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WHITNEY FORMULA IN HIGHER DIMENSIONS

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Abstract

The classical Whitney formula relates the algebraic number of times that a generic immersed plane curve cuts itself to the index ("rotation number") of this curve. Both of these invariants are generalized to higher dimension for the immersions of an n-dimensional manifold into an open $(n+1)$-manifold with the null-homologous image. We give a version of the Whitney formula if n is even. We pay special attention to immersions of S^2 into $\mathbb{R}^3$. In this case the formula is stated in the same terms which were used by Whitney for immersions of S^1 into $\mathbb{R}^2$.

1. Introduction

Let $f : S^1 \to \mathbb{R}^2$ be a generic immersion (i.e., an immersion without triple points and self-tangencies). The *index* of f is the degree of the Gauss map (which maps S^1 to the direction of $df(v)$ where v is a tangent vector field positive with respect to the standard orientation of S^1). Whitney in [7] showed that the index is the only invariant of f up to deformation in the class of immersions.

Fix a generic point $x \in S^1$. The cyclic order on S^1 determined by the orientation defines a linear order on $S^1 - \{x\}$. This determines an ordering of the positive vectors tangent to the two branches of f at every double point d of f. Following Whitney we define the sign $\epsilon_x(d)$ of d to be $+1$ (resp. -1) if the frame composed of these tangent vectors is *negative* (resp. *positive*) in $\mathbb{R}^2$.

We define the function ind : $\mathbb{R}^2 \to \frac{1}{2}\mathbb{Z}$ in the following way. The (integer) value of ind at $y \in \mathbb{R}^2 - f(S^1)$ is defined as the linking number of the oriented cycle $f(S^1)$ and the 0-dimensional cycle composed of the point y taken with the positive orientation and a point near infinity taken with the negative

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orientation. The value of ind at $y \in f(S^1)$ is defined as the average of the indices of the components of $\mathbb{R}^2 - f(S^1)$ adjacent to y.

Theorem 1 (Whitney [7]).

$$\text{index}(f) = \sum_d \epsilon_x(d) + 2\,\text{ind}(f(x)).$$

This formula was found in 1937. However, no high-dimensional versions have been known though the problem of generalization of the Whitney formula is not new (see Arnold [2]). Both the left-hand side and the right-hand side can be defined for codimension-1 immersions of n-manifolds $f : S \to \mathbb{R}^{n+1}$. A straightforward approach to generalize the left-hand side is to define it as the degree of the Gauss map (i.e., the map $S \to S^n$ defined by the coorienting unit vector field normal to $f(S) \subset \mathbb{R}^{n+1}$). Unfortunately, already for $n = 2$ this number does not depend on immersion — it equals to $\frac{1}{2}\chi(S)$ for any even n. This reveals the important difference between the immersions of even- and odd-dimensional manifolds. We use another natural way of generalizing the left-hand side of the Whitney formula; the outcome coincides with the degree of the Gauss map for odd n (when it is non-trivial), but it is also non-trivial for even n. Our generalization makes sense not only for immersions to $\mathbb{R}^{n+1}$ but also for the immersions to an open $(n+1)$-manifold with null-homologous image. For its definition we use the integral calculus based on the Euler characteristic χ (developed by Viro [6]).

Let M be a simplicial complex. A *stratification* of M is a decomposition of M into a disjoint finite union of (open) strata where each stratum τ is a union of open simplices of M. Let $F : M \to \mathbb{R}$ be a function constant on each stratum (and, therefore, on each open simplex) which vanishes on all but finitely many simplices. The *integral* $\int_M F d\chi$ is defined by the following summation over all strata τ of M

$$\int_M F d\chi = \sum_\tau F(\tau)\chi(\tau),$$

where by $\chi(\tau)$ we mean the combinatorial Euler characteristic of τ — the alternated (by dimension) number of simplices of τ.

