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ON THE LOWEST EIGENVALUE OF THE HODGE LAPLACIAN

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Abstract

The article gives a lower bound for the least positive eigenvalue of the Hodge Laplacian acting on p-forms in a compact $\mathcal{C}^\infty$ manifold without boundary. The bound is related to the parameters of a certain kind of covering $\{\mathcal{U}_i\}$ of the base manifold by finitely many geodesically convex sets. The parameters are the number of sets $\mathcal{U}_i$, their geodesic radii and bounds on the differential of the exponential map in each $\mathcal{U}_i$. The estimate is combined with the Lojasiewiscz inequality to show that the positive eigenvalues of the Hodge-Laplacian on the regular level sets of a $\mathcal{C}^\omega$ function $\Phi : \mathbf{S}^n \to \mathbb{R}$ are $\geq$ a constant times a *power* of the distance between the value of Φ and the set of critical values of Φ.

1. Introduction. Main theorems

Let $(\mathcal{M}^n, g)$ be a compact, connected and orientable Riemannian manifold of class $\mathcal{C}^\infty$, without a boundary. Our main aim is to obtain lower bounds for the first eigenvalue of the Hodge Laplacian on $\mathcal{M}$. Such eigenvalue estimates prove useful in establishing the local exactness in certain differential complexes defined by complex vector fields (see [2]).

We begin by fixing the notation most often used in the sequel. All norms (on tangent or cotangent bundles, Grassmannian algebras, etc.) are defined by means of the metric on the base manifold. On $\mathcal{M}$ we denote by $d\tau$ the volume element defined by g. We denote by Λ^p the pth exterior power of the cotangent bundle $T^*\mathcal{M}$; Λ^0 is identified to $\mathcal{M} \times \mathbb{R}$, and $\Lambda^p = 0$ if $p < 0$ or if $p > n$. If $\mathcal{U}$ is an open subset of $\mathcal{M}$ we denote by $\mathcal{C}^k(\mathcal{U}; \Lambda^p)$ the space of $\mathcal{C}^k$ p-forms in $\mathcal{U}(0 \leq k \leq +\infty)$ and by $L^2(\mathcal{U}; \Lambda^p)$ the Hilbert space of L^2 p-forms. The norm in $L^2(\mathcal{U}; \Lambda^p)$

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will be denoted by $\| \cdot \|_{L^2}$. Of course, when $p = 0$ we write $\mathcal{C}^k(\mathcal{U})$ and $L^2(\mathcal{U})$. We denote by d the *exterior derivative* in $\mathcal{M}$, by d^* the formal transpose of d for the inner product in $L^2(\mathcal{U}; \Lambda^p)$, by $*$ the *Hodge star operation*: for any $p, 0 \le p \le n, d$ and d^* are differential operators

$$d : \mathcal{C}^\infty(\mathcal{U}; \Lambda^p) \to \mathcal{C}^\infty(\mathcal{U}; \Lambda^{p+1}), d^* : \mathcal{C}^\infty(\mathcal{U}; \Lambda^p) \to \mathcal{C}^\infty(\mathcal{U}; \Lambda^{p-1});$$

$*$ is the linear isometry $L^2(\mathcal{U}; \Lambda^p) \to L^2(\mathcal{U}; \Lambda^{n-p})$ such that, for any compactly supported $f \in \mathcal{C}^\infty(\mathcal{U}; \Lambda^p)$,

$$\| f \|_{L^2}^2 = \int_{\mathcal{M}} \| f \|^2 \, d\tau = \int_{\mathcal{M}} f \wedge *f.$$

Also recall that

$$d^* = (-1)^{np+n+1} * d * .$$

The *Hodge Laplacian* is defined as the operator

$$\Delta = dd^* + d^*d : \mathcal{C}^\infty(\mathcal{U}; \Lambda^p) \to \mathcal{C}^\infty(\mathcal{U}; \Lambda^p).$$

We shall use the following terminology: a form $f \in \mathcal{C}^\infty(\mathcal{U}; \Lambda^p)$ will be said to be *d-closed* if $df = 0$, *d^*-closed* if $d^*f = 0$, *d-exact* if there is $u \in \mathcal{C}^\infty(\mathcal{U}; \Lambda^{p-1})$ such that $du = f$, *d^*-exact* if there is $u \in \mathcal{C}^\infty(\mathcal{U}; \Lambda^{p+1})$ such that $d^*u = f$, *harmonic* if $df = d^*f = 0$, which is equivalent to $\Delta f = 0$.

An essential ingredient in our main theorem is a certain kind of finite open covering of the base manifold. Let r_0 denote the injectivity radius of $\mathcal{M}$.

Definition 1.0. We shall call **admissible** any covering $\{\mathcal{U}_1, ..., \mathcal{U}_N\}$ of $\mathcal{M}$ by open sets if there are two positive numbers $A, r, r << \min(1, r_0)$, such that the following properties hold, for some $R > 4^n r$, and every $i = 1, \dots, N$,

(1.1) $\exists \; t_i \in \mathcal{U}_i$ such that the exponential map $E_i : T_{t_i}\mathcal{M} \to \mathcal{M}$ is a diffeomorphism of the ball $B(t_i, R) = \{v \in T_{t_i}\mathcal{M}; \| v \| < R\}$ onto $E_i(B(t_i, R))$;

$$A^{-1} \le \| dE_i(v) \| \le A \text{ for all } v \in B(t_i, R); \tag{1.2}$$

$$\mathcal{U}_i = E_i(B(t_i, r)). \tag{1.3}$$

We shall refer to the numbers N, r and A as the parameters of the admissible covering $\{\mathcal{U}_1, \dots, \mathcal{U}_N\}$.

