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ON THE L^2-INDEX OF DIRAC OPERATORS ON MANIFOLDS WITH CORNERS OF CODIMENSION TWO. I

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0. Introduction

The purpose of this paper is to generalize the Atiyah–Patodi–Singer index theorem (or APS theorem for briefness) [1] to compact manifolds with corners of codimension two. To explain this in more detail, we first recall the results of [1].

Let X be an even–dimensional compact oriented manifold with smooth boundary M and assume that X is endowed with a metric which is a product near the boundary. Let $E \to X$ be a Clifford bundle over X. We also assume that the metric and the connection of E are products near the boundary. Let $D^+ : C^\infty(X, E^+) \to C^\infty(X, E^-)$ be the associated chiral Dirac operator. Then near the boundary, D^+ takes the form

$$D^+ = \gamma\Big(\frac{\partial}{\partial u} + A\Big), \tag{0.1}$$

where γ denotes Clifford multiplication by the inward unit normal vector field, u is the inward unit normal coordinate and A is a Dirac operator on M. Let P be the nonnegative spectral projection of A and denote by $C^\infty(X, E^+; P)$ the space of smooth sections of E^+ satisfying the boundary conditions

$$P(\varphi|_M) = 0. \tag{0.2}$$

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Then $D^+ : C^\infty(X, E^+; P) \to C^\infty(X, E^-)$ is a Fredholm operator and its index is given by

$$\text{Ind } D^+ = \int_X \omega_D - \frac{1}{2}(\eta(A) + \dim\ker A), \tag{0.3}$$

where ω_D is the Atiyah–Singer index density of D^+, and $\eta(A)$ is the eta invariant of the self–adjoint operator A. Recall that the eta invariant is defined by the eta function

$$\eta_A(s) = \sum_{\lambda \neq 0} \frac{\operatorname{sign}\lambda}{|\lambda|^s}, \quad \operatorname{Re}(s) > \dim M, \tag{0.4}$$

where λ runs over the nonzero eigenvalues of A. The series is absolutely convergent in the half–plane $\operatorname{Re}(s) > \dim M$ and has a meromorphic continuation to $\mathbb{C}$ with no pole at $s = 0$. Then the eta invariant $\eta(A)$ is defined as $\eta_A(0)$.

In the sequel, the APS theorem has been rederived by many different approaches. First, Cheeger [5], [6] gave a new proof of the APS theorem for the signature operator using analysis on spaces with conical singularities. If one attaches a cone $C(M)$ to the boundary M of X, then $X \cup C(M)$ becomes a space with a conical singularity. The boundary conditions (0.2) are now replaced by the L^2–conditions in the complement of the cone tip and the computation of the L^2–index of the signature operator reproduces (0.3) in this case. In fact, as emphasized by Cheeger [5], this should not be considered as a rederivation of the APS theorem, but rather as the natural signature formula for a class of singular spaces. This approach was extended in [2] to twisted Dirac operators.

In place of a cone one can also attach a half–cylinder to the boundary of X, endow $\mathbb{R}^+ \times M$ with certain warped product metrics, and rederive the APS theorem as L^2–index theorem for the corresponding Dirac operator on the enlarged manifold [18], [15], [22]. In particular, we may consider the manifold $\widehat{X} = X \cup_M (\mathbb{R}^+ \times M)$ where the cylinder is equipped with the product metric. Then $\widehat{X}$ is a complete manifold, and it was already observed in [1] that the index of the APS boundary value problem and the L^2–index of the canonically extended Dirac operator on $\widehat{X}$ are closely related. After changing coordinates, we may think of $\widehat{X}$ as being the interior of a compact manifold with boundary, endowed with a complete metric which sends the boundary to infinity. This is the point of view adopted by Melrose in [15].

In the present paper we study similar index problems on manifolds with corners of codimension two. Here, we follow [15] and define a manifold with corners to be a topological manifold X with boundary together with an embedding $\iota : X \hookrightarrow \widetilde{X}$ into a closed C^∞ manifold for which there exists a finite collection of functions $\rho_i \in C^\infty(\widetilde{X})$, $i \in I$, such that $\iota(X) = \{x \in \widetilde{X} \mid \rho_i(x) \geq 0, i \in I\}$ and for each subset $J \subset I$, the $d\rho_i$, $i \in J$, are linearly independent at each point $x \in \widetilde{X}$ where all ρ_i, $i \in J$, vanish. It follows from this definition that the boundary of X is the union of embedded hypersurfaces Y_i, $i \in I$. Let $Y_{i_1 \cdots i_k} = Y_{i_1} \cap \cdots \cap Y_{i_k}$, $i_j \in I$. Then we say that $Y_{i_1 \cdots i_k}$ is a corner of codimension k. We assume that X is endowed with a metric which is a product near all hypersurfaces and also near all corners. This means that for any corner $Y_{i_1 \cdots i_k}$ of codimension k, the metric is a product on a neighborhood of the form $(-\varepsilon, 0]^k \times Y_{i_1 \cdots i_k}$. Let D^+ be a Dirac operator on X, which is adopted to the product structure near the boundary. Then the goal is to generalize the APS theorem to this case. There are several reasons to expect that such an extension will be of interest. For example, by investigating index problems on manifolds with smooth boundary one is led very naturally to new spectral invariants on odd–dimensional manifolds, namely the eta invariants. Therefore, the presence of corners may lead to other new invariants attached to the corners. Furthermore, an index formula is also closely related with a gluing formula for eta invariants (see §8).

We do not know if there exists any generalization of the APS boundary conditions to the case of manifolds with corners. However, as explained above, the APS boundary conditions can be replaced by the L^2–conditions on the corresponding manifold with cylindrical ends. This is the approach we are going to use for a manifold with corners X_0. To get a complete manifold, we may either enlarge X_0 by gluing successively cylinders to boundary components or, we may endow X_0 with a complete metric of the type used by Melrose [15]. One may even think of more general geometric structures at infinity so that, for example, locally symmetric manifolds of finite volume are included naturally into the setting.

Working in the L^2–setting introduces new difficulties which are connected with the presence of the continuous spectrum. But this should not be considered as being necessarily a disadvantage, because the L^2–approach also opens up new perspectives of the whole subject. We have to study the spectral theory of Dirac operators on such manifolds. In particular, we have to investigate the structure of the continuous spectrum of these operators and to establish the link with scattering theory. There is a close relation of these problems with both the analysis of the N–body problem in quantum mechanics and the study of the spectral resolution of the Casimir operator on locally symmetric manifolds of finite volume [13]. This may be a lot more interesting than simply the derivation of an index formula.

In the present paper we consider only manifolds with corners of codimension ≤ 2. The reason for this assumption is obvious because, in order to treat the continuous spectrum of Dirac operators on the corresponding complete manifolds, we need to know as much as possible about the spectral resolution of the induced Dirac operators on the boundary hypersurfaces. In the codimension-two case, the boundary hypersurfaces are manifolds with cylindrical ends for which the spectral theory is well understood.

For simplicity, we assume that the boundary of our manifold with corners X_0 is the union of exactly two hypersurfaces M_1 and M_2, intersecting in a closed manifold Y which is the corner in this case (see Fig.1). The extension of our results to several corners of codimension two is straightforward. We enlarge X_0 by gluing first half-cylinders to the boundary components M_i and then filling in $(\mathbb{R}^+)^2 \times Y$ (see Fig.2). In this way, we construct a complete manifold X which is canonically associated with X_0. Let $Z_i = M_i \cup_Y (\mathbb{R}^+ \times Y)$, $i = 1, 2$, be the manifolds obtained from M_i by attaching half-cylinders to their boundary Y. Then Z_i are manifolds with cylindrical ends which may be regarded as the components of the ideal boundary of X. Note that X is the union of $\mathbb{R}^+ \times Z_1$, $\mathbb{R}^+ \times Z_2$ and X_0.

In §2 we study Dirac operators $D : C^\infty(X, E) \to C^\infty(X, E)$ on X. We assume that D is adopted to the product structure of X near infinity, that is, we assume that on $\mathbb{R}^+ \times Z_i$, D takes the form

$$D = \gamma_i \Big(\frac{\partial}{\partial u_i} + A_i \Big), \tag{0.5}$$

and on $(\mathbb{R}^+)^2 \times Y$, it can be written as

$$D = \gamma_1 \frac{\partial}{\partial u_1} + \gamma_2 \frac{\partial}{\partial u_2} + D_Y,$$

where conditions (2.1) – (2.4) are satisfied. One of our main results in this section is that the space of L^2 solutions of D is finite-dimensional. Hence, if X is even-dimensional, the chiral Dirac operator $D^+ : C^\infty(X, E^+) \to C^\infty(X, E^-)$ has a well-defined L^2 index

$$L^2\text{-}\operatorname{Ind} D^+ = \dim(\ker(D^+) \cap L^2) - \dim(\ker(D^-) \cap L^2).$$

In §3 we study the space $\mathcal{H}^*_{(2)}(X)$ of L^2 harmonic forms on X. The main result is Proposition 3.13 which states that the canonical map

$\mathcal{H}^*_{(2)}(X) \to H^*(X)$ induces an isomorphism

$$j : \mathcal{H}^*_{(2)}(X) \xrightarrow{\sim} \operatorname{Im}(H^*_c(X) \to H^*(X)), \tag{0.6}$$

where $H^*_c(X)$ denotes the de Rham cohomology with compact supports. Suppose that $\dim X_0 = 4k$ and let $\operatorname{Sign}(X_0)$ be the signature of the compact manifold with boundary X_0. As a consequence of (0.6) we get that the L^2 index of the signature operator on X equals $\operatorname{Sign}(X_0)$, which should be expected to hold for the right choice of boundary conditions.

Let $\Delta = D^2$. In §4, we study the heat equation for Δ and construct a parametrix for the fundamental solution of $\partial/\partial t + \Delta$.

For the derivation of the index formula we need to describe the continuous spectrum of $\overline{\Delta}$ near zero. In §5 we study the resolvent $(\overline{\Delta} - \lambda^2)^{-1}$, $\operatorname{Im}(\lambda) > 0$. If the Dirac operator D_Y on the corner is invertible, then we prove that $(\overline{\Delta} - \lambda^2)^{-1}$, regarded as operator in certain weighted L^2 spaces, has an analytic continuation to a neighborhood of 0. We believe that the condition $\ker D_Y = 0$ can be removed. Then, however, $(\overline{\Delta} - \lambda^2)^{-1}$ does not extend analytically to a small disc around $\lambda = 0$, but rather to the logarithmic covering of such a disc. The investigation of the analytic continuation of the resolvent in general requires a more thorough study of the continuous spectrum, which we postpone to a forthcoming paper.

Let $A_i : C^\infty(Z_i, E|_{Z_i}) \to C^\infty(Z_i, E|_{Z_i})$ be the Dirac operator defined by (0.5) and let $\mathcal{A}_i$ be its unique self-adjoint extension in L^2. Using the analytic continuation of the resolvent to a neighborhood of the origin, we construct in §6 generalized eigensections $E_i(\phi, \lambda)$, $\operatorname{Im}(\lambda) > 0$, of Δ which are attached to $\phi \in \ker \mathcal{A}_i$, $i = 1, 2$. If $\ker D_Y = 0$, then the generalized eigensections $E_i(\phi, \lambda)$ can be extended to meromorphic functions of λ for $|\lambda| < c$. We establish a number of properties, including the functional equations, satisfied by the generalized eigensections. The continuous spectrum of $\overline{\Delta}$ near zero can be completely described in terms of the generalized eigensections $E_i(\phi, \lambda)$, $\phi \in \ker \mathcal{A}_i$, $i = 1, 2$.

Then in §7, we prove our index formula. Our approach is based on the local version of the McKean–Singer formula. Let $\dim X = 2k$ and let $\tau : E \to E$ be the canonical involution of the Clifford bundle. Then we have $\tau D = -D\tau$. Let $e^{-tD^2}(x, y)$ be the kernel of the heat operator e^{-tD^2}. Then the local McKean–Singer formula states that

$$\frac{\partial}{\partial t} \operatorname{tr}(\tau\, e^{-tD^2}(x, x)) = \operatorname{div} V_D, \tag{0.7}$$

where V_D is the vector field on X which is given locally, with respect to an orthonormal moving frame $\{e_i\}_{i=1}^n$, by

$$V_D = \sum_{i=1}^n \frac{1}{2} \operatorname{tr}\big(e_j \cdot \tau D e^{-tD^2}(x,x)\big)\, e_j.$$

For a closed manifold, (0.7) implies the usual statement that the supertrace $\operatorname{Tr}(\tau e^{-tD^2})$ is independent of t and equals Ind D^+. Since our manifold is noncompact, we exhaust it by compact submanifolds X_T, $T \geq 0$, with piecewise smooth boundary. For $T \geq 0$, let $Z_{i,T} = M_i \cup_Y ([0,T] \times Y)$. Then the boundary of X_T is the union of $Z_{1,T}$ and $Z_{2,T}$, which intersect in $\{T\} \times Y \simeq Y$. Using (0.7) together with the local index theorem for Dirac operators [10], we get

$$\begin{aligned} &L^2\text{-}\operatorname{Ind} D^+ \\ (0.8)\qquad &= \int_X \omega_D + \lim_{T\to\infty} \frac{1}{2} \int_0^\infty \int_{\partial X_T} \operatorname{tr}\big(e_n \cdot \tau D e^{-tD^2}(x,x)\big)\, dx\, dt, \end{aligned}$$

where ω_D is the Atiyah–Singer index density of D^+ and e_n is the outward unit normal vector field to the boundary. To compute the limit on the right-hand side of (0.8), we split the t-integral as $\int_0^{\sqrt{T}} + \int_{\sqrt{T}}^\infty$ and study the corresponding double integrals separately. The limit of the first double integral, where t runs from 0 to $\sqrt{T}$, can be described in terms of eta invariants. Since $\tau D = -D\tau$, it follows from (0.5) that the involution τ commutes with $\mathcal{A}_j$, $j = 1,2$. Let $\mathcal{A}_j^+$ be the restriction of $\mathcal{A}_j$ to $L^2(Z_i, E^+|_{Z_i})$. Then the eta invariant $\eta(\mathcal{A}_j^+)$ of $\mathcal{A}_j^+$ is defined by

$$(0.9)\qquad \eta(\mathcal{A}_j^+) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \int_{Z_j} \operatorname{tr}\big(A_j^+ e^{-t(\mathcal{A}_j^+)^2}(z_j, z_j)\big)\, dz_j\, dt,$$

where $e^{-t(\mathcal{A}_j^+)^2}(x,y)$ denotes the kernel of $e^{-t(\mathcal{A}_j^+)^2}$. The absolute convergence of (0.9) is proved in [16]. From the results of [16], it follows that as $T \to \infty$, the first double integral converges to $1/2(\eta(\mathcal{A}_1^+) + \eta(\mathcal{A}_2^+))$.

Let $R(T)$ be the remaining double integral, where t runs from $\sqrt{T}$ to ∞. The behaviour as $T \to \infty$, of $R(T)$ is determined by the continuous spectrum of $\overline{\Delta}$ near zero. If the continuous spectrum has a positive lower bound, then $R(T)$ decays exponentially as $T \to \infty$. Our analysis of the continuous spectrum shows that this case occurs if and only if $\ker D_Y = 0$ and $\ker \mathcal{A}_j = 0$, $j = 1,2$. We assume only that $\ker D_Y = 0$, which may be regarded as intermediate case. Then the generalized eigensections $E_i(\phi, \lambda)$, $\phi \in \ker \mathcal{A}_i$, determine scattering matrices $C_i(\lambda)$: $\ker \mathcal{A}_i \to \ker \mathcal{A}_i$ which are meromorphic functions on a disc $|\lambda| < c$ and satisfy the functional equations $C_i(\lambda)C_i(-\lambda)$ =Id, $|\lambda| < c$, $i = 1,2$. In particular, $C_i(\lambda)$ is regular at $\lambda = 0$ and $C_i(0)^2$ =Id. The canonical involution τ of E induces an involution of $\ker \mathcal{A}_i$, which we also denote by τ. The scattering matrix $C_i(\lambda)$ commutes with τ. Let $C_i^+(\lambda)$ be the restriction of $C_i(\lambda)$ to the $+1$-eigenspace of τ. Then it follows that as $T \to \infty$, $R(T)$ converges to $-1/2\,\mathrm{Tr}(C_1^+(0)) - 1/2\,\mathrm{Tr}(C_2^+(0))$, and our final index formula can now be stated as follows:

Theorem 0.1. *Let X_0 be an even–dimensional Riemannian manifold with a corner of codimension two such that the boundary of X_0 is the union of two components M_1 and M_2, intersecting in a closed manifold Y. Let X be the associated complete manifold constructed above with ideal boundary components $Z_i = M_i \cup_Y (\mathbb{R}^+ \times Y)$, $i = 1, 2$. Let*

$$D^+ : C^\infty(X, E^+) \to C^\infty(X, E^-)$$

be a chiral Dirac operator on X and assume that on $\mathbb{R}^+ \times Z_i$, D^+ takes the form

$$D^+ = \gamma_i\Big(\frac{\partial}{\partial u_i} + A_i^+\Big), \quad i = 1, 2,$$

where γ_i denotes Clifford multiplication by the inward unit normal vector field and A_i^+ is a Dirac operator on Z_i. Suppose that the corresponding Dirac operator D_Y on the corner Y is invertible. Then we have

$$\begin{aligned} L^2\text{-}\mathrm{Ind}\, D^+ = \int_X \omega_D &- \frac{1}{2}\{\eta(\mathcal{A}_1^+) + \mathrm{Tr}(C_1^+(0))\} \\ &- \frac{1}{2}\{\eta(\mathcal{A}_2^+) + \mathrm{Tr}(C_2^+(0))\}, \end{aligned} \tag{0.10}$$

where $\eta(\mathcal{A}_i^+)$ is the eta invariant, defined by (0.9), of the unique self–adjoint extension $\mathcal{A}_i^+$ of A_i^+ in L^2, and $C_i^+(\lambda) : \ker \mathcal{A}_i^+ \to \ker \mathcal{A}_i^+$, $|\lambda| < c$, is the scattering matrix associated with $\mathcal{A}_i^+$.

This index formula can be rewritten such that the right-hand side involves only terms which are defined on X_0. First observe that near the boundary of M_j, A_j^+ takes the form

$$A_j^+ = \sigma_j\Big(\frac{\partial}{\partial v_j} + B_j\Big),$$

where B_j is some Dirac operator on Y and σ_j denotes Clifford multiplication by the outward unit normal vector field. Let P_j be the negative spectral projection for B_j. Using P_j, we impose APS boundary conditions for A_j^+ at $\partial M_j = Y$. Since $\ker D_Y = \ker B_j = 0$, we get a self–adjoint extension $(A_j^+)_{P_j}$. In [16] we proved that $(A_j^+)_{P_j}$ has pure point spectrum and the eta invariant $\eta(A_j^+, P_j)$ of $(A_j^+)_{P_j}$ can be defined by analytic continuation of a series which is analogous to (0.4).

Moreover, by Theorem 0.1 of [16], we have

$$\eta(\mathcal{A}_j^+) = \eta(A_j^+, P_j), \quad j = 1, 2.$$

Let $h_j^\pm$ be the dimension of the subspace of $\ker \mathcal{A}_j^\pm$ consisting of all limiting values of extended L^2 solutions of $D^\pm$ (see the end of §7 for the definition). Then we have

$$\operatorname{Tr}(C_j^+(0)) = h_j^+ - h_j^- \quad \text{and} \quad \dim\ker \mathcal{A}_j^+ = h_j^+ + h_j^-.$$

In general, the L^2 index is not stable under compactly supported perturbations. However, as the index formula shows, $L^2\text{-}\operatorname{Ind} D^+ - h_1^- - h_2^-$ is stable under perturbations supported on a compact subset of X. This suggests to define

$$\widetilde{\operatorname{Ind}}\, D^+ = L^2\text{-}\operatorname{Ind}\, D^+ - h_1^- - h_2^-. \tag{0.11}$$

If the boundary of X_0 is smooth, it is proved in Corollary 3.13 of [1] that (0.10) equals the index of the APS boundary value problem. This index can also be interpreted as Fredholm index in weighted Sobolev spaces [15]. Therefore we think that $\widetilde{\operatorname{Ind}}\, D^+$, as defined above, has a similar interpretation which justifies the notation. Now we can reformulate Theorem 0.1 as follows:

Theorem 0.2. *Let the assumptions be the same as in Theorem 0.1. Suppose that near the boundary of M_i, A_i^+ has the form*

$$A_i^+ = \sigma_i\Big(\frac{\partial}{\partial v_i} + B_i\Big), \quad i = 1, 2,$$

where B_i is a Dirac operator on Y and σ_i denotes Clifford multiplication by the outward unit normal vector field. Let P_i be the negative spectral projection with respect to B_i. Then we have

$$\begin{aligned}\widetilde{\operatorname{Ind}}\, D^+ = \int_{X_0} \omega_D &- \frac{1}{2}\Big\{\eta(A_1^+, P_1) + \dim\ker(A_1^+)_{P_1}\Big\} \\ &- \frac{1}{2}\Big\{\eta(A_2^+, P_2) + \dim\ker(A_2^+)_{P_2}\Big\},\end{aligned}$$

where $\widetilde{\operatorname{Ind}}\, D^+$ is defined by (0.11), ω_D is the Atiyah–Singer index density for D^+ and $\eta(A_j^+, P_j)$ is the eta invariant of the self-adjoint extension $(A_j^+)_{P_j}$ of $A_j^+|_{M_j}$ with respect to the APS boundary conditions defined by P_j.

The elimination of the condition $\ker D_Y = 0$ requires a better understanding of the continuous spectrum of $\overline{\Delta}$. There will be no significant change of the index formula. Again, the contribution of the continuous spectrum in the index formula will be given as combination of traces of scattering matrices at energy zero. This will be discussed elsewhere.

In §8 we use the index theorem to derive a splitting formula for eta invariants. A number of authors [4], [8], [14], [25] have proved splitting formulas mod $\mathbb{Z}$. We identify explicitly the integer part as combination of indices of certain Dirac operators.

Finally, in §9 we discuss as an example the case where X is the product of two even–dimensional manifolds with cylindrical ends, say X_1 and X_2. We also assume that the Clifford bundle is the exterior tensor product of Clifford bundles over X_i. Then the L^2 index of the corresponding Dirac operator D^+ is the product of the L^2 indices of the Dirac operators D_i^+ on X_i. Using the index formula for Dirac operators on manifolds with cylindrical ends, we get a formula for L^2-Ind(D^+). We compare this formula with the answer given by Theorem 0.1. The boundary term in this index formula displays a natural decomposition where each term is associated with a particular stratum of the boundary at infinity. In the present case, the corner Y is the product of two odd–dimensional closed Riemannian manifolds. The term which seems to be naturally attached to the corner is the product of the eta–invariants of the induced Dirac operators on Y_i. At the end we briefly discuss a possible approach to obtain such a decomposition in general.

1. Manifolds with corners of codimension two

To simplify notation, we shall only consider the simplest case of a single corner of codimension two. The extension of our results to manifolds with several corners of codimension two is straightforward.

Let M be a closed oriented $(n-1)$-dimensional C^∞ Riemannian manifold and let $Y \subset M$ be a closed oriented submanifold of codimension 1, which separates M in two submanifolds, say M_1 and M_2. We also assume that, near Y, M is isometric to the product $(-\varepsilon, \varepsilon) \times Y$, $\varepsilon > 0$. Let X_0 be a compact oriented n–dimensional Riemannian manifold with boundary M. We assume that the metric on X_0 has the following properties:

(1) In a neighborhood $(-\varepsilon, 0] \times M_i$ of the boundary component M_i, $i = 1, 2$, X_0 is isometric to the product metric on $(-\varepsilon, 0] \times M_i$.
(2) In a neighborhood of the corner Y, X_0 is isometric to $(-\varepsilon, 0]^2 \times Y$, equipped with the product metric.

We shall call X_0 a *manifold with a corner at* Y. More generally, we may consider a compact oriented Riemannian manifold X which has k boundary components Y_i, and near each boundary component Y_i, the metric has a product structure as described above.

Example 1. Let M_i, $i = 1, 2$, be two compact oriented Rieman-

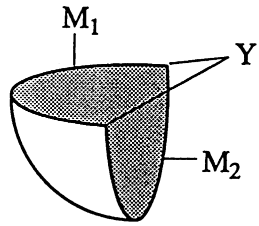

FIGURE 1. A 2–dimensional manifold with a corner at Y.

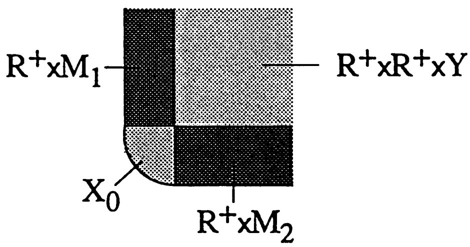

FIGURE 2. The complete manifold X.

nian manifolds with C^∞ boundary B_i. Suppose that in a neighborhood $(-\varepsilon, 0] \times B_i$ of the boundary B_i, the metric of M_i is isometric to the product metric on this neighborhood. Then $X_0 = M_1 \times M_2$ is a manifold with a corner at $Y = B_1 \times B_2$.

We associate with X_0 a noncompact complete Riemannian manifold X as follows. Let

$$Z_i = M_i \cup_Y (\mathbb{R}^+ \times Y), \quad i = 1, 2, \tag{1.1}$$

where the bottom $\{0\} \times Y$ of the half-cylinder is identified with $\partial M_i = Y$. Then Z_i is a manifold with a cylindrical end. Furthermore, let

$$W_1 = X_0 \cup_{M_2} (\mathbb{R}^+ \times M_2), \quad W_2 = X_0 \cup_{M_1} (\mathbb{R}^+ \times M_1). \tag{1.2}$$

Observe that W_i is an n–dimensional manifold with boundary Z_i. Set

$$X = W_1 \cup_{Z_1} (\mathbb{R}^+ \times Z_1) = W_2 \cup_{Z_2} (\mathbb{R}^+ \times Z_2), \tag{1.3}$$

where $\{0\} \times Z_i$ is identified with the boundary Z_i of W_i, $i = 1, 2$. We equip $\mathbb{R}^+ \times M_i$ and $\mathbb{R}^+ \times Z_i$, $i = 1, 2$, with the product metric and extend in this way the metric on X_0 to a complete C^∞ Riemannian metric on X. We call X a *complete manifold with a corner at* Y. If we unravel (1.1) and (1.2), we get a further decomposition of X as

$$X = X_0 \cup (\mathbb{R}^+ \times M_1) \cup (\mathbb{R}^+ \times M_2) \cup ((\mathbb{R}^+)^2 \times Y), \tag{1.4}$$

where the boundaries are identified correspondingly. See Fig.2 for an illustration.

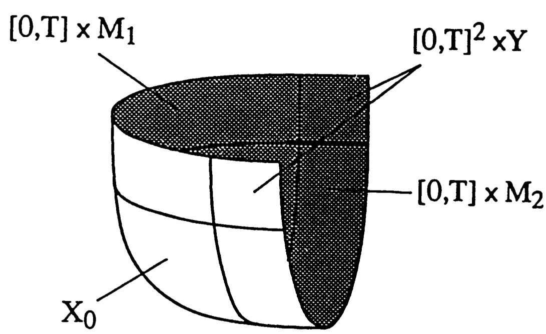

FIGURE 3. The extended manifold X_T.

Example 2. Let Z_1 and Z_2 be two Riemannian manifolds with cylindrical ends, that is, $Z_i = M_i \cup (\mathbb{R}^+ \times B_i)$ where M_i is a compact Riemannian manifold with boundary B_i. Then $X = Z_1 \times Z_2$ is a complete manifold with a corner at $B_1 \times B_2$.

There exists a distinguished exhaustion of X by compact submanifolds X_T, $T \geq 0$, which we shall now describe. Let $T \geq 0$ be given and set

$$Z_{i,T} = M_i \cup_Y ([0,T] \times Y), \quad i = 1, 2. \tag{1.5}$$

Here it is understood that $\{0\} \times Y$ is identified with ∂M_i. Then $Z_{i,T}$, $T \geq 0$, is a family of compact manifolds with boundary which exhaust Z_i. Next we attach the finite cylinder $[0,T] \times M_1$ to X_0 by identifying $M_1 \subset \partial X_0$ and $\{0\} \times M_1$ in the obvious way. The resulting manifold $W_{2,T} = X_0 \cup M_1([0,T] \times M_1)$ is a manifold with a corner. Note that the boundary of $W_{2,T}$ is the union of M_1 and $Z_{2,T}$. Now we glue the finite cylinder $[0,T] \times Z_{2,T}$ to $W_{2,T}$ where $\{0\} \times Z_{2,T}$ is identified with the corresponding piece $Z_{2,T}$ of the boundary of $W_{2,T}$. The resulting manifold is called X_T, that is,

$$X_T = W_{2,T} \cup_{Z_{2,T}} ([0,T] \times Z_{2,T}), \quad T \geq 0. \tag{1.6}$$

The manifold X_T is again a manifold with a corner at Y. Moreover, the boundary is given by

$$\partial X_T = Z_{1,T} \cup_Y (-Z_{2,T}). \tag{1.7}$$

We may also construct X_T by a different gluing process, namely

$$X_T = X_0 \cup ([0,T] \times M_1) \cup ([0,T] \times M_2) \cup ([0,T]^2 \times Y), \tag{1.8}$$

where the boundaries are identified correspondingly (see Fig. 3).