Lemma 1.1 (cf. Pukhlikov-Khovanskii [5]). *Let M be a simplicial manifold. Then $\int_M F d\chi$ depends neither on the stratification of M nor on the simplicial structure of M.*

Proof. By additivity of the combinatorial Euler characteristic

$$\int_M F d\chi = \sum_\sigma (-1)^{\dim(\sigma)} F(\sigma),$$

where the sum is taken over all the simplices σ of M. Therefore, $\int_M F d\chi$ does not depend on the stratification. The independence on the symplicial structure follows from the Alexander theorem [1] connecting any triangulation with the star moves, since $\int_M F d\chi$ is invariant under the Alexander moves.

We may express index(f) for $f : S^1 \to \mathbb{R}^2$ in terms of such integral. Denote by $\widetilde{f(S^1)}$ the smoothening of the curve $f(S^1)$ respecting the orientation. The singularities of a generic f are ordinary double points, so in local coordinates (x, y) $f(S^1)$ is given by $xy = 0$, and $\widetilde{f(S^1)}$ is given by $xy - \epsilon = 0$. Define $\widetilde{\text{ind}}(y)$, $y \in \mathbb{R}^2 - \widetilde{f(S^1)}$ as the linking number of the oriented cycle $\widetilde{f(S^1)}$ and the 0-dimensional cycle composed of y taken with the positive orientation and a point near infinity taken with the negative orientation.

Lemma 1.2 (cf. McIntyre-Cairns [4]).

$$\text{index}(f) = \int_{\mathbb{R}^2 - \widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi.$$

Proof. Note that index(f) does not change after smoothening (by the index of a multicomponent curve we mean the sum of indices of its components). To establish the equality index $\tilde{f} = \int_{\mathbb{R}^2 - \widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi$ we use induction on the number of components of $\widetilde{f(S^1)}$.

This allows us to rewrite the Whitney formula.

Theorem 1'.

$$\int_{\mathbb{R}^2 - \widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi = \sum_d \epsilon_x(d) + 2\,\text{ind}(f(x)).$$

The following corollary is a well-known application of the Whitney formula. Let n be the number of the double points of $f : S^1 \to \mathbb{R}^2$.

Corollary 1.

$$|\,\text{index}(f)| \leq n + 1.$$

Proof. To deduce the corollary from Theorem 1 it suffices to choose the base point $x \in S^1$ with exterior image (sitting on the boundary of the component of $\mathbb{R}^2 - f(S^1)$ with the non-compact closure) so that $|\,\text{ind}(f(x))| = \frac{1}{2}$.

Remark 1.3. Presentation of the Whitney formula in the form of Theorem 1' helps to generalize the formula to generic planar fronts. The *front* is a smooth map $f : S^1 \to \mathbb{R}^2$ equipped with a coorienting normal direction defined on $f(S^1)$, where f is an immersion except for a finite set of (semicubical) cusp points. We define index(f) as the degree of the Gauss map given by the coorientation. To obtain the "smoothened" (multicomponent) front (which has cusps but no double points) $\widetilde{f(S^1)}$ we smoothen the double points of $f(S^1)$ respecting both the orientation and the coorientation; see Figure 1. Other definitions stay the same.

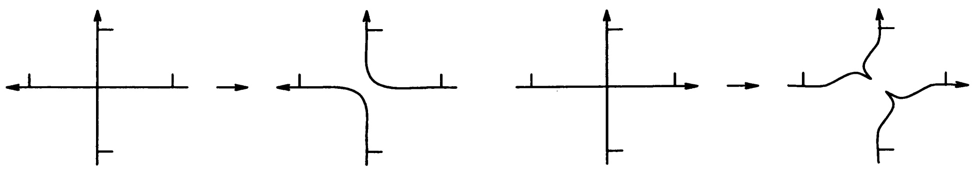

FIGURE 1.Smoothening of a double point of a front

Define the sign $\epsilon(c)$ of a cusp c to be $+1$ if the coorienting vector turns in the positive direction while going through a neighborhood of c in the orientation direction and -1 otherwise. Then

$$\text{index}(f) = \int_{\mathbb{R}^2-\widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi + \frac{1}{2}\sum_c \epsilon(c),$$

