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LIMITS OF SOLUTIONS TO THE KÄHLER-RICCI FLOW

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1. Introduction

We consider the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} \tag{1.1}$$

on a complex manifold X. Following [5], we have

Definiton 1.1. A complete solution $g_{i\bar{j}}$ to Eq.(1.1) is called a Type II limit solution if it is defined for $-\infty < t < \infty$ with uniformly bounded curvature, nonnegative holomorphic bisectional curvature and positive Ricci curvature where the scalar curvature R assumes its maximum in space-time.

Definition 1.2. A complete solution $g_{i\bar{j}}$ to Eq.(1.1) is called a Type III limit solution if it is defined for $0 < t < \infty$ with uniformly bounded curvature, nonnegative holomorphic bisectional curvature and positive Ricci curvature where tR assumes its maximum in space-time.

To understand the behavior of Type II or Type III limit solutions is very important because they arise as limits of blow ups of singularities in the Ricci flow, as pointed out by Richard Hamilton in [5].

In the Riemannian case, Hamilton [4] proved that any Type II limit with positive curvature operator is necessarily a gradient Ricci soliton.

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This is an essential step in classifying singularities for the Ricci flow. Later in [5] he conjectured that a similar result should hold in the Kähler case and that any type III limit should be a homothetically expanding gradient Ricci soliton. Our main results in this paper give an affirmative anwser to his conjectures.

Let us first recall the definitions of Ricci solitons. A solution $g_{i\bar{j}}$ to Eq.(1.1) is called a (translating) Kähler-Ricci soliton if it moves along Eq.(1.1) under a one-parameter family of automorphisms of X generated by some holomorphic vector field. More precisely, this means that the Ricci tensor of $g_{i\bar{j}}$ can be expressed as

$$R_{i\bar{j}} = V_{i,\bar{j}} + V_{\bar{j},i}$$

for some holomorphic vector field $V = (V^i)$. When the holomorphic vector field V comes from the gradient of a function on X, we say we have a gradient Ricci soliton. In this case, we have

$$R_{i\bar{j}} = f_{,i\bar{j}} \qquad \text{and} \qquad f_{,ij} = 0$$

for some (real-valued) function f on X. Note that the condition $f_{,ij} = 0$ is equivalent to saying that the vector field $V = \bigtriangledown f$ is holomorphic. Similarly, a solution $g_{i\bar{j}}$ to Eq.(1.1) is called a homothetically expanding (shrinking) gradient Kähler-Ricci soliton if there exist a real-valued function f on X and a constant $\rho > 0$ ($\rho < 0$) such that

$$R_{i\bar{j}} + \rho g_{i\bar{j}} = f_{,i\bar{j}} \qquad \text{and} \qquad f_{,ij} = 0.$$

Now we can state our main results:

Theorem 1.3. *Let X be a simply connected non-compact complex manifold of dimension n. Then any Type II limit solution to Eq.(1.1) is necessarily a gradient Kähler-Ricci soliton.*

Theorem 1.4. *Let X be a simply connected non-compact complex manifold of dimension n. Then any Type III limit solution to Eq.(1.1) is necessarily a homothetically expanding gradient Kähler-Ricci soliton.*

The proof of both theorems is a consequence of our Harnack estimate in [1] for the Kähler-Ricci flow and the strong maximum principle. We essentially follow the idea of Hamilton as in [4]. The first difficulty appears to be that while the curvature operator is a quadratic form, the holomorphic bisectional curvature is not. It turns out that we can overcome this difficulty by working on the Harnack estimate for the scalar curvature instead of the full Harnack estimate. This simplifies the proof a great deal.

Examples of complete gradient Kähler-Ricci solitons of Eq.(1.1) have been found recently by the author [2]. They are radially symmetric about an origin and have positive sectional curvature. In this paper we also provide the first example of a one-parameter family of expanding gradient Kähler-Ricci solitons

Theorem 1.5. *For each $\lambda > 0$, there exists on $\mathbf{C}^n$ a complete rotationally symmetric homothetically expanding gradient Kähler-Ricci soliton g_λ. The sectional curvature of g_λ is positive for $\lambda > 1$ and negative for $\lambda < 1$. Moreover, the Kähler potential function of g_λ is aymptotic to $|z|^{2/\lambda}$ as $|z|$ tends to the infinity.*

We are very grateful to Professor Richard Hamilton for bringing these problems to our attention. We also would like to thank Professor S.-T. Yau for his constant encouragement.

2. The Harnack estimate

In [1] we proved a Harnack estimate for the (normalized) Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}$$

on a compact Kähler manifold of positive bisectional curvature.

For the solution of Eq.(1.1) we have the following corresponding Harnack estimate.