2. Dirac operators on complete manifolds with corners

Let X be as above and let $E \to X$ be a Clifford bundle over X (cf. [11]). Let $D : C^\infty(X, E) \to C^\infty(X, E)$ be the (generalized) Dirac operator associated with E. We assume that the Hermitian metric and the connection ∇^E of the Clifford bundle E are compatible with the product structure of X. Let R^E be the curvature tensor of E. Then $|(\nabla^E)^k R^E(x)|$ is uniformly bounded on X for all $k \in \mathbb{N}$. Furthermore, D has the following properties:

(i) There exist Clifford bundles E_i over Z_i such that $E|_{\mathbb{R}^+ \times Z_i}$ is the pull–back of E_i, and on $\mathbb{R}^+ \times Z_i$ we have

$$D = \gamma_i\Big(\frac{\partial}{\partial u_i} + A_i\Big), \quad i = 1, 2, \tag{2.1}$$

where A_i is the Dirac operator of E_i, and γ_i denotes Clifford multiplication by the outward unit normal vector field. The γ_i satisfy the following relation

$$\gamma_i^2 = -\mathrm{Id}, \quad \gamma_i^* = -\gamma_i \quad \text{and} \quad \gamma_i A_i = -A_i\gamma_i, \quad i = 1, 2. \tag{2.2}$$

(ii) There exists a Clifford bundle S over Y such that $E|_{(\mathbb{R}^+)^2 \times Y}$ is the pull–back of S, and on $(\mathbb{R}^+)^2 \times Y$ we have

$$D = \gamma_1\frac{\partial}{\partial u_1} + \gamma_2\frac{\partial}{\partial u_2} + D_Y, \tag{2.3}$$

where D_Y is the Dirac operator of S, and γ_1, γ_2 are Clifford multiplications by the outward unit normal vector fields. In addition to (2.2), the following relations hold

$$\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0, \quad \gamma_i D_Y = -D_Y\gamma_i, \quad i = 1, 2. \tag{2.4}$$

We shall now describe some of the basic properties of D. Since X is a complete Riemannian manifold, from Theorem 1.17 of [11] it follows that $D : C_c^\infty(E) \to L^2(E)$ is essentially self–adjoint. For $\varphi \in C_c^\infty(E)$, set

$$\| \varphi \|_k^2 = \sum_{j=0}^{k} \| \nabla^j \varphi \|_{L^2}^2 \tag{2.5}$$

and denote by $H^k(E)$ the completion of $C_c^\infty(E)$ in this norm. The Sobolev space $H^k(E)$ coincides with the space of all $\varphi \in L^2(E)$ such that the distributional image $\nabla^l\varphi$ is also in $L^2(E)$ for all $l \le k$. The connection ∇ gives rise to an elliptic second order differential operator $\nabla^*\nabla : C^\infty(E) \longrightarrow C^\infty(E)$. Recall that the following Bochner–Weitzenböck formula holds:

$$D^2 = \nabla^*\nabla + R^E, \tag{2.6}$$

where R^E is defined by the curvature tensor of E. More precisely, if $e_1, ..., e_n$ is an orthonormal basis of T_xX, then

$$R^E = \frac{1}{2}\sum_{i,j=1}^{n} e_i \cdot e_j \; R^E_{e_i,e_j},$$

and $R^E_{v,w}$ is the curvature transformation of E. Due to our assumption on E, the curvature tensor is uniformly bounded on X. Therefore, by (2.6) there exist constants $C_1, C_2 > 0$ such that

$$C_1 \parallel \phi \parallel_1^2 \leq \parallel \phi \parallel^2 + \parallel D\phi \parallel^2 \leq C_2 \parallel \phi \parallel_1^2$$

for all $\phi \in H^1(E)$. This implies that an equivalent norm in $H^1(E)$ is given by

$$|||\, \phi \,|||_1^2 = \parallel \phi \parallel^2 + \parallel D\phi \parallel^2 .$$

A similar result holds for all $H^k(E)$.

Proposition 2.7. *For each $k \in \mathbb{N}$, there exist $C_1(k), C_2(k) > 0$ such that*

$$C_1(k) \parallel \phi \parallel_k^2 \leq \sum_{l=0}^{k} \parallel D^l\phi \parallel^2 \leq C_2(k) \parallel \phi \parallel_k^2$$

for all $\phi \in H^k(E)$.

To prove Proposition 2.7 one uses that the injectivity radius of X has a positive lower bound, and all covariant derivatives of the curvature tensor of E are uniformly bounded in absolute value. Then the claimed inequalities follow, as on a compact manifold, from the elliptic estimate for D.

In other words, an equivalent norm in $H^k(E)$ is given by

$$|||\, \phi \,|||_k^2 = \sum_{l=0}^{k} \parallel D^l\phi \parallel^2 .$$

For the same reason, it also follows that the Sobolev embedding theorem holds for X [9, Corollary 1.14]. Namely, we have

Proposition 2.8. *For $l > n/2 + k$, there exists a continuous embedding $H^l(X,E) \to C^k(X,E)$, i.e., there exists $C_{l,k} > 0$ such that $\parallel \varphi \parallel_{C^k} \leq C_{l,k} \parallel \varphi \parallel_{H^l}$ for all $\varphi \in H^l(X,E)$.*

Let $\mathcal{D}$ denote the unique self–adjoint extension of D in $L^2(E)$. We shall now investigate the kernel of $\mathcal{D}$. Let $\varphi \in L^2(E)$ and assume that

$D\varphi = 0$. By elliptic regularity, φ is a C^∞–section of E. Furthermore, Proposition 2.7 implies that

$$\text{(2.9)}\quad \nabla^k\varphi \in L^2(E \otimes (T^*X)^{\otimes k}) \quad \text{and} \quad \| \varphi \|_k \le C(k) \| \varphi \|_0, \quad k \in \mathbb{N},$$

for some constant $C(k) > 0$ and all $\varphi \in \ker \mathcal{D}$.

Now consider the restriction of φ to $\mathbb{R}^+ \times Z_1 \subset X$. On this submanifold we have

$$D = \gamma_1\Big(\frac{\partial}{\partial v} + A_1\Big),$$

where $A_1 : C^\infty(Z_1, E_1) \to C^\infty(Z_1, E_1)$ is a generalized Dirac operator on the manifold Z_1. Thus on $\mathbb{R}^+ \times Z_1$,

$$\text{(2.10)}\qquad \Big(\frac{\partial}{\partial v} + A_1\Big)\varphi(v,z) = 0.$$

Next recall that the manifold Z_1, defined by (1.1), is a manifold with a cylindrical end. Moreover, the connection ∇^{E_1} and the Hermitian metric of the Clifford bundle E_1 are compatible with the product structure of Z_1 on $\mathbb{R}^+ \times Y$. Hence on $\mathbb{R}^+ \times Y$, A_1 takes the form

$$\text{(2.11)}\qquad A_1 = \gamma_2\Big(\frac{\partial}{\partial u} + B_1\Big),$$

where $u \in \mathbb{R}^+$, γ_2 is Clifford multiplication by the outward unit normal vector field to Y, and $B_1 : C^\infty(Y, S) \to C^\infty(Y, S)$ is a Dirac operator on Y.

Since Z_1 is complete, A_1 is essentially self–adjoint in $L^2(Z_1, E_1)$ [11]. Let $\mathcal{A}_1$ be the unique self–adjoint extension of A_1. In §4 of [16], we have described the spectral resolution of such operators. It follows that $\mathcal{A}_1$ has only a point spectrum and an absolutely continuous spectrum. The point spectrum consists of a sequence of eigenvalues $\cdots \le \lambda_j \le \lambda_{j+1} \le \cdots$ of finite multiplicity, and the continuous spectrum has an explicit description in terms of the generalized eigensections.

Since for almost all $v \ge 0$, $\varphi(v,\cdot)$ belongs to $L^2(Z_1, E_1)$, we may expand $\varphi(v,\cdot)$ in terms of the L^2–eigensections and the generalized eigensections of $\mathcal{A}_1$. Let $L^2_d(E_1)$ and $L^2_c(E_1)$ denote the discrete and continuous subspace of $\mathcal{A}_1$, respectively. Denote by $\varphi_d(v,\cdot)$ (resp. $\varphi_c(v,\cdot)$) the orthogonal projection of $\varphi(v,\cdot)$ onto $L^2_d(E_1)$ (resp. $L^2_c(E_1)$). Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be an orthonormal basis of $L^2_d(E_1)$ consisting of eigensections of $\mathcal{A}_1$ with eigenvalues $\cdots \le \lambda_j \le \lambda_{j+1} \le \cdots$. Then we have

$$\varphi_d(v,z) = \sum_j a_j(v)\, \varphi_j(z)$$

and the a_j's satisfy

$$\Big(\frac{\partial}{\partial v} + \lambda_j\Big) a_j = 0.$$

Thus $a_j(v) = c_j e^{-\lambda_j v}$. Since φ_d is square integrable, it follows that $c_j = 0$ for $\lambda_j \leq 0$ and

$$\varphi_d(v,z) = \sum_{\lambda_j > 0} c_j e^{-\lambda_j v}\, \varphi_j(z). \tag{2.12}$$

Suppose that the enumeration of the eigenvalues of $\mathcal{A}_1$ is chosen such that $\lambda_1 > 0$ is the smallest positive eigenvalue. Let $T \geq 1$. Then we get

$$\begin{aligned} \int_T^\infty \int_{Z_1} |\varphi_d(v,z)|^2 \, dz\, dv &= \sum_{\lambda_j>0} \frac{|c_j|^2}{2\lambda_j}\, e^{-2T\lambda_j} \\ &\leq e^{-2T\lambda_1} \int_{\mathbb{R}^+ \times Z_1} |\varphi_d(v,z)|^2 \, dz\, dv \\ &\leq e^{-2T\lambda_1} \|\varphi\|^2. \end{aligned} \tag{2.13}$$

In the same way, we can derive a pointwise estimate. By the Sobolev embedding theorem, we obtain

$$\sup_{z \in Z_1} |\varphi_j(z)| \leq C(1+\lambda_j)^{n-1}$$

for some constant $C > 0$, independent of j. Furthermore, we also have

$$\sum \frac{|c_j|^2}{2\lambda_j} = \int_0^\infty \int_{Z_1} |\varphi_d(v,z)|^2 \, dz\, dv \leq \| \varphi \|^2 .$$

Hence $|c_j| \leq \sqrt{2\lambda_j}\, \| \varphi \|$ and, for $v \geq 1$,

$$|\varphi_d(v,z)| \leq C \sum_{\lambda_j>0} (1+\lambda_j)^n\, e^{-\lambda_j v}\, \| \varphi \| \leq C_1 e^{-\lambda_1 v/2}\, \| \varphi \| . \tag{2.14}$$

Now we shall investigate φ_c. Let P_+ be the positive spectral projection of the self-adjoint extension of the Dirac operator $B_1 : C^\infty(Y,S) \to C^\infty(Y,S)$, defined by (2.11). Furthermore, let $\tilde{\Pi}$ denote the orthogonal projection of $L^2(Y,S)$ onto the $+i$-eigenspace of $\gamma_1\gamma_2 : \ker B_1 \to \ker B_1$. Set $\Pi_+ = P_+ + \tilde{\Pi}$. Let $\{\phi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{Ran}(\Pi_+)$ consisting of eigensections of B_1 with corresponding eigenvalues $0 \leq \mu_1 \leq \mu_2 \leq \cdots$.

Let Σ^s be the Riemann surface associated with the functions $\sqrt{\lambda \pm \mu_j}$, $j \in \mathbb{N}$, such that $\sqrt{\lambda \pm \mu_j}$ has positive imaginary part for μ_j sufficiently large. The Riemann surface Σ^s is a ramified double covering $\pi^s : \Sigma^s \to \mathbb{C}$

of $\mathbb{C}$ with ramification locus $\{\pm\mu_j \mid j \in \mathbb{N}\}$. To each ϕ_j there corresponds a generalized eigensection $E(\phi_j, \Lambda) \in C^\infty(Z_1, E_1)$ of $\mathcal{D}$, which is a meromorphic function of $\Lambda \in \Sigma^s$ and satisfies

$$DE(\phi_j, \Lambda) = \pi^s(\Lambda) E(\phi_j, \Lambda), \quad \Lambda \in \Sigma^s,$$

(cf. [12]). The half-plane $\mathrm{Im}(\lambda) > 0$ can be identified with an open subset FP^s of Σ^s, the physical sheet, and each section $E(\phi_j, \Lambda)$ is regular on $\partial FP^s \cong \mathbb{R}$. Then φ_c has an expansion of the form

$$\begin{aligned} \varphi_c(v,z) = \sum_{j=1}^{\infty} \Big\{ \int_{\mu_j}^{\infty} & E(\phi_j, \lambda, z) \alpha_j(v, \lambda)\, d\tau_j(\lambda) \\ & + \int_{\mu_j}^{\infty} E(\phi_j, -\lambda, z) \beta_j(v,\lambda)\, d\tau_j(\lambda) \Big\}, \end{aligned} \tag{2.15}$$

where

$$d\tau_j(\lambda) = \frac{\sqrt{\lambda^2 - \mu_j^2}}{2\pi\lambda}\, d\lambda,$$

and $\alpha_j, \beta_j \in L^2(\mathbb{R}^+ \times [\mu_j, \infty); dv\, d\tau_j)$. Convergence of (2.15) has to be understood in the L^2 sense. By (2.10), α_j and β_j are smooth functions of v satisfying

$$\frac{\partial}{\partial v} \alpha_j(v,\lambda) + \lambda \alpha_j(v,\lambda) = 0 \quad \text{and} \quad \frac{\partial}{\partial v} \beta_j(v,\lambda) - \lambda \beta_j(v,\lambda) = 0.$$

Hence $\alpha_j(v,\lambda) = a_j(\lambda) e^{-\lambda v}$ and $\beta_j(v,\lambda) = b_j(\lambda) e^{\lambda v}$. Since each β_j is square integrable, it follows that $b_j \equiv 0$ for all $j \in \mathbb{N}$, and (2.15) leads to

$$\varphi_c(v,z) = \sum_{j=1}^{\infty} \int_{\mu_j}^{\infty} a_j(\lambda) e^{-\lambda v} E(\phi_j, \lambda, z)\, d\tau_j(\lambda). \tag{2.16}$$

Let $T \geq 1$. Then (2.16) implies

$$\begin{aligned} \int_T^\infty \int_{Z_1} |\varphi_c(v,z)|^2\, dz\, dv &= \sum_{j=1}^{\infty} \int_T^\infty \int_{\mu_j}^\infty |a_j(v,\lambda)|^2\, d\tau_j(\lambda)\, dv \\ &= \sum_{j=1}^{\infty} \frac{1}{2} \int_{\mu_j}^\infty \frac{|a_j(\lambda)|^2}{\lambda} e^{-2T\lambda}\, d\tau_j(\lambda) \\ &\leq \sum_{j=1}^{\infty} \frac{1}{2} e^{-2T\mu_j} \int_{\mu_j}^\infty \frac{|a_j(\lambda)|^2}{\lambda}\, d\tau_j(\lambda). \end{aligned} \tag{2.17}$$

First, assume that $\ker B_1 = 0$. Then $\mu_1 > 0$ is the smallest positive eigenvalue of B_1, and by (2.17) we get

$$\int_T^\infty \int_{Z_1} |\varphi_c(v,z)|^2 \, dz \, dv \ \le \ Ce^{-T\mu_1} \parallel \varphi_c \parallel^2 \le \ Ce^{-T\mu_1} \parallel \varphi \parallel^2, \tag{2.18}$$

where $C > 0$ is independent of φ. Similar estimates hold for the restriction of φ to $\mathbb{R}^+ \times Z_2$.

Let D_Y be the Dirac operator defined by (2.3). Then we have $B_1 = -\gamma_2 D_Y$. In particular, $\ker D_Y = \ker B_1$. If we combine (2.13), (2.18) and the corresponding estimates with respect to Z_2, we obtain

Proposition 2.19. *Assume that* $\ker D_Y = 0$*. Let* $T \ge 1$ *and let* X_T *be the manifold defined by* (1.6)*. Then there exist constants* $C, c > 0$ *such that for* $\varphi \in \ker \mathcal{D}$*, we have*

$$\int_{X - X_T} |\varphi(x)|^2 \, dx \ \le \ Ce^{-cT} \parallel \varphi \parallel^2 .$$

Proposition 2.19 combined with (2.9) implies that $\ker \mathcal{D}$ is finite-dimensional.

We shall now relax the assumption $\ker D_Y = 0$. For this purpose we introduce an auxiliary differential operator $L : C^\infty(E) \to C^\infty(E \otimes T^*X)$ as follows. Let $\omega \in \Lambda_0^1(X)$ be a 1-form with compact support such that $|\omega(x)| = 1$ for all $x \in X_0$. Since X_0 has a nonempty boundary, such a 1-form always exists. Let $\psi_0 \in C_0^\infty(X_1)$ and suppose that $\psi_0(x) = 1$ for $x \in X_0$. Furthermore, let $\psi_i \in C^\infty(\mathbb{R}), i = 1, 2$, be such that $\psi_i(u) = 1$ for $u \ge 1$ and $\psi_i(u) = 0$ for $u \le 0$. We regard ψ_i as a smooth function on $\mathbb{R}^+ \times Z_i$ in the obvious way. Given $\varphi \in C^\infty(E)$, let φ_i denote the restriction of φ to $\mathbb{R}^+ \times Z_i$. Then we set

$$L\varphi = \psi_0 \varphi \otimes \omega + \psi_1 \frac{\partial \varphi_1}{\partial v} \otimes dv + \psi_2 \frac{\partial \varphi_2}{\partial u} \otimes du. \tag{2.20}$$

By (2.9), L induces a bounded linear operator $\tilde{L} : \ker \mathcal{D} \to L^2(E \otimes T^*X)$.

Lemma 2.21. *We have* $\ker \tilde{L} = \{0\}$.

Proof. Let $\varphi \in \ker \mathcal{D}$ and suppose that $L\varphi = 0$. Then it follows from (2.20) that

$$\varphi|_{X_0} = 0, \quad \frac{\partial}{\partial v} \varphi_1 = 0 \quad \text{and} \quad \frac{\partial}{\partial u} \varphi_2 = 0.$$

By (2.12) and (2.16), the second equality implies that $\varphi_1 = 0$. In the same way we get $\varphi_2 = 0$. Hence $\varphi = 0$. q. e. d.

Let $T \geq 1$ and let $\varphi \in \ker \mathcal{D}$. By (2.20), we obtain

$$\int_{X-X_T} |L\varphi(x)|^2 dx \leq \int_T^\infty \int_{Z_1} \left|\frac{\partial}{\partial v}\varphi_1(v,z_1)\right|^2 dz_1 dv + \int_T^\infty \int_{Z_2} \left|\frac{\partial}{\partial u}\varphi_2(u,z_2)\right|^2 dz_2 du.$$

The integrals on the right-hand side can be estimated in the same way as above. From (2.12) and (2.16), it follows that the first integral is bounded by

$$\begin{aligned} Ce^{-cT} \parallel \varphi \parallel^2 &+ \frac{1}{2}\int_0^\infty \lambda^2 \frac{|a_0(\lambda)|^2}{\lambda} e^{-2T\lambda} d\lambda \\ &\leq Ce^{-cT} \parallel \varphi \parallel^2 + \frac{1}{2T^2}\int_0^\infty \frac{|a_0(\lambda)|^2}{\lambda} d\lambda \\ &\leq Ce^{-cT} \parallel \varphi \parallel^2 + \frac{1}{T^2}\int_0^\infty \int_0^\infty |a_0(\lambda)|^2 e^{-2\lambda v} d\lambda dv \\ &\leq \frac{C_1}{T^2} \parallel \varphi \parallel^2, \end{aligned}$$

where $C, c > 0$ are constants, independent of φ. A similar estimate holds for the second integral. Thus we have proved.

Lemma 2.22. *There exists a constant $C > 0$ such that for $T \geq 1$ and $\varphi \in \ker \mathcal{D}$, we have*

$$\int_{X-X_T} |L\varphi(x)|^2 dx \leq \frac{C}{T^2} \parallel \varphi \parallel^2 .$$

Corollary 2.23. *Suppose that there exists $C > 0$ such that $\parallel \varphi \parallel \leq C \parallel L\varphi \parallel$ for all $\varphi \in \ker D$. Then $\ker \mathcal{D}$ is finite-dimensional.*

Proof. By Lemma 2.21, it is sufficient to show that $L(\ker \mathcal{D})$ is finite-dimensional. Let

$$I : L(\ker \mathcal{D}) \to L^2(E \otimes T^*X)$$

be the inclusion. For $T \geq 0$ we denote by $I_T : L(\ker \mathcal{D}) \to L^2(E \otimes T^*X)$ the composition of the restriction of sections to X_T and the canonical inclusion. By Lemma 2.22, we have

$$\parallel I - I_T \parallel \leq \frac{C}{T}, \quad T \geq 1. \tag{2.24}$$

Let $H^1(X_T, E \otimes T^*X)$ denote the Sobolev Space of the restriction of $E \otimes T^*X$ to X_T. Since X_T is compact, the canonical map

$$H^1(X_T, E \otimes T^*X) \to L^2(X_T, E \otimes T^*X)$$

is a compact operator. Furthermore, from (2.9) it follows that $L(\ker \mathcal{D})$ is contained in $H^1(E \otimes T^*X)$ and

$$\| L\varphi \|_1 \leq C \| L\varphi \|_0, \quad \varphi \in \ker \mathcal{D},$$

for some constant $C > 0$, independent of φ. By Rellich's compactness theorem, I_T is a compact operator and hence, by (2.24), I is compact too. Therefore, $\ker \mathcal{D}$ is finite-dimensional. q.e.d.

Next we shall estimate the supremum norm of any $\varphi \in \ker \mathcal{D}$. As above, we consider the restriction of a given $\varphi \in \ker \mathcal{D}$ to $\mathbb{R}^+ \times Z_1$. By Proposition 2.8, we have

$$\sup_{z \in Z_1} |\varphi_c(v, z)| \leq C \| (I + A_1)^n \varphi_c(v) \|,$$

where A_1 is the Dirac operator considered above. Let $v \geq 1$. Employing (2.17), we get

$$\begin{aligned} \| (I + A_1)^n \varphi_c(v) \|^2 =& \sum_{j=1}^{\infty} \int_{\mu_j}^{\infty} (1 + \lambda)^{2n} |a_j(\lambda)|^2 e^{-2\lambda v} d\tau_j(\lambda) \\ \leq& C_1 \sum_{\mu_j > 0} e^{-\mu_j v} \int_{\mu_j}^{\infty} \frac{|a_j(\lambda)|^2}{\lambda} d\tau_j(\lambda) \\ &+ C_2 \frac{1}{v} \sum_{\mu_j = 0} \int_0^{\infty} \frac{|a_j(\lambda)|^2}{\lambda} d\lambda \\ \leq& C_3 \frac{1}{v} \sum_{j=1}^{\infty} \int_0^{\infty} \int_{\mu_j}^{\infty} |a_j(\lambda)|^2 e^{-2\lambda v} d\tau_j(\lambda) dv \\ =& C_4 \frac{1}{v} \| \varphi \|^2 . \end{aligned}$$

Combining this with (2.14) gives

Lemma 2.25. *There exists $C > 0$ such that for all $v \geq 1$ and $\varphi \in \ker \mathcal{D}$, the following inequality holds*

$$\sup_{z \in Z_i} |\varphi(v, z)| \leq \frac{C}{\sqrt{v}} \| \varphi \|, \quad i = 1, 2.$$

If $\ker D_Y = 0$, then we have exponential decay.

Now suppose that $n = 2k, k \in \mathbb{N}$. Let $\tau = i^k \gamma_1 \ldots \gamma_{2_k}$ be the canonical involution of the Clifford bundle E and let

$$E = E_+ \oplus E_-$$

be the parallel orthogonal splitting of E into the ± 1-eigenbundles of τ. Since n is even, τ anticommutes with D and we get a pair of elliptic first order operators

$$D_\pm = C^\infty(E_\pm) \to C^\infty(E_\mp),$$

called chiral Dirac operators. Let $\mathcal{D}_\pm$ denote the closure of $D_\pm$ in L^2. Then we have

$$\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \quad \text{and} \quad \mathcal{D}_+ = \mathcal{D}_-^*.$$

By Corollary 2.24, $\ker \mathcal{D}_+$ and $\ker \mathcal{D}_-$ are finite-dimensional. Therefore, we can define the L^2-index of D_+ by

$$L^2\text{-}\operatorname{Ind} D_+ = \dim(\ker \mathcal{D}_+) - \dim(\ker \mathcal{D}_-). \tag{2.26}$$

Remark. In general, $\mathcal{D}_+$ is not a Fredholm operator. We observe that $\mathcal{D}_+$ is Fredholm if and only if 0 is not in the continuous spectrum of $\mathcal{D}_-\mathcal{D}_+$ or, what is the same, if the continuous spectrum of $\mathcal{D}_-\mathcal{D}_+$ has a positive lower bound. In this case, the L^2-index of D_+ equals the index of the Fredholm operator $\mathcal{D}_+$. This implies that the L^2-index of D_+ is stable under compactly supported pertubations of D. If $\mathcal{D}_+$ is not Fredholm, then the L^2-index will be unstable in general. This makes it difficult to compute the index for these cases.

3. L^2-harmonic forms and cohomology

Let $\Lambda^*(X)$ be the space of C^∞–differential forms on X. In this section we consider the Gauß–Bonnet operator $d + d^* : \Lambda^*(X) \to \Lambda^*(X)$. This is a generalized Dirac operator on X, which obviously satisfies (2.1) - (2.4). Therefore, the results of the previous section can be applied to this operator. Let $\Delta = (d + d^*)^2$ be the Laplace–Beltrami operator on forms, and let Δ_p be the restriction of Δ to the space $\Lambda^p(X)$ of C^∞ p–forms. We shall denote the self–adjoint extensions of $d + d^*$ and Δ in $L^2\Lambda^*(X)$ by $\overline{d} + \overline{d}^*$ and $\overline{\Delta}$, respectively.

Let $\mathcal{H}^*_{(2)}(X)$ denote the space of square integrable harmonic forms on X, that is,

$$\mathcal{H}^*_{(2)}(X) = \{\varphi \in \Lambda^*(X) \mid \Delta\varphi = 0, \ \ \|\varphi\| < \infty\}. \tag{3.1}$$

Correspondingly, $\mathcal{H}^p_{(2)}(X)$ will denote the space of square integrable harmonic p–forms on X. Since X is complete, $\mathcal{H}^*_{(2)}(X)$ equals the kernel of $\overline{d} + \overline{d}^*$ (cf. [11]). In other words,

$$\mathcal{H}^*_{(2)}(X) = \{\varphi \in \Lambda^*(X) \mid d\varphi = d^*\varphi = 0, \ \ \|\varphi\| < \infty\} = \ker(\overline{d} + \overline{d}^*).$$

In this section, we prove

Proposition 3.2. *The space $\mathcal{H}^*_{(2)}(X)$ is finite–dimensional.*

Let $H^*_{(2)}(X;\mathbb{C})$ be the L^2–cohomology of X. Recall that $H^*_{(2)}(X;\mathbb{C})$ is the cohomology of the L^2–de Rham complex consisting of all C^∞–forms which together with their exterior derivative are square integrable [26]. Then $\mathcal{H}^p_{(2)}(X)$ equals $H^p_{(2)}(X;\mathbb{C})$ if and only if the essential spectrum of $\overline{\Delta}_p$ has a positive lower bound. As we shall see in §6, this depends on the cohomology of Y, M_1 and M_2. If the essential spectrum of $\overline{\Delta}_p$ contains zero, then $H^*_{(2)}(X;\mathbb{C})$ is infinite–dimensional.

Now we shall study the relation of $\mathcal{H}^*_{(2)}(X)$ with the de Rham cohomology of X. Let $H^*(X)$ be the de Rham cohomology of X with complex coefficients and let $H^*_c(X)$ be the de Rham cohomology of X with compact supports and complex coefficients. Set

$$H^*_!(X) = \operatorname{Im}(H^*_c(X) \xrightarrow{\iota} H^*(X)),$$

where ι is the canonical map. As mentioned above, a harmonic L^2-form ϕ satisfies $d\phi = 0$ and $d^*\phi = 0$. In particular, ϕ defines a cohomology class $[\phi]$ in $H^*(X)$. In this way we obtain a canonical map

$$j : \mathcal{H}^*_{(2)}(X) \to H^*(X).$$

For a general complete manifold, this map will neither be injective, nor surjective. In the present case, however, we can describe this map completely.