We shall need the following direct consequences of (1.1) and (1.2):

(1.4) There is a constant $B \geq 1$ depending solely on A and n, such that, if $d\lambda_i$ denotes the Lebesgue measure on $B(t_i, R) \subset T_{t_i}\mathcal{M}$, and $d\mu_i$ denotes the pullback under the map E_i of the volume element $d\tau$, then

$$B^{-1} \leq |d\mu_i/d\lambda_i| \leq B.$$

All the results in this article follow from

Theorem 1.1. *Suppose there exists an admissible covering $\{\mathcal{U}_1, \dots, \mathcal{U}_N\}$ of $\mathcal{M}$ with parameters N, r and A. Then there is a constant $C > 0$ depending solely on n and A, such that the following is true, whatever the integer $p, 1 \leq p \leq n$:*

If $f \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^p)$ is d-exact then the equation $du = f$ has a solution $u \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^{p-1})$ such that

$$\| u \|_{L^2} \leq CrN^{4p} \| f \|_{L^2} . \tag{1.5}$$

It is elementary to solve the equation $du = f$ with unspecified (but finite) upper bounds on $\| u \|_{L^2} / \| f \|_{L^2}$. But the applications of Theorem 1.1 in Section 3 (see also [2]) require that such a bound grow polynomially with r^{-1} and N. It should be emphasized that the present work does not attempt to obtain an "optimal" bound for the ratio $\| u \|_{L^2} / \| f \|_{L^2}$. The bound CrN^{4p} in (1.5) is sufficient for the application in Section 3. Actually one might need estimates of the $C^k (0 \leq k < +\infty)$ norm of u in terms of that of f. In the cases under consideration in Section 3 such estimates are consequences of Theorem 1.1 (see [2]).

Of course Theorem 1.1 does not provide estimates of the L^2 norm of an arbitrary solution of the equation $du = f$, since we can add to u any closed $\mathcal{C}^\infty (p-1)$-form. But it does yield an estimate for the *canonical solution*, that is, the solution orthogonal in $L^2(\mathcal{M}; \Lambda^{p-1})$ to all *closed* forms. Indeed this solution minimizes the L^2 norm over the affine subspace of all L^2 solutions.

In order to get a lower bound for the lowest eigenvalue of the Hodge-Laplacian the following immediate consequence of Theorem 1.1 is needed:

Theorem 1.2. *Same hypotheses as in Theorem 1.1; let p be an integer, $0 \leq p \leq n-1$. If $f \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^p)$ is d^*-exact, then the equation $d^*u = f$ has a solution $u \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^{p+1})$ such that*

$$\| u \|_{L^2} \leq CrN^{4(n-p)} \| f \|_{L^2},$$

where C is the same positive constant as in (1.5).

Proof. Apply Theorem 1.1 to the equation $dv = (-1)^{np+1} *^{-1} f$ and set $u = *^{-1}v$. q.e.d.

Theorem 1.3. *Same hypotheses as in Theorem 1.1; let p be an integer, $0 \leq p \leq n$. Then the smallest positive eigenvalue of the Hodge Laplacian Δ acting on p-forms is $\geq C^{-2}r^{-2}N^{-4(n+1)}$.*

Proof. Let $f \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^p)$ be orthogonal to the harmonic forms on $\mathcal{M}$, and $u \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^p)$ satisfy $dd^*u + d^*du = f$ and be also orthogonal to the harmonic forms. Then we have

$$\| f \|_{L^2} \geq \| dd^*u \|_{L^2} + \| d^*du \|_{L^2} .$$

Now d^*u is the solution of the equation $dv = dd^*u$ which is orthogonal to all d-closed forms, and du is the solution of the equation $dv = d^*du$ which is orthogonal to all d^*-closed forms. It follows from Theorems 1.1 and 1.2 that

$$\| f \|_{L^2} \geq C^{-1}r^{-1} \left[N^{-4p} \| d^*u \|_{L^2} + N^{-4(n-p)} \| du \|_{L^2} \right] . \tag{1.6}$$

Let u_1 (resp., u_2) be the orthogonal projection of u onto the subspace of $L_2(\mathcal{M}; \Lambda^{p-1})$ orthogonal to the d-closed (resp., d^*-closed) forms. Again by Theorems 1.1 and 1.2 we have

$$\begin{aligned} \| du \|_{L^2} &\geq C^{-1}r^{-1}N^{-4(p+1)} \| u_1 \|_{L^2}, \\ \| d^*u \|_{L^2} &\geq C^{-1}r^{-1}N^{-4(n-p+1)} \| u_2 \|_{L^2} . \end{aligned}$$

Putting this into (1.6) yields the conclusion in Theorem 1.3. q.e.d.