Theorem 2.1. *Let $g_{i\bar{j}}$ be a complete solution of Eq.(1.1) on a complex manifold X with bounded curvature and nonnegtive bisectional curvature and $0 \le t < T$. For any point x in X and any vector V in the holomorphic tangent bundle T_xX of X at x, let*

$$Z_{i\bar{j}} = \frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},k} V_{\bar{k}} + R_{i\bar{j},\bar{k}} V_k + R_{i\bar{j}k\bar{l}} V_{\bar{k}} V_l + \frac{1}{t} R_{i\bar{j}}.$$

Then we have

$$Z_{i\bar{j}} W^i W^{\bar{j}} \ge 0$$

for all $W \in T_xX$.

The proof of Theorem 2.1 follows from essentially the same calculations as in [1] plus a perturbation arguement of Hamilton for strong maximum principle as in [3].

Taking the trace in Theorem 2.1, we derive the following Harnack estimate for the scalar curvature R.

Corollary 2.2. *Under the assumptions of Theorem 2.1, the scalar curvature R satisfies the estimate*

$$\frac{\partial R}{\partial t} + R_{,k}V^k + R_{,\bar{k}}V^{\bar{k}} + R_{k\bar{l}}V^iV^{\bar{j}} + \frac{1}{t}R \geq 0$$

for all $x \in X$ and $V \in T_xX$.

If our solution is defined for $-\infty < t < 0$, we can drop the term $\frac{1}{t}R_{i\bar{j}}$ in Theorem 2.1, as Hamilton does in [4]. When we have a solution on $\alpha < t < T$, we can replace t by $t - \alpha$ in the Harnack estimate. Then if $\alpha \to -\infty$, the expression $1/(t-\alpha) \to 0$ and the term $\frac{1}{t}R_{i\bar{j}}$ disappears. Thus we have

Theorem 2.3. *Let $g_{i\bar{j}}$ be a complete solution of Eq.(1.1) on a complex manifold X with bounded curvature and nonnegtive bisectional curvature and $-\infty < t < T$. For any point x in X and any vector V in the holomorphic tangent bundle T_xX of X at x, let*

$$Q_{i\bar{j}} = \frac{\partial}{\partial t}R_{i\bar{j}} + R_{i\bar{k}}R_{k\bar{j}} + R_{i\bar{j},k}V_{\bar{k}} + R_{i\bar{j},\bar{k}}V_k + R_{i\bar{j}k\bar{l}}V_{\bar{k}}V_l.$$

Then we have

$$Q_{i\bar{j}}W^iW^{\bar{j}} \geq 0$$

for all $W \in T_xX$.

Taking the trace again in Theorem 2.3, we have

Corollary 2.4. *Under the assumptions of Theorem 2.3, the scalar curvature R satisfies the estimate*

$$\frac{\partial R}{\partial t} + R_{,k}V^k + R_{,\bar{k}}V^{\bar{k}} + R_{k\bar{l}}V^iV^{\bar{j}} \geq 0$$

for all $x \in X$ and $V \in T_xX$.

To conclude this section, we list two important calculations which are crucial in the proof of Theorem 2.1 and Theorem 2.3, respectively. They will also be used in the next section.

Lemma 2.5. (See pp. 258-259 in [1].) *The expression $Z_{i\bar{j}}$ satisfies the evolution equation*

$$(\frac{\partial}{\partial t} - \Delta)Z_{i\bar{j}} = R_{i\bar{j}k\bar{l}}Z_{l\bar{k}} + P_{i\bar{l}k}P_{l\bar{j}\bar{k}} - P_{i\bar{l}\bar{k}}P_{l\bar{j}k} - \frac{1}{2}(R_{i\bar{k}}Z_{k\bar{j}} + Z_{i\bar{k}}R_{k\bar{j}}) - \frac{2}{t}Z_{i\bar{j}}$$

at a point where

$$V_{l,\bar{k}} = V_{\bar{k},l} = R_{l\bar{k}} + \frac{1}{t} g_{l\bar{k}}, \quad V_{l,p} = V_{\bar{l},\bar{p}} = 0, \tag{2.1}$$

and

$$(\frac{\partial}{\partial t} - \Delta) V_k = \frac{1}{2} R_{k\bar{p}} V_p - \frac{1}{t} V_k. \tag{2.2}$$

Here we denote

$$P_{i\bar{j}k} = R_{i\bar{j},k} + R_{i\bar{j}k\bar{l}} v^{\bar{l}}, \quad P_{i\bar{j}\bar{l}} = R_{i\bar{j},\bar{l}} + R_{i\bar{j}k\bar{l}} v^{k}.$$