Lemma 3.3. *The image of j is contained in $H^*_!(X)$.*

Proof. First observe that by the construction of X, there is a canonical retraction $X \to X_0$. Hence, $H^*_!(X)$ can be identified with the image of $H^*(X_0, \partial X_0)$ in $H^*(X_0) = H^*(X)$. Let $\varphi \in \mathcal{H}^*_{(2)}(X)$. In order to see that the cohomology class $[\varphi]$ is contained in $H^*_!(X)$, it is sufficient to show that

$$\int_\alpha \varphi = 0$$

for all cycles α in $\partial X_0 = M_1 \cup_Y M_2$. Using this decomposition of the boundary, it follows that $H_*(\partial X_0)$ has a basis which can be represented by cycles α of the following form: There exist a cycle α_0 in Y and relative cycles α_i in (M_i, Y) such that

$$\partial\alpha_i = \alpha_0, \quad i = 1,2, \quad \text{and} \quad \alpha = \alpha_1 \cup_{\alpha_0} (-\alpha_2).$$

Note that α_0 may be zero. In this case, α_i is a cycle in M_i and α is the disjoint union of α_1 and α_2.

Let $T \geq 0$ and let $Z_{i,T}$ be the manifold defined by (1.5). Then we define relative cycles $\alpha_{i,T}$ in $(Z_{i,T}, \partial Z_{i,Z})$ by

$$\alpha_{i,T} = \alpha_i \cup_{\alpha_0} ([0,T] \times \alpha_0), \quad i = 1,2,$$

where we identify $\partial\alpha_i$ with $\{0\} \times \alpha_0$. Set

$$\alpha_T = \alpha_{1,T} \cup (-\alpha_{2,T}).$$

Then α_T is a cycle in $\partial X_T = Z_{1,T} \cup Z_{2,T}$. If we regard α and α_T as cycles in X_T, then the construction of α_T implies that α and α_T are homologous. Since φ is closed, it follows that

$$\int_\alpha \varphi = \int_{\alpha_T} \varphi = \int_{\alpha_{1,T}} \varphi - \int_{\alpha_{2,T}} \varphi, \quad T \geq 0.$$

We shall now estimate the integrals on the right-hand side. For this purpose we use the expansion of φ on $\mathbb{R}^+ \times Z_i$ in terms of the eigensections of A_i, $i = 1, 2$. It is sufficient to consider the integral over $\alpha_{1,T}$. We have to specialize the eigensection expansion (2.12) and (2.15) to the present case. Let Δ_{Z_1} be the Laplacian on $\Lambda^*(Z_1)$ and let Δ_Y be the Laplacian on $\Lambda^*(Y)$. Note that

$$\Lambda^*(\mathbb{R}^+ \times Y) = (C^\infty(\mathbb{R}^+)\hat{\otimes}\Lambda^*(Y)) \oplus (\Lambda^1(\mathbb{R}^+)\hat{\otimes}\Lambda^*(Y)).$$

Therefore, to each eigenform ϕ of Δ_Y there correspond two generalized eigenforms of Δ_{Z_1}; namely $E(\phi, \lambda, z)$ and $E(du \wedge \phi, \lambda, z)$. Let $i_v : \{v\} \times Z_1 \subset \mathbb{R}^+ \times Z_1$ be the inclusion and let

$$\varphi_1(v) = i_v^*(\varphi|_{\mathbb{R}^+ \times Z_1}).$$

Let $\phi_1, ..., \phi_m$ be an orthonormal basis of $\mathcal{H}^*(Y)$. Then from the above remarks combined with (2.12) and (2.16) (specialized to the present case), it follows that

$$\begin{aligned} \varphi_1(v, z) = \sum_{j=1}^m \Big\{ &\int_0^\infty a_j(\lambda) e^{-\lambda v} E(\phi_j, \lambda, z)\, d\lambda \\ &+ \int_0^\infty b_j(\lambda) e^{-\lambda v} E(du \wedge \phi_j, \lambda, z)\, d\lambda \Big\} + \psi_1(v, z), \end{aligned} \tag{3.4}$$

where ψ_1 satisfies

$$\sup_{z \in Z_1} | \psi_1(T, z) | \leq C e^{-cT}, \tag{3.5}$$

with constants $C, c > 0$, independent of T. The integrals converge in the L^2 sense. Moreover, the functions $a_j(\lambda)$ and $b_j(\lambda)$ are square integrable with respect to the measure $\lambda^{-1} d\lambda$. From (3.5) follows that

$$\left| \int_{\alpha_{1,T}} \psi_1 \right| \leq C_1 T e^{-cT}. \tag{3.6}$$

It remains to investigate the differential forms defined by the infinite integrals on the right-hand side of (3.4). We consider the first type of integrals. Let $\phi \in \mathcal{H}^*(Y)$ and $a \in L^2(\mathbb{R}^+; \lambda^{-1}d\lambda)$. Put

$$\begin{aligned} \omega_1(\phi, v) &= \int_0^{1/\sqrt{v}} a(\lambda)e^{-\lambda v}E(\phi,\lambda)d\lambda, \\ \omega_2(\phi, v) &= \int_{1/\sqrt{v}}^{\infty} a(\lambda)e^{-\lambda v}E(\phi,\lambda)d\lambda. \end{aligned} \tag{3.7}$$

It is clear that $\omega_1(\phi, v)$ is a smooth differential form. The convergence of the infinite integral is understood in the L^2 sense. Therefore, it is not obvious that $\omega_2(\phi, v)$ is a smooth differential form. To verify smoothness, let $v \geq 1$ and $m \in \mathbb{N}$. By definition, we have

$$\begin{aligned} \| (I + A_1)^m \omega_2(\phi, v) \|^2 &= \int_{1/\sqrt{v}}^{\infty} |a(\lambda)|^2 (1+\lambda)^{2m} e^{-2\lambda v} d\lambda \leq \\ &\leq C_m e^{-\sqrt{v}} \int_{1/\sqrt{v}}^{\infty} |a(\lambda)|^2 \frac{d\lambda}{\lambda} \leq C'_m e^{-\sqrt{v}}. \end{aligned}$$

By the Sobolev embedding theorem, this implies that $\omega_2(\phi, v)$ is a smooth form which satisfies

$$\sup_{z \in Z_1} | \omega_2(\phi, v, z) | \leq C e^{-\sqrt{v}}, \quad v \geq 1, \tag{3.8}$$

for some constant $C > 0$. Put

$$\omega(\phi, v) = \omega_1(\phi, v) + \omega_2(\phi, v).$$

Our goal is to estimate $\int_{\alpha_{1,T}} \omega(\phi, T)$ as $T \to \infty$. By (3.8), $\int_{\alpha_{1,T}} \omega_2(\phi, T)$ decays exponentially as $T \to \infty$. To deal with $\omega_1(\phi, T)$, we observe that on $\mathbb{R}^+ \times Y$, $E(\phi, \lambda)$ has an expansion of the form

$$\begin{aligned} E(\phi, \lambda, (u, y)) =& e^{-i\lambda u}\phi(y) + e^{i\lambda u}(C_1(\lambda)\phi + du \wedge C_2(\lambda)\phi) \\ &+ \sum_{\mu_j \neq 0} e^{-\sqrt{\mu_j^2 - \lambda^2}u}(T_{1,j}(\lambda)\phi + du \wedge T_{j,2}(\lambda)\phi), \end{aligned} \tag{3.9}$$

where $C_i(\lambda)\phi \in \mathcal{H}^*(Y)$, and $T_{i,j}(\lambda)\phi$ is contained in the μ_j-eigenspace of Δ_Y. The existence of the expansion (3.9) follows from (4.20) in [16], specialized to the Laplace operator. Let χ be the characteristic function of $\mathbb{R}^+ \times Y \subset Z_1$ and set

$$\tilde{E}(\phi, \lambda) = E(\phi, \lambda) - \chi[e^{-i\lambda u}\phi + e^{i\lambda u}(C_1(\lambda)\phi + du \wedge C_2(\lambda)\phi)].$$

It follows from (3.9) that there exists $\varepsilon > 0$ such that for $\lambda \leq \varepsilon$, we have

$$|\tilde{E}(\phi, \lambda, (u, y))| \leq C e^{-cu}, \quad (u, y) \in \mathbb{R}^+ \times Z_1.$$

Put

$$\tilde{\omega}_1(\phi, T) = \int_0^{1/\sqrt{T}} a(\lambda) e^{-\lambda T} \tilde{E}(\phi, \lambda) d\lambda.$$

Let $1/\sqrt{T} < \varepsilon$. Since $a \in L^2(\mathbb{R}^+, \lambda^{-1} d\lambda)$, we get

$$\begin{aligned} \left| \int_{\alpha_{1,T}} \tilde{\omega}_1(\phi, T) \right| &\leq C \int_0^{1/\sqrt{T}} |a(\lambda)| e^{-\lambda T} d\lambda \\ &\leq C_1 \Big(\int_0^{1/\sqrt{T}} \lambda e^{-2\lambda T} d\lambda \Big)^{1/2} \leq C_2 T^{-3/4}. \end{aligned}$$

It remains to study the integral of the differential form

$$\int_0^{1/\sqrt{T}} a(\lambda)\, e^{-\lambda T} \chi \Big[e^{-i\lambda u} \phi \,+\, e^{i\lambda u} (C_1(\lambda)\phi \,+\, du \wedge C_2(\lambda)\phi) \Big]\, d\lambda.$$

If we integrate this form over $\alpha_{1,T}$, we get

$$\begin{aligned} &\int_0^{1/\sqrt{T}} \Big\{ \Big(\int_{\alpha_0} C_2(\lambda)\phi \Big)\, a(\lambda) e^{-\lambda T} \int_0^T e^{i\lambda u}\, du \Big\}\, d\lambda \\ &\qquad = i \int_0^{1/\sqrt{T}} \frac{1 - e^{i\lambda T}}{\lambda} \Big(\int_{\alpha_0} C_2(\lambda)\phi \Big)\, a(\lambda) e^{-\lambda T}\, d\lambda. \end{aligned}$$

By Schwarz's inequality, this integral can be estimated by

$$\begin{aligned} &C \left(\int_0^{1/\sqrt{T}} \frac{|1 - e^{i\lambda T}|^2}{\lambda} e^{-2\lambda T}\, d\lambda \right)^{1/2} \left(\int_0^{1/\sqrt{T}} |a(\lambda)|^2 \frac{d\lambda}{\lambda} \right)^{1/2} \\ &\qquad \leq C_1 \left(\int_0^{1/\sqrt{T}} |a(\lambda)|^2 \frac{d\lambda}{\lambda} \right). \end{aligned}$$

Since $a(\lambda)$ is square integrable with respect to the measure $\lambda^{-1} d\lambda$, the right-hand side converges to zero as $T \to \infty$. If we replace ϕ in (3.7) by $du \wedge \phi$, we get the second type of forms which we have to consider. The investigation of these forms integrated over $\alpha_{1,T}$ is completely analogous to the previous case. The corresponding integrals tend also to 0 as $T \to \infty$. Together with (3.4) and (3.6), we get

$$\lim_{T \to \infty} \int_{\alpha_{1,T}} \varphi = 0.$$

The same holds for $\int_{\alpha_{2,T}} \varphi$. Hence, $\int_\alpha \varphi = 0$ for all cycles α in ∂X_0.

q.e.d.

By Lemma 3.3, j induces a map

(3.10) $$j : \mathcal{H}^*_{(2)}(X) \to H^*_!(X).$$

Now suppose that $[\phi] \in H_!^*(X)$ is represented by a closed C^∞ form ϕ with compact support. In particular, ϕ is square integrable. Therefore, by a theorem of de Rham-Kadaira [20, p. 169], we have

$$\phi = \psi + d\theta,$$

where $\psi \in L^2, d\psi = d^*\psi = 0$ and θ is a current. Since $H^*(X)$ can be computed from the complex of currents, it follows that the map (3.10) is surjective.

To deal with the injectivity, we observe that the manifold X has a natural compactification $\overline{X}$ obtained by adjoining copies of Z_1, Z_2 and Y at infinity. Putting $r = 1/u$ and $w = 1/v$, we get natural coordinates near the boundary.

Lemma 3.11. *Each $\varphi \in \ker \mathcal{D}$ extends to a C^1 form on $\overline{X}$.*

Proof. Let $\varphi \in \ker \mathcal{D}$. Denote by φ_i the restriction of φ to $\mathbb{R}^+ \times Z_i, i = 1, 2$. Then φ_i can be written in the form (3.4) with forms ψ_i satisfying (3.5). Using Proposition 2.8, it is easy to generalize (3.5) as follows: For all $k \in \mathbb{N}$, there exist $C_k > 0$ and $c > 0$ such that

$$\sup_{z \in Z_i} \mid \nabla^k \psi_i(v, z) \mid \leq C_k e^{-cv}, \quad k \in \mathbb{N},\ i = 1, 2.$$

Hence ψ_1 and ψ_2 extend to C^∞ forms on $\overline{\mathbb{R}}^+ \times Z_1$ and $\overline{\mathbb{R}}^+ \times Z_2$, respectively. To finish the argument, we have to consider the forms $\omega_1(\phi, v)$ and $\omega_2(\phi, v)$ defined by (3.7). Again, by referring to the Sobolev embedding theorem, it is easy to show that

$$\mid \nabla^k \omega_2(\phi, v, z) \mid \leq C_k e^{-cv}, \quad k \in \mathbb{N},$$

for constants $C_k, c > 0$. Thus, ω_2 extends also to a C^∞ form on $\overline{\mathbb{R}}^+ \times Z_1$. Since $E(\phi, \lambda, z)$ is analytic in λ and smooth in z, we get

$$\begin{aligned}
|(d + d^*)\omega_1(\phi, v, z)| \leq & C \int_0^{1/\sqrt{v}} \lambda |a(\lambda)|\, e^{-v\lambda}\, d\lambda \\
\leq & C \left(\int_0^{1/\sqrt{v}} \lambda^3 e^{-2v\lambda} d\lambda \right)^{1/2} \left(\int_0^{1/\sqrt{v}} |a(\lambda)|^2 \frac{d\lambda}{\lambda} \right)^{1/2} \\
\leq & \frac{C_1}{v^2} \left(\int_0^{\sqrt{v}} \lambda^3\, e^{-2\lambda} d\lambda \right)^{1/2} \leq \frac{C_2}{v^2},
\end{aligned}$$

which implies that ω_1 extends to a C^1 form on $\overline{\mathbb{R}}^+ \times Z_1$ and hence, φ_1 does so. By the same argument φ_2 extends to a C^1-form on $\overline{\mathbb{R}}^+ \times Z_2$.

q.e.d.

Now recall that $H^*(X) = H^*(\overline{X})$ can be computed from the de Rham complex of C^1 forms on $\overline{X}$. Let $\varphi \in \mathcal{H}^*_{(2)}(X)$ and suppose that $j(\varphi) = 0$.

Then there exists a C^1 form on $\overline{X}$ such that $\varphi = d\theta$. In particular, we may assume that θ is bounded. Now we apply Green's formula to the compact manifold X_T, $T \geq 0$. Since $d^*\varphi = 0$ and $\varphi = d\theta$, we get

$$\begin{aligned}\int_{X_T} \varphi \wedge *\varphi &= \int_{X_T} d\theta \wedge *\varphi = \int_{\partial X_T} \theta \wedge *\varphi \\ &= \int_{Z_{1,T}} \theta_1 \wedge *\varphi_1 + \int_{Z_{2,T}} \theta_2 \wedge *\varphi_2,\end{aligned} \tag{3.12}$$

where θ_i and φ_i are the restrictions of θ and φ, respectively, to $\mathbb{R}^+ \times Z_i$. To estimate the boundary integrals we use again (3.4) and its analogue for φ_2. Since $\psi_i(T)$ decays exponentially while θ is bounded as $T \to \infty$, we get

$$\int_{Z_{i,T}} \theta_i \wedge *\psi_i \to 0 \quad \text{as} \quad T \to \infty.$$

The same argument applies to the forms $\omega_2(\phi, T)$ and $\omega_2(du \wedge \phi, T)$. To determine the contribution of the forms $\omega_1(\phi, T)$, we observe that $|E(\phi, \lambda, z)| \leq C$, for $0 \leq \lambda \leq \varepsilon$ and $z \in Z_1$. Hence

$$\begin{aligned}\left|\int_{Z_{1,T}} \theta_1 \wedge *\omega_1(\phi)\right| &\leq CT \int_0^{1/\sqrt{T}} |a(\lambda)| e^{-\lambda T}\, d\lambda \\ &\leq CT \left(\int_0^{1/\sqrt{T}} \lambda\, e^{-2\lambda T}\, d\lambda\right)^{1/2} \left(\int_0^{1/\sqrt{T}} |a(\lambda)|^2 \frac{d\lambda}{\lambda}\right)^{1/2} \\ &= C \left(\int_0^{\sqrt{T}} \lambda\, e^{-\lambda}\, d\lambda\right)^{1/2} \left(\int_0^{1/\sqrt{T}} |a(\lambda)|^2 \frac{d\lambda}{\lambda}\right)^{1/2}.\end{aligned}$$

Since $a(\lambda)$ is square integrable with respect to the measure $\lambda^{-1} d\lambda$, the right-hand side converges to zero as $T \to \infty$. The same holds for $\omega_1(du \wedge \phi, T)$. By (3.12), we deduce that

$$\int_{X_T} \varphi \wedge *\varphi \to 0$$

as $T \to \infty$. This implies that $\varphi = 0$. Thus j is injective. We can now summarize our results about the L^2 harmonic forms by

Proposition 3.13. *The canonical map $\mathcal{H}^*_{(2)}(X) \to H^*(X)$ induces an isomorphism*

$$j : \mathcal{H}^*_{(2)}(X) \tilde{\to} H^*_!(X).$$

In particular, this proves Proposition 3.2.

Next we shall investigate the L^2-index of the signature operator. Suppose that $n = 2l$ and let τ be the involution of $\Lambda^*(X)$ which is defined by

$$\tau\phi = i^{p(p-1)+l} * \phi \quad \text{for} \quad \phi \in \Lambda^p(X).$$

Let $\Lambda^*_\pm(X)$ denote the ± 1-eigenspaces of τ. Since $d + d^*$ anticommutes with $\tau, d + d^*$ interchanges $\Lambda^*_+(X)$ and $\Lambda^*_-(X)$ and hence, defines by restriction operators

$$D_\pm : \Lambda^*_\pm(X) \to \Lambda^*_\mp(X).$$

The operator D_+ is usually called *signature operator.* The involution τ acts on $\mathcal{H}^*_{(2)}(X)$ and we denote the ± 1-eigenspaces of τ by $\mathcal{H}^*_{(2),\pm}(X)$. Then it is easy to see that

$$L^2\text{-Ind}\, D_+ = \dim \mathcal{H}^*_{(2),+}(X) - \dim \mathcal{H}^*_{(2),-}(X).$$

By definition, τ maps $\mathcal{H}^p_{(2)}(X)$ onto $\mathcal{H}^{2l-p}_{(2)}(X)$. Let $\mathcal{H}^l_{(2),\pm}(X)$ denote the ± 1-eigenspaces of τ acting in $\mathcal{H}^l_{(2)}(X)$, and for $p < l$ set

$$\mathcal{H}^p_{(2),\pm}(X) = \{\varphi \pm \tau\varphi \mid \varphi \in \mathcal{H}^p_{(2)}(X)\}.$$

Then it is clear that

$$\mathcal{H}^*_{(2),\pm}(X) = \bigoplus_{p \le l} \mathcal{H}^p_{(2),\pm}(X).$$

Since $\dim \mathcal{H}^p_{(2),+}(X) = \dim \mathcal{H}^p_{(2),-}(X)$ for $p < l$, it follows that

$$L^2\text{-Ind}\, D_+ = \dim \mathcal{H}^l_{(2),+}(X) - \dim \mathcal{H}^l_{(2),-}(X). \tag{3.14}$$

There are two cases that we have to distinguish depending on whether l is odd or even. First, suppose that $l = 2k+1$. In this case, the mapping $\tau : \mathcal{H}^l_{(2)}(X) \to \mathcal{H}^l_{(2)}(X)$ coincides with $i*$. Since $*$ is a real operator, it follows that the map $\varphi \mapsto \overline{\varphi}$ induces an isomorphism of $\mathcal{H}^l_{(2),+}$ onto $\mathcal{H}^l_{(2),-}$. Thus

$$L^2\text{-Ind}\, D_+ = 0, \quad \text{if} \quad l = 2k+1.$$

So we can assume that $n = 4k$. Then on $\mathcal{H}^{2k}_{(2)}(X)$, τ coincides with $*$ which is a real operator. Furthermore under the isomorphism $\mathcal{H}^{2k}_{(2)}(X) \to H^{2k}_!(X)$, the quadratic form $\varphi \mapsto \langle \varphi, *\varphi \rangle$ on $\mathcal{H}^{2k}_{(2)}(X)$ corresponds to the intersection form on $H^{2k}_!(X)$. This quadratic form is induced by the degenerate quadratic form on $H^{2k}_c(X) \cong H^{2k}(X_0, \partial X_0)$ given by the cup product. Poincaré duality for $(X_0, \partial X_0)$ shows that the radical is precisely the kernel of $H^{2k}(X_0, \partial X_0) \to H^{2k}(X_0)$. The signature $\operatorname{Sign}(X_0)$

of the $4k$–dimensional manifold X_0 is defined to be the signature of the intersection form on $H_!^{2k}(X;\mathbb{R})$. Then the above argument shows that

$$\operatorname{Sign}(X_0) = \dim \mathcal{H}^{2k}_{(2),+}(X) \;-\; \dim \mathcal{H}^{2k}_{(2),-}(X),$$

which together with (3.14) yields

Proposition 3.15. *Let $D_+ : \Lambda^*_+(X) \to \Lambda^*_-(X)$ be the signature operator. Then we have*

$$L^2\text{-}\operatorname{Ind} D_+ = \operatorname{Sign}(X_0).$$

4. The heat kernel

Let $D : C^\infty(X,E) \to C^\infty(X,E)$ be a Dirac operator satisfying (2.1) – (2.4) and consider the spinor Laplacian

$$\Delta = D^2.$$

The purpose of this section is to construct the fundamental solution for the heat equation $(\partial/\partial t + \Delta)\varphi = 0$.

Let X_1 be the manifold defined by (1.6) where $T = 1$. Then X_1 is also a manifold with a corner at Y, and the boundary of X_1 is the union of $Z_{1,1}$ and $Z_{2,1}$, where $Z_{i,1} = M_i \cup ([0,1] \times Y)$ (see Fig. 3). Let $V = X_1 \cup_{Z_{2,1}} (-X_1)$ be the C^∞ manifold obtained by gluing two copies of X_1 along the submanifold $Z_{2,1} \subset \partial X_1$ of the boundary. Note that V is an oriented C^∞ manifold with smooth boundary, and the Riemannian metric on X_1 induces a smooth Riemannian metric on V which is a product near the boundary. Let $\tilde{V}$ be the double of this manifold. Then $\tilde{V}$ is a closed oriented C^∞ Riemannian manifold, and we may identify X_1 with a submanifold of $\tilde{V}$. The bundle $E_1 = E|_{X_1}$ also extends to a Clifford bundle $\tilde{E}$ over $\tilde{V}$. Let $\tilde{D}$ be the corresponding generalized Dirac operator and set $\tilde{\Delta} = \tilde{D}^2$. Let $\tilde{K}(x,y,t)$ be the fundamental solution for $\partial/\partial t + \tilde{\Delta}$ on $\tilde{V}$, and let $K_0(x,y,t)$ be the restriction of the kernel $\tilde{K}$ to X_1. The kernel K_0 is the interior part of the parametrix.

Next we have to construct the exterior part of the parametrix. By (2.1) and (2.2), it follows that on $\mathbb{R}^+ \times Z_i$, we have

$$D^2 = -\frac{\partial^2}{\partial u_i^2} + A_i^2, \quad i = 1,2. \tag{4.1}$$

We extend the right-hand side in the obvious way to a differential operator Δ_i on $\mathbb{R} \times Z_i$. Let $K_i(x,y,t)$ be the fundamental solution for

$\partial/\partial t + \Delta_i$. By (4.1), we have

$$K_i((u,w),(v,z),t) = \frac{\exp\left(-\frac{(u-v)^2}{4t}\right)}{\sqrt{4\pi t}} \tilde{K}_i(w,z,t), \quad i = 1,2, \tag{4.2}$$

where $\tilde{K}_i$ is the heat kernel for A_i^2 acting on $C^\infty(Z_i, E_i)$. Finally, by (2.3) and (2.4) it follows that on $(\mathbb{R}^+)^2 \times Y$, we have

$$D^2 = -\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + D_Y^2. \tag{4.3}$$

As above, we extend the right-hand side in the obvious way to a differential operator Δ_3 on $\mathbb{R}^2 \times Y$, and we denote the fundamental solution for $\partial/\partial t + \Delta_3$ by $K_3(x,y,t)$. Then we have

$$\begin{aligned} &K_3((u_1,u_2,w),(v_1,v_2,z),t) \\ &\quad = \frac{\exp\left(-\frac{(u_1-v_1)^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\exp\left(-\frac{(u_2-v_2)^2}{4t}\right)}{\sqrt{4\pi t}} \tilde{K}_3(w,z,t), \end{aligned} \tag{4.4}$$

where $\tilde{K}_3$ is the heat kernel for D_Y^2.

The heat kernels K_i satisfy the standard short time asymptotic. Let $d(x,y)$ denote the geodesic distance of $x, y \in X$.

Proposition 4.5. *For all $p, l \in \mathbb{N}$, there exist constants $C, c_1, c_2 > 0$ such that*

$$\left| \frac{\partial^l}{\partial t^l} \nabla_x^p K_i(x,y,t) \right| \le C\, t^{-(n+l+p)/2} e^{c_1 t} e^{-c_2 d^2(x,y)/t}$$

for $t > 0$, $i = 0,1,2,3$.

Proof. It is well–known that the heat kernel on a compact manifold satisfies the estimate claimed by Proposition 4.5. Therefore, our statement is obvious for K_0. Since Y is compact, we can use (4.4) to derive the required estimate for K_3. We are left with K_1 and K_2. By (4.2), it is sufficient to prove the corresponding estimate for $\tilde{K}_1$ and $\tilde{K}_2$, respectively. Since $\tilde{K}_1$ and $\tilde{K}_2$ are the heat kernels for spinor Laplacians on manifolds with cylindrical ends, the required estimate follows from (3.5) and (3.3) of [16]. q.e.d.

We shall now use the kernels K_i to construct a parametrix for the fundamental solution of $\partial/\partial t + \Delta$. Let $\rho(a,b)$ denote an increasing C^∞–function of the real variable u, such that $\rho = 0$ for $u \le a$ and $\rho = 1$ for $u \ge b$. Define C^∞–functions as follows:

$$\varphi = 1 - \rho(3/4, 7/8), \quad \chi = \rho(0, 1/4), \quad \xi = \rho(3/8, 5/8), \quad \psi = 1 - \xi.$$

Let u_i be the normal direction to $\{0\} \times Z_i \subset X$. We consider $\varphi(u_1)$, $\psi(u_1)$ as functions on the cylinder $[0,1] \times Z_1 \subset X$ and extend them in the obvious way to functions φ_1, ψ_1 on X. Similarly, we regard $\varphi(u_2)$, $\psi(u_2)$ as functions on the cylinder $[0,1] \times Z_2$. Again we extend these functions in the obvious way to functions φ_2, ψ_2 on X. Then we set

$$\Phi_0 = \varphi_1\, \varphi_2, \quad \Psi_0 = \psi_1\, \psi_2.$$

Observe that the support of Φ_0 and Ψ_0 is contained in X_1.