In this article we shall give a single application of the preceding theorems: to a real-valued, real analytic function Φ on the unit sphere $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ for $n \geq 2$. We shall denote by $\mathfrak{R}$ the subset of the compact interval $\Phi(\mathbf{S}^n)$ consisting of the *noncritical* values of Φ. The critical values of Φ make up a finite set, obviously equal to the *boundary* $\partial\mathfrak{R}$ of $\mathfrak{R}$. The preimage of $\mathfrak{R}$ under Φ is a disjoint union of finitely many open and connected subsets of $\mathbf{S}^n$ whose boundaries are (pieces of) analytic varieties. If $\mathcal{O}$ is a connected component of $\Phi^{-1}(\mathfrak{R})$, and $y \in \Phi(\mathcal{O})$, then the level set of Φ in $\mathcal{O}, L_y = \Phi^{-1}(y) \cap \mathcal{O}$, is a compact, connected (Lemma 3.1) submanifold of $\mathbf{S}^n$, of class $\mathcal{C}^\omega$ and codimension one (and without a boundary). If $y', y'' \in \Phi(\mathcal{O})$, then the gradient flow of Φ defines a diffeomorphism of $L_{y'}$ onto $L_{y''}$. Let each manifold L_y be equipped with the metric induced by $\mathbf{S}^n$, and denote by $\lambda_1^{(p)}(y)$ the smallest positive eigenvalue of the Hodge Laplacian Δ on L_y acting on p-forms. With these definitions and notation we can state

Theorem 1.4. *Let p be an integer, $0 \leq p \leq n$. Then there are numbers $\mu, \alpha > 0$ such that, for every $y \in \mathfrak{R}$,*

$$\lambda_1^{(p)}(y) \geq \mu[\operatorname{dist}(y, \partial\mathfrak{R})]^\alpha.$$

So far as we know this result has been proved only for $p = 0$, i.e., for the connection Laplacian (i.e, the Laplace-Beltrami operator acting on functions). In the case $p = 0$ it was proved in [4] under the hypothesis that Φ is the restriction to $\mathbf{S}^n$ of a polynomial in $\mathbb{R}^{n+1}$, and in full generality in [1]. The derivation of Theorem 1.4 from Theorems 1.1, 1.2 and 1.3 can be found in Section 3.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on solving the equation $du = f$ in a bounded convex open subset of Euclidean space with good bounds on the L^2 norm of the solution (but without any pretense to obtain the best possible bounds).

Lemma 2.1. *Let Ω be a bounded and convex open subset of $\mathbb{R}^n$. Then there is a constant $C_n > 0$, which depends only on the dimension n, such that, given any d-closed form $f \in \mathcal{C}^\infty(\bar{\Omega}; \Lambda^p)(1 \leq p \leq n)$, the equation $du = f$ in Ω has a solution $u \in \mathcal{C}^\infty(\bar{\Omega}; \Lambda^{p-1})$ which satisfies*

$$\| u \|_{L^2(\Omega)} \leq C_n (\operatorname{diam}\Omega)^{n+1} (\operatorname{Vol}\Omega)^{-1} \| f \|_{L^2(\Omega)} . \tag{2.1}$$

Proof. Let be given a d-closed form $f \in \mathcal{C}^\infty(\bar{\Omega}; \Lambda^p)$. Let $J = \{j_1, \dots, j_p\}$ be any multi-index of length p (thus $1 \leq j_1 < \cdots < j_p \leq n$); if $y \in \Omega$ we write

$$\omega_J(x, y, dx) = \sum_{\alpha=1}^{p} (-1)^{\alpha-1} (x_{j_\alpha} - y_{j_\alpha}) dx_{j_1} \wedge \cdots \wedge dx_{j_{\alpha-1}} \wedge dx_{j_{\alpha+1}} \wedge \cdots \wedge dx_{j_q}$$

and consider the element of $\mathcal{C}^\infty(\bar{\Omega}; \Lambda^{p-1})$,

$$v(x, y, dx) = - \sum_{|J|=p} \left[\int_0^1 f_J((1-\lambda)x + \lambda y)(1-\lambda)^{p-1} d\lambda \right] \omega_J(x, y, dx).$$

It is well known and easy to check that $d_x v = f$. We introduce the average over Ω,

$$u(x, dx) = (\operatorname{Vol}\Omega)^{-1} \int_\Omega v(x, y, dx) dy;$$

of course, $du = f$. We carry out the following change of variables in the integral with respect to y:

$$z = (1-\lambda)x + \lambda y (\in \Omega)$$

whence $x - y = (x-z)/\lambda$, and (setting $d = \operatorname{diam}\Omega$)

$$\begin{aligned}
&(\operatorname{Vol}\Omega)\parallel u(x,dx)\parallel \\
&\quad\le C\int_0^1\int_{|z-x|<\lambda d, z\in\Omega} |z-x| \parallel f(z)\parallel (1-\lambda)^{p-1}\lambda^{-n-1}d\lambda dz \\
&\quad\le C\int_{|z-x|<d, z\in\Omega} |z-x| \parallel f(z)\parallel \left[\int_{|z-x|/d}^1 \lambda^{-n-1}d\lambda\right]dz \\
&\quad\le Cn^{-1}d^n\int_{|z-x|<d, z\in\Omega} \parallel f(z)\parallel |z-x|^{1-n}dz.
\end{aligned}$$

Applying the Young inequalities for convolution yields (2.1). q.e.d.