where c goes over all cusps of $f(S^1)$. Note that $\sum_c(\epsilon(c)$ is equal to the sum of the signs of all cusps of $\widetilde{f(S^1)}$ since the cusps appearing after smoothening are of opposite signs. Theorem 1', which also works for fronts, produces $\int_{\mathbb{R}^2-\widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi = \sum_d \epsilon_x(d) + 2\,\text{ind}(f(x))$, so

$$\text{index}(f) = \frac{1}{2}\sum_c \epsilon(c) + \sum_d \epsilon_x(d) + 2\,\text{ind}(f(x)).$$

Note that one can also incorporate the contribution of cusps into the integral by the following modification χ' of the Euler characteristics. For a component τ of $\mathbb{R}^2 - \widetilde{f(S^1)}$ we add $+\frac{1}{2}$ to $\chi(\tau)$ for each cusp of $\partial\tau$ turned inwards τ and $-\frac{1}{2}$ for each cusp turned outwards. Then $\text{index}(f) = \int_{\mathbb{R}^2-\widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi'$.

Remark 1.4. The definitions of the function ind and the signs $\epsilon_x(d)$ make sense as well for a generic immersion f of S^1 into a connected open oriented surface F other than $\mathbb{R}^2$ if $f(S^1)$ is homologous to zero. This leads to a new integer-valued invariant gen defined on the set of classes of null-homologous loops on F up to free homotopy. We define

$$\text{gen}(f) = \frac{1}{2}(\sum_d \epsilon_x(d) + 2\,\text{ind}(f(x)) - \int_{F-\widetilde{f(S^1)}} \widetilde{\text{ind}} d\chi)$$

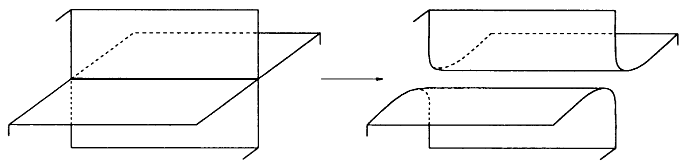

FIGURE 2. Smoothening of a double curve

for any choice of a base point $x \in S^1$. Note that if f is an embedding, then $|\operatorname{gen}(f)|$ equals the genus of the compact surface in F bounded by $f(S^1)$, so gen can be viewed as an "algebraic" version of genus which makes sense for immersed curves as well.

2. Immersions $S^2 \to R^3$

Let $f : S^2 \to \mathbb{R}^3$ be a generic immersion. Denote $\Sigma = f(S^2)$.

The inverse image of the double points $\Delta \subset \Sigma \subset \mathbb{R}^3$ is an immersed (multicomponent) curve $D \subset S^2$. The orientation of $\mathbb{R}^3$ and the orientation of S^2 determine a coorientation of the image $\Sigma - \Delta = f(S^2 - D)$, i.e., an orientation of the normal bundle $N_{\mathbb{R}^3}(\Sigma - \Delta)$ of $\Sigma - \Delta$ in $\mathbb{R}^3$, via the identity

$$T\mathbb{R}^3 = N_{\mathbb{R}^3}(\Sigma - \Delta) \oplus T(\Sigma - \Delta).$$

The set of non-singular points D' of D is equipped with the free involution $j : D' \to D'$ such that $fj = f$. The curve D' admits a natural coorientation in S^2 which comes from the coorientation of $\Sigma - \Delta$ via the identity

$$N_{S^2}(D') = N_{\mathbb{R}^3}\Sigma|_{jD'}.$$

The singular surface Σ admits a canonical smoothening $\tilde{\Sigma}$ respecting the coorientation (see Figure 2 and Figure 3). Choose local coordinates (x, y, z) at a point of D' so that Σ is given by $xy = 0$, and the coorientation of Σ is positive (given by the gradient of the coordinates). Then $\tilde{\Sigma}$ is given by $xy - \epsilon = 0$ for a small $\epsilon > 0$. Similarily, at a triple point Σ is given by $xyz = 0$ and $\tilde{\Sigma}$ is given by $xyz - \epsilon(x + y + z) = 0$.