Next we consider $\chi(u_1)$, $\xi(u_1)$ as functions on $[0,1] \times Z_1$ and extend them by 1 to C^∞–functions Φ_1, Ψ_1 on $\mathbb{R}^+ \times Z_1$. In the same way we define Φ_2, Ψ_2 on $\mathbb{R}^+ \times Z_2$. Note that we may extend Φ_1, Ψ_1, Φ_2 and Ψ_2 by zero to C^∞–functions on X. Since $(\mathbb{R}^+)^2 \times Y$ is contained in both $\mathbb{R} \times Z_1$ and $\mathbb{R} \times Z_2$, we may restrict Φ_1, Ψ_1, Φ_2, Ψ_2 to C^∞–functions $\tilde{\Phi}_1$, $\tilde{\Psi}_1$, $\tilde{\Phi}_2$, $\tilde{\Psi}_2$ on $(\mathbb{R}^+)^2 \times Y$. Set

$$\Phi_3 = \tilde{\Phi}_1 \cdot \tilde{\Phi}_2, \quad \Psi_3 = \tilde{\Psi}_1 \cdot \tilde{\Psi}_2.$$

Again we extend Φ_3, Ψ_3 by zero to C^∞–functions on X. Note that

$$\Psi_1 = 1 - \psi_1, \quad \Psi_2 = 1 - \psi_2, \quad \Psi_3 = \Psi_1\, \Psi_2 = (1-\psi_1)\,(1-\psi_2),$$

implying that

$$\Psi_0 + \Psi_1 + \Psi_2 - \Psi_3 = 1. \tag{4.6}$$

Set

$$Q(x,y,t) = \sum_{i=0}^{2} \Phi_i(x)\, K_i(x,y,t)\, \Psi_i(y) \;-\; \Phi_3(x)\, K_3(x,y,t)\, \Psi_3(y). \tag{4.7}$$

Lemma 4.8. *For every $f \in C_0^\infty(X,E)$ we have*

$$\lim_{t\to 0^+} \int_X Q(x,y,t)\, f(y)\, dy = f(x).$$

The proof follows immediately from the construction of Q and (4.6). Set

$$Q_1(x,y,t) = \Big(\frac{\partial}{\partial t} + \Delta_x\Big)\, Q(x,y,t), \tag{4.9}$$

where Δ is applied to the first variable. For every $y \in X$, the support of $Q_1(\cdot, y, t)$ is contained in

$$([0,1] \times Z_1) \cup ([0,1] \times Z_2).$$

Lemma 4.10. *Let $x_0 \in X_0$. There exist $C, c_1, c_2 > 0$ such that*

$$|Q_1(x,y,t)| \le C\, e^{c_1 t - c_2/t}\, e^{-c_2(d^2(x_0,x)+d^2(x_0,y))/t} \tag{4.11}$$

for all $x, y \in X$ *and* $0 < t$.

Proof. We shall estimate $Q_1(x,y,t)$ for $x \in [0,1] \times Z_1$. The case $x \in [0,1] \times Z_2$ is similar. Fix $T > 0$. First observe that by Proposition 4.5, we have

$$|Q_1(x,y,t)| \leq C\, t^{-(n+1)/2}\, e^{c_1 t}\, e^{-c_2\, d^2(x,y)/t}. \tag{4.12}$$

Moreover, the definition of Φ_i, Ψ_i implies that there exists $\delta > 0$ such that

$$d(\operatorname{supp}(\nabla\Phi_i), \operatorname{supp}(\Psi_i)) \geq \delta, \quad i = 0, ..., 3.$$

Hence

$$d(x,y) \geq \delta \quad \text{whenever} \quad Q_1(x,y,t) \neq 0. \tag{4.13}$$

Let $x \in X_1$. We also fix $x_0 \in X_0$. Since X_1 is compact, (4.11) follows from (4.12) and (4.13). Next assume that $x \in [0,1] \times (Z_1 - Z_{1,1})$, where $Z_{1,1}$ is defined by (1.5). For such x we have

$$\begin{aligned} Q_1(x,y,t) = \ & \Big(\frac{\partial}{\partial t} + \Delta\Big)\{\Phi_1(x)\, K_1(x,y,t)\, \Psi_1(y) \\ & \qquad - \Phi_3(x)\, K_3(x,y,t)\, \Psi_3(y)\}. \end{aligned} \tag{4.14}$$

Therefore, we can assume that $y \in \mathbb{R}^+ \times Z_1$. We distinguish two cases:

a) $y \in \mathbb{R}^+ \times Z_{1,1}$.

Let $x = (u,w)$ and $y = (v,z)$ where $u \in [0,1]$, $v \in \mathbb{R}^+$, $w \in Z_1$, and $z \in Z_{1,1}$. From (4.2), (4.4), (4.12) and (4.13), it follows that

$$|Q_1((u,w),(v,z),t)| \leq C_1\, e^{c_1 t - c_2/t}\, e^{-c_2(v^2 + d^2(w,z))/t} \tag{4.15}$$

for certain constants $C_1, c_1, c_2 > 0$. Using the compactness of X_1 again, we can easily see that there exists $c_3 > 0$ such that

$$c_3(d^2(x_0,x) + d^2(x_0,y) + 1) \leq 1 + v^2 + d^2(w,z),$$

which implies (4.11) in this case:

b) $y \in [1,\infty) \times Z_2$.

By definition, we have $\Phi_3(x) = \Phi_1(x)$ and $\Psi_3(x) = \Psi_1(x)$ for $x \in [1,\infty) \times Z_2$. Thus (4.14) yields

$$Q_1(x,y,t) = \Big(\frac{\partial}{\partial t} + \Delta\Big)\{\Phi_1(x)\,[K_1(x,y,t) \ - \ K_3(x,y,t)]\,\Psi_1(y)\}.$$

Moreover, we may assume that $x = (u_1, u_2, w)$, $y = (v_1, v_2, z)$ where $u_1, u_2, v_1, v_2 \in \mathbb{R}^+$ and $w, z \in Y$. From (4.2) and (4.4) we have

$$\begin{aligned}
&K_1((u_1,u_2,w),(v_1,v_2,z),t) \;-\; K_3((u_1,u_2,w),(v_1,v_2,z),t)\\
&= \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{(u_2-v_2)^2}{4t}\Big)\\
&\times \Big\{\tilde{K}_1((u_1,w),(v_1,z),t) \;-\; \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{(u_1-v_1)^2}{4t}\Big)\tilde{K}_3(w,z,t)\Big\}.
\end{aligned}$$

Furthermore, by the definition of Φ_1, we obtain $\Phi_1((u_1, u_2, w)) = \Phi_1(u_2)$. Combining these observations leads to

$$\begin{aligned}
&Q_1((u_1,u_2,w),(v_1,v_2,z),t)\\
&\quad = \Big\{-\Phi_1''(u_2) + \Phi_1'(u_2)\,\frac{(u_2-v_2)}{2t}\Big\}\,\frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{(u_2-v_2)^2}{4t}\Big)\\
&\quad \times \Big\{\tilde{K}_1((u_1,w),(v_1,z),t) \;-\; \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{(u_1-v_1)^2}{4t}\Big)\tilde{K}_3(w,z,t)\Big\}.
\end{aligned} \tag{4.16}$$

By (3.5) of [16], we have

$$\begin{aligned}
&\Big|\tilde{K}_1((u_1,w),(v_1,z),t) \;-\; \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{(u_1-v_1)^2}{4t}\Big)\tilde{K}_3(w,z,t)\Big|\\
&\qquad \le C_2 \exp\Big(c_1 t - \frac{c_2(1 + d^2(m_0,(u_1,w)) + d^2(m_0,(v_1,z)))}{t}\Big)
\end{aligned} \tag{4.17}$$

for some $m_0 \in M_1$ and constants $C_2, c_2 > 0$. Since $x_0 \in X_0$, (4.16) and (4.17) imply that (4.11) holds in this case too. q.e.d.

We can now proceed in the standard way and construct the fundamental solution K from Q. We define inductively

$$Q_{m+1}(x,y,t) = \int_0^t \int_X Q_1(x,w,t-\tau)\,Q_m(w,y,\tau)\,dw\,d\tau, \quad m \ge 1.$$

Usually, a kernel obtained in this way is denoted by $Q_1 * Q_m$. By Lemma 4.10, the w–integral is absolutely convergent and the following estimate holds

$$\begin{aligned}
&|Q_{m+1}(x,y,t)|\\
&\qquad \le C^m \frac{t^m}{m!} \exp\Big(c_1 t - \frac{c_2(1 + d^2(x_0,x) + d^2(x_0,y))}{t}\Big).
\end{aligned} \tag{4.18}$$

Set

$$P = \sum_{m=1}^{\infty} (-1)^m Q_m.$$

By (4.18), this series is absolutely convergent and defines a smooth kernel. Set

$$K = Q + Q * P. \tag{4.19}$$

Then we have

Proposition 4.20. *The kernel K is the fundamental solution for $\partial/\partial t + \Delta$ on X. Moreover, the following estimate holds*

$$\begin{aligned} &|K(x,y,t) - Q(x,y,t)| \\ &\quad \leq C \exp\Big(c_1 t - \frac{c_2(1 + d^2(x_0,x) + d^2(x_0,y))}{t}\Big) \end{aligned} \tag{4.21}$$

for $t > 0$ and certain constants $C, c_1, c_2 > 0$. A similar estimate holds for $D_x K(x,y,t) - D_x Q(x,y,t)$.

The estimate (4.21) follows from (4.18). If we use Proposition 4.5, then it is easy to extend (4.21) so that the derivatives are included. In particular, (4.21) implies that $K - Q$ is the kernel of a Hilbert–Schmidt operator.

We shall now modify the heat kernels K_1, K_2 and K_3 by introducing Dirichlet boundary conditions. Let $\Delta_{i,D}$, $i = 1,2$, be the self-adjoint extension of

$$-\frac{\partial^2}{\partial u_i^2} + A_i^2 : C_0^\infty(\mathbb{R}^+ \times Z_i, E) \to L^2(\mathbb{R}^+ \times Z_i, E),$$

which is obtained by introducing Dirichlet boundary conditions. Then the kernel $K_{i,D}$ of $\exp(-t\Delta_{i,D})$ is given by

$$\begin{aligned} &K_{i,D}((u,w),(v,z),t) \\ &= \frac{1}{\sqrt{4\pi t}}\Big\{\exp(-\frac{(u-v)^2}{4t}) - \exp(-\frac{(u+v)^2}{4t})\Big\}\tilde{K}_i(w,z,t), \\ &\qquad i = 1,2, \end{aligned} \tag{4.22}$$

where $\tilde{K}_i$ is the heat kernel for A_i^2. Next consider

$$-\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + D_Y^2 : C_0^\infty((\mathbb{R}^+)^2 \times Y, E) \to L^2((\mathbb{R}^+)^2 \times Y, E)$$

and introduce Dirichlet boundary conditions. Let $\Delta_{3,D}$ be the corresponding self–adjoint extension and let $K_{3,D}$ be the kernel of the heat operator $\exp(-t\Delta_{3,D})$. Then $K_{3,D}$ is given by

$$
\begin{aligned}
K_{3,D}&((u_1,u_2,w),(v_1,v_2,z),t)\\
&= \frac{1}{4\pi t}\Big\{\exp\big(-\frac{(u_1-v_1)^2}{4t}\big) - \exp\big(-\frac{(u_1+v_1)^2}{4t}\big)\Big\}\\
&\quad\times \Big\{\exp\big(-\frac{(u_2-v_2)^2}{4t}\big) - \exp\big(-\frac{(u_2+v_2)^2}{4t}\big)\Big\}\tilde{K}_3(w,z,t),
\end{aligned}
\tag{4.23}
$$

where $\tilde{K}_3$ has the same meaning as in (4.4). We extend the heat operators $\exp(-t\Delta_{1,D})$, $\exp(-t\Delta_{2,D})$ and $\exp(-t\Delta_{3,D})$ to bounded operators in $L^2(X,E)$, and put them equal to zero on the orthogonal complement of the subspaces $L^2(\mathbb{R}^+\times Z_1,E)$, $L^2(\mathbb{R}^+\times Z_2,E)$ and $L^2((\mathbb{R}^+)^2\times Y,E)$, respectively. We can now prove

Theorem 4.24. *Let the notation be as above. Then for each $t>0$, the operator*

$$e^{-t\overline{\Delta}} - e^{-t\Delta_{1,D}} - e^{-t\Delta_{2,D}} + e^{-t\Delta_{3,D}}$$

is a Hilbert–Schmidt operator.

Proof. We extend the kernels $K_{i,D}$, $i=1,2,3$, by zero to kernels on $X\times X$ and we denote these kernels also by $K_{i,D}$. To prove the theorem, we have to show that

$$\int_X\int_X \left|K(x,y,t)-K_{1,D}(x,y,t)-K_{2,D}(x,y,t)+K_{3,D}(x,y,t)\right|^2 dx\,dy < \infty. \tag{4.25}$$

By (4.21), we may replace K by the parametrix Q. Since X_1 is compact, we can remove K_0 from the parametrix. Let χ_i, $i=1,2$, be the characteristic function of $\mathbb{R}^+\times Z_i\subset X$ and let χ_3 denote the characteristic function of $(\mathbb{R}^+)^2\times Y\subset X$. Let K_i, $i=1,2,3$, be the kernels defined by (4.2) and (4.4), respectively, and set

$$\tilde{Q}(x,y,t) = \sum_{i=1}^{2}\chi_i(x)K_i(x,y,t)\chi_i(y) - \chi_3(x)K_3(x,y,t)\chi_3(y).$$

Now observe that

$$\Phi_iK_i\Psi_i - \chi_iK_i\chi_i = (\Phi_i-\chi_i)K_i\Psi_i+\chi_iK_i(\Psi_i-\chi_i),\quad i=1,2,3. \tag{4.26}$$

Furthermore, by definition, the support of each of the functions $\Phi_i-\chi_i$ and $\Psi_i-\chi_i$, $i=1,2,3$, is contained in $([0,1]\times Z_1)\cup([0,1]\times Z_2)$.

Therefore, we may use Proposition 4.5 and proceed in essentially the same way as in the proof of Lemma 4.10 to show that

$$(4.27)\quad \left|Q(x,y,t) - \tilde{Q}(x,y,t)\right| \le C\exp(-c(d^2(x_0,x)+d^2(x_0,y))/t),$$

for some $x_0 \in X_0$. Hence, in order to prove (4.25), we can replace K by $\tilde{Q}$, that is, we have to investigate

$$(4.28)\qquad \tilde{Q}(x,x',t) - K_{1,D}(x,x',t) - K_{2,D}(x,x',t) + K_{3,D}(x,x',t).$$

This kernel vanishes, unless $x, x' \in \mathbb{R}^+ \times Z_1$ or $x, x' \in \mathbb{R}^+ \times Z_2$. Consider the first case, that is, $x = (u_1, w)$, $x' = (v_1, z)$, $w, z \in Z_1$. Suppose that $w \in M_1$. Then the kernel (4.28) equals

$$\frac{1}{\sqrt{4\pi t}}\exp(-\frac{(u_1+v_1)^2}{4t})\,\tilde{K}_1(w,z,t).$$

By (3.5) of [16], $|\tilde{K}_1(w,z,t)|$ belongs to $L^2(M_1 \times Z_1)$. Hence, the kernel (4.28) is square integrable on $(\mathbb{R}^+ \times M_1) \times (\mathbb{R}^+ \times Z_1)$, and by symmetry, it is also square integrable on $(\mathbb{R}^+ \times Z_1) \times (\mathbb{R}^+ \times M_1)$. It remains to consider the case $x = (u_1, u_2, y)$ and $x' = (v_1, v_2, y')$, $y, y' \in Y$. Then (4.28) equals

$$\begin{aligned}
&\frac{1}{\sqrt{4\pi t}}\exp(-\frac{(u_1+v_1)^2}{4t})\\
&\quad\times\left\{\tilde{K}_1((u_2,y),(v_2,y'),t) - \frac{1}{\sqrt{4\pi t}}\exp(-\frac{(u_2-v_2)^2}{4t})\,\tilde{K}_3(y,y',t)\right\}\\
&\quad+\frac{1}{\sqrt{4\pi t}}\exp(-\frac{(u_2+v_2)^2}{4t})\\
&\quad\times\left\{\tilde{K}_2((u_1,y),(v_1,y'),t) - \frac{1}{\sqrt{4\pi t}}\exp(-\frac{(u_1-v_1)^2}{4t})\,\tilde{K}_3(y,y',t)\right\}\\
&\quad+\frac{1}{4\pi t}\exp(-\frac{(u_1+v_1)^2}{4t})\exp(-\frac{(u_2+v_2)^2}{4t})\,\tilde{K}_3(y,y',t).
\end{aligned}$$

Since Y is compact, the third term is square integrable on $(\mathbb{R}^+)^2 \times Y$. To deal with the first two terms, we refer again to the estimate (3.5) of [16] from which we deduce that these two terms are also square integrable on $(\mathbb{R}^+)^2 \times Y$. Combining our results yields the required (4.25). q.e.d.

By a more elaborate method one can improve the statement of Theorem 4.24 by showing that the combination of the heat operators is the trace class. This result will be important for the investigation of the continuous spectrum.

5. The analytic continuation of the resolvent

The notation will be the same as in the previous section. In particular, $\Delta = D^2$ is the spinor Laplacian associated with some Dirac operator D. Our purpose is to extend the resolvent $(\overline{\Delta} - \lambda^2)^{-1}$ analytically as a function of λ to a neighborhood of $\lambda = 0$. In the present paper we study this problem only under the additional assumption that $\ker D_Y = 0$, where D_Y is the Dirac operator attached to the corner Y.

Let $\mathcal{A}_j$, $j = 1, 2$, be the self–adjoint extension of the Dirac operator $A_j : C_0^\infty(Z_j, E_j) \to L^2(Z_j, E_j)$ defined by (2.1). To begin with, we construct a parametrix for $(\mathcal{A}_j^2 - \lambda^2)^{-1}$, $\operatorname{Im}(\lambda) > 0$. Recall that on $\mathbb{R}^+ \times Y \subset Z_j$, A_j^2 takes the form

$$A_j^2 = -\frac{\partial^2}{\partial u^2} + D_Y^2.$$

This follows immediately from (2.1)–(2.4). Let $\{\phi_l\}_{l \in \mathbb{N}}$ be an orthonormal basis for $L^2(Y, S)$ consisting of eigensections of D_Y^2 with corresponding eigenvalues $0 \le \mu_1 \le \mu_2 \le \cdots$. For $u \ne v$, we put

$$\begin{aligned} &H((u,y),(v,y'),\lambda) \\ &\quad = \sum_{l=1}^{\infty} \frac{e^{-\sqrt{\mu_l - \lambda^2}\,|u-v|}}{\sqrt{\mu_l - \lambda^2}} \phi_l(y) \otimes \phi_l(y'), \quad \operatorname{Im}(\lambda) > 0. \end{aligned} \tag{5.1}$$

Then $H(\lambda)$ is the kernel of the parametrix at infinity. We glue it to an interior parametrix which we construct as follows. Let $\hat{Z}_{j,1} = Z_{j,1} \cup (-Z_{j,1})$ be the double of the compact manifold $Z_{j,1}$ which is defined by (1.5). The operator A_j^2, restricted to $Z_{j,1}$, has a natural extension to an elliptic operator on $\hat{Z}_{j,1}$, and we denote its resolvent by $Q_{j,1}(\lambda)$. Let $\rho(a,b) \in C^\infty(\mathbb{R})$ be the function introduced in §4 and put

$$\Phi_{j2} = \rho(1/4, 5/16), \qquad \Psi_{j2} = \rho(3/8, 5/8),$$
$$\Phi_{j1} = 1 - \rho(7/8, 1), \qquad \Psi_{j1} = 1 - \Psi_{j2}.$$

We regard Φ_{ji}, Ψ_{ji}, $i, j \in \{1, 2\}$, as functions on $[0,1] \times Y$ and extend them to C^∞ functions on Z_j in the obvious way. Put

$$P_j(\lambda) = \Psi_{j1} Q_{j1}(\lambda) \Phi_{j1} + \Psi_{j2} H(\lambda) \Phi_{j2}, \quad \operatorname{Im}(\lambda) > 0. \tag{5.2}$$

Then we have

$$P_j(\lambda)(A_j^2 - \lambda^2) = \operatorname{Id} + K_j(\lambda), \tag{5.3}$$

where $K_j(\lambda)$ has a smooth kernel $K_j(z, z', \lambda)$ which satisfies

$$\operatorname{supp}_{z'} K_j(z, z', \lambda) \subset (0,1) \times Y$$

and

$$K_j(z,z',\lambda)=0 \quad \text{for} \quad d(z,z')<1/16.$$

This implies that $K_j(\lambda)$ is a holomorphic family of compact operators in L^2. Moreover, from (5.1), it follows that there exists $C>0$ such that $\| K_j(i\lambda) \| \leq C/\lambda$, $\lambda \geq 1$. Thus $\mathrm{Id}+K_j(i\lambda)$ is invertible for $\lambda \gg 0$ and hence, $\left(\mathrm{Id}+K_j(\lambda)\right)^{-1}$ is a meromorphic function on $\mathrm{Im}(\lambda)>0$ [21]. By (5.3), we get

$$(5.4) \qquad (\mathcal{A}_j^2-\lambda^2)^{-1}=\left(\mathrm{Id}+K_j(\lambda)\right)^{-1}P_j(\lambda), \quad \mathrm{Im}(\lambda)>0.$$

Lemma 5.5. *For each λ in the half-plane* $\mathrm{Im}(\lambda) > 0$, $(\mathcal{A}_j^2-\lambda^2)^{-1} - P_j(\lambda)$ *is a compact operator in L^2.*

Proof. By (5.4), we have

$$(5.6) \quad (\mathcal{A}_j^2-\lambda^2)^{-1}-P_j(\lambda)=-\left(\mathrm{Id}+K_j(\lambda)\right)^{-1}K_j(\lambda)P_j(\lambda), \quad \mathrm{Im}(\lambda)>0,$$

and the claimed result follows from the compactness of $K_j(\lambda)$. q.e.d.

Now we construct a parametrix for $(\overline{\Delta}-\lambda^2)^{-1}$. Let Δ_i, $i=1,2,3$, and $\tilde{\Delta}$ be the differential operators introduced at the beginning of §4. Let $\overline{\Delta}_i$ be the unique self-adjoint extension of Δ_i. Note that $\overline{\Delta}_i$, $i=1,2$, are self-adjoint operators in $L^2(\mathbb{R}\times Z_i,\widetilde{E}_i)$, and $\overline{\Delta}_3$ is a self-adjoint operator in $L^2(\mathbb{R}^2\times Y,\widetilde{S})$. Here $\widetilde{E}_i$ and $\widetilde{S}$ denote the pullbacks of the corresponding bundles over Z_i and Y, respectively, to vector bundles over $\mathbb{R}\times Z^i$ and $\mathbb{R}^2\times Y$, respectively. Put

$$R_i(\lambda)=(\overline{\Delta}_i-\lambda^2)^{-1}, \quad \mathrm{Im}(\lambda)>0,\ i=1,2,3.$$

Furthermore, let

$$R_0(\lambda)=(\tilde{\Delta}-\lambda^2)^{-1}, \quad \mathrm{Im}(\lambda)>0.$$

Let $\Phi_i,\Psi_i\in C^\infty(X)$, $i=0,\dots,3$, be the functions introduced in §4. Put

$$(5.7) \qquad P(\lambda)=\sum_{i=0}^{2}\Psi_i R_i(\lambda)\Phi_i \;-\; \Psi_3R_3(\lambda)\Phi_3, \quad \mathrm{Im}(\lambda)>0.$$

Then $P(\lambda)$ is a bounded operator in L^2, and we shall now verify that $P(\lambda)$ is a parametrix for $(\overline{\Delta}-\lambda^2)^{-1}$. Put

$$(5.8) \qquad G(\lambda)=P(\lambda)(\Delta-\lambda^2)\;-\;\mathrm{Id}, \quad \mathrm{Im}(\lambda)>0.$$

Then we have to show that $G(\lambda)$ is a compact operator in L^2. By (5.7) we may write

$$G(\lambda)=\sum_{j=0}^{2}G_j(\lambda)\;-\;G_3(\lambda),$$

where

$$G_j(\lambda) = (\Psi_j R_j(\lambda)\Phi_j)(\Delta - \lambda^2) \; - \; \Psi_j \,\mathrm{Id}. \tag{5.9}$$

Since Φ_0, Ψ_0 have compact support, it follows from Rellich's compactness theorem that $G_0(\lambda)$ is a holomorphic function on the upper half–plane with values in the compact operators in L^2. For $j = 1, 2$ we have

$$G_j(\lambda) = -\Psi_j\Big(R_j(\lambda) \circ \frac{\partial}{\partial v_j}\Big)\frac{\partial \Phi_j}{\partial v_j} \; - \; \Psi_j R_j(\lambda)\frac{\partial^2 \Phi_j}{\partial v_j^2}, \tag{5.10}$$

and $R_j(\lambda)$ is given by the operator valued kernel

$$R_j(u, v, \lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(u-v)}\,(\mathcal{A}_j^2 - \lambda^2 + \xi^2)^{-1}\,d\xi. \tag{5.11}$$

Since $\| \, (\mathcal{A}_j^2 - \lambda^2 + \xi^2)^{-1} \, \| = 1/\operatorname{dist}(\mathbb{R}^+, \lambda^2 - \xi^2)$, the integral is absolutely convergent. If $u \neq v$, we can integrate by parts for $N \in \mathbb{N}$, to get

$$\begin{aligned} &R_j(u, v, \lambda) \\ (5.12)\qquad &= \frac{(-i)^N}{2\pi}\,(u - v)^{-N}\int_{-\infty}^{\infty} e^{i\xi(u-v)}\Big(\frac{\partial}{\partial \xi}\Big)^N\Big[(\mathcal{A}_j^2 - \lambda^2 + \xi^2)^{-1}\Big]\,d\xi. \end{aligned}$$

If $k > n$, then $(\mathcal{A}_j^2 - \lambda^2 + \xi^2)^{-k}$ has a continuous kernel. Let $T \geq 0$ and let $Z_{j,T}$ be defined by (1.5). Since $\partial\Phi_j/\partial v_j$ and Ψ_j have disjoint supports, from (5.10) and (5.12) it follows that the restriction of $G_j(\lambda)$ to $\mathbb{R}^+ \times Z_{j,T}$ is Hilbert–Schmidt. Let $\chi_{j,T}$ be the characteristic function of $\mathbb{R}^+ \times Z_{j,T} \subset \mathbb{R}^+ \times Z_j$. Suppose that $z \in M_j \subset Z_j$ and $z' \in [1, \infty) \times Y \subset Z_j$. Then by (5.2) we get $P_j(z, z', \lambda) = 0$. Hence, for $T > 1$,

$$\begin{aligned} &\chi_{j,0} R_j(u, v, \lambda)(1 - \chi_{j,T}) \\ (5.13)\qquad &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(u-v)}\Big\{(\mathcal{A}_j^2 - \lambda^2 + \xi^2)^{-1} \; - \; P_j(\sqrt{\lambda^2 - \xi^2})\Big\}\,d\xi. \end{aligned}$$

First integrating by parts and then applying Lemma 5.5 show that $\chi_{j,0} G_j(\lambda)(1 - \chi_{j,T})$ is a compact operator in L^2.

Let θ be the characteristic function of $(\mathbb{R}^+)^2 \times Y \subset X$. Then our investigation of $G(\lambda)$ is reduced to the study of $\sum_{j=1}^2 \theta G_j(\lambda)\theta - \theta G_3(\lambda)\theta$.

Using the definiton of Φ_3 and Ψ_3, we get

$$
\begin{aligned}
\sum_{j=1}^{2} \theta G_j(\lambda)\theta - \theta G_3(\lambda)\theta = &- \Psi_1\{R_1(\lambda) - \Psi_2 R_3(\lambda)\Phi_2\} \frac{\partial^2\Phi_1}{\partial v_1^2} \\
&- \Psi_1\Big[\{R_1(\lambda) - \Psi_2 R_3(\lambda)\Phi_2\} \circ \frac{\partial}{\partial v_1}\Big] \frac{\partial \Phi_1}{\partial v_1} \\
&- \Psi_2\{R_2(\lambda) - \Psi_1 R_3(\lambda)\Phi_1\} \frac{\partial^2\Phi_2}{\partial v_2^2} \\
&- \Psi_2\Big[\{R_2(\lambda) - \Psi_1 R_3(\lambda)\Phi_1\} \circ \frac{\partial}{\partial v_2}\Big] \frac{\partial \Phi_2}{\partial v_2}.
\end{aligned} \tag{5.14}
$$

We consider the first term. Let $H(\lambda)$ be the operator in $L^2(\mathbb{R}^+ \times Y, E_1)$, which is defined by the kernel (5.1). Then $R_3(\lambda)$ can be represented by the operator valued kernel

$$
R_3(u,v,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(u-v)}\, H\big(\sqrt{\lambda^2-\xi^2}\big)\, d\xi, \quad \operatorname{Im}(\lambda) > 0.
$$

The integral is absolutely convergent. Now observe that the parametrices $P_1(\lambda)$ and $H(\lambda)$ differ by a compact operator. Hence, up to a compact operator, the first term on the right of (5.14) can be written as

$$
\frac{i}{2\pi}\Psi_1\Big\{\frac{1}{u_1-v_1}\int_{-\infty}^{\infty} e^{i\xi(u_1-v_1)}\,\frac{\partial}{\partial\xi}\Big[(\mathcal{A}_1^2-\lambda^2+\xi^2)^{-1} - P_1\big(\sqrt{\lambda^2-\xi^2}\big)\Big]\, d\xi\Big\}\,\frac{\partial\Phi_1}{\partial v_1}. \tag{5.15}
$$

The integral is absolutely convergent. Since

$$
\operatorname{supp}(\partial\Phi_1/\partial v_1) \subset (0,1)\times Z_1,
$$

and $\partial\Phi_1/\partial v_1$, Ψ_1 have disjoint supports, it follows from Lemma 5.5 that (5.15) is a compact operator in L^2. The remaining terms in (5.14) can be treated in the same way. Using (5.9) one can show that $\| G(i\lambda) \| \le C/\lambda$, $\lambda \ge 1$. Thus we proved

Lemma 5.16. *Let $P(\lambda)$ be defined by* (5.7). *Then we have*

$$
P(\lambda)(\Delta - \lambda^2) = \operatorname{Id} + G(\lambda), \quad \operatorname{Im}(\lambda) > 0,
$$

where $G(\lambda)$ is a holomorphic function on the upper half-plane with values in the compact operators in $L^2(X,E)$. Moreover, there exists $C > 0$ such that $\| G(i\lambda) \| \le C/\lambda$ for $\lambda \ge 1$.