We now return to our manifold $\mathcal{M}$ and to the admissible covering $\{\mathcal{U}_1,\dots,\mathcal{U}_N\}$ of $\mathcal{M}$. We shall make systematically use of the notation in Definition 1.0 and in (1.4). It is also convenient to adopt the following notation: if $I = \{i_0, i_1, \dots, i_q\}$ is a multi-index, which for us will always mean that $1 \le i_0 < i_1 < \cdots < i_q \le N$, we shall write

$$\mathcal{U}_I = \mathcal{U}_{i_0}\cap\mathcal{U}_{i_1}\cap\cdots\cap\mathcal{U}_{i_q}.$$

If q is an integer, $1 \le q \le N$, we shall call S_q the set of multi-indices $I = \{i_0, i_1, \dots, i_q\}$ such that $\mathcal{U}_I \neq \emptyset$.

We define

$$\mathcal{U}_i^{(\lambda)} = E_i\left[B(t_i, 4^\lambda r)\right], \quad \lambda = 0,\dots,n.$$

In particular $\mathcal{U}_i^{(0)} = \mathcal{U}_i$. If $I = \{i_0, i_1, \dots, i_q\} \in S_q$ we define the intersections $\mathcal{U}_I^{(\lambda)} = \mathcal{U}_{i_0}^{(\lambda)}\cap\mathcal{U}_{i_1}^{(\lambda)}\cap\cdots\cap\mathcal{U}_{i_q}^{(\lambda)}$. The following observation will be crucial in the argument that follows:

Note that by (1.3) each $\mathcal{U}_i$ is a geodesic ball of radius r. If $I = \{i_0, i_1, \dots, i_q\} \in S_q$ (hence $\mathcal{U}_I \neq \emptyset$) the centers of the balls $\mathcal{U}_{i_\alpha} (\alpha = 0, 1, \dots, q)$ lie at a mutual distance $\le 2r$. It follows that if $0 \le \lambda \le n-1$, then $\mathcal{U}_{i_\alpha}^{(\lambda)} \subset\subset \mathcal{U}_{i_\beta}^{(\lambda+1)}$ whatever $\alpha, \beta = 0, 1, \dots, q$, and therefore $\mathcal{U}_{i_\alpha}^{(\lambda)} \subset\subset \mathcal{U}_I^{(\lambda+1)}$ whatever $\alpha = 0, 1, \dots, q$. To each multi-index $I \in S_q$ we associate one of its elements i (selected at random); we may assert:

(2.2) *There is a number* $\epsilon, 0 < \epsilon < 1$, *such that*

$$\cup_{j\in I}\mathcal{U}_j^{(\lambda)} \subset\subset E_i(B(t_i, 4^{\lambda+\epsilon}r)) \subset\subset \mathcal{U}_I^{(\lambda+1)}.$$

Below we write $\mathcal{O}_I^{(\lambda)} = E_i(B(t_i, 4^{\lambda+\epsilon}r))$; of course, if $1 \leq \lambda \leq n-1$, then $\mathcal{U}_j \subset\subset \mathcal{U}_I^{(\lambda)} \subset\subset \mathcal{O}_I^{(\lambda)} \subset\subset \mathcal{U}_I^{(\lambda+1)}$ for every $j \in I$.

Lemma 2.2. *There is a constant* $C > 0$, *depending only on* n *and on* A, *such that to each d-closed form* $f \in \mathcal{C}^\infty(\mathcal{U}_I^{(\lambda+1)}; \Lambda^p)(1 \leq p \leq n)$ *there is a solution* $u \in \mathcal{C}^\infty(\mathcal{O}_I^{(\lambda)}; \Lambda^{p-1})$ *of the equation* $du = f$ *in* $\mathcal{O}_I^{(\lambda)}$ *that satisfies the inequality*

$$\| u \|_{L^2(\mathcal{O}_I^{(\lambda)})} \leq Cr \| f \|_{L^2(\mathcal{O}_I^{(\lambda)})} . \tag{2.3}$$

Proof. We know, by Lemma 2.1, that the inequality (2.3) would be valid, if $B(t_i, 4^{\lambda+\epsilon}r)$ were substituted for $\mathcal{O}_I^{(\lambda)}$, and the forms f and u were replaced by their pullbacks under the map E_i (in which case the constant C would only depend on n). But

$$0 < c \leq \| f \|_{L^2(\mathcal{O}_I^{(\lambda)})} / \| E_i^* f \|_{L^2(B(t_i, 4^{\lambda+\epsilon}r))} \leq c^{-1}$$

with c depending only on the constant B in (1.4), whence the assertion. q.e.d.

We can now proceed with the proof of Theorem 1.1. We combine the line of reasoning followed in proofs of the De Rham theorem (see [3]) with L^2 estimates.