Lemma 2.6. *The expression $Q_{i\bar{j}}$ satisfies the equation*

$$(\frac{\partial}{\partial t} - \Delta) Q_{i\bar{j}} = R_{i\bar{j}k\bar{l}} Q_{l\bar{k}} + P_{i\bar{l}k} P_{l\bar{j}\bar{k}} - P_{i\bar{l}\bar{k}} P_{l\bar{j}k} - \frac{1}{2}(R_{i\bar{k}} Q_{k\bar{j}} + Q_{i\bar{k}} R_{k\bar{j}})$$

at a point where

$$V_{l,\bar{k}} = V_{\bar{k},l} = R_{l\bar{k}}, \quad V_{l,p} = V_{\bar{l},\bar{p}} = 0, \tag{2.3}$$

and

$$(\frac{\partial}{\partial t} - \Delta) V_k = \frac{1}{2} R_{k\bar{p}} V_p. \tag{2.4}$$

3. The strong maximum principle

Throughout this section we assume that $g_{i\bar{j}}$ is a complete solution of Eq.(1.1) defined for $A < t < \infty$ with bounded curvature, nonnegative bisectional curvature and positive Ricci curvature. Here either $A = -\infty$ or $A = 0$. We are going to deduce the strong maximum principle for the quadratic forms

$$Q = g^{i\bar{j}} Q_{i\bar{j}} = \frac{\partial R}{\partial t} + R_{,k} V^k + R_{,\bar{k}} V^{\bar{k}} + R_{k\bar{l}} V^i V^{\bar{j}}$$

and

$$Z = g^{i\bar{j}} Z_{i\bar{j}} = \frac{\partial R}{\partial t} + R_{,k} V^k + R_{,\bar{k}} V^{\bar{k}} + R_{k\bar{l}} V^i V^{\bar{j}} + \frac{1}{t} R.$$

The basic idea of proof comes from Hamilton's work [4]. We note that Corollary 2.4 (respectively Corollary 2.2) implies that Q (respectively Z) is semi-positive definite in V. Moreover, from Lemma 2.6 and Lemma 2.5 it follows respectively that at a point where V satisfying (2.3) and (2.4),

$$(\frac{\partial}{\partial t}-\Delta)Q = (\frac{\partial}{\partial t}-\Delta)(g^{i\bar{j}}Q_{i\bar{j}})$$
$$= g^{i\bar{j}}(\frac{\partial}{\partial t}-\Delta)Q_{i\bar{j}} + Q_{i\bar{j}}(\frac{\partial}{\partial t}-\Delta)g^{i\bar{j}} = Q_{i\bar{j}}R_{j\bar{i}}, \tag{3.1}$$

and that at a point where V satisfying (2.1) and (2.2),

$$(\frac{\partial}{\partial t}-\Delta)Z = (\frac{\partial}{\partial t}-\Delta)(g^{i\bar{j}}Z_{i\bar{j}})$$
$$= g^{i\bar{j}}(\frac{\partial}{\partial t}-\Delta)Z_{i\bar{j}} + Z_{i\bar{j}}(\frac{\partial}{\partial t}-\Delta)g^{i\bar{j}} = Z_{i\bar{j}}R_{j\bar{i}} - \frac{2}{t}Z. \tag{3.2}$$

Proposition 3.1. *Let $g_{i\bar{j}}$ be a complete solution to Eq.(1.1) defined for $-\infty < t < \infty$ with bounded curvature, nonnegative bisectional curvature and positive Ricci curvature. If Q is positive for all $V \in T_{x_0}X$ at some point x_0 at $t = 0$, then it is positive for all $V \in T_xX$ at every point $x \in X$ for any time $t > 0$.*

Proof. Suppose that Q is positive for all $V \in T_{x_0}X$ at some point x_0 at $t = 0$. Then we can find a function F on X with support in a neighberhood of x_0 so that $F(x_0) > 0$ and $Q \geq F$ for all V everywhere at $t = 0$. Let F evolve by the heat equation

$$(\frac{\partial}{\partial t}-\Delta)F = 0. \tag{3.3}$$

It then follows from the usual strong maximum principle that $F > 0$ everywhere for any $t > 0$. Now it suffices to prove that $Q \geq F$ for $t > 0$. In order to show this, we use the perturbation arguement as in [3] and [4]. From the work of Shi [5], we can pick a function $u(x)$ on X so that $u(x) \geq 1$, $u(x) \to \infty$ as $x \to \infty$ and $|\Delta u| \leq C$ for a constant C. Next we put

$$\phi(x,t) = \epsilon e^{At}u(x), \tag{3.4}$$

where $\epsilon > 0, A > C$ are constants. Therefore

$$(\frac{\partial}{\partial t}-\Delta)\phi = \epsilon e^{At}(Au - \Delta u) > 0. \tag{3.5}$$