By Lemma 5.16, $\operatorname{Id} + G(i\lambda)$ is invertible for $\lambda \gg 0$. Hence $\lambda \mapsto (\operatorname{Id} + G(\lambda))^{-1}$ is a meromorphic function on the upper half-plane with values in the bounded operators in $L^2(X,E)$ [21]. Thus we get

$$
(\overline{\Delta} - \lambda^2)^{-1} = (\operatorname{Id} + G(\lambda))^{-1} P(\lambda), \quad \operatorname{Im}(\lambda) > 0. \tag{5.17}
$$

We shall use (5.17) to extend the resolvent to a meromorphic function in a neighborhood of zero.

Let $W_i \subset X$, $i = 1, 2$, be the submanifolds defined by (1.2) and recall the decomposition (1.3). Let $\rho_i \in C^\infty(X)$ be such that $\rho_i|W_i = 1$, $\rho(u_i, z_i) = \rho_i(u_i)$ for $(u_i, z_i) \in \mathbb{R}^+ \times Z_i$ and $\rho_i(u_i, z_i) = u_i$ for $u_i \geq 1$. Set

$$\rho = \rho_1 + \rho_2.$$

Given $\delta \in \mathbb{R}$, we define a weighted L^2–space by

$$L^2_\delta(X, E) = \{\varphi : X \to E \mid \varphi \textit{ a measurable section and} \tag{5.18}$$
$$\int_X |\varphi(x)|^2 e^{2\delta\rho(x)}\, dx < \infty\}.$$

Note that for $\delta > 0$ the following inclusions hold:

$$L^2_\delta(X, E) \subset L^2(X, E) \subset L^2_{-\delta}(X, E). \tag{5.19}$$

Given $\delta, \delta' \in \mathbb{R}$, we denote by $\mathcal{L}(L^2_\delta(X, E), L^2_{\delta'}(X, E))$ the space of bounded linear operators from $L^2_\delta(X, E)$ into $L^2_{\delta'}(X, E)$. Let $\kappa_i > 0$ be the smallest positive eigenvalue of $\mathcal{A}_i^2$, $i = 1, 2$. Put

$$\delta_0 = \frac{1}{2} \min\{\sqrt{\mu_1}, \sqrt{\kappa_1}, \sqrt{\kappa_2}\},$$

and let

$$\Omega = \{\lambda \in \mathbb{C} \mid \operatorname{Im}(\lambda) > 0\} \cup \{\lambda \in \mathbb{C} \mid |\lambda| < \delta_0\}. \tag{5.20}$$

Lemma 5.21. *Let $0 < \delta < \delta_0$. Suppose that $\ker D_Y = 0$. Then the parametrix $P(\lambda)$ extends from the upper half-plane to a meromorphic function on Ω with values in the space $\mathcal{L}(L^2_\delta(X, E), L^2_{-\delta}(X, E))$.*

Proof. Let $\delta < \delta_0$. We have to show that each term on the right-hand side of (5.7) extends to a meromorphic function on Ω. Since $\tilde{\Delta}$ is an elliptic operator on a closed manifold, $\Psi_0 R_0(\lambda)\Phi_0$ is a meromorphic function on the whole complex plane with values in the bounded operators in $L^2(X, E)$. Using the inclusions (5.19), $\Psi_0 R_0(\lambda)\Phi_0$ becomes a meromorphic family on $\mathbb{C}$ with values in $\mathcal{L}(L^2_\delta(X, E), L^2_{-\delta}(X, E))$. Next consider $R_3(\lambda)$. Let $\tilde{S}$ be the pullback of $S \to Y$ to $\mathbb{R}^2 \times Y$. Given $\varphi \in L^2(\mathbb{R}^2 \times Y, \tilde{S})$, let $\hat{\varphi}(\xi, y)$ denote the Fourier transform of $\varphi(u, y)$ with respect to the u–variables. Recall that $\Delta_3 = -\partial^2/\partial u_1^2 - \partial^2/\partial u_2^2 + D_Y^2$. Hence, we may write $R_3(\lambda)$ in the form

$$\begin{aligned}&(R_3(\lambda)\varphi)(u, y)\\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{i\langle \xi, u\rangle} \left(D_Y^2 + \|\xi\|^2 - \lambda^2\right)^{-1} (\hat{\varphi}(\xi, y))\, d\xi, \quad \operatorname{Im}(\lambda) > 0.\end{aligned} \tag{5.22}$$

Since D_Y^2 is self-adjoint and nonnegative, we have

$$\| (D_Y^2 + \| \xi \|^2 - \lambda^2)^{-1} \| = (\operatorname{dist}(\mathbb{R}^+, \lambda^2 - \| \xi \|^2))^{-1},$$

which implies that the integral (5.22) is absolutely convergent for $\operatorname{Im}(\lambda) > 0$. If $\ker D_Y = 0$, then the spectrum of D_Y^2 is contained in $[\mu_1, \infty)$, $\mu_1 > 0$. Therefore, the right-hand side of (5.22) defines a bounded operator in L^2 for all $\lambda \in \Omega$. Let $\delta_0 > \delta \geq 0$ be given. Using (5.19), we obtain a holomorphic function on Ω with values in $\mathcal{L}(L^2_\delta(X,E), L^2_{-\delta}(X,E))$.

It remains to investigate $R_j(\lambda)$, $j = 1, 2$. If $\ker D_Y = 0$, then it follows from Theorem 4.10 of [16] that the continuous spectrum of $\mathcal{A}_j^2$ equals $[\mu_1, \infty)$, where $\mu_1 > 0$. Let $L^2(Z_j, E_j)^\perp$ be the orthogonal complement of $\ker \mathcal{A}_j$ in $L^2(Z_j, E_j)$, and let $\mathcal{A}_{j,1}$ denote the restriction of $\mathcal{A}_j$ to $L^2(Z_j, E_j)^\perp$. Then the spectrum of $(\mathcal{A}_{j,1})^2$ is contained in $[\mu_1, \infty)$. Since $\Delta_j = -\partial^2/\partial u_j^2 + A_j^2$, we get a corresponding decomposition for $\overline{\Delta}_j$; namely

$$\overline{\Delta}_j = \overline{\Delta}_{j,1} \oplus \overline{\Delta}_{j,2}. \tag{5.23}$$

The spectrum of $\overline{\Delta}_{j,1}$ is also contained in $[\mu_1, \infty)$, and $\overline{\Delta}_{j,2}$ is the self-adjoint extension of $\partial^2/\partial u^2 \otimes \mathrm{Id}$, acting in $C_0^\infty(\mathbb{R}) \otimes \ker \mathcal{A}_j$. Let

$$R_{j,i}(\lambda) = (\overline{\Delta}_{j,i} - \lambda^2)^{-1},$$

$i, j \in \{1, 2\}$. Then we have

$$R_j(\lambda) = R_{j,1}(\lambda) \oplus R_{j,2}(\lambda), \quad j = 1, 2, \quad \operatorname{Im}(\lambda) > 0. \tag{5.24}$$

Since $\sigma(\overline{\Delta}_{j,1}) \subset [\mu_1, \infty)$, it follows that $R_{j,1}(\lambda)$ is a holomorphic function on Ω with values in the bounded operators in $L^2(X,E)$. Using (5.19), we get a holomorphic function on Ω with values in $\mathcal{L}(L^2_\delta(X,E), L^2_{-\delta}(X,E))$. Let $\varphi_1, \ldots, \varphi_{m_j}$ be an orthonormal basis for $\ker \mathcal{A}_j$. Then for $u_j \neq v_j$, the kernel of $R_{j,2}(\lambda)$ is given by

$$\begin{aligned} &R_{j,2}((u_j, z_j), (v_j, z_j'), \lambda) \\ &\quad = \frac{1}{\lambda} e^{i\lambda|u_j - v_j|} \sum_{l=1}^{m_j} \varphi_l(z_j) \otimes \overline{\varphi_l(z_j')}, \quad \operatorname{Im}(\lambda) > 0. \end{aligned} \tag{5.25}$$

This kernel has an obvious extension to a meromorphic function on $\mathbb{C}$, and for $0 < \delta < \delta_0$ the extended kernel defines a meromorphic function on Ω with values in the space $\mathcal{L}(L^2_\delta(X,E), L^2_{-\delta}(X,E))$. Thus for all terms on the right-hand side of (5.7), we have constructed analytic extensions with the desired properties. q.e.d.

Let

$$\Omega_1 = \{\lambda \in \mathbb{C} \mid \operatorname{Im}(\lambda) > |\operatorname{Re}(\lambda)|\} \cup \{\lambda \in \mathbb{C} \mid |\lambda| < \delta_0\}.$$

Lemma 5.26. *Let* $0 < \delta < \delta_0$ *and suppose that* $\ker D_Y = 0$. *Then the operator* $G(\lambda)$*, defined by* (5.8)*, extends from the upper half–plane to a meromorphic function* $\lambda \in \Omega_1 \mapsto \tilde{G}(\lambda)$ *of compact operators in* $L^2_{-\delta}(X, E)$*. Moreover, there exists* $C > 0$ *such that for* $\lambda \geq 1$, $\| \tilde{G}(i\lambda) \|_{-\delta} \leq C/\lambda$.

Proof. Let $0 < \delta < \delta_0$. First we show that $G(\lambda)$ extends to a meromorphic function $\lambda \in \Omega_1 \mapsto \tilde{G}(\lambda) \in \mathcal{L}((L^2_{-\delta}(X, E))$. As above, we write $G(\lambda) = \sum_{j=0}^{2} G_j(\lambda) - G_3(\lambda)$, where $G_j(\lambda)$ is defined by (5.9). The statement of the lemma holds obviously for $G_0(\lambda)$. To treat $R_3(\lambda)$, we consider the weighted L^2 space $L^2_{-\delta}(\mathbb{R}^2 \times Y, \tilde{S})$ defined as the space of measurable sections which are square integrable with respect to the weight function $e^{2\delta(u_1+u_2)}$, $\delta \in \mathbb{R}$. For $\varphi \in C_0^\infty(\mathbb{R}^2 \times Y, \tilde{S})$ and $\delta_0 > \delta \geq 0$, put

$$\tilde{R}_3(\lambda)\varphi = \frac{1}{4\pi^2} \int_{\mathrm{Im}(\xi)=\delta} e^{i\langle u,\xi\rangle} \left(D_Y^2 + \xi_1^2 + \xi_2^2 - \lambda^2\right)^{-1} (\hat{\varphi}(\xi, y))\, d\xi, \quad \lambda \in \Omega_1.$$

Since the spectrum of D_Y^2 is contained in $[\mu_1, \infty)$, the integral is absolutely convergent for $\lambda \in \Omega_1$. Then $\tilde{R}_3(\lambda)$ extends to a bounded operator in $L^2_{-\delta}(\mathbb{R}^2 \times Y, \tilde{S})$, which coincides with $R_3(\lambda)$ on the subspace $L^2(\mathbb{R}^2 \times Y, \tilde{S})$. By (5.9) we get a holomorphic function

$$\lambda \in \Omega_1 \to \tilde{G}_3(\lambda) \in \mathrm{L}(L^2_{-\delta}(X, E)).$$

Next consider $G_j(\lambda)$, $j = 1, 2$. We use (5.10) to express $G_j(\lambda)$ in terms of $R_j(\lambda)$. If we substitute the decomposition (5.24) on the right-hand side of (5.10), we obtain a corresponding decomposition

$$G_j(\lambda) = G_{j,1}(\lambda) \oplus G_{j,2}(\lambda). \tag{5.27}$$

Consider the kernel (5.25) of $R_{j,2}(\lambda)$. We observe that by (4.1) of [16], each $\phi \in \ker \mathcal{A}_j$, $\| \phi \| = 1$, satisfies $|\phi(u, y)| \leq Ce^{-\mu_1 u}$ for $u \in \mathbb{R}^+$, $y \in Y$. Using (5.25), it follows that $G_{j,2}(\lambda)$ extends from the upper half–plane to a meromorphic function $\lambda \in \Omega \mapsto \tilde{G}_{j,2}(\lambda) \in \mathcal{L}(L^2_{-\delta}(X, E))$. The resolvent $R_{j,1}(\lambda)$ has the following operator valued kernel

$$\begin{aligned} &R_{j,1}(u, v, \lambda) \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(u-v)} \left((\mathcal{A}_{j,1})^2 + \xi^2 - \lambda^2\right)^{-1} d\xi, \quad \mathrm{Im}(\lambda) > 0. \end{aligned} \tag{5.28}$$

By definition, the spectrum of $(\mathcal{A}_{j,1})^2$ is contained in $[\mu_1, \infty)$. Therefore the right-hand side of (5.28) is convergent for $\lambda \in \Omega$. To study (5.28) we introduce weighted L^2 spaces for Z_j. Let $\sigma_j \in C^\infty(Z_j)$ be such that

$\sigma_j(u,y) = u$ for $(u,y) \in [1,\infty) \times Y$. For $\delta \in \mathbb{R}$ let

$$L^2_\delta(Z_j,E_j) = \{\varphi : Z_j \to E_j \mid \varphi \text{ a measurable section and } \int_{Z_j} |\varphi(z)|^2\, e^{2\delta\sigma_j(z)}\, dz < \infty\}.$$

Note that for $\delta \geq 0$, we have the following inclusions:

$$L^2_\delta(Z_j,E_j) \subset L^2(Z_j,E_j) \subset L^2_{-\delta}(Z_j,E_j). \tag{5.29}$$

Our goal is to extend $\left((\mathcal{A}_{j,1})^2 + \xi^2 - \lambda^2\right)^{-1}$ to a bounded operator in $L^2_{-\delta}(Z_j,E_j)$ for $\lambda \in \Omega$. Let $L^2_{-\delta}(\mathbb{R}\times Y,\tilde{S})$ be the corresponding local L^2 space with weight function $e^{-2\delta u}$. For $\psi \in C_0^\infty(\mathbb{R}\times Y,\tilde{S})$ let $\hat{\psi}(\xi,y)$ be the Fourier transform of $\psi(u,y)$ with respect to the u–variable, and set

$$\begin{aligned}(\tilde{H}(\lambda)\psi)(u,y) &\\ &= \frac{1}{2\pi}\int_{\operatorname{Im}(\xi)=\delta} e^{iu\xi}\left(D_Y^2 + \xi^2 - \lambda^2\right)^{-1}(\hat{\psi}(\xi,y))\, d\xi,\\ &\qquad \lambda \in \Omega_1.\end{aligned} \tag{5.30}$$

Then $\tilde{H}(\lambda)$ extends to a bounded operator in $L^2_{-\delta}(\mathbb{R}\times Y,\tilde{S})$, which coincides with $H(\lambda)$ on the subspace $L^2(\mathbb{R}\times Y,\tilde{S})$. Being the resolvent of an elliptic operator on a compact manifold, $Q_{j,1}(\lambda)$ is a meromorphic function on $\mathbb{C}$. If $Q_{j,1}(\lambda)$ has a pole at $\lambda = 0$, we remove the contribution of this pole and denote the resulting kernel by $\tilde{Q}_{j,1}(\lambda)$. If we pick $\delta_0 > 0$ sufficiently small, then $\tilde{Q}_{j,1}(\lambda)$ is holomorphic in Ω. Put

$$\tilde{P}_j(\lambda) = \Psi_{j1}\tilde{Q}_{j,1}(\lambda)\Phi_{j1} + \Psi_{j2}\tilde{H}(\lambda)\Phi_{j2}, \quad \lambda \in \Omega_1.$$

Then $\tilde{P}_j(\lambda)$, $\lambda \in \Omega_1$, is a holomorphic family of bounded operators in $L^2_{-\delta}(Z_j,E_j)$. It follows from (5.30) that there exists $C > 0$ such that

$$\| \tilde{P}_j(\lambda) \|_{-\delta} \leq C\,(1+|\lambda|^2)^{-1}, \quad \lambda \in \Omega_1.$$

By the same argument one can prove that the operators $K_j(\lambda)$ in (5.3) extend to a holomorphic family $\lambda \in \Omega \mapsto \tilde{K}_j(\lambda)$ of compact operators in $L^2_{-\delta}(Z_j,E_j)$, and for $\lambda \geq 1$ the norm of $\tilde{K}_j(i\lambda)$ is bounded by $C\lambda^{-1}$. Hence, $\lambda \in \Omega \mapsto \left(\operatorname{Id} + \tilde{K}_j(\lambda)\right)^{-1}$ is a meromorphic function of bounded operators in $L^2_{-\delta}(Z_j,E_j)$ [21]. Moreover, for $\lambda \in \Omega_1$, $|\lambda| \geq C$ we have $\| \left(\operatorname{Id} + \tilde{K}_j(\lambda)\right)^{-1} \|_{-\delta} \leq 2$. Put

$$R_{Z_j}(\lambda) = \left(\operatorname{Id} + \tilde{K}_j(\lambda)\right)^{-1}\tilde{P}_j(\lambda).$$

Combining our results with (5.4), we obtain

Lemma 5.31. *Let $0 < \delta < \delta_0$. The resolvent $(\mathcal{A}_j^2 - \lambda^2)^{-1}$ extends from the upper half-plane to a meromorphic function*

$$\lambda \in \Omega_1 \mapsto R_{Z_j}(\lambda) \in \mathcal{L}(L^2_{-\delta}(Z_j, E_j)).$$

Moreover, there exists $C > 0$ such that $R_{Z_j}(\lambda)$ is holomorphic in $\Omega_1 \cap \{\lambda \in \mathbb{C} \mid |\lambda| \geq C\}$ and satisfies

$$\| R_{Z_j}(\lambda) \|_{-\delta} \leq C(1 + |\lambda|^2)^{-1}, \quad \lambda \in \Omega_1.$$

Suppose that $\lambda \in \Omega_1$ is a pole of $R_{Z_j}(\lambda)$. It follows from (5.3) that there exists $\varphi \in L^2_{-\delta}(Z_j, E_j)$ such that $A_j^2\varphi = \lambda^2\varphi$. On $\mathbb{R}^+ \times Y$, we may expand φ in terms of the eigensections $\{\phi_l\}_{l \in \mathbb{N}}$ of D_Y^2:

$$\varphi(u, y) = \sum_{l=1}^{\infty} \left(a_l e^{-\sqrt{\mu_l - \lambda^2}\, u} + b_l e^{\sqrt{\mu_l - \lambda^2}\, u} \right) \phi_l(y).$$

Here the square root has been chosen such that $\operatorname{Im}(\sqrt{\mu_l - \lambda^2}) > 0$ for all λ in the upper half-plane. If $\operatorname{Im}(\lambda) \neq 0$, then $\operatorname{Re}(\sqrt{\mu_l - \lambda^2}) \neq 0$ and we may pick $\delta > 0$ such that $\left|\operatorname{Re}(\sqrt{\mu_l - \lambda^2})\right| > \delta$ for all $l \in N$. Thus φ is square integrable and ,therefore, vanishes if $\operatorname{Im}(\lambda) \neq 0$. By Lemma 5.31, $R_{Z_j}(\lambda)$ has only finitely many poles in Ω_1. Hence, we may pick $\delta_0 > \delta > 0$ such that the only poles of $R_{Z_j}(\lambda)$ in Ω_1 are real. But, by our choice of δ_0, the only possible pole can occur at $\lambda = 0$. Let $R_{Z_j,1}(\lambda)$ be the operator obtained by removing the contribution of the pole at $\lambda = 0$. Then $R_{Z_j,1}(\lambda)$ is still a bounded operator in $L^2_{-\delta}(Z_j, E_j)$, which is a holomorphic function of $\lambda \in \Omega_1$. Put

$$\tilde{R}_{j,1}(u, v, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(u-v)} R_{Z_j,1}(\sqrt{\lambda^2 - \xi^2})\, d\xi, \quad \lambda \in \Omega_\varepsilon. \tag{5.32}$$

By Lemma 5.31, the integral is absolutely convergent, and for all $u, v \in \mathbb{R}$ the kernel defines a bounded operator in $L^2_{-\delta}(Z_j, E_j)$. From the construction it is obvious that for $\lambda \in \Omega_1$, $\tilde{R}_{j,1}(\lambda)$ is an extension of $R_{j,1}(\lambda)$. Let $\tilde{R}_{j,2}(\lambda)$ be the extension of the operator defined by the kernel (5.25), and put

$$\tilde{R}_j(\lambda) = \tilde{R}_{j,1}(\lambda) \oplus \tilde{R}_{j,2}(\lambda).$$

We replace $R_j(\lambda)$ by $\tilde{R}_j(\lambda)$ on the right-hand side of (5.10), and denote the resulting kernel by $\tilde{G}_j(\lambda)$. Since the support of $\partial\Phi_j/\partial v_j$ is contained in $[0, 1] \times Z_j$, it follows from Lemma 5.31 that $\tilde{G}_j(\lambda)$ defines a bounded operator in $L^2_{-\delta}(X,E)$. Let

$$\tilde{G}(\lambda) = \sum_{j=0}^{2} \tilde{G}_j(\lambda) - \tilde{G}_3(\lambda).$$

Summarizing our results, we have proved that

$$\lambda \in \Omega_1 \mapsto \tilde{G}(\lambda) \in \mathcal{L}(L^2_{-\delta}(X,E))$$

is a meromorphic function which extends $G(\lambda)$.

It remains to verify that $\tilde{G}(\lambda)$ is compact. To establish compactness, we may proceed in essentially the same way as in the proof of Lemma 5.16. Let θ be the characteristic function of $(\mathbb{R}^+)^2 \times Y \subset X$. Let $\lambda \in \Omega$. Then $\theta\tilde{G}(\lambda)\theta$ is given by

$$\begin{aligned} \theta\tilde{G}(\lambda)\theta = &- \Psi_1\{\tilde{R}_1(\lambda) - \Psi_2\tilde{R}_3(\lambda)\Phi_2\} \frac{\partial^2\Phi_1}{\partial v_1^2} \\ &- \Psi_1\Big[\{\tilde{R}_1(\lambda) - \Psi_2\tilde{R}_3(\lambda)\Phi_2\} \circ \frac{\partial}{\partial v_1}\Big] \frac{\partial \Phi_1}{\partial v_1} \\ &- \Psi_2\{\tilde{R}_2(\lambda) - \Psi_1\tilde{R}_3(\lambda)\Phi_1\} \frac{\partial^2\Phi_2}{\partial v_2^2} \\ &- \Psi_2\Big[\{\tilde{R}_2(\lambda) - \Psi_1\tilde{R}_3(\lambda)\Phi_1\} \circ \frac{\partial}{\partial v_2}\Big] \frac{\partial \Phi_2}{\partial v_2}, \end{aligned} \tag{5.33}$$

where $\tilde{R}_j(\lambda)$, $j = 1,2,3$, are the operators introduced above. Consider the first term. Observe that $\tilde{R}_3(\lambda)$ can be written as

$$\tilde{R}_3(u,v,\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(u-v)}\, \tilde{H}(\sqrt{\lambda^2 - \xi^2})\, d\xi.$$

Now replace $\tilde{H}(\lambda)$ by the parametrix $\tilde{P}_1(\lambda)$, (5.30) and (5.32) to show that, up to a compact operator in $L^2_{-\delta}(X,E)$, $\tilde{R}_1(\lambda) - \Psi_2\tilde{R}_3(\lambda)\Phi_2$ equals

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(u-v)} \Big(\mathrm{Id} + \tilde{K}_1(\sqrt{\lambda^2 - \xi^2})\Big)^{-1} \tilde{K}_1(\sqrt{\lambda^2 - \xi^2}) \tilde{P}_1(\sqrt{\lambda^2 - \xi^2})\, d\xi.$$

Since $\tilde{K}_1(\lambda)$, $\lambda \in \Omega$, is a compact operator in $L^2_{-\delta}(Z_j, E_j)$, it follows that the first term on the right-hand side of (5.33) is a compact operator in $L^2_{-\delta}(Z_j, E_j)$. The other terms can be treated in the same way. Thus, we obtain that $\theta\tilde{G}(\lambda)\theta$ is a compact operator. Let $\chi_{j,T}$ be the characteristic functions defined above. By a similar argument, it can be shown that $\chi_{j,0}\tilde{R}_j(\lambda)(1 - \chi_{j,T})$, $j = 1,2$, is compact. This completes the proof of the compactness of $\tilde{G}(\lambda)$. The claimed estimate of the norm follows directly from the definiton of the analytic continuation of the operators $R_j(\lambda)$. q.e.d.

By Lemma 5.26, $\operatorname{Id} + \tilde{G}(i\lambda)$ is invertible for $\lambda \gg 0$. Since $\tilde{G}(\lambda)$ is compact, we get a meromorphic function

$$\lambda \in \Omega_1 \mapsto \left(\operatorname{Id} + \tilde{G}(\lambda)\right)^{-1} \in \mathcal{L}(L^2_{-\delta}(X, E))$$

[21]. Together with (5.17) and Lemma 5.21, we obtain

Theorem 5.34. *Suppose that* $\ker D_Y = 0$. *For every* $\varepsilon > 0$ *there exists* δ, $0 < \delta < \delta_0$, *such that the resolvent* $(\overline{\Delta} - \lambda^2)^{-1}$ *admits an analytic continuation from the upper half-plane to a meromorphic function* $\lambda \in \Omega_1 \mapsto R(\lambda) \in \mathcal{L}(L^2_\delta(X, E), L^2_{-\delta}(X, E))$.

6. The continuous spectrum near zero

In this section we shall investigate the continuous spectrum of $\Delta = D^2$ near zero. Since our approach depends on the analytic continuation of the resolvent, we can treat this problem only under the assumption that $\ker D_Y = 0$.

Let $\mathcal{A}_i$ be the self-adjoint extension of the Dirac operator $A_i : C_0^\infty(Z_i, E_i) \to L^2(Z_i, E_i)$ defined by (2.1). Our first result concerning the continuous spectrum of $\overline{\Delta}$ is

Proposition 6.1. *Suppose that* $\ker D_Y = 0$ *and* $\ker \mathcal{A}_i = 0$, $i = 1, 2$. *Then the essential spectrum of* Δ *has a positive lower bound.*

Proof. Let $\Delta_{i,D}$, $i = 1, 2, 3$, be the Dirichlet Laplacians introduced in §4 (cf. Theorem 4.24). If $\ker D_Y = 0$, then the spectrum of D_Y^2 has a positive lower bound $\mu_1 > 0$, and from the definition of $\Delta_{3,D}$ it follows that

$$\langle \Delta_{3,D}\varphi, \varphi\rangle \geq \mu_1 \parallel \varphi \parallel^2, \quad \varphi \in C_0^\infty((\mathbb{R}^+)^2 \times Y, E).$$

Thus, the spectrum of $\Delta_{3,D}$ is contained in $[\mu_1, \infty)$. Next observe that by Theorem 4.10 of [16], the assumption $\ker D_Y = 0$ implies also that the continuous spectrum of $\mathcal{A}_i$ has a gap at zero. Since $\ker \mathcal{A}_i = 0$, the spectrum of $\mathcal{A}_i^2$ has a positive lower bound. From the definition of $\Delta_{i,D}$, we deduce that the same holds for the spectrum of $\Delta_{i,D}$, $i = 1, 2$. Therefore, there exists $c > 0$ such that for every $\lambda > 0$, we have

$$\parallel (\Delta_{i,D} + \lambda)^{-1} \parallel \leq \frac{1}{c + \lambda}, \quad i = 1, 2, 3.$$

This estimate of the norm implies that the spectrum of

$$(\Delta_{1,D} + \lambda)^{-1} + (\Delta_{2,D} + \lambda)^{-1} - (\Delta_{3,D} + \lambda)^{-1}$$

is contained in $(-3/(c + \lambda), 3/(c + \lambda))$. Let A be a positive self-adjoint operator in some Hilbert space. Then for $\operatorname{Re}(\lambda) > 0$, the resolvent of A

is given by

$$(A+\lambda)^{-1} = \int_0^\infty e^{-t\lambda}\, e^{-tA}\, dt.$$

From this observation and Theorem 4.24, it follows that the operator

$$(\Delta+\lambda)^{-1} - [(\Delta_{1,D}+\lambda)^{-1} + (\Delta_{2,D}+\lambda)^{-1} - (\Delta_{3,D}+\lambda)^{-1}]$$

is compact. Applying Lemma 3, Ch. XIII, §4, of [19], we conclude that the essential spectrum of $(\overline{\Delta}+\lambda)^{-1}$ is contained in $[0, 3/(c+\lambda))$. This fact combined with Lemma 2, Ch. XIII, §4, of [19] yields that the essential spectrum of $\overline{\Delta}$ is contained in $[(c-2\lambda)/3, \infty)$. Since $\lambda > 0$ is arbitrary, the essential spectrum of $\overline{\Delta}$ has the lower bound $c/3 > 0$. q.e.d.