Definition 2.1. The value of the function $\widetilde{\operatorname{ind}} : \mathbb{R}^3 - \tilde{\Sigma} \to \mathbb{Z}$ at $y \in \mathbb{R}^3 - \tilde{\Sigma}$ is defined as the linking number of the cooriented surface $\tilde{\Sigma}$ and the 0-dimensional cycle $[y] - [\infty]$ composed of y taken with the positive orientation and a point near infinity taken with the negative orientation.

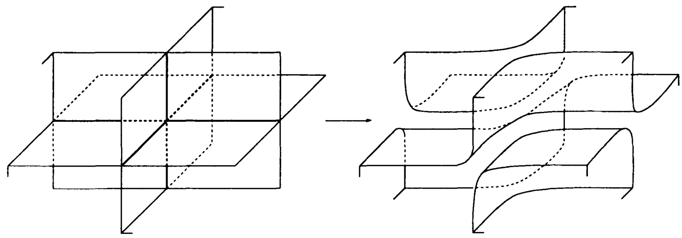

FIGURE 3. Smoothening of a triple point

Fix a base point $x \in S - D$. Define $\operatorname{ind}(f(x))$ as the average of the indices of the components of $\mathbb{R}^3 - \Sigma$ adjacent to $f(x)$.

The singular curve $D \subset \Sigma$ admits a canonical smoothening $\tilde{D} \subset \Sigma$ respecting the coorientation.

Definition 2.2. The *sign* $\epsilon_x(\tilde{d})$ of a component $\tilde{d}$ of $\tilde{D}$ is 1 (resp. -1) if the coorientation of $\tilde{d}$ induced from Σ coincides with (resp. opposite to) the coorientation of $\tilde{d}$ determined by the outer vector field of the component of $S^2 - \tilde{d}$ not containing x (i.e., by the normal vector field to $\tilde{d}$ pointing out to x).

Theorem 2.

$$-\int_{\mathbb{R}^3-\tilde{\Sigma}} \widetilde{\operatorname{ind}} d\chi = \sum_{\tilde{d}} \epsilon_x(\tilde{d}) + 2\operatorname{ind}(f(x)).$$

This theorem is a special case of Theorem 4 proven in Section 4.

Remark 2.3. Recall that the left-hand side of the original Whitney formula (Theorem 1) is the only degree-0 Vassiliev invariant of immersions of S^1 to $\mathbb{R}^2$. In the paper of Gorunov [3] $\int_{\mathbb{R}^3-\tilde{\Sigma}} \widetilde{\operatorname{ind}} d\chi$ appeared as the only non-trivial (apart from the number of double curves and triple points) degree-1 Vassiliev invariant of immersions of S^2 to $\mathbb{R}^3$; note that there are no non-trivial degree-0 invariants since the space of immersions $S^2 \to \mathbb{R}^3$ is connected.

The following corollary is similar to Corollary 1. Let n_δ be the number of double curves of f (i.e., the number of components of Δ after normalization). Let n_τ be the number of triple points of f.

Corollary 2.

$$\left|\int_{\mathbb{R}^3-\tilde{\Sigma}} \widetilde{\operatorname{ind}} d\chi\right| \le 2n_\delta + 2n_\tau + 1.$$

Proof. Similarily to the proof of Corollary 1 we choose an exterior base point x so that $|\operatorname{ind}(f(x))| = \frac{1}{2}$. Theorem 2 implies that

$$|\int_{\mathbb{R}^3-\tilde{\Sigma}} \widetilde{\operatorname{ind}} d\chi| \le |\sum_{\tilde{d}} \epsilon_x(\tilde{d})| + 1.$$