As in Theorem 1.1, p will be an integer, $1 \leq p \leq n$. We select a $\mathcal{C}^\infty$ partition of unity in $\mathcal{M}$, $\{\varphi_1, \ldots, \varphi_N\}$, subordinate to the covering $\{\mathcal{U}_1, \ldots, \mathcal{U}_N\}$; we assume $\varphi_i \geq 0$ for every $i = 1, \ldots, N$; and also, as is permitted, that there is a constant $C' > 0$ which only depends on the parameter A and is such that

$$\| d\varphi_i \|_{L^\infty} \leq C' r^{-1}. \tag{2.4}$$

We apply Lemma 2.2 for $\lambda = p$ and $I \in S_0$ (i.e., I consists of a single element i). We obtain, for each $i = 1, \ldots, N$, a form $u_i \in \mathcal{C}^\infty(\mathcal{U}_i^{(p)}; \Lambda^{p-1})$ which satisfies $du_i = f$ in $\mathcal{U}_i^{(p)}$, and is such that

$$\| u_i \|_{L^2(\mathcal{U}_i^{(p)})} \leq Cr \| f \|_{L^2(\mathcal{U}_i^{(p+1)})} \leq Cr \| f \|_{L^2} .$$

We define $v^{(1)} = \sum_{i=1}^N \varphi_i u_i$ and we derive, from the preceding estimate,

$$\| v^{(1)} \|_{L^2} \leq CrN \| f \|_{L^2} .$$

Since $\sum_{i=1}^N \varphi_i \equiv 1$ and therefore $\sum_{i=1}^N d\varphi_i \equiv 0$ in $\mathcal{M}$, we have

$$\begin{aligned} f - dv^{(1)} &= \sum_{i=1}^N d\varphi_i \wedge u_i = \sum_{i,j=1}^N \varphi_j d\varphi_i \wedge (u_i - u_j) \\ &= \sum_{i<j} \varphi_j d\varphi_i \wedge (u_i - u_j) + \sum_{i>j} \varphi_j d\varphi_i \wedge (u_i - u_j) \\ &= \sum_{i<j} (\varphi_j d\varphi_i - \varphi_i d\varphi_j) \wedge u_{ij}, \end{aligned}$$

where $u_{ij} = u_i - u_j$ if $i < j$; $u_{ij} \in \mathcal{C}^\infty(\mathcal{U}_{ij}^{(p)}; \Lambda^{p-1})$ and $d(u_i - u_j) = 0$ in $\mathcal{U}_{ij}^{(p)} = \mathcal{U}_i^{(p)} \cap \mathcal{U}_j^{(p)}$.

We state now the induction hypothesis: to each $I \in S_q$ there is a form $u_I \in \mathcal{C}^\infty(\mathcal{U}_I^{(p-q+1)}; \Lambda^{p-q})$ such that

$$f + \sum_{I \in S_q} \sum_{i \in I} \epsilon(i, I \backslash i) \varphi_i d\varphi_{I \backslash i} \wedge u_I = dv^{(q)}, \tag{2.5$_q$}$$

where $v^{(q)} \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^{p-1})$ satisfies, for some $C'' > 0$ that depends solely on n and A,

$$\| v^{(q)} \|_{L^2} \leq C'' r N^q \| f \|_{L^2} . \tag{2.6$_q$}$$

We have used the notation

$$d\varphi_J = d\varphi_{j_0} \wedge \cdots \wedge d\varphi_{j_s} \quad \text{if} \quad J = \{j_0 \ldots , j_s\};$$

$\epsilon(i, I \backslash i) = +1$ or -1 depending on whether the permutation that orders the set $\{i, I \backslash \{i\}\}$ is even or odd. Moreover, for every multi-index $I \in S_q$,

$$du_I = 0, \tag{2.7$_q$}$$

$$\| u_I \|_{L^2(\mathcal{U}_I^{(p-q+1)})} \leq [(q+1)!]^2 C^q r^q \| f \|_{L^2}; \tag{2.8$_q$}$$

and for every $J \in S_{q+1}$,

$$(2.9)_q \qquad \sum_{j\in J} \epsilon(j, J\backslash j) u_{J\backslash j} = 0.$$

We have seen that the induction hypothesis is satisfied for $q = 1$. Assume it is satisfied for some $q, 1 \le q \le p-1$. We apply Lemma 2.2: given any $I \in S_q$ there is $v_I \in \mathcal{C}^\infty(\mathcal{O}_I^{(p-q)}; \Lambda^{p-q-1})$ such that $dv_I = u_I$ in $\mathcal{O}_I^{(p-q)}$, and

$$(2.10) \qquad \begin{aligned} \| v_I \|_{L^2(\mathcal{O}_I^{(p-q)})} &\le Cr \| u_I \|_{L^2(\mathcal{O}_I^{(p-q)})} \\ &\le [(q+1)!]^2 C^{q+1} r^{q+1} \| f \|_{L^2} . \end{aligned}$$

We have

$$f - dv^{(q)} = \sum_{I\in S_q} \sum_{i\in I} \epsilon(i, I\backslash i)\varphi_i d\varphi_{I\backslash i} \wedge dv_I.$$