As Proposition 6.1 shows, the continuous spectrum of $\overline{\Delta}$ near zero is completely determined by $\ker D_Y$, $\ker \mathcal{A}_1$ and $\ker \mathcal{A}_2$. We shall now study the case where $\ker D_Y = 0$, but at least one of the spaces $\ker \mathcal{A}_i$, $i = 1, 2$, is nonzero. Then the argument used in the proof of Proposition 6.1 implies that the continuous spectrum of $\overline{\Delta}$ extends down to zero. Our purpose is to construct generalized eigensections of $\overline{\Delta}$, associated with elements of $\ker \mathcal{A}_j$, and to describe explicitly the continuous spectrum of $\overline{\Delta}$ near zero in terms of these eigensections.

Let $\phi \in \ker \mathcal{A}_j$ and let $\lambda \in \mathbb{C}$. Define $h_j(\phi,\lambda) \in C^\infty(\mathbb{R}^+ \times Z_j, E)$ by

$$h_j(\phi, \lambda, (u_j, z_j)) = e^{-i\lambda u_j}\phi(z_j).$$

Note that $h_j(\phi,\lambda)$ satisfies

$$\left(-\frac{\partial^2}{\partial u_j^2} + A_j^2\right) h_j(\phi,\lambda,(u_j,z_j)) = \lambda^2\, h_j(\phi,\lambda,(u_j,z_j)). \tag{6.2}$$

Let $f \in C^\infty(\mathbb{R})$ be such that $f(u) = 0$ for $u \le 1$ and $f(u) = 1$ for $u \ge 2$. Define $f_j \in C^\infty(\mathbb{R}^+ \times Z_j)$ by $f_j(u_j, z_j) = f(u_j)$ and then extend this function by zero to a smooth function on X. Using (4.1) together with (6.2), we obtain that $(\Delta - \lambda^2)(f_j h_j(\phi,\lambda))$ is a smooth section of E which belongs to $L^2(X, E)$. Hence we can apply the resolvent $(\overline{\Delta} - \lambda^2)^{-1}$ to this section. Put

$$F_j(\phi,\lambda) = f_j\, h_j(\phi,\lambda) - (\overline{\Delta}-\lambda^2)^{-1}\Big((\Delta-\lambda^2)(f_j h_j(\phi,\lambda))\Big), \quad \mathrm{Im}(\lambda) > 0. \tag{6.3}$$

Then $F_j(\phi,\lambda)$ belongs to $C^\infty(X,E)$ and satisfies

$$\Delta F_j(\phi,\lambda) = \lambda^2 F_j(\phi,\lambda), \quad \mathrm{Im}(\lambda) > 0. \tag{6.4}$$

Since $\phi \in \ker \mathcal{A}_j$, it follows from (4.6) in [16] that there exists $C > 0$ such that

$$|\phi(u,y)| \leq C\, e^{-\mu_1 u/2}, \quad u \geq 0,\, y \in Y,$$

where $\mu_1 > 0$ is the smallest positive eigenvalue of D_Y. Let $0 < \delta < \min\{\mu_1/2, \delta_0\}$. Then $(\Delta - \lambda^2)(f_j h_j(\phi,\lambda))$ is contained in $L^2_\delta(X,E)$ for all $\lambda \in \mathbb{C}$. Since ker $D_Y = 0$, we can apply Theorem 5.34 which implies that the right-hand side of (6.3) extends to a meromorphic function $\lambda \in \Omega \mapsto F_j(\phi,\lambda) \in L^2_{-\delta}(X,E)$. In particular, $F_j(\phi,\lambda)$ is locally integrable, and therefore we can apply Δ in the distributional sense. By (6.4) we get $(\Delta - \lambda^2)F_j(\phi,\lambda) = 0$ for $\mathrm{Im}(\lambda) > 0$. Since $(\Delta - \lambda^2)F_j(\phi,\lambda)$ is a meromorphic function, it vanishes for all $\lambda \in \Omega$. By elliptic regularity, it follows that $F_j(\phi,\lambda) \in C^\infty(X,E)$. Thus we have proved

Theorem 6.5. *The section $F_j(\phi,\lambda)$ defined by* (6.3) *extends to a meromorphic function $\lambda \in \Omega \mapsto F_j(\phi,\lambda) \in L^2_{-\delta}(X,E)$ with the following properties:*

1) $F_j(\phi,\lambda,x)$ is smooth in $x \in X$ and satisfies

$$(\Delta - \lambda^2)F_j(\phi,\lambda) = 0, \quad \lambda \in \Omega.$$

2) For $\mathrm{Im}(\lambda) > 0$, *$f_j h_j(\phi,\lambda) - F_j(\phi,\lambda)$ is square integrable.*

Now consider the restriction of $F_j(\phi,\lambda)$ to $\mathbb{R}^+ \times Z_j$. For $\mathrm{Im}(\lambda) > 0$, $F_j(\phi,\lambda,(u,\cdot))$ is square integrable on Z_j. Hence, we can expand $F_j(\phi,\lambda,(u,\cdot))$ in terms of the eigensections of $\mathcal{A}_j^2$. Let $L^2_d(Z_j,E_j)$ be the subspace of $L^2(Z_j,E_j)$, which is spanned by all L^2 eigensections of $\mathcal{A}_j^2$. In the orthogonal complement of $\ker \mathcal{A}_j$ in $L^2_d(Z_j,E_j)$, we pick an orthonormal basis $\{\varphi_k\}_{k\in I}$ consisting of eigensections of $\mathcal{A}_j^2$ with corresponding eigenvalues λ_k, $k \in I$. Furthermore, let $\{\phi_l\}_{l\in\mathbb{N}}$ be an orthonormal basis of $L^2(Y,S)$ consisting of eigensections of D_Y^2 with eigenvalues $0 < \mu_1 \leq \mu_2 \leq \cdots \to \infty$. For each $l \in \mathbb{N}$, let $E_j(\phi_l,\Lambda)$ be the generalized eigensection of $\mathcal{A}_j^2$ associated with ϕ_l (cf. [16, §4]). Using the fact that $(\Delta - \lambda^2)F_j(\phi,\lambda) = 0$, we obtain the following expansion of $F_j(\phi,\lambda)$ on $\mathbb{R}^+ \times Z_j$:

$$\begin{aligned} F_j(\phi,\lambda,(u,z)) &= e^{-i\lambda u}\phi(z) + e^{i\lambda u}(C_j(\lambda)\phi)(z) \\ &\quad + \sum_{k\in I} a_k(\lambda)\, e^{-\sqrt{\lambda_k-\lambda^2}\,u}\varphi_k(z) \\ &\quad + \sum_{l=1}^{\infty}\int_{\mu_l}^{\infty} b_l(\lambda,\Lambda)\, e^{-\sqrt{\Lambda^2-\lambda^2}\,u}\, E_j(\phi_l,\Lambda,z)\, d\tau_l(\Lambda). \end{aligned} \tag{6.6}$$

Here $C_j(\lambda) : \ker \mathcal{A}_j \to \ker \mathcal{A}_j$ is a linear operator which is a meromorphic function of $\lambda \in \Omega$. We call $C_j(\lambda)$ "scattering matrix". The measure $d\tau_l$

is given by

$$d\tau_l(\Lambda) = \frac{\sqrt{\Lambda^2 - \mu_l}}{2\pi\Lambda}\, d\Lambda.$$

The convergence of the series and integrals on the right-hand side of (6.6) is understood in the L^2 sense. Moreover, the expansion (6.6) holds for all $\lambda \in \Omega$. We define the constant term $F_{j,0}(\phi, \lambda) \in C^\infty(\mathbb{R}^+ \times Z_j, E)$ of $F_j(\phi, \lambda)$ by

$$\begin{aligned} F_{j,0}(\phi, \lambda, (u_j, z_j)) &= e^{-i\lambda u_j}\phi(z_j) + e^{i\lambda u_j}(C_j(\lambda)\phi)(z_j), \\ &\qquad (u_j, z_j) \in \mathbb{R}^+ \times Z_j. \end{aligned} \tag{6.7}$$

Suppose that $|\lambda| < 1/2\min\{\mu_1, \lambda_1\}$. Let $m \geq 1$. Using (6.6), we get

$$\begin{aligned} &\| (\mathcal{A}_j)^m F_j(\phi, \lambda, (u, \cdot)) \|^2 \\ &\quad = \sum_{k \in I} |a_k(\lambda)|^2\, e^{-2\operatorname{Re}\left(\sqrt{\lambda_k - \lambda^2}\right)u}\, \lambda_k^{2m} \\ &\qquad + \sum_{l=1}^{\infty} \int_{\mu_l}^{\infty} |b_l(\lambda, \Lambda)|^2\, e^{-2\operatorname{Re}\left(\sqrt{\Lambda^2 - \lambda^2}\right)u}\, \Lambda^{4m}\, d\tau_k(\Lambda) \\ &\quad \leq C_l\, e^{-cu}. \end{aligned}$$

We observe that the injectivity radius of Z_j has a positive lower bound and all covariant derivatives of the curvature tensor of E are uniformly bounded on Z_j. Hence, the norm of the Sobolev space $H^m(Z_j, E_j)$ is equivalent to the norm $\| (I + \mathcal{A}_j^2)^{m/2}\varphi \|$, and the Sobolev embedding theorem holds [9]. This implies that (6.6) is pointwise convergent. Moreover, by the Sobolev embedding theorem we get

$$\begin{aligned} &\sup_{z \in Z_j} \left| F_j(\phi, \lambda, (u, z)) - \{e^{-i\lambda u}\phi(z) + e^{i\lambda u}(C_j(\lambda)\phi)(z)\} \right| \leq C\, e^{-cu}, \\ &\qquad |\lambda| < \frac{1}{2}\min\{\mu_1, \lambda_1\}. \end{aligned} \tag{6.8}$$

Next consider the restriction of $F_j(\phi, \lambda)$ to $\mathbb{R}^+ \times Z_l$, $l \neq j$. We shall now expand $F_j(\phi, \lambda, (u_l, \cdot))$ in terms of the eigensections of $\mathcal{A}_l^2$. Let $L_d^2(Z_l, E_l)$ be the subspace of $L^2(Z_l, E_l)$, which is spanned by all eigensections of $\mathcal{A}_l^2$. Let $\{\psi_p\}_{p \in J}$ be an orthonormal basis of $L_d^2(Z_l, E_l)$ consisting of eigensections of $\mathcal{A}_l^2$ and denote the corresponding eigenvalues by ν_p, $p \in J$. Let $\{\phi_k\}_{k \in N}$ and $0 < \mu_1 \leq \mu_2 \leq \cdots \to \infty$ be as above and let $E_l(\phi_k, \Lambda)$ be the generalized eigensection of $\mathcal{A}_l^2$ associated with ϕ_k (cf. [16, §4]). Suppose that $\operatorname{Im}(\lambda) > 0$. Then by Theorem 6.5, $F_j(\phi, \lambda)$ −

$f_jF_{j,0}(\phi,\lambda)$ is square integrable. Therefore on $\mathbb{R}^+\times Z_l$, we may write $F_j(\phi,\lambda)$ as

$$F_j(\phi,\lambda,(u_l,z_l)) = f_jF_{j,0}(\phi,\lambda)(u_l,z_l) + \sum_{p\in J}\alpha_p(u_l,\lambda)\,\psi_p(z_l)$$

$$+\sum_{k=1}^{\infty}\int_{\mu_k}^{\infty}\beta_k(u_k,\lambda,\Lambda)\,E_l(\phi_k,\Lambda,z_l)\,d\tau_k(\Lambda). \tag{6.9}$$

Again, the convergence of the series and integrals has to be understood in the L^2 sense. As functions of u, both α_p and β_k satisfy certain differential equations which we describe now. First note that on Z_l, the section $(\Delta-\lambda^2)(f_jF_{j,0}(\phi,\lambda))(u,\cdot)$ is square integrable for each $u\in\mathbb{R}^+$. Let

$$g_p(u,\lambda) = \langle(\Delta-\lambda^2)(f_jF_{j,0}(\phi,\lambda))(u,\cdot),\psi_p\rangle. \tag{6.10}$$

To proceed further, we need to know the asymptotic behaviour of $g_p(u\lambda)$ as $u\to\infty$. By definition, we have

$$\begin{aligned}&(\Delta-\lambda^2)(f_jF_{j,0}(\phi,\lambda))(u_l,u_j,y)\\ &= \xi_1(u_j)\,e^{-i\lambda u_j}\phi(u_l,y) + \xi_2(u_j)\,e^{i\lambda u_j}\,(C_j(\lambda)\phi)(u_l,y),\end{aligned} \tag{6.11}$$

where $\operatorname{supp}\xi_j\subset[1,2]$, $j=1,2$. Since ϕ is square integrable on Z_j and satisfies $A_j\phi=0$, we can expand ϕ on $\mathbb{R}^+\times Y\subset Z_j$ in terms of the eigensections ϕ_k of D_Y^2. Using that $A_j^2=-\partial^2/\partial u_l^2+D_Y^2$ on $\mathbb{R}^+\times Y$, we obtain the following expansion

$$\phi(u_l,y) = \sum_{k=1}^{\infty}c_k\,e^{-\sqrt{\mu_k}\,u_l}\,\phi_k(y). \tag{6.12}$$

A similar expansion holds for $C_j(\lambda)\phi$. Moreover, the eigensections ψ_p have also an expansion of this type on $\mathbb{R}^+\times Y\subset Z_l$:

$$\psi_p(u_j,y) = \sum_{\mu_m>\nu_p}d_m\,e^{-\sqrt{\mu_m-\nu_p}\,u_j}\,\phi_m(y). \tag{6.13}$$

Using (6.10) - (6.13), we get the estimate

$$|g_p(u,\lambda)|\le C\begin{cases}e^{-\sqrt{\mu_1}\,u/2}, & \nu_p=0;\\ e^{-\sqrt{\nu_p}\,u/2}, & \nu_p>0;\end{cases} \tag{6.14}$$

which holds uniformly for λ in a compact subset of $\mathbb{C}$.

Since $(\Delta - \lambda^2)F_j(\phi, \lambda) = 0$, (6.9) implies that the functions α_p satisfy the following differential equation:

$$-\frac{d^2}{du^2}\alpha_p(u,\lambda) = (\lambda^2 - \nu_p)\alpha_p(u,\lambda) + g_p(u,\lambda). \tag{6.15}$$

Let $\nu_1 > 0$ be the smallest positive eigenvalue of $\mathcal{A}_l^2$ and put

$$\delta = \frac{1}{2}\min\{\mu_1, \nu_1\}.$$

Let $\nu_p > 0$ and suppose that $|\lambda| < \delta$. In view of (6.14), the general solution of (6.15) has the form

$$\begin{aligned} \alpha_p(u,\lambda) = &\frac{e^{-\sqrt{\nu_p-\lambda^2}\,u}}{2\sqrt{\nu_p-\lambda^2}} \int_0^u e^{\sqrt{\nu_p-\lambda^2}\,v}\, g_p(v,\lambda)\,dv \\ &+ \frac{e^{\sqrt{\nu_p-\lambda^2}\,u}}{2\sqrt{\nu_p-\lambda^2}} \int_u^\infty e^{-\sqrt{\nu_p-\lambda^2}\,v}\, g_p(v,\lambda)\,dv \\ &+ c_{p1}(\lambda)\, e^{-\sqrt{\nu_p-\lambda^2}\,u} + c_{p2}(\lambda)\, e^{\sqrt{\nu_p-\lambda^2}\,u}. \end{aligned} \tag{6.16}$$

The branch of the square root has been chosen such that $\mathrm{Re}(\sqrt{\nu_p - \lambda^2}) > 0$ for λ as above. Since α_p is square integrable as a function of u, we get $c_{p2}(\lambda) \equiv 0$. Hence, for each $\nu_p > 0$, there exists $C_p > 0$ such that

$$|\alpha_p(u,\lambda)| \le C_p\, e^{-\sqrt{\nu_p}\,u}, \quad u \in \mathbb{R}^+,\ |\lambda| < \delta,$$

and $\sum_{p\in J} |C_p|^2 < \infty$.

Now assume that $\nu_p = 0$. Then for $|\lambda| < \delta$, the general solution of (6.15) is given by

$$\begin{aligned} \alpha_p(u,\lambda) = &d_p(\lambda)\, e^{i\lambda u} + i\frac{e^{i\lambda u}}{2\lambda} \int_0^u e^{-i\lambda v}\, g_p(v,\lambda)\,dv \\ &+ i\,\frac{e^{-i\lambda u}}{2\lambda} \int_u^\infty e^{i\lambda v}\, g_p(v,\lambda)\,dv. \end{aligned} \tag{6.17}$$

Put

$$c_p(\lambda) = d_p(\lambda) + \frac{i}{2\lambda}\int_0^\infty e^{-i\lambda v}\, g_p(v,\lambda)\,dv.$$

Then (6.17) can be rewritten as

$$\begin{aligned} \alpha_p(u,\lambda) = &c_p(\lambda)\, e^{i\lambda u} - i\frac{e^{i\lambda u}}{2\lambda} \int_u^\infty e^{-i\lambda v}\, g_p(v,\lambda)\,dv \\ &+ i\,\frac{e^{-i\lambda u}}{2\lambda} \int_u^\infty e^{i\lambda v}\, g_p(v,\lambda)\,dv. \end{aligned} \tag{6.18}$$

By (6.14) we have

$$(6.19)\qquad \left|\alpha_p(u,\lambda) - c_p(\lambda)\,e^{i\lambda u}\right| \le C_p\,e^{-\delta u}, \quad u\in\mathbb{R}^+,\ |\lambda|<\delta,$$

for some constant $C_p > 0$. The coefficients $\beta_k(u,\lambda,\Lambda)$ in (6.9) can be determined in a similar way. If we proceed as above and use the Sobolev embedding theorem, we get

$$(6.20)\qquad \begin{aligned}\sup_{z\in Z_l} \Big|F_j(\phi,\lambda,(u,z)) &- f_jF_{j,0}(\phi,\lambda,(u,z)) \\ &- \sum_{\nu_p=0} c_p(\lambda)\,\psi_p(z)\,e^{i\lambda u}\Big| \le C\,e^{-cu}, \quad u\in\mathbb{R}^+,\ |\lambda|<\delta,\end{aligned}$$

for certain constants $C, c > 0$. Combined with (6.8), this estimate implies that $c_p(\lambda) = 0$ for $\nu_p = 0$ and $\operatorname{Im}(\lambda) < 0$. But $c_p(\lambda)$ is a meromorphic function of λ and therefore vanishes identically. Putting together (6.8) and (6.20), we can summarize our results by

Theorem 6.21. *Suppose that* $\ker D_Y = 0$. *Let*

$$\delta_0 = 1/2\min\{\mu_1,\nu_1,\lambda_1\}.$$

Let χ_j *be the characteristic function of* $\mathbb{R}^+\times Z_j \subset X$, *and for* $\phi\in\ker\mathcal{A}_j$ *let* $F_{j,0}(\phi,\lambda)$ *be the constant term of* $F_j(\phi,\lambda)$, *defined by* (6.7). *For each* $\phi\in\ker\mathcal{A}_j$, *the restriction of the generalized eigensection* $F_j(\phi,\lambda)$ *to* $\mathbb{R}^+\times Z_l\subset X$, $l=1,2$, *satisfies*

$$\sup_{z\in Z_l}\left|F_j(\phi,\lambda,(u,z)) - \chi_jF_{j,0}(\phi,\lambda,(u,z))\right| \le C\,e^{-cu}, \quad u\in\mathbb{R}^+,\ |\lambda|<\delta_0,$$

for some constants $C, c > 0$.

We can now proceed in essentially the same way as in [17, §7] and derive the basic properties satisfied by the generalized eigensections. Suppose that $|\lambda| < \delta_0$ and $\operatorname{Im}(\lambda) < 0$. Then from the estimations proved above it follows that $F_j(\phi,\lambda) - f_jh_j(C_j(\lambda)\phi,-\lambda)$ is square integrable. Put $u(\lambda) = F_j(C_j(\lambda)\phi,-\lambda) - F_j(\phi,\lambda)$ and assume that $\operatorname{Im}(\lambda)<0$ and $|\lambda|<\delta_0$. Then $u(\lambda)$ is square integrable and satisfies $(\Delta-\lambda^2)u(\lambda) = 0$. Since Δ is essentially self–adjoint, $u(\lambda) = 0$, i.e., $F_j(C_j(\lambda)\phi,-\lambda) = F_j(\phi,\lambda)$. Comparing both sides of expansion (6.6) yields

Theorem 6.22. *Let* $j=1,2$. *The generalized eigensections* $F_j(\phi,\lambda)$, $\phi\in\ker\mathcal{A}_j$, *satisfy the following functional equations*

$$(6.23)\qquad F_j(C_j(\lambda)\phi,-\lambda) = F_j(\phi,\lambda),\ \ C_j(\lambda)C_j(-\lambda) = \mathrm{Id}, \quad |\lambda|<\delta_0.$$

Given $T \geq 0$, let χ_T be the characteristic function of $[T,\infty) \times Z_j \subset X$, where $[T,\infty) \times Z_j$ is regarded as a submanifold with respect to the decomposition (1.3). Put

$$F_j^T(\phi,\lambda) = F_j(\phi,\lambda) - \chi_T F_{j,0}(\phi,\lambda), \tag{6.24}$$

where $F_{j,0}(\phi,\lambda)$ is defined by (6.7). If $\lambda \in \Omega$, then $F_j^T(\phi,\lambda)$ is square integrable by Theorem 6.21, and the inner product of the F_j^T's can be computed as follows. Let $\phi,\psi \in \ker \mathcal{A}_j$ and let $\lambda,\lambda' \in \Omega$ be such that $\lambda \neq \pm\overline{\lambda'}$. Integrating by parts gives

$$\begin{aligned}
&\langle F_j^T(\phi,\lambda), F_j^T(\psi,\lambda')\rangle \\
&\quad = (\lambda^2 - \overline{\lambda'}^2)^{-1}\Big\{\langle \Delta F_j^T(\phi,\lambda), F_j^T(\psi,\lambda')\rangle \\
&\quad\quad - \langle F_j^T(\phi,\lambda), \Delta F_j^T(\psi,\lambda')\rangle\Big\} \\
&\quad = \frac{i}{\lambda - \overline{\lambda'}}\Big\{e^{-iT(\lambda-\overline{\lambda'})}\langle\phi,\psi\rangle - e^{iT(\lambda-\overline{\lambda'})}\langle C_j(\lambda)\phi, C_j(\lambda')\psi\rangle\Big\} \\
&\quad\quad + \frac{i}{\lambda + \overline{\lambda'}}\Big\{e^{-iT(\lambda+\overline{\lambda'})}\langle\phi, C_j(\lambda')\psi\rangle - e^{iT(\lambda+\overline{\lambda'})}\langle C_j(\lambda)\phi,\psi\rangle\Big\}.
\end{aligned} \tag{6.25}$$

Put $T = 0$, assume that $0 \neq \lambda \in \mathbb{R}$, $|\lambda| < \delta_0$, and let $\lambda' \to \lambda$. Then the left-hand side stays bounded, and therefore the right-hand side must stay bounded as well. This fact implies that $C_j(r)$ is unitary for $r \in \mathbb{R}$, $0 < |r| < \delta_0$. For $r = 0$, the functional equation gives $C_j(0)^2 =\mathrm{Id}$. Hence, $C_j(r)$ is regular for $r \in (-\delta_0,\delta_0)$. Now let $0 \neq r, r' \in \mathbb{R}$ and suppose that $|r|, |r'| < \delta_0$ and $r \neq r'$. Let $r \to r'$ and apply the functional equation (6.23). Then (6.25) leads to

$$\begin{aligned}
\langle F_j^T(\phi,r), F_j^T(\psi,r)\rangle &= 2T\langle\phi,\psi\rangle - i\Big\langle C_j(-r)\Big(\frac{d}{dr}C_j(r)\Big)\phi,\psi\Big\rangle \\
&\quad + \frac{i}{2r}\Big\{e^{-2iTr}\langle\phi, C_j(r)\psi\rangle - e^{2iTr}\langle C_j(r)\phi,\psi\rangle\Big\}, \\
&\qquad\qquad r \in \mathbb{R},\ 0 < |r| < \delta_0.
\end{aligned} \tag{6.26}$$

Since $C_j(r)$ is regular on $(-\delta_0,\delta_0) - \{0\}$, $F_j(\phi,\lambda)$ is also regular on $(-\delta_0,\delta_0)$. Using $C_j(0)^2 =\mathrm{Id}$, one may derive a similar formula for $r = 0$. Summarizing, we have proved

Proposition 6.27. *The scattering matrix $C_j(\lambda)$, $j = 1,2$, is unitary for $\lambda \in (-\delta_0,\delta_0)$. Both $C_j(\lambda)$ and $F_j(\phi,\lambda)$, $\phi \in \ker \mathcal{A}_j$, have no poles on $(-\delta_0,\delta_0)$.*

We can now use the generalized eigensections $F_j(\phi,r)$, $r \in (-\delta_0,\delta_0)$, to describe the continuous spectrum of $\overline{\Delta}$ near zero. Let $0 < \delta \leq \delta_0$

and let $\phi \in \ker \mathcal{A}_j$. By Proposition 6.27, $F_j(\phi, \lambda)$ is square integrable as function of $\lambda \in [0, \delta]$. Let $f \in L^2([0, \delta])$ and put

$$W_{j,\phi}(f) = \frac{1}{\sqrt{2\pi}} \int_0^\delta F_j(\phi, r)\, f(r)\, dr.$$

Lemma 6.28. *1) For all $f \in L^2([0, \delta])$, $W_{j,\phi}(f)$ belongs to $L^2(X, E)$, and for any $f, g \in L^2([0, \delta])$, $\phi, \psi \in \ker \mathcal{A}_j$, the inner product of $W_{j,\phi}(f)$ and $W_{j,\psi}(g)$ is given by*

$$\langle W_{j,\phi}(f), W_{j,\psi}(g) \rangle = \langle f, g \rangle \langle \phi, \psi \rangle.$$

2) Let L be the bounded operator in $L^2([0, \delta])$, which is defined by $Lf(r) = r^2 f(r)$. Then we have

$$(\overline{\Delta} - z)^{-1} W_{j,\phi}(f) = W_{j,\phi}((L - z)^{-1} f), \quad z \in \mathbb{C} - \mathbb{R}^+,\ f \in L^2([0, \delta]).$$

3) For all $\phi \in \ker \mathcal{A}_1$, $\psi \in \ker \mathcal{A}_2$ and $f, g \in L^2([0, \delta])$, we have

$$\langle W_{1,\phi}(f), W_{2,\psi}(g) \rangle = 0.$$

Proof. Let $T \geq 0$ and put

$$W_{j,\phi}^T(f) = \frac{1}{\sqrt{2\pi}} \int_0^\delta F_j^T(\phi, r)\, f(r)\, dr,$$

where $F_j^T(\phi, \lambda)$ is defined by (6.24). Since $F_j^T(\phi, \lambda)$ is square integrable, $W_{j,\phi}^T(f)$ is also square integrable. Using the inner product formula (6.25) and the Riemann–Lebesgue lemma, we get

$$\lim_{T \to \infty} \| W_{j,\phi}^T(f) \|^2 = \int_0^\delta |f(r)|^2\, dr\ \| \phi \|^2 .$$

Applying Lebesgue's theorem, it follows that $W_{j,\phi}(f)$ is square integrable. The inner product formula can be derived in the same way. This proves 1). Since $\Delta F_j(\phi, \lambda) = \lambda^2 F_j(\phi, \lambda)$, we get 2). By Theorem 6.21, we obtain

$$\lim_{T \to \infty} \langle F_1^T(\phi, \lambda), F_2^T(\psi, \lambda) \rangle = 0,$$

which implies 3). q.e.d.