Note that $\sum_{\tilde{d}} \epsilon_x(\tilde{d})$ is equal to $\int_{S^2-\{x\}} \widetilde{\operatorname{ind}}' d\chi$, where $\widetilde{\operatorname{ind}}'(y)$, $y \in S^2 - \{x\}$, is the linking number of $\tilde{D}$ and $[y] - [\infty]$ in $S^2 - \{x\} \approx \mathbb{R}^2$. By Lemma 1.2 the latter is equal to the sum $\sum_d \operatorname{index}(d)$ over all the components $d \subset S^2 - \{x\} \approx \mathbb{R}^2$ of D. Corollary 1 yields that $|\operatorname{index}(d)|$ is not greater than one plus the number of self-intersections of d. Combining all this we obtain

$$|\int_{\mathbb{R}^3-\tilde{\Sigma}} \widetilde{\operatorname{ind}} d\chi| \le |\sum_{d} \operatorname{index}(d)| + 1 \le n_d + n_t + 1,$$

where n_d is the number of components of D after normalization, and n_t is the total number of self-intersections of components of D. The following lemmas imply that $n_d = 2n_\delta$ and $n_t \le 2n_\tau$ finishing the proof of the corollary.

Lemma 2.4. *The inverse image of every component δ of Δ consists of two components.*[1]

Proof. Let $p \in \delta$ be a generic point. The coorientation of Σ equips p with two vectors normal to δ and allows us to translate these vectors over δ. Since $\mathbb{R}^3$ does not contain disorienting loops, the monodromy at p does not swap the vectors and therefore they correspond to different components of the inverse image of δ.

Lemma 2.5. *Not more than two out of the three points in the inverse image of a triple point τ of f correspond to self-intersection points of components of D.*

Proof. Suppose all the three points t_x, t_y, t_z of the inverse image of τ correspond to self-intersection points of the components of D. Let t_x be a self-intersection point of a component a of D. Then Lemma 2.4 implies that t_y and t_z are self-intersection points of a component $b \neq a$ of D which maps onto the same component of Δ as a. In a similar way Lemma 2.4 leads to that t_x and t_z are self-intersection points of a and we get a contradiction.

Remark 2.6. Theorem 2 extends to generic maps $f : S^2 \to \mathbb{R}^3$ which are not necessarily immersions. The definitions of this section make also sense in this situation. The (integer) number $\operatorname{ind}(u)$, where u is a Whitney umbrella point is the average of the indices of the 3 components of $\mathbb{R}^3 - \Sigma$ adjacent to u (it equals the index of the component which is "the most

[1] recall that we consider components in "algebro-geometrical", not in "point-set-topological" sense

adjacent" to u). The coorientation does not extend to the Whitney umbrella points, but the smoothening $\tilde{\Sigma}$ of $\Sigma = f(S^2)$ is still a smooth surface which is defined by the coorientation at other points. Theorem 2 extends to

$$-\int_{\mathbb{R}^2-\tilde{\Sigma}} \widetilde{\operatorname{ind}} d\chi = \sum_u \operatorname{ind}(u) + \sum_{\tilde{d}} \epsilon_x(\tilde{d}) + 2\operatorname{ind}(f(x)),$$

where u and $\tilde{d}$ go over respectively all Whitney umbrellas and all components of $\tilde{D}$ (some of them contain Whitney points).

3. Indices and smoothening of the image of immersion in higher dimensions

Let $f : S \to R$ be a generic immersion of an oriented n-dimensional manifold S to an open oriented $(n+1)$-manifold R, and assume that $\Sigma = f(S)$ is homologous to zero in R. The definitions from the previous section generalize in the following way.

The inverse image of the double points $\Delta \subset \Sigma \subset R$ is a singular hypersurface $D \subset S$ equipped with the free involution $j : D' \to D'$ defined by the property $fj = f$ on the set D' of the non-singular points of D. The orientation of R and the orientation of S determine a coorientation of $\Sigma - \Delta = f(S - D)$ via the identity $TR = N_R(\Sigma) \oplus T\Sigma$ at non-singular points of Σ.

The hypersurface D' admits a natural coorientation in S, which comes from the coorientation of Σ via the identity $N_S(D') = N_R\Sigma|_{jD'}$. The coorientation of D' determines an orientation of D via $TS|_D = N_S(D) \oplus TD$.