Now we modify Q by setting

$$\hat{Q} = Q - F + \phi.$$

Then $\hat{Q}$ is strictly positive everywhere at $t = 0$, and strictly positive outside some compact set for $t > 0$. We claim that $\hat{Q}$ is strictly positive everywhere for $t > 0$. Suppose this is not true. Then there is a first time $t_0 > 0$ such that $\hat{Q}$ is zero at some point $x_0 \in X$ in the direction V_0. Extend V_0 to a local vector field V in space-time satisfying (2.3) and (2.4) at (x_0, t_0). Now it follows from (3.1), (3.3) and (3.5) that, at (x_0, t_0),

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\hat{Q} &= (\frac{\partial}{\partial t} - \Delta)Q + (\frac{\partial}{\partial t} - \Delta)\phi \\ &= Q_{i\bar{j}}R_{j\bar{i}} + (\frac{\partial}{\partial t} - \Delta)\phi > 0.\end{aligned} \tag{3.6}$$

On the other hand $\hat{Q}$ achieves a minimum at (x_0, t_0), hence $\frac{\partial}{\partial t}\hat{Q} \leq 0$ and $\Delta\hat{Q} \geq 0$ which contradict (3.6). Therefore $\hat{Q}$ is strictly positive everywhere for $t > 0$ and it follows that $Q \geq F$ by letting the constant $\epsilon \to 0$ in the definition of ϕ. This proves that $Q \geq F$ for all $t > 0$ and hence Proposition 3.1.

Proposition 3.2. *Let $g_{i\bar{j}}$ be a complete solution of Eq.(1.1) defined for $0 < t < \infty$ with bounded curvature, nonnegative bisectional curvature and positive Ricci curvature. If Z is positive for all $V \in T_{x_0}X$ at some point x_0 at $t = t_0 > 0$, then it is positive for all $V \in T_xX$ at every point $x \in X$ for any time $t > t_0$.*

Proof. The proof is similar to that of Proposition 3.1. Since Z is positive for all $V \in T_{x_0}X$ at some point x_0 at $t = t_0 > 0$, we can find a function F on X with support in a neighborhood of x_0 so that $F(x_0) > 0$ and $Z \geq F/t_0^2$ for all V everywhere at $t = t_0$. We then let F to evolve by the heat equation (3.3). Again we need to prove that $Z \geq F/t^2$ for $t > t_0$.

We modify Z by setting $\hat{Z} = Z - F/t^2 + \phi$ where ϕ is defined as in (3.4). Then $\hat{Z}$ is strictly positive everywhere at $t = t_0$, and strictly positive outside some compact set for $t > t_0$. We claim that $\hat{Z}$ is strictly positive everywhere for $t > t_0$. Suppose this is not true, then there is a first time $t_1 > t_0$ such that $\hat{Z}$ is zero at some point $x_0 \in X$ in the direction V_0. Extend V_0 to a local vector field V in space-time satisfying

(2.1) and (2.2) at (x_0, t_1). Now it follows from (3.2), (3.3) and (3.5) that, at (x_0, t_1),

$$(\frac{\partial}{\partial t} - \Delta)\hat{Z} = (\frac{\partial}{\partial t} - \Delta)Z + \frac{2F}{t^3} + (\frac{\partial}{\partial t} - \Delta)\phi$$
$$= Z_{i\bar{j}}R_{\bar{j}\bar{i}} - 2\frac{\hat{Z}}{t} + 2\frac{\phi}{t} + (\frac{\partial}{\partial t} - \Delta)\phi > 0.$$

This contradicts the fact that $\hat{Z}$ achieves a minimum at (x_0, t_1). Therefore $\hat{Z}$ is strictly positive everywhere for $t > t_0$ and it follows that $Z \geq F/t$ for $t > t_0$. This proves Proposition 3.2.

4. The proof of main theorems

In this section we prove our main results (Theorem 1.3 and Theorem 1.4) as stated in the introduction.

Theorem 4.1. *Let X be a simply connected non-compact complex manifold of dimension n. Let $g_{i\bar{j}}$ be a complete solution to Eq.(1.1) defined for $-\infty < t < \infty$ with uniformly bounded curvature, nonnegative bisectional curvature and positive Ricci curvature where the scalar curvature R assumes its maximum in space-time. Then $g_{i\bar{j}}$ is necessarily a gradient Kähler-Ricci soliton.*