Let $m_j = \dim(\ker \mathcal{A}_j)$. Let $\psi_{j1}, ..., \psi_{jm_j}$ be an orthonormal basis for $\ker \mathcal{A}_j$. Then we define the operator

$$W_j : \bigoplus_{k=1}^{m_j} L^2([0, \delta]) \to L^2(X, E)$$

by

$$W_j(\{f_k\}) = \sum_{k=1}^{m_j} \frac{1}{\sqrt{2\pi}} \int_0^\delta F_j(\psi_{jk}, r)\, f_k(r)\, dr.$$

By Lemma 6.28, W_j is an isometry onto a closed subspace $\mathcal{H}_j^\delta \subset L^2(X,E)$ and $\mathcal{H}_1^\delta$ is orthogonal to $\mathcal{H}_2^\delta$. Moreover, we have

$$(6.29) \qquad W_j^* W_j = \mathrm{Id} \quad \text{and} \quad W_j W_j^* = P_j^\delta,$$

where P_j^δ is the orthogonal projection of $L^2(X,E)$ onto $\mathcal{H}_j^\delta$. Lemma 6.28 implies that $\mathcal{H}_j^\delta$ is an invariant subspace for $\overline{\Delta}$. Let Δ_j^δ denote the restriction of $\overline{\Delta}$ to $\mathcal{H}_j^\delta$. Let $\alpha \in C^\infty(\mathbb{R})$ be such that $\alpha(u) = 1$ for $|u| \leq \delta/2$ and $\alpha(u) = 0$ for $|u| \geq \delta$. From (6.29) it follows that the kernel of $\alpha(\Delta_j^\delta)\exp(-t\Delta_j^\delta)$ is given by
(6.30)

$$K_j^\alpha(x,x',t) = \sum_{k=1}^{m_j} \frac{1}{2\pi} \int_0^\infty \alpha(\lambda^2)\, e^{-t\lambda^2}\, F_j(\psi_{jk},\lambda,x) \otimes F_j(\psi_{jk},\lambda,x')\, d\lambda.$$

We extend $\alpha(\Delta_j^\delta)\exp(-t\Delta_j^\delta)$, $j = 1,2$, to operators in $L^2(X,E)$, and put them equal to zero in the orthogonal complement of $\mathcal{H}_j^\delta$.

Lemma 6.31. *Let $0 < \delta < \delta_0$ and let $\alpha \in C_0^\infty(\mathbb{R})$ be as above. For $j = 1,2$ and $t \geq 0$, the operators*

$$\alpha(\Delta_j^\delta)\exp(-t\Delta_j^\delta) \; - \; \alpha(\Delta_{j,D})\exp(-t\Delta_{j,D})$$

are Hilbert–Schmidt.

Proof. Let $F_j^0(\psi_{jk},\lambda)$ be defined by (6.24) where $T = 0$. Then we may write

$$\begin{aligned} F_j(\psi_{jk},\lambda,x) \otimes F_j(\psi_{jk},\lambda,x') &= F_j(\psi_{jk},\lambda,x) \otimes F_j^0(\psi_{jk},\lambda,x') \\ &+ F_j^0(\psi_{jk},\lambda,x) \otimes F_{j,0}(\psi_{jk},\lambda,x') \; + \; F_{j,0}(\psi_{jk},\lambda,x) \otimes F_{j,0}(\psi_{jk},\lambda,x'), \end{aligned}$$

which induces a corresponding decomposition of the kernel (6.30), say

$$K_j^\alpha(t) = K_{j,1}^\alpha(t) \; + \; K_{j,2}^\alpha(t) \; + \; K_{j,3}^\alpha(t).$$

Now consider the individual kernels. Let $\varphi \in L^2(X,E)$. Then from Lemma 6.28, 1) it follows that

$$\| \, K_{j,1}^\alpha(t)\varphi \, \|^2 = \sum_{k=1}^{m_j} \int_0^\infty \alpha(\lambda^2)\, e^{-t\lambda^2} \left| \langle F_j^0(\psi_{jk},\lambda), \varphi \rangle \right|^2 d\lambda.$$

Hence, the Hilbert–Schmidt norm $|||\; K_{j,1}^\alpha(t) \;|||$ of $K_{j,1}^\alpha(t)$ is finite and given by

$$|||\; K_{j,1}^\alpha(t) \;|||^2 = \sum_{k=1}^{m_j} \int_0^\infty \alpha(\lambda^2)\, e^{-t\lambda^2} \;\| \, F_j^0(\psi_{jk},\lambda) \, \|^2 \; d\lambda.$$

In the same way one can show that $K^{\alpha}_{j,2}(t)$ has finite Hilbert–Schmidt norm. Thus, it is sufficient to prove that $K^{\alpha}_{j,3}(t) - \alpha(\Delta_{j,D})\exp(-t\Delta_{j,D})$ is a Hilbert–Schmidt operator. Since $\ker D_Y = 0$, the spectrum of $\Delta_{j,D}$, restricted to the orthogonal complement of $L^2(\mathbb{R}^+) \otimes \ker \mathcal{A}_j$ in $L^2(\mathbb{R}^+ \times Z_j, E)$, is contained in $[\delta_0, \infty)$. Therefore, the kernel of $\alpha(\Delta_{j,D})\exp(-t\Delta_{j,D})$ is given by

$$\frac{2}{\pi}\int_0^{\infty} \alpha(\lambda^2)\, e^{-t\lambda^2} \sin(\lambda u)\sin(\lambda v)\, d\lambda \sum_{k=1}^{m_j} \psi_{jk}(z) \otimes \overline{\psi_{jk}(z')}. \tag{6.34}$$

Since $C_j(\lambda)$ is unitary for λ real, we have

$$\sum_{k=1}^{m_j} C_j(\lambda)\psi_{jk} \otimes \overline{C_j(\lambda)\psi_{jk}} = \sum_{k=1}^{m_j} \psi_{jk} \otimes \overline{\psi_{jk}}.$$

From (6.7), (6.33) and (6.34) it follows that the kernel of $K^{\alpha}_{j,3}(t) - \alpha(\Delta_{j,D})\exp(-t\Delta_{j,D})$ equals

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\infty} \alpha(\lambda^2)e^{-t\lambda^2}\, e^{i\lambda(u+v)}\, \big(C_j(\lambda)\psi_{jk} - \psi_{jk}\big) \otimes \overline{\psi_{jk}}\, d\lambda \\
&\quad - \frac{1}{2\pi}\int_0^{\infty} \alpha(\lambda^2)e^{-t\lambda^2}\, e^{-i\lambda(u+v)}\, \psi_{jk} \otimes \overline{\big(C_j(\lambda)\psi_{jk} - \psi_{jk}\big)}\, d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(\lambda^2)e^{-t\lambda^2}\, e^{i\lambda(u+v)}\, \big(C_j(\lambda)\psi_{jk} - \psi_{jk}\big) \otimes \overline{\psi_{jk}}\, d\lambda.
\end{aligned} \tag{6.35}$$

To obtain the equality, we made use of the relation $C_j(\lambda)^* = C_j(-\lambda)$ which is a consequence of the functional equation (6.23). Integrating by parts yields that the right-hand side can be estimated by $C_N(u+v)^{-N}$ for every $N \in \mathbb{N}$. This proves our claim. q.e.d.

If we use Theorem 4.24 together with Lemma 6.31 and proceed as in the proof of Proposition 6.1, we obtain

Proposition 6.36. *Suppose that* $\ker D_Y = 0$. *Let*

$$0 < \delta < \frac{1}{2}\min\{\mu_1, \lambda_1, \nu_1\}$$

and let $\mathcal{H}^{\delta}_j \subset L^2(X, E)$, $j = 1, 2$, *be the* $\overline{\Delta}$*-invariant subspaces introduced above. Let* $\hat{\Delta}$ *be the restriction of* $\overline{\Delta}$ *to the orthogonal complement of* $\mathcal{H}^{\delta}_1 \oplus \mathcal{H}^{\delta}_2$ *in* $L^2(X, E)$. *Then the essential spectrum of* $\hat{\Delta}$ *is contained in* $[\delta, \infty)$.

This result implies that for the case $\ker D_Y = 0$, the generalized eigensections constructed above give a complete description of the continuous spectrum of $\overline{\Delta}$ near zero. From the spectral theorem together with Proposition 6.36, it follows that for every $p \in \mathbb{N}$, there exists $C_p > 0$ such that

$$\| \hat{\Delta}^p e^{-t\hat{\Delta}} \| \leq C_p t^{-p} e^{-t\delta}, \quad t > 0.$$

Applying Proposition 2.7 and Proposition 2.8, we get

Corollary 6.37. *Suppose that* $\ker D_Y = 0$. *Let* $K_j^\alpha(x, x', t)$, $j = 1, 2$, *be defined by* (6.30) *and let* $K(x, x', t)$ *be the kernel of* $\exp(-t\Delta)$. *Then there exist* $C, c > 0$ *such that*

$$\left| D_x K(x, x', t) - \sum_{j=1}^{2} D_x K_j^\alpha(x, x', t) \right| \leq C e^{-ct}, \quad \textit{for all} \quad x, x' \in X, \, t \geq 1.$$

7. The L²-index formula

Let $D : C^\infty(X, E) \to C^\infty(X, E)$ be a generalized Dirac operator satisfying (2.1) – (2.4) and suppose that $\ker D_y = 0$. Assume that $n = 2k$, $k \in \mathbb{N}$. Then the Clifford bundle E splits into the ± 1–eigenspaces $E_\pm$ of the canonical involution τ, and our goal is to derive a formula for the L^2-index of $D_+ : C^\infty(X, E_+) \to C^\infty(X, E_-)$. The method that we shall employ to prove the index theorem is based on the local version of the McKean–Singer formula. This formula has been used, for example, by Stern [22], [23] to derive a formula for the L^2-index of the signatur operator on locally symmetric spaces of finite volume.

Let $h(x, y)$ be the kernel of the orthogonal projection of $L^2(X, E)$ onto $\ker \mathcal{D}$ where, as above, $\mathcal{D}$ denotes the unique self–adjoint extension of $D : C_0^\infty(X, E) \to L^2(X, E)$. Recall that by Corollary 2.23, $\ker \mathcal{D}$ is finite–dimensional. Let $\phi_1, ..., \phi_m$ be an orthonormal basis of $\ker \mathcal{D}$. Then h is given by

$$h(x, y) = \sum_{j=1}^{m} \phi_j(x) \otimes \overline{\phi_j(y)}.$$

Let $K(x, y, t)$ be the heat kernel for $\Delta = D^2$ which was constructed in §4. Then we have the following result.

Lemma 7.1. *We have pointwise convergence of kernels*

$$\lim_{t \to \infty} K(x, y, t) = h(x, y).$$

The convergence is uniform in the C^∞ *topology on compact subsets of* $X \times X$.

Proof. We may follow essentially the proof of Lemma 6.3 in [7]. For the sake of completeness we include details. Pick a parametrix P

for $\exp(-t\Delta)$ which is compactly supported in space and time, that is, $P(x,y,t)=0$ if $d(x,y)\geq\varepsilon>0$ or $t\geq t_0>0$. Set

$$P_1(x,y,t)=\Big(\frac{\partial}{\partial t}+\Delta_x\Big)P(x,y,t).$$

By Duhamel's principle we can write

$$K(x,y,t)=P(x,y,t)\;-\;\int_0^t e^{-(t-s)\Delta}\circ P_1(s)\,ds.$$

As y varies in a compact subset Θ of X the functions $P_1(z,y,t)$ (viewed, for each s, as function of z) vary in a compact subset of $L^2(X,E)$. Thus, using the spectral theorem and the fact that $P(t)$ is compactly supported in time, we have pointwise convergence as $t\to\infty$:

$$K(x,y,t)\to-\int_0^\infty\int_X h(x,z)P_1(z,y,s)\,dz\,ds. \tag{7.2}$$

Since X has uniformly bounded C^k geometry for all $k\in\mathbb{N}$ (see [7] for the definition), it follows that the convergence is uniformly C^∞ as y varies over Θ and x varies over X. The right-hand side of (7.2) can be written as

$$\begin{aligned}
-\lim_{\varepsilon\to0}&\int_\varepsilon^\infty\int_X h(x,z)\Big(\frac{\partial}{\partial s}+\Delta_z\Big)P(z,y,s)\,dz\,ds\\
&=-\lim_{\varepsilon\to0}\int_\varepsilon^\infty\int_X h(x,z)\frac{\partial}{\partial s}P(z,y,s)\,dz\,ds\\
&=\lim_{\varepsilon\to0}\int_X h(x,z)P(z,y,\varepsilon)\\
&=h(x,y). \qquad\qquad \text{q.e.d.}
\end{aligned}$$

Next recall the local index theorem for generalized Dirac operators [10]. Note that our Dirac operators are compatible. Therefore, we can apply Theorem 5 of [10]. Let $\omega_D(x)$ be the local index density for D_+. Then we have as $t\to0$

$$\operatorname{tr}(\tau K(x,x,t))\,dx=\omega_D(x)+O(t). \tag{7.3}$$

The constant occurring in $O(t)$ is uniformly bounded on compact subsets.

Now consider the compact submanifolds X_T, $T\geq0$, of X defined by (1.6). Using Lemma 7.1 and (7.3), we obtain

$$\begin{aligned}
\int_{X_T}\operatorname{tr}(\tau h(x,x))\,dx&=\lim_{t\to\infty}\int_{X_T}\operatorname{tr}(\tau K(x,x,t))\,dx\\
&=\int_{X_T}\omega_D\;+\;\int_0^\infty\int_{X_T}\frac{\partial}{\partial t}\operatorname{tr}(\tau K(x,x,t))\,dx\,dt.
\end{aligned} \tag{7.4}$$

If the spectrum of Δ has a gap at 0, then the t–integral is absolutely convergent. In the following we shall prove that this holds in general.

As $T \to \infty$, the left-hand side of (7.4) converges to the L^2–index of D_+. Now consider the right-hand side. From Proposition 4.20 it follows that in (7.3) we may replace K by the parametrix Q and we still get the same asymptotic expansion as $t \to 0$. In particular, ω_D is determined by Q. Since γ_i commutes with $-\partial^2/\partial u_i^2 + A_i^2$, $i = 1, 2$, and anticommutes with τ, it follows that $\omega_D \equiv 0$ on $\mathbb{R}^+ \times Z_i$, $i = 1, 2$. Hence the limit as $T \to \infty$ of the double integral on the right-hand side of (7.4) exists and we have

$$(7.5) \qquad L^2\text{-Ind}\, D_+ = \int_X \omega_D + \lim_{T\to\infty} \int_0^\infty \int_{X_T} \frac{\partial}{\partial t} \operatorname{tr}(\tau K(x,x,t))\, dx\, dt.$$

To treat the double integral we use the following lemma which is the local version of the McKean–Singer formula.

Lemma 7.6. *Let $D : C^\infty(X,E) \to C^\infty(X,E)$ be a generalized Dirac operator and let $\tau : E \to E$ be a bundle isomorphism which satisfies $\tau^2 = \mathrm{Id}$ and $\tau D = -D\tau$. Let $e^{-tD^2}(x,y)$ and $De^{-tD^2}(x,y)$ be the kernels of e^{-tD^2} and De^{-tD^2}, respectively. Then*

$$\frac{\partial}{\partial t} \operatorname{tr}(\tau e^{-tD^2}(x,x)) = \operatorname{div} V_D,$$

where V_D is the vector field on X whose j-th component with respect to an orthonormal moving frame $\{e_i\}_{i=1}^n$ is given by

$$\frac{1}{2} \operatorname{tr}(e_j \cdot \tau D e^{-tD^2}(x,x)).$$

Proof. We have

$$\begin{aligned}
\frac{\partial}{\partial t} \operatorname{tr}\Big(\tau\, e^{-tD^2}(x,x)\Big) &= \operatorname{tr}\Big(\tau \frac{\partial}{\partial t} e^{-tD^2}(x,x)\Big) \\
&= \operatorname{tr}\Big(\tau \frac{\partial}{\partial t} e^{-tD^2}(x,y)|_{x=y}\Big) \\
&= -\operatorname{tr}(\tau D_x^2\, e^{-tD^2}(x,y)|_{x=y}) \\
&= \frac{1}{2} \operatorname{tr}(D_x \tau D_x\, e^{-tD^2}(x,y)|_{x=y}) \\
&\quad + \frac{1}{2} \operatorname{tr}(D_y \tau D_x\, e^{-tD^2}(x,y)|_{x=y}).
\end{aligned}$$

Choose normal coordinates at x_0 and pick a local frame field $\{e_i\}_{i=1}^n$ such that $(\nabla_{e_i} e_j)(x_0) = 0$ and $e_i(x_0) = \dfrac{\partial}{\partial x_i}|_{x=x_0}$. Then the right-hand side can be rewritten as

$$\frac{1}{2}\sum_{i=1}^{n} \operatorname{tr}\left(e_i(x)\cdot \nabla_{e_i(x)}\tau D_x\, e^{-tD^2}(x,y)|_{x=y}\right)$$
$$+\frac{1}{2}\sum_{i=1}^{n} \operatorname{tr}\left(e_i(y)\cdot \nabla_{e_i(y)}\tau D_x\, e^{-tD^2}(x,y)|_{x=y}\right)$$
$$= -\sum_{i=1}^{n} \operatorname{tr}\left((\nabla_{e_i(x)}e_i(x))\cdot \tau D\, e^{-tD^2}(x,x)\right)$$
$$+\frac{1}{2}\sum_{i=1}^{n} \operatorname{tr}\Big(\nabla_{e_i(x)}\big(e_i(x)\cdot \tau D_x\, e^{-tD^2}(x,y)\big|_{x=y}\big)\Big)$$
$$+\frac{1}{2}\sum_{i=1}^{n} \operatorname{tr}\Big(\nabla_{e_i(y)}\big(e_i(y)\cdot \tau D_x\, e^{-tD^2}(x,y)|_{x=y}\big)\Big).$$

The first sum on the right-hand side vanishes at $x = x_0$, and the remaining two terms, evaluated at $x = x_0$, give

$$\frac{1}{2}\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\operatorname{tr}\left(e_i(x)\tau D\, e^{-tD^2}(x,y)\right)\big|_{x_0=x=y}$$
$$+\frac{1}{2}\sum_{i=1}^{n} \frac{\partial}{\partial y_i}\operatorname{tr}\left(e_i(y)\tau D\, e^{-tD^2}(x,y)\right)\big|_{x_0=x=y}$$
$$=\frac{1}{2}\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\operatorname{tr}\left(e_i(x)\tau\, e^{-tD^2}(x,x)\right)\big|_{x_0=x} = \operatorname{div} V_D(x_0),$$

where V_D is the vector field which is given by

$$V_D = \frac{1}{2}\sum_{i=1}^{n} \operatorname{tr}\left(e_i(x)\tau D\, e^{-tD^2}(x,x)\right)e_i(x).$$

q.e.d.

The corresponding statement for a power of the resolvent has been used by Stern in [22], [23]. We now apply Lemma 7.6 to (7.5). Let e_n be the outward unit normal vector to the smooth part of ∂X_T. Then

$$L^2\text{-Ind}\, D_+ = \int_X \omega_D + \lim_{T\to\infty}\frac{1}{2}\int_0^\infty\int_{\partial X_T} \operatorname{tr}\left(e_n\cdot \tau D e^{-tD^2}(x,x)\right)dx\,dt. \tag{7.7}$$

To compute the limit, we split the t-integral as follows:

$$\int_0^\infty = \int_0^{\sqrt{T}} + \int_{\sqrt{T}}^\infty.$$

Put

$$R(T) = \frac{1}{2} \int_{\sqrt{T}}^{\infty} \int_{\partial X_T} \operatorname{tr}(e_n \cdot \tau D e^{-tD^2}(x, x))\, dx\, dt. \tag{7.8}$$

The convergence of this integral follows from the manipulations above. In the following we shall see that the integral is indeed absolutely convergent. Moreover recall that

$$\partial X_T = (\{T\} \times Z_{1,T}) \cup (\{T\} \times Z_{2,T}),$$

and that γ_i is the outward unit normal vector field to $Z_i, i = 1, 2$. Hence, the double integral on the right-hand side of (7.7) can be written as

$$\begin{aligned} &\frac{1}{2} \int_0^{\sqrt{T}} \int_{Z_{1,T}} \operatorname{tr}\Big(\gamma_1 \tau D e^{-tD^2}((T, z_1), (T, z_1))\Big)\, dz_1\, dt \\ &+ \frac{1}{2} \int_0^{\sqrt{T}} \int_{Z_{2,T}} \operatorname{tr}\Big(\gamma_2 \tau D e^{-tD^2}((T, z_2), (T, z_2))\Big)\, dz_2\, dt \, + \, R(T), \end{aligned} \tag{7.9}$$

and we have to investigate the limit as $T \to \infty$ of the individual terms. We begin with the first two terms.

As above, let $\mathcal{A}_i$ be the self–adjoint extension of the Dirac operator $A_i : C^\infty(Z_i, E_i) \to C^\infty(Z_i, E_i)$ defined by (2.1). Since $\dim X$ is even, we have

$$\tau A_i = A_i \tau \quad \text{and} \quad \gamma_i \tau = -\tau \gamma_i, \quad i = 1, 2. \tag{7.10}$$

Let $E_i = E_i^+ \oplus E_i^-$ be the decomposition into the ± 1-eigenspaces of τ. By (7.10), A_i and γ_i take the following form with respect to this decomposition:

$$A_i = \begin{pmatrix} A_i^+ & 0 \\ 0 & A_i^-, \end{pmatrix}. \qquad \gamma_i = \begin{pmatrix} 0 & \gamma_i^- \\ \gamma_i^+ & 0 \end{pmatrix}. \tag{7.11}$$

In particular, on $\mathbb{R}^+ \times Z_i$ we have

$$D_\pm = \gamma_i^\pm \left(\frac{\partial}{\partial u_i} + A_i^\pm \right),$$

and $A_i^\pm$ is the Dirac operator associated with the Clifford bundle $E_i^\pm$. Let $\mathcal{A}_i^\pm$ be the self–adjoint extension of $A_i^\pm$. Since A_i^+ is a compatible Dirac operator [10], from [16, §6] it follows that the eta invariant of $\mathcal{A}_i^+$ can be defined by

$$\eta(0, \mathcal{A}_i^+) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \int_{Z_i} \operatorname{tr}(A_i^+ e^{-t(\mathcal{A}_i^+)^2}(z, z))\, dz\, dt. \tag{7.12}$$

Proposition 7.13. *We have*

$$\lim_{T\to\infty}\int_0^{\sqrt{T}}\int_{Z_{i,T}}\operatorname{tr}\Big(\gamma_i\tau D\, e^{-tD^2}((T,z_i),(T,z_i))\Big)\,dz_i\,dt=\eta(0,\mathcal{A}_i^+),\qquad i=1,2.$$

Proof. Let $Q(x,y,t)$ be the parametrix defined by (4.7). It follows from Proposition 4.20 that

$$\begin{aligned}&\Bigg|\int_0^{\sqrt{T}}\int_{Z_{i,T}}\Big\{\operatorname{tr}\Big(\gamma_i\tau De^{-tD^2}((T,z_i),(T,z_i))\Big)\\&\qquad-\operatorname{tr}\Big(\gamma_i\tau D_x Q((T,z_i),(T,z_i),t)\Big)\Big\}\,dz_i\,dt\Bigg|\\&\le C_1\operatorname{Vol}(Z_{i,T})\int_0^{\sqrt{T}}e^{ct}e^{-cT^2/t}\,dt\\&\le C_2T^{3/2}e^{c\sqrt{T}}e^{-cT^{3/2}}\to 0\end{aligned}$$

as $T\to\infty$. Hence we may replace the heat kernel by the parametrix Q.

Now consider the integral over $Z_{1,T}$. Suppose that $T>1$. Since the supports of ϕ_0 and ψ_0 are contained in X_1, the term $\phi_0K_0\psi_0$ in (4.7) makes no contribution. Hence, using the definition of Q, we get

$$\begin{aligned}&\int_{Z_{1,T}}\operatorname{tr}\Big(\gamma_1\tau D_xQ((T,z),(T,z),t)\Big)\,dz\\&=\frac{1}{\sqrt{4\pi t}}\int_{Z_{1,T}}\operatorname{tr}\big(\tau A_1e^{-t\mathcal{A}_1^2}(z,z)\big)\,dz\,dt\\&\quad+\frac{1}{\sqrt{4\pi t}}\int_0^T\psi_2(u_2)du_2\int_Y\operatorname{tr}\Big(\gamma_1\tau\gamma_2(A_2)_{(u_1,Y)}\{\tilde{K}_2((u_1,y),(T,y'),t)\\&\qquad\qquad-\frac{1}{\sqrt{4\pi t}}e^{-(u_1-T)^2/4t}\tilde{K}_3(y,y',t)\}\Big|_{\substack{y=y'\\u_1=T}}\Big)dy.\end{aligned}$$

Here we have used that ϕ_2' and ψ_2 have disjoint support. By (3.5) of [16], the second integral on the right-hand side can be estimated by $T\,e^{-cT^2/t}e^{ct}$. Next observe that by (7.11), we have

$$\operatorname{tr}\big(\tau A_1\,e^{-t\mathcal{A}_1^2}(z,z)\big)=\operatorname{tr}\big(A_1^+\,e^{-t(\mathcal{A}_1^+)^2}(z,z)\big)-\operatorname{tr}\big(A_1^-\,e^{-t(\mathcal{A}_1^-)^2}(z,z)\big).$$

Moreover, using (2.2) and (7.11), we get $A_1^- = -(\gamma_1^-)^{-1}A_1^+\gamma_1^-$. Thus

$$\begin{aligned}&\int_0^{\sqrt{T}}\int_{Z_{1,T}} \operatorname{tr}\Big(\gamma_1\cdot \tau D_x Q((T,z),(T,z),t)\Big)\,dz\,dt\\ (7.14)\qquad &= \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{T}} t^{-1/2}\int_{Z_{1,T}} \operatorname{tr}\big(A_1^+ e^{-t(A_1^+)^2}(z,z)\big)\,dz\,dt \,+\, O\big(e^{-C_1T^{3/4}}\big).\end{aligned}$$

We are now exactly in the situation of Proposition 7.6 of [16], which implies that the right-hand side of (7.14) tends to $\eta(0,\mathcal{A}_1^+)$ as $T\to\infty$. The same holds for $i=2$. q.e.d.

We are left with the third term, $R(T)$, in (7.9). First, we shall derive two further properties of the scattering matrix.

Lemma 7.15. *Let $j=1,2$. Then the scattering matrix $C_j(\lambda)$ satisfies*

$$\tau C_j(\lambda) = C_j(\lambda)\tau,\quad \gamma_j C_j(\lambda) = -C_j(\lambda)\gamma_j,\quad \lambda\in\Omega.$$

Proof. Let $\phi\in\ker\mathcal{A}_j$ and suppose that $\operatorname{Im}(\lambda)>0$. Then it follows from Theorem 6.5 that $\tau F_j(\phi,\lambda) - F_j(\tau\phi,\lambda)$ and $DF_j(\phi,\lambda) + i\lambda F_j(\gamma_j\phi,\lambda)$ are square integrable eigensections of Δ with eigenvalue λ^2 and both therefore must be zero. Hence, we get

$$(7.16)\qquad \tau F_j(\phi,\lambda) = F_j(\tau\phi,\lambda)\quad\text{and}\quad DF_j(\phi,\lambda) = -i\lambda F_j(\gamma_j\phi,\lambda),$$

which hold for all $\lambda\in\Omega$. Comparing the constant terms of both sides of these equations, the desired relations follow. q.e.d.

Given $\phi\in\ker\mathcal{A}_j$, put

$$I_j(\phi,\lambda,x) = \langle\gamma_j\cdot\tau(DF_j)(\phi,\lambda,x), F_j(\phi,\lambda,x)\rangle\,.$$

Lemma 7.17. *Let $\phi\in\ker\mathcal{A}_j$ and suppose that $C_j(0)\phi=\pm\phi$. Then for every compact subset $U\subset X$, there exists $C_U>0$ such that*

$$|I_j(\phi,\lambda,x)|\le C_U\lambda^2,\quad |\lambda|<\delta_0,\quad x\in U.$$

Proof. First note that by (7.16), we have

$$(7.18)\qquad I_j(\phi,\lambda,x) = -i\lambda\big\langle\gamma_j F_j(\tau\gamma_j\phi,\lambda,x), F_j(\phi,\lambda,x)\big\rangle.$$

Let $\psi\in\ker\mathcal{A}_j$ and suppose that $C_j(0)\psi=-\psi$. Then the functional equation (6.23) implies that $F_j(\psi,0)=0$. Furthermore, if ϕ satisfies $C_j(0)\phi=\pm\phi$, then by Lemma 7.15, we get $C_j(0)(\tau\gamma_j\phi)=\mp\tau\gamma_j\phi$. Hence, under the given assumption, it follows that either $F_j(\phi,0)=0$ or $F_j(\tau\gamma_j\phi,0)=0$. Therefore, by (7.18), we get $I_j(\phi,\lambda,x)=O(\lambda^2)$, uniformly on compact subsets. q.e.d.