The singular hypersurface $\Sigma \subset R$ admits a canonical smoothening $\tilde{\Sigma}$ respecting the coorientation. We may obtain this smoothening by the following inductive procedure.

The *multiplicity* of $x \in \Sigma$ is the cardinality of $f^{-1}(x)$. The multiplicity induces a stratification of Σ. A stratum Σ_k of multiplicity k is a smooth open manifold of dimension $n - k + 1$. The stratum Σ_2 is the singular locus of $U_2 = \Sigma_1 \cup \Sigma_2$. The proper regular neighborhood of Σ_2 in (R, U_2) is isomorphic to $\Sigma_2 \times (D^2, C_2)$, where D^2 is the 2-disk and C_2 is the cone over 4 points. The coorientation of C_2 in D^2 induced by the coorientation of Σ determines a smoothening of C_2 in D^2 and, therefore, it determines a smoothening of the regular neighborhood of Σ_2 (similar to the previous section, see Figure 2). Denote the resulting smoothening of U_2 by $\tilde{U}_2$.

Inductively we assume that $\tilde{U}_m$ is the smoothening of U_m. Denote $U_{m+1} = \tilde{U}_m \cup \Sigma_{m+1}$. The singular locus of U_{m+1} is Σ_{m+1}. The regular

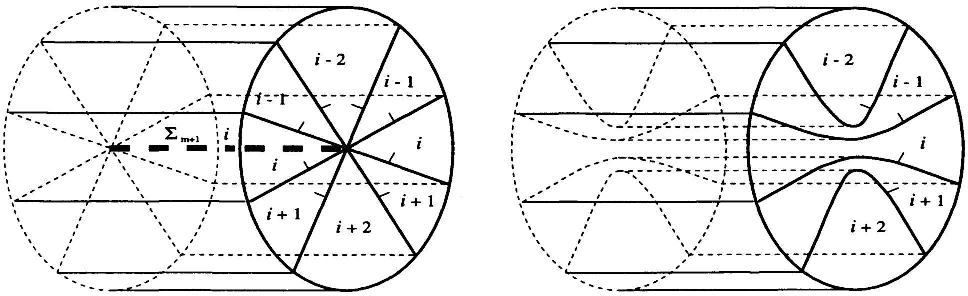

FIGURE 4.The smoothening of Σ_{m+1}

neighborhood of Σ_{m+1} in (R, U_{m+1}) is isomorphic to $\Sigma_{m+1} \times (D^{m+1}, C_{m+1})$, where D^{m+1} is the $(m+1)$-disk, and C_{m+1} is the cone over $m+1$ copies of S^{m-1} (see Figure 4). The coorientation of C_{m+1} in D^{m+1} induced by the coorientation of Σ determines a smoothening of C_{m+1} in D^{m+1} (see Figure 4) and therefore a smoothening of the regular neighborhood of Σ_{m+1}. Finally $\tilde{\Sigma} = \tilde{U}_{n+1}$ is a smooth (multicomponent) manifold.

Remark 3.1. We can also describe the smoothening of Σ locally without going through the above inductive procedure. Choose local coordinates $(x_1, \dots, x_{n+1})$ at $x \in \Sigma_k$ so that Σ is given by equation $x_1 \dots x_k = 0$, and the coorientation of Σ is positive (given by the gradient of the coordinates). Then $\tilde{\Sigma}$ is described by

$$x_1 \dots x_k + \sum_{m=1}^{[\frac{k}{2}]} (-\epsilon)^m \Big(\sum_{j_1 < \dots < j_{k-2m}} x_{j_1} \dots x_{j_{k-2m}} \Big) = 0$$

for a small $\epsilon > 0$.