Proof. Let $g_{i\bar{j}}$ be a complete solution to Eq.(1.1) defined for $-\infty < t < \infty$ with bounded curvature, nonnegative bisectional curvature and positive Ricci curvature where the scalar curvature R assumes its maximum at a point (x_0, t_0) in space time. Then at this point, $\frac{\partial}{\partial t}R = 0$ and hence the quadratic form

$$Q = \frac{\partial R}{\partial t} + R_{,k}V^k + R_{,\bar{k}}V^{\bar{k}} + R_{k\bar{l}}V^iV^{\bar{j}}$$

vanishes in the direction $V = 0$. It follows from Proposition 3.1 that at any earlier time, there is a V at every point such that $Q = 0$ in the direction V. In fact, by considering the first variation of Q in V, null vectors must satisfy the equation

$$R_{,k} + R_{k\bar{l}}V_l = 0. \tag{4.1}$$

Since the Ricci tensor is positive definite, we see that such a null vector V is unique at each point and varies smoothly in space-time. Hence it gives rise to a time-dependent smooth section of the holomorphic tangent bundle TX. For this vector field V, we have $Q = 0$. Together with (4.1) this implies that

$$\frac{\partial R}{\partial t} + R_{,\bar{k}} V_k = 0. \tag{4.2}$$

Moreover, since $Q_{i\bar{j}} \geq 0$ and its trace $Q = 0$, we must have

$$Q_{i\bar{j}} = \frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},k} V^k + R_{i\bar{j},\bar{k}} V^{\bar{k}} + R_{i\bar{j}k\bar{l}} V^k V^{\bar{l}} = 0.$$

Again from the first variation of $Q_{i\bar{j}}$ in V, it follows that

$$R_{i\bar{j},k} + R_{i\bar{j}k\bar{l}} V^{\bar{l}} = 0, \tag{4.3}$$

and hence

$$\frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},\bar{k}} V^{\bar{k}} = 0. \tag{4.4}$$

Note that equations (4.3) and (4.4) are the first order and the second order equations satisfied by a gradient Kähler-Ricci soliton (see equations (2.3) and (2.5) in [1]). We are going to show that these indentities indeed imply that V is a holomorphic gradient vector field and the solution $g_{i\bar{j}}$ is a gradient soliton.

First, we apply the heat operator $(\frac{\partial}{\partial t} - \Delta)$ to (4.1). By using the evolution equations for the Ricci curvature and the scalar curvature and (4.3) we obtain

$$\begin{aligned} 0 =& (\frac{\partial}{\partial t} - \Delta)(R_{,k} + R_{k\bar{l}} V_l) \\ =& R_{i\bar{j},k}(R_{j\bar{i}} - V_{j\bar{i}}) + \frac{1}{2} R_{k\bar{l}} R_{,l} + R_{k\bar{l}}(\frac{\partial}{\partial t} - \Delta) V_l - R_{k\bar{l},\bar{p}} V_{l,p}. \end{aligned} \tag{4.5}$$

Next, we apply $(\frac{\partial}{\partial t} - \Delta)$ to (4.2). By direct computations and (4.4) we get

$$\begin{aligned} 0 =& (\frac{\partial}{\partial t} - \Delta)(\frac{\partial R}{\partial t} + R_{,\bar{k}} V_k) \\ =& 2R_{,k\bar{l}} R_{l\bar{k}} + R_{k\bar{l}} R_{l\bar{p}} R_{p\bar{k}} + R_{p\bar{l},\bar{k}} R_{l\bar{p}} V_k - \frac{1}{2} R_{k\bar{l}} V_l R_{,\bar{k}} \\ &+ R_{,\bar{k}}(\frac{\partial}{\partial t} - \Delta) V_k - R_{,k\bar{l}} V_{l,\bar{k}} - R_{,\bar{k}\bar{l}} V_{k,l}. \end{aligned} \tag{4.6}$$

From (4.1) it follows that

$$
\begin{aligned}
R_{,k\bar{l}} &= -R_{k\bar{p}}V_{p,\bar{l}} - R_{k\bar{p},\bar{l}}V_p \\
&= -R_{p\bar{l}}V_{\bar{p},k} - R_{k\bar{l},p}V_{\bar{p}}
\end{aligned} \tag{4.7}
$$

When we substitute (4.7) into (4.6) and use (4.5), there are many cancellations and we are left with

$$R_{k\bar{l}}(R_{l\bar{p}} - V_{l,\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p}) + R_{k\bar{l}}V_{l,p}V_{\bar{k},\bar{p}} = 0. \tag{4.8}$$

Now we can diagonalize the Ricci tensor at an arbitrary point so that $R_{k\bar{l}} = R_{k\bar{k}}\delta_{kl}$. Thus (4.8) becomes

$$R_{k\bar{k}}\left[|R_{k\bar{l}} - V_{k,\bar{l}}|^2 + |V_{k,l}|^2\right] = 0.$$

Since the Ricci curvature is positive we must have

$$V_{k,l} = V_{\bar{k},\bar{l}} = 0 \quad \text{and} \quad R_{k\bar{l}} = V_{k,\bar{l}} = V_{\bar{l},k}, \quad \text{for all } k, l. \tag{4.9}$$

Therefore V is a holomorphic vector field and the solution metric $g_{i\bar{j}}$ is a soliton. Furthermore, the second identity in (4.9) and the assumption that X is simply connected imply that the vector field V is the gradient of some function f on X. So our soliton $g_{i\bar{j}}$ is indeed a gradient soliton. This completes the proof of Theorem 4.1.