If we set

$$v^{(q+1)} = v^{(q)} + \sum_{I\in S_q} \sum_{i\in I} \epsilon(i, I\backslash i)\varphi_i d\varphi_{I\backslash i} \wedge v_I,$$

then

$$\begin{aligned} f-&dv^{(q+1)} \\ &= -(q+1) \sum_{I\in S_q} d\varphi_I \wedge v_I \\ &= -(q+1) \sum_{I\in S_q} \sum_{j=1}^{N} \Big[\varphi_j d\varphi_I - \sum_{i\in I} \epsilon(i, I\backslash i)\varphi_i d\varphi_j \wedge d\varphi_{I\backslash i}\Big] \wedge v_I \\ &= -(q+1) \sum_{I\in S_q} \sum_{j\notin I} \Big[\varphi_j d\varphi_I - \sum_{i\in I} \epsilon(i, I\backslash i)\varphi_i d\varphi_j \wedge d\varphi_{I\backslash i}\Big] \wedge v_I \\ &\quad + (q+1) \sum_{I\in S_q} \Big[\sum_{\substack{i,j\in I \\ i\neq j}} \epsilon(i, I\backslash i)\varphi_i d\varphi_j \wedge d\varphi_{I\backslash i}\Big] \wedge v_I \\ &= -(q+1) \sum_{I\in S_q} \sum_{j\notin I} \Big[\varphi_j d\varphi_I \\ &\qquad\qquad - \sum_{i\in I} \epsilon(i, I\backslash i)\epsilon(j, I\backslash i)\varphi_i \wedge d\varphi_{J\backslash i}\Big] \wedge v_I, \end{aligned}$$

since $d\varphi_j \wedge d\varphi_{I\backslash i} = 0$ when $i, j \in I$ and $i \neq j$. When $j \notin I$ we have called J the multi-index obtained by ordering $\{j\} \cup I$; in this case we avail ourselves of the formula

$$\epsilon(i, I\backslash i)\epsilon(j, I\backslash i) = -\epsilon(i, J\backslash i)\epsilon(j, J\backslash j).$$

We get

$$-\sum_{i\in I}\epsilon(i,I\backslash i)\varphi_i d\varphi_j\wedge d\varphi_{I\backslash i}=\epsilon(j,J\backslash j)\sum_{i\in I}\epsilon(i,J\backslash i)\varphi_i d\varphi_{J\backslash i}.$$

and therefore, with the same meaning of J as before,

$$f-dv^{(q+1)}=-(q+1)\sum_{I\in S_q}\sum_{j\notin I}\Big[\sum_{i\in J}\epsilon(i,J\backslash i)\varphi_i\wedge d\varphi_{J\backslash i}\Big]\wedge\epsilon(j,J\backslash j)v_{J\backslash j}.$$

But we observe that there is a one-to-one correspondence between pairs (I,j) such with $|I|=q$ and $j\notin I$, and pairs (J,j) with $|J|=q+1, j\in J$, whence

$$\begin{aligned}&f-dv^{(q+1)}\\&=-(q+1)\textstyle\sum_{J\in S_{q+1}}\Big[\sum_{i\in J}\epsilon(i,J\backslash i)\varphi_i\wedge d\varphi_{J\backslash i}\Big]\wedge\sum_{j\in J}\epsilon(j,J\backslash j)v_{J\backslash j}.\end{aligned}$$

If then we set

$$u_J=-(q+1)\sum_{j\in J}\epsilon(j,J\backslash j)v_{J\backslash j}\in\mathcal{C}^\infty(\mathcal{U}_J^{(p-q)};\Lambda^{p-q-1}),$$

we obtain $(2.5)_{q+1}$. Thanks to $(2.9)_q$ we have

$$du_J=-(q+1)\sum_{j\in J}\epsilon(j,J\backslash j)u_{J\backslash j}=0,$$

whence $(2.7)_{q+1}$. If K is a multi-index such that $|K|=q+2$ we have

$$\sum_{k\in K}\sum_{j\in K\backslash k}\epsilon(k,K\backslash k)\epsilon(j,K\backslash\{j,k\})u_{K\backslash\{j,k\}}=0,$$

since $\epsilon(k,K\backslash k)\epsilon(j,K\backslash\{j,k\})=-\epsilon(j,K\backslash j)\epsilon(k,K\backslash\{j,k\})$. This proves $(2.9)_{q+1}$.

From the inequalities

$$\begin{aligned}\| u_J\|_{L^2(\mathcal{U}_J^{(p-q)})}&\le(q+1)Cr\sum_{j\in J}\| v_{J\backslash j}\|_{L^2(\mathcal{U}_J^{(p-q)})}\\&\le\big[(q+1)!\big]^2(q+1)(q+2)C^{q+1}r^{q+1}\| f\|_{L^2},\end{aligned}$$

$(2.8)_{q+1}$ follows. Lastly, observing that, for any $i\in I$,

$$\operatorname{supp}\varphi_i d\varphi_{I\backslash i}\subset\bigcap_{j\in I}\mathcal{U}_j\subset\bigcap_{j\in I}\mathcal{U}_j^{(p-q)}\subset\subset\mathcal{O}_I^{(p-q)},$$

we obtain

$$\| v^{(q+1)} \|_{L^2} \leq \| v^{(q)} \|_{L^2} + \sum_{|I|=q} \sum_{i \in I} \| \varphi_i d\varphi_{I\setminus i} \|_{L^\infty} \| v_I \|_{L^2(\mathcal{O}_I^{(p-q)})},$$

and by combining this with (2.4) and (2.10),

$$\| v^{(q+1)} \|_{L^2} \leq \| v^{(q)} \|_{L^2} + C' r [(q+1)!]^2 C^{q+1} N^{q+1} \| f \|_{L^2},$$

we obtain $(2.6)_{q+1}$. [Keep in mind that $q < p \leq n$.]

