main issue here is to check that the domain of the operator is the same as the domain of its adjoint. Since our manifold M is non-compact and non-complete, this issue is non-trivial, for the Laplacian on forms with compact support may have distinct maximal and minimal extensions (in L^2). Not all ways of extending the operator in L^2 would be self-adjoint. Note also that if it is not essentially self-adjoint, there will be a continuum of choices for a self-adjoint extension. We have chosen the one used for Theorem 2.2 because of the nice geometric representatives for E-valued forms that it gives us in the Hodge theorem on cone-manifolds. It also turns out that this domain is the natural one from the point of view of Cheeger's analysis [7], when applied to *real*-valued 1-forms.

First, we describe some extensions of the operator d on smooth forms. Think of d as defined first on smooth forms which are in L^2 and d of which are in L^2. Following [6], define the "strong closure" of this operator to mean that $d\alpha = \eta$ when α and η are in L^2, and there are smooth forms $\alpha_i \to \alpha$ such that $d\alpha_i \to \eta$. (Convergence in this section will always mean in L^2 unless stated otherwise.) Using a smoothing operator (see [6, p.141]) this is equivalent to having distributional derivatives in L^2 or the statement that $d\alpha = \eta$ weakly; i.e., that $\langle\eta,\beta\rangle = \langle\alpha,\delta\beta\rangle$ for all β smooth, with compact support. This is called the "maximal" extension of the operator d in L^2, which we will denote by d_{max}. Similarly, we can define δ_{max}. The Stokes' theorem proved above holds for the strong closure and hence for d_{max} and δ_{max} by taking limits.

Another closure which is important to us is the graph closure of d on smooth forms with compact support, which we will denote by $\overline{d}_0$. In other words, $\overline{d}_0\alpha = \eta$, if there is a sequence of smooth α_i *with compact support* which converge to α such that $d\alpha_i \to \eta$. This is the "minimal extension" of d, denoted d_{min}. Similarly, we define δ_{min}.

It is a general fact, proved by Gaffney [10] that the adjoint of d_{max} is δ_{min} and that of δ_{max} is d_{min}. Stokes' theorem implies that the domain of the adjoint of d_{max} *contains* that of δ_{max}. Thus, $\delta_{max} = \delta_{min}$. Similarly, $d_{max} = d_{min}$.

To apply this to the Laplace operator, we need another result of Gaffney [10]. Define two extensions of the Laplacian on smooth forms with compact support

$$\triangle_D = d_{min}\ \delta_{max} + \delta_{max}\ d_{min}$$

and

$$\triangle_N = d_{max}\ \delta_{min} + \delta_{min}\ d_{max},$$

called the Dirichlet and Neumann extensions, respectively. We are using the convention in writing the operators in this manner, that the domain of a composition $A \circ B$ of operators are those objects α in the domain of B, such $B(\alpha)$ is in the domain of A, and the domain of a sum is the intersection of the domains of the summands. Gaffney proved that these extensions are always self-adjoint. From the fact that $d_{max} = d_{min}$ and $\delta_{max} = \delta_{min}$, we get that

$$\triangle_D = \triangle_N = d_{max}\ \delta_{max} + \delta_{max}\ d_{max}.$$

The latter operator is precisely the one we are using; hence it is self-adjoint.

We have now essentially proved our desired result:

Theorem 5.4. *Let $\overline{M}$ denote a 3-dimensional hyperbolic cone-manifold with singular locus Σ. Then the Laplacian on real-valued forms for the smooth, incomplete manifold $M = \overline{M} - \Sigma$ whose domain consists of those forms α such that $\alpha, d\alpha, \delta\alpha, d\delta\alpha, \delta d\alpha$ are all in L^2 is a closed, non-negative, self-adjoint, elliptic operator.*

Proof. The derivatives in the statement of the theorem are generally to be interpreted in the distributional sense. Thus, the operator is just the one above, $d_{max}\,\delta_{max} + \delta_{max}\,d_{max}$, which is closed by definition and is clearly elliptic since it has the same symbol as the Laplacian. We have seen that it is self-adjoint since it equals both the Dirichlet and the Neumann extensions. Finally, it is non-negative since

$$\langle (d\delta + \delta d)\alpha, \beta\rangle = \langle d\alpha, d\beta\rangle + \langle \delta\alpha, \delta\beta\rangle$$

for α, β in our domain, by the above Stokes' theorem. q.e.d.