Theorem 4.2. *Let X be a simply connected non-compact complex manifold of dimension n. Let $g_{i\bar{j}}$ be a complete solution to Eq.(1.1) defined for $0 \le t < \infty$ with uniformly bounded curvature, nonnegative bisectional curvature and positive Ricci curvature where tR assumes its maximum in space-time. Then $g_{i\bar{j}}$ is necessarily a homothetically expanding gradient Kähler-Ricci soliton.*

Proof. At the point (x_0, t_0) in space time where tR assumes its maximum, we must have $R + t\frac{\partial}{\partial t}R = 0$. Hence, at (x_0, t_0), the quadratic form

$$Z = \frac{\partial R}{\partial t} + R_{,k}V^k + R_{,\bar{k}}V^{\bar{k}} + R_{k\bar{l}}V^iV^{\bar{j}} + \frac{R}{t}$$

vanishes in the direction $V = 0$. Note that necessarily $t_0 > 0$. It follows from Proposition 3.2 that at any earlier time, there is a V at every point such that $Z = 0$ in the direction V. By considering the first variation of Z in V, null vectors V must satisfy the equation

$$R_{,k} + R_{k\bar{l}}V_l = 0. \tag{4.10}$$

It again gives rise to a time-dependent smooth section V of the holomorphic tangent bundle TX. For this vector field V, we have $Z = 0$. Together with (4.1) this implies that

$$\frac{\partial R}{\partial t} + R_{,\bar{k}} V_k + \frac{R}{t} = 0. \tag{4.11}$$

Also, since $Z_{i\bar{j}} \geq 0$ and its trace $Z = 0$, we have

$$Z_{i\bar{j}} = \frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},k} V^k + R_{i\bar{j},\bar{k}} V^{\bar{k}} + R_{i\bar{j}k\bar{l}} V^k V^{\bar{l}} + \frac{1}{t} R_{i\bar{j}} = 0.$$

Again from the first variation of $Z_{i\bar{j}}$ in V, it follows that

$$R_{i\bar{j},k} + R_{i\bar{j}k\bar{l}} V^{\bar{l}} = 0, \tag{4.12}$$

so that

$$\frac{\partial}{\partial t} R_{i\bar{j}} + R_{i\bar{k}} R_{k\bar{j}} + R_{i\bar{j},\bar{k}} V^{\bar{k}} + \frac{1}{t} R_{i\bar{j}} = 0 \tag{4.13}$$

From (4.12) we obtain

$$\begin{aligned} 0 =& \left(\frac{\partial}{\partial t} - \Delta\right)(R_{,k} + R_{k\bar{l}} V_l) \\ =& R_{i\bar{j},k}(R_{j\bar{i}} - V_{j\bar{i}}) + \frac{1}{2} R_{k\bar{l}} R_{,l} + R_{k\bar{l}}\left(\frac{\partial}{\partial t} - \Delta\right) V_l - R_{k\bar{l},\bar{p}} V_{l,p} \end{aligned} \tag{4.14}$$

Next, by direct computations and (4.13) we get

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\partial R}{\partial t} + R_{,\bar{k}} V_k + \frac{R}{t}\right) \\ &= 2R_{,k\bar{l}} R_{l\bar{k}} + R_{k\bar{l}} R_{l\bar{p}} R_{p\bar{k}} + R_{p\bar{l},\bar{k}} R_{l\bar{p}} V_k \\ &\quad - \frac{1}{2} R_{k\bar{l}} V_l R_{,\bar{k}} + R_{,\bar{k}}\left(\frac{\partial}{\partial t} - \Delta\right) V_k - R_{,k\bar{l}} V_{l,\bar{k}} \\ &\quad - R_{,\bar{k}\bar{l}} V_{k,l} - \frac{R}{t^2}. \end{aligned} \tag{4.15}$$

When we substitute (4.7) into (4.15) and use (4.14) we are left with

$$R_{k\bar{l}}(R_{l\bar{p}} - V_{l,\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p}) - \frac{R}{t^2} + R_{k\bar{l}} V_{l,p} V_{\bar{k},\bar{p}} = 0. \tag{4.16}$$

We claim that

$$R_{k\bar{l}}(R_{l\bar{p}} - V_{l,\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p}) - \frac{R}{t^2} = R_{k\bar{l}}(R_{l\bar{p}} - V_{l,\bar{p}} + \frac{1}{t}g_{l\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p} + \frac{1}{t}g_{p\bar{k}}). \tag{4.17}$$