Definition 3.2. The value of the function $\widetilde{\operatorname{ind}}_R : R \to \frac{1}{2}\mathbb{Z}$ at y is defined as the linking number of the cooriented (null-homologous) hypersurface $\tilde{\Sigma}$ and the 0-dimensional cycle $[y] - [\infty]$ composed of y taken with the positive orientation and a point near infinity taken with the negative orientation (if $y \in \Sigma$ then $\widetilde{\operatorname{ind}}_R(y)$ is the average of the indices of the components of $R - \tilde{\Sigma}$ adjacent to y). By the *point near infinity* we mean any point in a component of $R - \Sigma$ with a non-compact closure. Since Σ is closed and homologous to zero, the linking number does not depend on the choice of ∞.

In a similar way we define $\operatorname{ind}_R(y)$, $y \in R$, as the linking number of Σ and $[y] - [\infty]$.

Lemma 3.3.
$$\int_R \operatorname{ind}_R d\chi = \int_R \widetilde{\operatorname{ind}}_R d\chi.$$

Proof. Recall our smoothening process. The m-th step smoothens the regular neighborhood $\Sigma_{m+1} \times (D^{m+1}, C_{m+1})$ of Σ_{m+1} in U_{m+1}. It suffices to prove that the integral of index does not change after this smoothening. Let a component A of Σ_{m+1} be of index j in R. In the regular neighborhood of A we have $(n+1)$-dimensional strata of indices $j - \frac{m+1}{2}, \dots, j + \frac{m+1}{2}$, n-dimensional strata of indices $j - \frac{m}{2}, \dots, j + \frac{m}{2}$ and the core m-dimensional stratum A. The smoothening adds $(-1)^m$ to the Euler characteristics of ($(n+1)$-dimensional) strata of indices $j - \frac{m-1}{2}, \dots, \widehat{j}, \dots, j + \frac{m-1}{2}$ and adds $(-1)^{m-1}$ to the Euler characteristics of (n-dimensional) strata of indices $j - \frac{m}{2}, \dots, \widehat{j}, \dots, j + \frac{m}{2}$. The Euler characteristics of stratum of index j decreases by $1 + (-1)^{m+1}$. Therefore the total change of the integral is 0.

4. Immersions of even-dimensional manifolds

Lemma 4.1. *The oriented hypersurface $D \subset S$ is homologous to zero in S.*

Proof.
$$D = \partial \sum_s (\operatorname{ind}_R(s) + \frac{1}{2})\bar{s}.$$

The sum is taken over all the components s of $S - D$, $\bar{s}$ is the closure of s equipped with the orientation induced from S, and $\operatorname{ind}_R(s)$ is the value of the (constant) function $\operatorname{ind}_R|_s$ ($\frac{1}{2}$ is added to make the coefficients of the chain integer).

Denote by $\tilde{D}$ the unique smoothening of $D \subset S$ respecting the coorientation. Fixing a base point $x \in S - D$ and substituting $S - \{x\}$, $\tilde{D}$ and D to the Definition 3.2 give the definitions of $\widetilde{\operatorname{ind}}_{S-\{x\}} : S - (\{x\} \cup \tilde{D}) \to \mathbb{Z}$ and $\operatorname{ind}_{S-\{x\}} : S - \{x\} \to \frac{1}{2}\mathbb{Z}$.

Theorem 3.
$$-\int_{R-\tilde{\Sigma}} \widetilde{\operatorname{ind}}_R d\chi = \int_{S-\tilde{D}} \widetilde{\operatorname{ind}}_{S-\{x\}} d\chi + \chi(S) \operatorname{ind}_R(f(x)).$$

Lemma 4.2.
$$\int_R \operatorname{ind}_R d\chi = 0.$$

Remark 4.3. The proof of the lemma works for any function $p_0 : R - \Sigma \to \mathbb{Z}$ extended to $p : R \to \frac{1}{2}\mathbb{Z}$ by averaging (cf. Definition 3.2).