It is, perhaps, important, to emphasize what we are *not* saying here as well. It still may be that $\Delta_{max} \neq \Delta_{min}$; indeed, this is the case where a cone angle is larger than 2π. In fact, even the Dirac operator $d + \delta$ is not essentially self-adjoint in this case. (See [18].) Thus, even though an L^2 form is closed and co-closed, it is *not* necessarily true that there is a sequence $\alpha_i \to \alpha$ of smooth forms with compact support such that $d\alpha_i \to 0$ and $\delta\alpha_i \to 0$. The fact that $d_{max} = d_{min}$ and $\delta_{max} = \delta_{min}$ only guarantees one such sequence for d and another for δ.

It is not hard to see that, if there were always such a single sequence of forms with compact support (in the case with E-valued forms), then the Calabi-Weil arguments would go through easily. This is the crux of Garland's argument and illustrates the essential difference between the the finite volume, complete case and the cone-manifold situation. In [18], it is shown that the assumption that there always is such a single sequence for closed and co-closed E-valued forms in the 2-dimensional cone-manifold setting actually leads to an incorrect computation of the L^2-cohomology $H^1_{L^2}(M; E)$. Thus, it seems highly unlikely that this phenomenon can be avoided directly in the 3-dimensional context.

References

[1] L. Ahlfors, *Invariant operators and integral representations in hyperbolic space*, Math. Scand. **36** (1975) 27-43.

[2] W. Ballmann, M. Gromov & V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser, Basel, 1985.

[3] A. Besse, *Einstein manifolds*, Springer, Berlin, 1987.

[4] S. A. Bleiler & C. D. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996) 809-833.

[5] E. Calabi, *On compact riemannian manifolds with constant curvature, I*, Amer. Math. Soc. Proc. Sympos. Pure Math. **3** (1961) 155-180.

[6] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geom. Laplace Operator, Amer. Math. Soc. Proc. Sympos. Pure Math. **36** (1980) 91-146.

[7] ———, *Spectral geometry of spaces with cone-like singularities*, Preprint.

[8] M. Culler & P. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983) 109-146.

[9] P. Eberlein, *Lattices in manifolds of non-positive curvature*, Ann. of Math. **111** (1980) 435-476.

[10] M. Gaffney, *The heat equation method of Milgram and Rosenbloom for open Riemannian manifolds*, Ann. of Math. **60** (1954) 458-466.

[11] ———, *Hilbert space methods in the theory of harmonic integrals*, Trans. Amer. Math. Soc. **78** (1955) 426-444.

[12] H. Garland, *A rigidity theorem for discrete subgroups*, Trans. Amer. Math. Soc. **129** (1967) 1-25.

[13] H. Garland & M S. Raghunathan, *Fundamental domains for lattices in (R)-rank 1 semi-simple Lie groups*, Ann. of Math. **78** (1963) 279-326.

[14] M. Gromov & W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987) 1-12.

[15] C. Hodgson, *Degeneration and regeneration of geometric structures on 3–manifolds*, Ph. D. thesis, Princeton Univ., 1986.

[16] J. Jost, *Riemannian geometry and geometric analysis*, Springer, Berlin, 1995.

[17] A. Lubotsky & A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **336** (1985).

[18] S. Kerckhoff, *The deformation theory of hyperbolic surfaces with conical singularities*, Preprint.

[19] S. Kojima, *Non-singular parts of hyperbolic 3-cone-manifolds*, Topology and Teichmüller Spaces, (eds. Kojima et. al), World Scientific, Singapore, 1996, 115-122.

[20] R. Mazzeo, *Elliptic theory of differential edge operators. I*, Comm. in Partial Differential Equations **16** (1991) 1615-1664.

[21] Y. Matsushima & S. Murakami, *Vector bundle valued harmonic forms and automorphic forms on a symmetric riemannian manifold*, Ann. of Math. **78** (1963) 365-416.

[22] C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*, Princeton University Press, Princeton, 1996.

[23] T. Sakai, *Geodesic knots in a hyperbolic 3-manifold*, Kobe J. Math. **8** (1991) 81-87.

[24] W. Thurston, *The geometry and topology of three-manifolds*, Princeton Univ., 1979.

[25] ———, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982) 357-381.

[26] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, Berlin, 1972.

[27] John G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer, Berlin, 1994.

[28] A. Weil, *On discrete subgroups of Lie groups*, Ann. of Math. **72** (1960) 369-384.

[29] ———, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964) 149-157.

[30] H. Wu, *The Bochner technique in differential geometry*, Math. Reports, London, 1987.

[31] K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker, New York, 1970.

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