To verify Eq. (4.17), we compute that

$$\begin{aligned} R_{k\bar{l}} \ &\left(R_{l\bar{p}} - V_{l,\bar{p}} + \frac{1}{t}g_{l\bar{p}}\right)\left(R_{p\bar{k}} - V_{\bar{k},p} + \frac{1}{t}g_{p\bar{k}}\right) \\ &= R_{k\bar{l}} \ (R_{l\bar{p}} - V_{l,\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p}) \\ &\quad + \frac{1}{t}R_{k\bar{l}}\left[(R_{l\bar{k}} - V_{l,\bar{k}}) + (R_{l\bar{k}} - V_{\bar{k},l})\right] + \frac{R}{t^2} \\ &= R_{k\bar{l}} \ (R_{l\bar{p}} - V_{l,\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p}) - \frac{R}{t^2} \\ &\quad + \frac{1}{t}\left[2|R_{k\bar{l}}|^2 - R_{k\bar{l}}(V_{l,\bar{k}} + V_{\bar{k},l}) + 2\frac{R}{t}\right]. \end{aligned} \tag{4.18}$$

On the other hand, (4.7), (4.13) and the fact that $R_{,i\bar{j}} = \frac{\partial}{\partial t}R_{i\bar{j}}$ imply that

$$-R_{i\bar{l}}V_{l,\bar{j}} + R_{i\bar{k}}R_{k\bar{j}} + \frac{R_{i\bar{j}}}{t} = 0.$$

Taking the trace, we obtain

$$|R_{k\bar{l}}|^2 + \frac{R}{t} = R_{k\bar{l}}V_{l,\bar{k}}.$$

So the last term in (4.18) vanishes. This gives Eq.(4.17) and hence

$$R_{k\bar{l}}(R_{l\bar{p}} - V_{l,\bar{p}} + \frac{1}{t}g_{l\bar{p}})(R_{p\bar{k}} - V_{\bar{k},p} + \frac{1}{t}g_{p\bar{k}}) + R_{k\bar{l}}V_{l,p}V_{\bar{k},\bar{p}} = 0, \tag{4.19}$$

which implies that

$$V_{k,l} = V_{\bar{k},\bar{l}} = 0 \tag{4.20}$$

and

$$R_{k\bar{l}} = V_{k,\bar{l}} + \frac{1}{t}g_{k\bar{l}} = V_{\bar{l},k} + \frac{1}{t}g_{\bar{l}k}, \quad \text{for all } k, l \tag{4.21}$$

Now Eq. (4.20) and the assumption that X is simply connected imply that the vector field V is holomorphic and is the gradient of some function f on X. Eq. (4.21) says that $g_{i\bar{j}}$ is a homothetically expanding soliton. This completes the proof of Theorem 4.2.

5. Homothetically expanding gradient Ricci solitons

In this section we provide a one-parameter family of homothetically expanding gradient Kähler-Ricci solitons.

Let $z = (z_1, z_2, \cdots, z_n)$ be the coordinate system of the n-dimensional complex Euclidean space $\mathbf{C}^n$. Any Kähler metric $g_{i\bar{j}} dz^i dz^{\bar{j}}$ on $\mathbf{C}^n$ invariant under the unitary group $U(n)$ corresponds to a Kähler potential function $\Phi(z, \bar{z})$ given by

$$\Phi(z, \bar{z}) = u(t), \qquad t = \log |z|^2,$$

where u is a smooth function on $(-\infty, \infty)$ satisfing the differential inequalities

$$u'(t) > 0, \quad u''(t) > 0, \qquad t \in (-\infty, \infty) \tag{5.1}$$

and the asymptotic condition

$$u(t) = a_0 + a_1 e^t + a_2 e^{2t} + \cdots, (\rightarrow a_1 > 0) \tag{5.2}$$

as $t \to -\infty$.

In fact, we have, for $t = \log |z|^2$,

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(t) = e^{-t} u'(t) \delta_{i\bar{j}} + e^{-2t} \bar{z}_i z_j \big(u''(t) - u'(t) \big).$$

Consequently,

$$g^{i\bar{j}} = e^t (u'(t))^{-1} \delta^{i\bar{j}} + z_i \bar{z}_j \big(u''(t)^{-1} - u'(t)^{-1} \big),$$

and

$$\det(g_{i\bar{j}}) = e^{-nt} (u'(t))^{n-1} u''(t).$$

Let

$$f(t) = -\log \det(g_{i\bar{j}}) = nt - (n-1) \log u'(t) - \log u''(t). \tag{5.3}$$

Then the Ricci tensor of the metric $g_{i\bar{j}}$ is

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} f(t)$$

and hence

$$R_{i\bar{j}} + g_{i\bar{j}} = \partial_i \partial_{\bar{j}} (f + u).$$

The gradient vector field of the function $f + u$ is given by

$$V^i = g^{i\bar{j}} \partial_{\bar{j}} (f + u) = g^{i\bar{j}} e^{-t} z_j (f' + u') = z_i \frac{f'(t) + u'(t)}{u''(t)}.$$

Since z_i is holomorphic and $f' + u'/u''$ is real valued, we see that the vector field V is holomorphic if and only if

$$f'(t) + u'(t) = \lambda u''(t) \tag{5.4}$$

for some constant λ.