The induction take us to the last step, $q = p$. We get $v = v^{(p)} \in \mathcal{C}^\infty(\mathcal{M}; \Lambda^{p-1})$ and for each $I \in S_p$, $u_I \in \mathcal{C}^\infty(\mathcal{U}_I^{(1)})$ satisfying

$$f - dv = \sum_{I \in S_p} \sum_{i \in I} \epsilon(i, I\setminus i) u_I \varphi_i d\varphi_{I\setminus i}, \tag{2.11}$$

$$\| v \|_{L^2} \leq C'' r N^p \| f \|_{L^2} . \tag{2.12}$$

According to $(2.7)_p$ and $(2.9)_p$ the $u_I (I \in S_p)$ are constants, such that

$$[\mathrm{Vol}\, \mathcal{U}_I^{(1)}]^{1/2} |u_I| \leq [(p+1)!]^2 C^p r^p \| f \|_{L^2}; \tag{2.13}$$

and for every multi-index $J \in S_{p+1}$,

$$\sum_{j \in J} \epsilon(j, J\setminus j) u_{J\setminus j} = 0. \tag{2.14}$$

We regard the system of constants $\{u_I\}_{I \in S_p}$ as a Čech cocycle for the covering $\{\mathcal{U}_i\}_{i=1,\dots,N}$. We know by (2.2) that $\mathcal{U}_i \subset\subset \mathcal{U}_I^{(1)}$ for all $i \in I$, hence

$$\mathrm{Vol}\, \mathcal{U}_I^{(1)} \geq c_0 r^n$$

for a suitable $c_0 > 0$, which depends only on n and A. According to (2.13) we obtain

$$|u_I| \leq c_0^{-1} [(p+1)!]^2 C^p r^{p-n/2} \| f \|_{L^2} \tag{2.15}$$

for all $I \in S_p$ such that $\mathcal{U}_I \neq \emptyset$.

The meaning of (2.11) is that the (Čech) cohomology class of this cocycle is equal to the (De Rham) cohomology class of f. But since f is

d–exact, the cocycle $\{u_I\}_{I\in S_p}$ must be a coboundary. This means that to each $H \in S_{p-1}$ there is a constant c_H such that, whatever $I \in S_p$,

$$u_I = \sum_{i\in I} \epsilon(i, I\backslash i) c_{I\backslash i}. \tag{2.16}$$

Putting this into the right-hand side of (2.11) gets us

$$\begin{aligned} f - dv &= \sum_{I\in S_p} \sum_{i,j\in I} \epsilon(i, I\backslash i)\epsilon(j, I\backslash j) c_{I\backslash j}\varphi_i d\varphi_{I\backslash i} \\ &= \sum_{I\in S_p} \sum_{i\in I} c_{I\backslash i}\varphi_i d\varphi_{I\backslash i} + \sum_{I\in S_p} \sum_{\substack{i,j\in I\\ i\neq j}} \epsilon(i, I\backslash i)\epsilon(j, I\backslash j) c_{I\backslash j}\varphi_i d\varphi_{I\backslash i}. \end{aligned}$$

We observe that, for $i, j \in I, i \neq j$,

$$\begin{aligned} \varphi_i d\varphi_{I\backslash i} =& d\big[\epsilon(j, I\backslash \{ij\}\varphi_i\varphi_j d\varphi_{I\backslash\{ij\}}\big] \\ & - \epsilon(i, I\backslash\{ij\})\epsilon(j, I\backslash\{ij\})\varphi_j d\varphi_{I\backslash\{j\}}, \end{aligned}$$

and

$$\epsilon(i, I\backslash\{ij\})\epsilon(j, I\backslash\{ij\}) = -\epsilon(i, I\backslash i)\epsilon(j, I\backslash j),$$

whence

$$\begin{aligned} f - dv =& (p+1) \sum_{I\in S_q} \sum_{i\in I} c_{I\backslash i}\varphi_i d\varphi_{I\backslash i} \\ & - d\Big[\sum_{I\in S_p} \sum_{I\in S_p} \sum_{\substack{i,j\in I\\ i\neq j}} \epsilon(i, I\backslash\{ij\}) c_{I\backslash j}\varphi_i\varphi_j d\varphi_{I\backslash\{ij\}}\Big] \\ =& (p+1) \sum_{I\in S_q} \sum_{H\in S_{p-1}} \sum_{i\notin H} c_H\varphi_i d\varphi_H \\ & - d\Big[\sum_{H\in S_{p-1}} \sum_{i\in H} \sum_{h\notin H} \epsilon(i, H\backslash i) c_H\varphi_i\varphi_h d\varphi_{H\backslash i}\Big]. \end{aligned}$$

We note that

$$\begin{aligned} \sum_{i\notin H} c_H\varphi_i d\varphi_H &= c_H d\varphi_H - \sum_{i\in H} c_H\varphi_i d\varphi_H \\ &= d\Big[\sum_{i\in H} \epsilon(i, H\backslash i) c_H(\frac{1}{p}\varphi_i - \frac{1}{2}\varphi_i^2) d\varphi_{H\backslash i}\Big]. \end{aligned}$$

In summary we have proved that $f = du$, with

$$(2.17)\quad u = v + \sum_{H\in S_{p-1}} \sum_{i\in H} \epsilon(i, H\backslash i) c_H \Big[1 + \frac{1}{p} - \frac{p+1}{2}\varphi_i - \sum_{j\notin H} \varphi_j\Big] \varphi_i d\varphi_{H\backslash i}.$$