Proof of Lemma 4.2. By Lemma 3.3, $\int_R \operatorname{ind}_R d\chi = \int_R \widetilde{\operatorname{ind}}_R d\chi$. Denote $M_{\pm j} = (\pm\widetilde{\operatorname{ind}}_R)^{-1}[\frac{j}{2}, +\infty)$, $j \in \mathbb{N}$. Following Lebesgue, we decompose

$$\begin{aligned}\int_R \widetilde{\operatorname{ind}}_R d\chi =& \sum_{j=1}^{\infty} \frac{1}{2}\chi(M_j) - \sum_{j=-\infty}^{-1} \frac{1}{2}\chi(M_j)\\ =& \sum_{k=1}^{\infty} \frac{1}{2}(\chi(M_{2k-1}) + \chi(M_{2k})) - \sum_{j=-\infty}^{-1} \frac{1}{2}(\chi(M_{2k+1}) + \chi(M_{2k})).\end{aligned}$$

Note that $M_{\pm(2k-1)}$, $k \in \mathbb{N}$, is a compact odd-dimensional manifold with the interior $\operatorname{int}(M_{\pm(2k-1)}) = M_{\pm 2k}$. The double $W_{\pm k}$ of $M_{\pm(2k-1)}$ is a closed odd-dimensional manifold, thus $\chi(W_{\pm k}) = 0$. On the other hand for the (combinatorial) Euler characteristic we have

$$\begin{aligned}0 = \chi(W_{\pm k}) =& \chi(M_{\pm(2k-1)}) + \chi(\operatorname{int}(M_{\pm(2k-1)}))\\ =& \chi(M_{\pm(2k-1)}) + \chi(M_{\pm 2k})\end{aligned}$$

and the lemma follows.

Proof of Theorem 3. Recall again our smoothening process. The mth step of the smoothening adds $m \cdot \int_{\Sigma_{m+1}} \operatorname{ind}_R d\chi$ to $-\int_{R-\Sigma} \operatorname{ind}_R d\chi$, thus

(4.1)
$$-\int_{R-\tilde{\Sigma}} \widetilde{\operatorname{ind}_R} d\chi = -\int_{R-\Sigma} \operatorname{ind}_R d\chi + \sum_{j=2}^{n+1} (j-1) \int_{\Sigma_j} \operatorname{ind}_R d\chi.$$

Lemma 4.2 implies that $-\int_{R-\Sigma} \operatorname{ind}_R d\chi = \int_\Sigma \operatorname{ind}_R d\chi$. Substituting this in (4.1) gives

$$\begin{aligned}-\int_{R-\tilde{\Sigma}} \widetilde{\operatorname{ind}_R} d\chi =& \int_\Sigma \operatorname{ind}_R d\chi + \sum_{j=2}^{n+1} (j-1) \int_{\Sigma_j} \operatorname{ind}_R d\chi\\ =& \sum_{j=1}^{n+1} j \int_{\Sigma_j} \operatorname{ind}_R d\chi. \qquad (4.2)\end{aligned}$$

By the Fubini theorem [6] we get

$$\sum_{j=1}^{n+1} j \int_{\Sigma_j} \operatorname{ind}_R d\chi = \int_S \operatorname{ind}_R \circ f d\chi.$$

Note that $\text{ind}_R \circ f = \text{ind}_{S-\{x\}} + \text{ind}_R(f(x))$, so

$$\int_S \text{ind}_R \circ f d\chi = \int_S \text{ind}_{S-\{x\}} d\chi + \chi(S)\,\text{ind}_R(f(x)).$$

By Lemma 3.3, $\int_S \text{ind}_{S-\{x\}}\, d\chi = \int_S \widetilde{\text{ind}}_{S-\{x\}} d\chi$; substituting this in the previous equality and noticing that

$$\int_S \widetilde{\text{ind}}_{S-\{x\}} d\chi = \int_{S-\tilde{D}} \widetilde{\text{ind}}_{S-\{x\}} d\chi,$$

since the dimension of a smooth manifold $\tilde{D}$ is odd, we finally get

$$-\int_{R-\Sigma} \text{ind}_R\, d\chi = \int_{S-\tilde{D}} \widetilde{\text{ind}}_{S-\{x\}} d\chi + \chi(S)\,\text{ind}_R(f(x)).$$

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