Plugging (5.3) into (5.4) and setting $\phi(t) = u'(t)$, we derive the following second order equation in ϕ:

$$\frac{\phi''}{\phi'} + \left[\frac{n-1}{\phi} + \lambda\right]\phi' = n + \phi. \tag{5.5}$$

Since the variable t does not appear in Eq.(5.5), we can solve for ϕ' and get

$$\begin{aligned}\phi' =& \frac{1}{\phi^{n-1}e^{\lambda\phi}}\left[n\int \phi^{n-1}e^{\lambda\phi}d\phi + \int \phi^n e^{\lambda\phi}d\phi\right]\\ =& \frac{1}{\lambda^{n+1}\phi^{n-1}}\left[\lambda^n\phi^n + \sum_{j=0}^{n-1}(-1)^{n-j}\frac{n!}{j!}(1-\lambda)\lambda^j\phi^j + ce^{-\lambda\phi}\right],\end{aligned}$$

where c is another constant.

An implicit solution $u(t)$ with $\phi(t) = u'(t)$ is given by

$$t = \int \frac{\lambda^{n+1}\phi^{n-1}d\phi}{\lambda^n\phi^n + \sum_{j=0}^{n-1}(-1)^{n-j}\frac{n!}{j!}(1-\lambda)\lambda^j\phi^j + ce^{-\lambda\phi}}. \tag{5.6}$$

The solution u satisfies the inequality (5.1) if and only if the constants λ and c are such that the function of ϕ,

$$h(\phi) = \frac{\lambda^{n+1}\phi^{n-1}e^{\lambda\phi}d\phi}{\lambda^n\phi^n e^{\lambda\phi} + e^{\lambda\phi}\sum_{j=0}^{n-1}(-1)^{n-j}\frac{n!}{j!}(1-\lambda)\lambda^j\phi^j + c} \tag{5.7}$$

is strictly positive in an interval I in the half-line $\phi > 0$, which is bounded by two poles of the function $h(\phi)$. Finally the asymptotic condition (5.2) corresponds to the property that the integrand in (5.6) has a simple pole at $\phi = 0$ with residue equal to 1. This in turn corresponds to that the constants λ and c must satisfy the equation

$$c = (-1)^{n+1}n!(1-\lambda). \tag{5.8}$$

One can verify that under the conditions (5.8) and $\lambda > 0$, the function $h(\phi)$ of (5.7) has poles only at $\phi = 0$ and $\phi = \infty$ and $h > 0$ for

$\phi \in (0, \infty)$. Thus, for each $\lambda > 0$ there exists a Kähler metric g_λ which is an expanding gradient Kähler-Ricci soliton.

To examine the asymptotic behavior of ϕ as $t \to \infty$, we set $x = 1/\phi$. Then from (5.6), we have

$$\begin{aligned} t &= -\int \frac{\lambda dx}{x + \text{higher order terms}} \\ &= -\lambda \log x + \cdots \end{aligned}$$

This implies that x has an expansion in powers of $e^{-t/\lambda}$, which in turn implies that ϕ is asymptotic to $e^{t/\lambda}$ as $t \to \infty$. In particular, this shows that the Kähler metric g_λ is complete.

By similar computations as we carried out in [2] (see Lemma 2.2 in [2]), one can show that, for $t \in (-\infty, \infty)$,

$$\phi - \phi' > 0 \qquad (\phi')^2 - \phi\phi'' > 0 \qquad (\phi'')^2 - \phi'\phi''' > 0$$

for $\lambda > 1$, and

$$\phi - \phi' < 0 \qquad (\phi')^2 - \phi\phi'' < 0 \qquad (\phi'')^2 - \phi'\phi''' < 0$$

for $0 < \lambda < 1$. It follows from our computation for the curvature tensor in [2] that the metric g_λ has positive sectional curvature for $\lambda > 1$ and negative sectional curvature for $0 < \lambda < 1$. In summary, we have

Proposition 5.1. *For each $\lambda > 0$, there exists on $\mathbf{C}^n$ a complete rotationally symmetric homothetically expanding gradient Kähler-Ricci soliton g_λ. The sectional curvature of g_λ is positive for $\lambda > 1$ and negative for $\lambda < 1$. Moreover, the Kähler potential function of g_λ is aymptotic to $|z|^{2/\lambda}$ as $|z|$ tends to the infinity.*

We remark that $\lambda = 1$ corresponds to the flat Euclidean metric on $\mathbf{C}^n$.

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