All that is left to do is to get a good estimate of a solution $\{c_H\}_{H\in S_{p-1}}$ of the equations (2.16). According to Lemma A.6 of [6] we can find a solution such that, for some constants $C'', C''' > 0$ depending only on p,

$$\sum_{H\in S_{p-1}} |c_H|^2 \le C'' N^{p+2} \sum_{I\in S_p} |u_I|^2 \le C''' N^{3p+4} r^{-n+2p} \parallel f \parallel_{L^2}^2 .$$

We put this into (2.17). Taking (2.4) and (2.12) into account yields

$$\begin{aligned}\parallel u \parallel_{L^2} / \parallel f \parallel_{L^2} &\le C^{iv} r N^p + C^v r N^{p+3p/2} \sum_{H\in S_{p-1}} [\mathrm{Vol}\,\mathcal{U}_H]^{1/2} / r^{n/2} \\ &\le C^{vi} r N^{2p+3p/2},\end{aligned}$$

whence (1.5). The proof of Theorem 1.1 is complete.

3. Application to the level sets of an analytic function

We follow the notation introduced at the end of Section 1. We consider a $\mathcal{C}^\omega$ function $\Phi : \mathbf{S}^n \to \mathbb{R}$, with $n \ge 2$; $\mathfrak{R}$ is the subset of the compact interval $\Phi(\mathbf{S}^n)$ consisting of the noncritical values of Φ; the set $\partial\mathfrak{R}$ of critical values of Φ is finite: $\partial\mathfrak{R} = \{y_0, y_1 \dots, y_k\} \subset \mathbb{R}$ with $\min_{\mathbf{S}^n} \Phi = y_0 < y_1 < \cdots < y_k = \mathrm{Max}_{\mathbf{S}^n} \Phi$. Let $\mathcal{O}$ be a connected component of $\Phi^{-1}(\mathfrak{R})$; $\Phi(\mathcal{O})$ is a connected component of $\mathfrak{R}$, i.e., $\Phi(\mathcal{O}) = (y_{j-1}, y_j)$ for some $j, 1 \le j \le k$. As already stated the level sets of Φ in $\mathcal{O}$ are compact submanifolds L_y of class $\mathcal{C}^\omega$ of $\mathcal{O}$; codim $L_y = 1$ and $\partial L_y = \emptyset$. To make sure that the geometric picture is clear we prove

Lemma 3.1. *Every level set L_y in a connected component $\mathcal{O}$ of $\Phi^{-1}(\mathfrak{R})$ is connected.*

Proof. Let Γ be a connected component of L_y and call $\hat{\Gamma}$ the set of points of $\mathcal{O}$ which lie on some integral curve of $\nabla\Phi$ that intersects Γ. Keeping in mind that $\nabla\Phi \neq 0$ at every point of $\mathcal{O}$ it is clear that $\hat{\Gamma}$ contains a tubular neighborhood of Γ. Since the flow of $\nabla\Phi$ defines local diffeomorphisms, $\hat{\Gamma}$ is relatively open and closed in $\mathcal{O}$. We conclude that $\hat{\Gamma} = \mathcal{O}$. If L_y had another connected component $\Gamma' \neq \Gamma$, each point of

Γ' should lie on an integral curve γ of $\nabla\Phi$ intersecting Γ, which would imply that Φ takes the value y at two distinct points of γ. This would contradict the general fact that Φ is a bijection of anyone of its integral curves in $\mathcal{O}$ onto $\Phi(\mathcal{O})$. q.e.d.

Clearly $\mathcal{O}$ can be identified to the product $(y_{j-1}, y_j) \times L_{y*}$ for any $y_* \in (y_{j-1}, y_j)$: a pair $(y, t_*), y_{j-1} < y < y_j, t_* \in L_{y*}$, is identified to the unique point $t \in L_y$ joined to t_* by an arc of an integral curve of $\nabla\Phi$.

We equip each L_y with the metric g induced by $\mathbf{S}^n$. We use the properties of the level sets L_y established in [1], some of which we recall now. The sought estimates will be simpler if we assume diam $\Phi(\mathbf{S}^n) < 1$, which can always be achieved by rescaling. If $y \in \Phi(\mathbf{S}^n)$ we shall write $\delta(y) = \text{dist}(y, \partial\mathfrak{R})$; note that $\delta(y) \leq \delta_0 < 1$. First of all (Lemma 2, *loc. cit.*) combining the implicit function theorem with the Lojacewicz inequality shows that the injectivity radius of L_y is $\geq \delta(y)^{\sigma_0}$ for a suitably large constant $\sigma_0 > 0$, independent of $y \in \mathfrak{R}$. According to Lemmas 2 and 5, *ibid.*, if N, r and A are the numbers in Definition 1.0 for $\mathcal{M} = L_y$, then we can take

$$N \leq \delta(y)^{-\rho}, \qquad r = \delta(y)^{\sigma}, \qquad A = 100 \tag{3.1}$$

with a suitably large number $\rho > 0$, also independent of y, and $\sigma = 100\sigma_0$. By Lemma 3, *ibid.*, we can also take $B = 100$ in (1.4).

If we apply Theorem 1.3 with these values of N, r, A we obtain rightaway Theorem 1.4.

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