Let $\phi \in \ker \mathcal{A}_j$ be an eigenvector of $C_j(0)$. Then Lemma 7.17 yields that

$$\left| \int_{Z_{j,T}} I_j(\phi, \lambda, (T, z))\, dz \right| \le C(T)\, \lambda^2, \quad |\lambda| < \delta_0, \tag{7.19}$$

for some constant $C(T) > 0$ depending on T. By the functional equation (6.23), we have $C_j(0)^2 =$Id. Thus $C_j(0)$ is a symmetric operator with eigenvalues equal to $+1$ or -1. Let $\psi_{j1}, ..., \psi_{jm_j}$ be an orthonormal basis for $\ker \mathcal{A}_j$ consisting of eigenvectors of $C_j(0)$, i.e., $C_j(0)\psi_{jk} = \pm\psi_{jk}$. From Corollary 6.37 and (6.30), it follows that as $T \to \infty$,

$$\begin{aligned} R(T) =& \frac{1}{4\pi} \sum_{j=1}^{2} \sum_{k=1}^{m_j} \int_{\sqrt{T}}^{\infty} \int_0^{\delta} e^{-t\lambda^2} \int_{Z_{j,T}} I_j(\psi_{jk}, \lambda, (T, z_j))\, dz_j\, d\lambda\, dt \\ &+ O(e^{-cT}). \end{aligned} \tag{7.20}$$

We note that by (7.19), each of the integrals is absolutely convergent.

Let $F_{j,0}(\phi, \lambda)$ be the constant term of $F_j(\phi, \lambda)$, defined by (6.7), and put

$$I_{j,0}(\phi, \lambda, x) = \langle \gamma_j \cdot \tau(DF_{j,0})(\phi, \lambda, x), F_{j,0}(\phi, \lambda, x) \rangle.$$

Consider the expansion (6.6). If $\phi \in \ker \mathcal{A}_j$ is such that $F_j(\phi, 0) = 0$, then all coefficients in this expansion must vanish, i.e., $a_k(0) = 0$, $k \in I$, and $b_l(0, \Lambda) \equiv 0$, $l \in \mathbb{N}$. Now suppose that $C_j(0)\phi = \pm\phi$. As above, from the functional equation (6.23) and Lemma 7.15 it follows that either $F_j(\phi, 0) = 0$ or $F_j(\gamma_j\phi, 0) = 0$. Hence, if we proceed as in the proof of (6.8), we may deduce that there exist $C, c > 0$ such that

$$\sup_{z \in Z_j} |I_j(\phi, \lambda, (u, z)) - I_{j,0}(\phi, \lambda, (u, z))| \le C\, \lambda^2 e^{-cu}, \; |\lambda| < \delta_0, \; u \in \mathbb{R}^+.$$

Therefore, in (7.20) we can replace $I_j(\phi, \lambda)$ by $I_{j,0}(\phi, \lambda)$, and the resulting expression equals $R(T)$ up to an exponentially small term in T as $T \to \infty$. Next observe that by (6.12), each $\phi \in \ker \mathcal{A}_j$ satisfies $|\phi(u, y)| \le C \exp(-\sqrt{\mu_1}\, u)$, $y \in Y$, for some constant $C > 0$. Using (6.7), this estimate implies

$$\left| \int_{Z_j} I_{j,0}(\phi, \lambda, (T, z))\, dz - \int_{Z_{j,T}} I_{j,0}(\phi, \lambda, (T, z))\, dz \right| \le C\, \lambda^2 T e^{-cT}, \quad |\lambda| < \delta_0.$$

Furthermore, by (6.7) and (2.1), we get

$$\begin{aligned}&\int_{Z_j} I_{j,0}(\phi, \lambda, (T, z))\, dz \\ &\quad = -i\lambda\{\langle \tau\phi, \phi\rangle - \langle \tau C_j(\lambda)\phi, C_j(\lambda)\phi\rangle + e^{-2i\lambda T}\langle \tau\phi, C_j(\lambda)\phi\rangle \\ &\qquad - e^{2i\lambda T}\langle \tau C_j(\lambda)\phi, \phi\rangle\}.\end{aligned} \tag{7.21}$$

Applying Lemma 7.15 and using the fact that $C_j(\lambda)$ is unitary for $\lambda \in (-\delta_0, \delta_0)$, we get

$$\langle \tau\phi, \phi\rangle = \langle \tau C_j(\lambda)\phi, C_j(\lambda)\phi\rangle, \quad \phi \in \ker \mathcal{A}_j,$$

i.e., the first two terms on the right-hand side of (7.21) cancel. The remaining terms on the right of (7.21) are equal to

$$(e^{-2i\lambda T} - e^{2i\lambda T})\langle \phi, \tau C_j(\lambda)\phi\rangle + e^{2i\lambda T}\langle \phi, \tau(C_j(\lambda) - C_j(-\lambda))\phi\rangle.$$

Putting these remarks together, we get

$$\begin{aligned}R(T) = -\frac{1}{2\pi}\sum_{j=1}^{2}\Big\{&\int_0^\delta e^{-\sqrt{T}\lambda^2}\, \mathrm{Tr}\big(\tau C_j(\lambda)\big)\, \frac{\sin(2\lambda T)}{\lambda}\, d\lambda \\ &+ \int_0^\delta e^{-\sqrt{T}\lambda^2} e^{2i\lambda T}\, \frac{\mathrm{Tr}(\tau C_j(\lambda)) - \mathrm{Tr}(\tau C_j(-\lambda))}{\lambda}\, d\lambda\Big\} + O(e^{-cT}).\end{aligned}$$

The first integral can be treated as follows:

$$\begin{aligned}&\int_0^\delta e^{-\sqrt{T}\lambda^2}\, \mathrm{Tr}(\tau C_j(\lambda))\, \frac{\sin(2\lambda T)}{\lambda}\, d\lambda \\ &\quad = \int_0^{\delta/T^{1/8}} e^{-\sqrt{T}\lambda^2}\, \mathrm{Tr}(\tau C_j(\lambda))\, \frac{\sin(2\lambda T)}{\lambda}\, d\lambda + O(e^{-T^{1/4}}) \\ &\quad = \mathrm{Tr}(\tau C_j(0)) \int_0^\infty e^{-\sqrt{T}\lambda^2}\, \frac{\sin(2\lambda T)}{\lambda}\, d\lambda + O(5(e^{-T^{1/4}}).\end{aligned}$$

Applying Fourier's integral formula, we get

$$\lim_{T\to\infty} R(T) = -\frac{1}{4}\, \mathrm{Tr}(\tau C_1(0)) - \frac{1}{4}\, \mathrm{Tr}(\tau C_2(0)).$$

Now observe that

$$\ker \mathcal{A}_j = \ker \mathcal{A}_j^+ \oplus \ker \mathcal{A}_j^-, \quad j = 1, 2, \tag{7.22}$$

is the decomposition of $\ker \mathcal{A}_j$ into the ± 1–eigenspaces of

$$\tau : \ker \mathcal{A}_j \to \ker \mathcal{A}_j.$$

By Lemma 7.15, $C_j(\lambda)$ preserves the decomposition (7.22). Let $C_j^{\pm}(\lambda)$ be the restriction of $C_j(\lambda)$ to ker $\mathcal{A}_j^{\pm}$. Then

$$\operatorname{Tr}(\tau C_j(0)) = \operatorname{Tr}(C_j^+(0)) - \operatorname{Tr}(C_j^-(0)).$$

By Lemma 7.15, we know that $\gamma_j C_j(0) = -C_j(0)\gamma_j$. Since $\gamma_j \tau = -\tau \gamma_j$, we obtain

$$C_j^-(0) = -\gamma_j C_j^+(0)\gamma_j^{-1}. \tag{7.23}$$

Hence, as the final result we have

$$\lim_{T\to\infty} R(T) = -\frac{1}{2}\operatorname{Tr}(C_1^+(0)) - \frac{1}{2}\operatorname{Tr}(C_2^+(0)).$$

We note that $C_j^+(0) : \ker \mathcal{A}_j^+ \to \ker \mathcal{A}_j^+$, $|\lambda| < \delta_0$, may be regarded as scattering matrix associated with the continuous spectrum of D_-D_+ near zero. Indeed, let $\phi \in \ker \mathcal{A}_j^+$. Then $F_j(\phi, \lambda)$ belongs to $C^\infty(X, E^+)$ and hence is a generalized eigenfunction for D_-D_+. The scattering operator $C_j^+(\lambda)$ is determined by the constant term of $F_j(\phi, \lambda)$.

Summarizing our results, we have proved Theorem 0.1.

According to Theorem 0.1 of [16], the eta invariant $\eta(0, \mathcal{A}_j^+)$ can also be described in terms of the restriction of A_j^+ to the compact submanifold $M_j \subset Z_j$. On $\mathbb{R}^+ \times Y \subset Z_j$, A_j^+ has the form

$$A_j^+ = \sigma_j\Big(\frac{\partial}{\partial v_j} + B_j\Big),$$

where v_j is the outward normal coordinate, $\sigma_j : E_j^+|Y \to E_j^-|Y$ is a bundle isomorphism and $B_j : C^\infty(Y, E_j^+|Y) \to C^\infty(Y, E_j^+|Y)$ is a generalized Dirac operator on Y. By (2.3), σ_j and B_j can be expressed in terms of γ_1, γ_2 and D_Y as follows: $\sigma_1 = (\gamma_2\gamma_1)^+$, $\sigma_2 = (\gamma_1\gamma_2)^+$, $B_1 = (D_Y\gamma_2)^+$ and $B_2 = (D_Y\gamma_1)^+$, where "+" denotes the restriction of the corresponding operator to the +1–eigenspace of τ. Let P_j be the negative spectral projection with respect to B_j. Using P_j, we impose spectral boundary conditions on ∂M_j. More precisely, put

$$H^1(M_j, E_j^+; P_j) = \{\varphi \in H^1(M_j, E_j^+) \mid P_j(\varphi|\partial M_j) = 0\},$$

and let

$$(A_j^+)_{P_j} : H^1(M_j, E_j^+; P_j) \to L^2(M_j, E_j^-)$$

be defined by $(A_j^+)_{P_j}\varphi = A_j^+\varphi$. Since ker $B_j = 0$, $(A_j^+)_{P_j}$ is self–adjoint. Moreover, $(A_j^+)_{P_j}$ has pure point spectrum, and the eta invariant of $(A_j^+)_{P_j}$, which we denote by $\eta(A_j^+; P_j)$, can be defined in the same way as in the closed case. Since A_j^+ is a compatible operator of Dirac type, it follows from Corollary 1.29 of [16] that

$$\eta(A_j^+;P_j) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{Tr}\Big((A_j^+)_{P_j}e^{-t\left((A_j^+)_{P_j}\right)^2}\Big)\,dt. \tag{7.24}$$

Thus by Theorem 0.1 of [16], we have

$$\eta(0,\mathcal{A}_j^+) = \eta(A_j^+,P_j).$$

The term $\mathrm{Tr}(C_j^+(0))$ has also an alternative description analogous to [1]. As observed above, the scattering matrix satisfies $C_j(0)^2 = \mathrm{Id}$. Let $\mathcal{E}_j^\pm$ be the ±1–eigenspaces of $C_j(0)$. Then

$$\ker\mathcal{A}_j = \mathcal{E}_j^+\oplus\mathcal{E}_j^-.$$

Let $\phi\in\mathcal{E}_j^+$ and put $\varphi = \frac{1}{2}F_j(\phi,0)$. Then we have

$$2D\varphi = DF_j(\phi,\lambda)\big|_{\lambda=0} = -i\lambda F_j(\gamma\phi,\lambda)\big|_{\lambda=0} = 0.$$

Let $\hat{\chi}_j$ be the characteristic function of $\mathbb{R}^+\times Z_j\subset X$. It follows from Theorem 6.21 that

$$\varphi = \hat{\chi}_j\phi + \psi,$$

where $\psi\in L^2(X,E)$. In analogy with [1, p.58], we call φ an extended L^2–solution of D with limiting value $\phi\in\ker\mathcal{A}_j$. Let $\mathcal{L}_j\subset\ker\mathcal{A}_j$ be the subspace consisting of all limiting values of extended L^2–solutions of D.

Lemma 7.25. *We have $\mathcal{L}_j = \mathcal{E}_j^+$, $j=1,2$.*

Proof. Above we have seen that $\mathcal{E}_j^+\subset\mathcal{L}_j$. To prove the reverse inclusion, let $\phi\in\mathcal{L}_j$ and suppose that $\varphi\in C^\infty(X,E)$ is an extended L^2–solution of D with limiting value ϕ. Write $\phi=\phi_++\phi_-$ where $C_j(0)\phi_\pm=\phi_\pm$. Put $\xi=\frac{1}{2}F_j(\gamma_j\phi_-,0)$. Then ξ satisfies $D\xi=0$ and $\xi-\hat{\chi}_j\gamma_j\phi_-\in L^2(X,E)$. Let $X_{j,T} = X-((T,\infty)\times Z_j)$. Applying Green's formula, we get

$$\begin{aligned}
0 &= \int_{X_{j,T}}\langle D\varphi(x),\xi(x)\rangle\,dx\\
&= \int_{Z_j}\langle\gamma_j\varphi(T,z),\xi(Z,z)\rangle\,dz + \int_{X_{j,T}}\langle\varphi(x),D\xi(x)\rangle\,dx\\
&= \|\ \gamma_j\phi_-\ \| + O(e^{-cT}).
\end{aligned}$$

Hence $\phi_-=0$. q.e.d.

The space $\mathcal{L}_j$ decomposes according to the decomposition (7.22). Let $\mathcal{L}_j^{\pm} \subset \ker \mathcal{A}_j^{\pm}$ be the subspace of all limiting values of extended L^2–solutions of $D_{\pm}$. Then we have

$$\mathcal{L}_j = \mathcal{L}_j^+ \oplus \mathcal{L}_j^- \quad j = 1, 2.$$

Lemma 7.26. *Let $h_j^{\pm} = \dim \mathcal{L}_j^{\pm}$, $j = 1, 2$. Then*

$$\operatorname{Tr}(C_j^+(0)) = h_j^+ - h_j^-, \quad j = 1, 2.$$

Proof. Using Lemma 7.25, we get

$$\mathcal{L}_j^{\pm} = \mathcal{L}_j \cap \ker \mathcal{A}_j^{\pm} = \mathcal{E}_j^+ \cap \ker \mathcal{A}_j^{\pm}, \ j = 1, 2.$$

Since $C_j^{\pm}(0)$ is the restriction of $C_j(0)$ to $\ker \mathcal{A}_j^{\pm}$, it follows that $h_j^{\pm} = \dim \ker(C_j^{\pm}(0) - \mathrm{Id})$. Moreover, (7.23) implies that $\dim \ker(C_j^-(0) - \mathrm{Id}) = \dim \ker(C_j^+(0) + \mathrm{Id})$. Putting everything together, we obtain

$$\begin{aligned} \operatorname{Tr}(C_j^+(0)) &= \dim \ker(C_j^+(0) - \mathrm{Id}) - \dim \ker(C_j^+(0) + \mathrm{Id}) \\ &= h_j^+ - h_j^-, j = 1, 2. \end{aligned}$$

q.e.d.

Finally observe that by Proposition 3.11 of [1], we have $\ker \mathcal{A}_j^+ = \ker(A_j^+)_{P_j}$, $j = 1, 2$. We can now rewrite the index formula (0.10) in the following way:

(7.27)

$$L^2\text{-}\operatorname{Ind} D_+ = \int_{X_0} \omega_{D_+} - \frac{1}{2}\sum_{j=1}^{2}\{\eta(A_j^+; P_j) + \dim(\ker(A_j^+)_{P_j}\} + h_1^- + h_2^-.$$

Theorem 0.2 follows directly from (7.27). There is an obvious extension of this result to the case of several corners of codimension two.

8. A splitting formula for eta invariants

In this section we apply our index formula to derive a splitting formula for eta invariants.

Let X be a $2k$–dimensional compact oriented Riemannian manifold with C^∞ boundary M. Let $Z \hookrightarrow X$ be a compact oriented hypersurface with C^∞ boundary Y such that Z intersects the boundary of X transversally in Y and devides X in two submanifolds X_1 and X_2 (see Fig.4). We assume that the metric on X is a product in a neighborhood $(-\varepsilon, 0] \times M$ of the boundary and in a tubular neighborhood $(-\varepsilon, \varepsilon) \times Z$ of Z. Let E be a Clifford bundle over X and assume that the metric and the connection of E are products near M and Z. Let $D : C^\infty(X, E) \to C^\infty(X, E)$

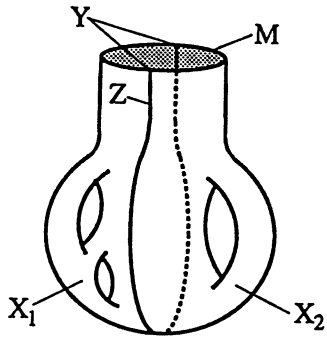

FIGURE 4. Cutting X in two pieces X_1 and X_2.

be the associated Dirac operator, and denote the restriction of D to X_i by $D_i : C^\infty(X_i, E_i) \to C^\infty(X_i, E_i)$, $i = 1, 2$. Then D_1 and D_2 are Dirac operators which satisfy (2.1)–(2.4). Let D^+ and D_i^+, $i = 1, 2$, be the restriction of D and D_i, to E^+ and E_i^+, respectively. Near the boundary and the hypersurface we have

$$D^+ = \gamma_1\Big(\frac{\partial}{\partial u_1} + A\Big), \quad \text{on} \quad (-\varepsilon, 0] \times M, \tag{8.1}$$

$$D^+ = \gamma_2\Big(\frac{\partial}{\partial u_2} + D_Z\Big), \quad \text{on} \quad (-\varepsilon, \varepsilon) \times Z, \tag{8.2}$$

where A and D_Z are the induced Dirac operators on M and Z, respectively, γ_1 denotes Clifford multiplication by the inward unit normal vector field to M, and γ_2 denotes Clifford multiplication by the unit normal vector field to Z, which points into X_1. Furthermore, on $(-\varepsilon, 0] \times (-\varepsilon, \varepsilon) \times Y$, D takes the form

$$D = \gamma_1 \frac{\partial}{\partial u_1} + \gamma_2 \frac{\partial}{\partial u_2} + D_Y, \tag{8.3}$$

where $D_Y : C^\infty(Y, E|Y) \to C^\infty(Y, E|Y)$ is a Dirac operator on Y, and the commutation relations (2.4) hold. We assume that $\ker D_Y = 0$. From (8.3) it follows that

$$D_Z = \sigma_1\Big(\frac{\partial}{\partial u_1} + B_1\Big) \quad \text{on} \quad (-\varepsilon, 0] \times Y,$$

and

$$A = \sigma_2\Big(\frac{\partial}{\partial u_2} + B_2\Big) \quad \text{on} \quad (-\varepsilon, \varepsilon) \times Y,$$

where B_1 and B_2 are the restrictions to $E^+|Y$ of $-\gamma_2 D_Y$ and $-\gamma_1 D_Y$, respectively. In particular, the assumption $\ker D_Y = 0$ implies that

$\ker B_i = 0$, $i = 1, 2$. Let P^+ (resp. P^-) be the positive (resp. negative) spectral projection for B_2, and let P be the negative spectral projection for B_1. Finally, let A_i be the restriction of A to M_i, $i = 1, 2$. If we apply our index formula of Theorem 0.2 to D_i^+, then we get

$$\begin{aligned}\widetilde{\operatorname{Ind}}\, D_1^+ = \int_{X_1} \omega_1 \;-\; & \frac{1}{2}\{\eta(A_1, P^-) \;+\; \dim\ker(A_1)_{P^-}\} \\ - \; & \frac{1}{2}\{\eta(D_Z, P) \;+\; \dim\ker(D_Z)_P\},\end{aligned} \tag{8.4}$$

$$\begin{aligned}\widetilde{\operatorname{Ind}}\, D_2^+ = \int_{X_2} \omega_2 \;-\; & \frac{1}{2}\{\eta(A_2, P^+) \;+\; \dim\ker(A_2)_{P^+}\} \\ + \; & \frac{1}{2}\{\eta(D_Z, P) \;+\; \dim\ker(D_Z)_P\},\end{aligned} \tag{8.5}$$

where ω_i is the Atiyah–Singer index density of D_i^+, and $(A_1)_{P^-}$, $(A_2)_{P^+}$ and $(D_Z)_P$ are the self–adjoint extensions of A_1, A_2 and D_Z, respectively, defined by the corresponding spectral projections. On the other hand, the index theorem of Atiyah, Patodi and Singer [1] applied to X gives

$$\operatorname{Ind} D^+ = \int_X \omega \;-\; \frac{1}{2}\{\eta(A) \;+\; \dim\ker A\}, \tag{8.6}$$

where ω is the Atiyah–Singer index density of D^+, and $\operatorname{Ind} D^+$ is the index of the APS–boundary value problem. The index formulae suggest the introduction of the ξ–invariant

$$\xi(A) = \frac{1}{2}[\eta(A) \;+\; \dim\ker A].$$

Similarly, we denote by $\xi(A_1, P^-)$ and $\xi(A_2, P^+)$ the ξ–invariants for A_1 and A_2 with respect to APS boundary conditions defined by P^- and P^+, respectively. Since ω, ω_1 and ω_2 are locally computable, we have

$$\int_X \omega = \int_{X_1} \omega_1 \;+\; \int_{X_2} \omega_2.$$

Hence, if we compare (8.4), (8.5) and (8.6), we obtain

Theorem 8.7. *Let the assumptions be as above. Then the following splitting formula holds for the ξ–invariants*

$$\begin{aligned}\xi(A) =\, & \xi(A_1, P^-) \;+\; \xi(A_2, P^+) \;+\; \widetilde{\operatorname{Ind}} D_1^+ \\ & + \widetilde{\operatorname{Ind}} D_2^+ \;-\; \operatorname{Ind}\, D^+ \;+\; \dim\ker(D_Z)_P.\end{aligned}$$

Note that the same result holds if X has additional C^∞ boundary components which are disjoint from the hypersurface Z.

We shall now employ this result to derive a mod Z splitting formula for the ξ–invariant. Such formulae were recently proved by various authors [4], [8], [14], [25].

Let M be a closed oriented $(2k-1)$–dimensional spin manifold. This means that M is equipped with a Riemannian metric and a spin structure is fixed. Let $Y \hookrightarrow M$ be a closed oriented hypersurface which devides M into two pieces M_1 and M_2, that is, M_1 and M_2 are submanifolds of M with boundary Y, and M is obtained by gluing M_1 and M_2 along the common boundary Y. We assume that the metric of M is a product in a tubular neighborhood $(-\varepsilon, \varepsilon) \times Y$ of the hypersurface Y. Let D_M be the Dirac operator on M. On $(-\varepsilon, \varepsilon) \times Y$ we have

$$D_M = \gamma\Big(\frac{\partial}{\partial u} + D_Y\Big),$$

where γ is Clifford multiplication by the unit normal vector field to Y which points into M_2.

We shall now construct two $2k$–dimensional manifolds X_1 and X_2 with a corner at Y. Let $M_{i,\varepsilon} = M_i - ((-\varepsilon, 0] \times Y)$, $i = 1, 2$, and let

$$X_\varepsilon = ([-\varepsilon, 0] \times M_{1,\varepsilon}) \cup ([-\varepsilon, 0]^2 \times Y) \cup ([-\varepsilon, 0] \times M_{2,\varepsilon}),$$

where the three manifolds are glued together along pieces of their boundaries in the following manner: We identify $[-\varepsilon, 0] \times \partial M_{1,\varepsilon}$ with $\{-\varepsilon\} \times [-\varepsilon, 0] \times Y$ and $[-\varepsilon, 0] \times \partial M_{2,\varepsilon}$ with $[-\varepsilon, 0] \times \{-\varepsilon\} \times Y$. Then X_ε is a manifold with two boundary components which are piecewise smooth. One component equals $M' = M_1 \cup M_2$ and the other component equals $M_\varepsilon = M_{1,\varepsilon} \cup_Y M_{2,\varepsilon}$. Both boundary components are homeomorphic to M. The product metrics on $[-\varepsilon, 0] \times M_1$ and $[-\varepsilon, 0] \times M_2$ coincide on the common submanifold $[-\varepsilon, 0]^2 \times Y$ and therefore extend to a metric on X_ε in the obvious way. We smooth X_ε at the corner of M_ε. The manifold X'_ε, obtained in this way, is isometric to X_ε in a neighborhood of M'. The other boundary component M'_ε of X'_ε is diffeomorphic to M. Next we glue the cylinder $[0, 1] \times M$ to X'_ε by identifying $\{1\} \times M$ and M'_ε by a diffeomorphism. Let X_1 be the manifold obtained in this way. Finally, we patch together the product metric on the cylinder and the given metric on X'_ε to get a smooth metric on X_1. Then ∂X_1 is the disjoint union of M and M'. Moreover, in a neighborhood of M, the metric of X_1 is a product, and in a neighborhood of M' the metric has the structure described in §1 (see Fig.5 below).

Then X_1 is also a spin manifold and we can pick the spin structure such that it extends the given spin structures on the boundaries.

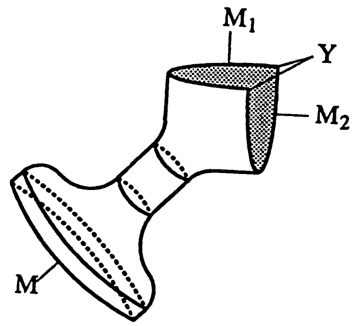

FIGURE 5. The manifold X_1 with corner Y.

Now let $N = M_2 \cup (-M_2)$. Then N is a spin manifold, and using the same construction as above we get a spin manifold X_2 with boundary $N \cup N'$, where N is the smooth boundary component and N' is homeomorphic to N with a corner at Y. We glue X_1 and X_2 at the corner along their boundary components $M_2 \subset M'$ and $-M_2 \subset N'$. In this way we get a spin manifold X with three smooth boundary components M, $-M$ and $-N$. Applying Theorem 8.7, we obtain

Theorem 8.8. *Let M be a closed odd-dimensional spin manifold. Let $Y \hookrightarrow M$ be a closed oriented hypersurface which devides M into two submanifolds M_1 and M_2. We assume that the metric is a product near Y and that $\ker D_Y = 0$. Let P^+ (resp. P^-) be the positive (resp. negative) spectral projection of D_Y. Furthermore, let D_M, D_{M_1} and D_{M_2} be the Dirac operators on M, M_1 and M_2, respectively. Then we have*

$$\begin{aligned}\xi(D_M) =& \xi(D_{M_1}, P^-) + \xi(D_{M_2}, P^+) + \widetilde{\operatorname{Ind}} D^+_{X_1} + \widetilde{\operatorname{Ind}} D^+_{X_2} \\ &- \operatorname{Ind} D^+_X + \dim\ker(D_{M_2})_{P^-},\end{aligned}$$

where D^+_X, $D^+_{X_1}$ and $D^+_{X_2}$ are the Dirac operators on half-spinors of X, X_1 and X_2, respectively, $(D_{M_2})_{P^-}$ denotes the self-adjoint extension of D_{M_2} with respect to P^-, and $\widetilde{\operatorname{Ind}}$ is defined by (0.11).

A similar result holds for twisted Dirac operators. In particular, we recover in this way the mod $\mathbb{Z}$ splitting formulae of [4], [8], [14], [25] in the case where the Dirac operator D_Y is invertible.

9. An example

In this section we consider the case where X is the product of two manifolds with cylindrical ends, and the Clifford bundle is the exterior tensor product of Clifford bundles over the factors. Then we compare our index formula with the result obtained by using the product structure.

Let X_1 and X_2 be two oriented Riemannian manifolds with cylindrical ends of dimension $2k_1$ and $2k_2$, respectively, and let $X = X_1 \times X_2$. The manifold X_i has a decomposition as $X_i = N_i \cup_{Y_i} (\mathbb{R}^+ \times Y_i)$ where N_i is a compact manifold with boundary Y_i. Let $E_i \to X_i$, $i = 1, 2$, be a Clifford bundle over X_i and assume that on $\mathbb{R}^+ \times Y_i$, the connection and the Hermitian metric of E_i are products. Let

$$D_i^{\pm} \colon C^\infty(X_i, E_i^{\pm}) \to C^\infty(X_i, E_i^{\mp})$$

be the corresponding chiral Dirac operators. Our assumption implies that on $\mathbb{R}^+ \times Y_i$, D_i^+ takes the form

$$D_i^+ = \gamma_i \left(\frac{\partial}{\partial u_i} + B_i \right), \tag{9.1}$$

where γ_i denotes Clifford multiplication by the outward unit normal vector field, and $B_i \colon C^\infty(Y_i, E_i^+|_{Y_i}) \to C^\infty(Y_i, E_i^+|_{Y_i})$ is a Dirac operator on Y_i. Let $E = E_1 \otimes E_2$ be the tensor product of E_1 and E_2 over $X = X_1 \times X_2$, that is, the fibres are given by $E_{(x,y)} = (E_1)_x \otimes (E_2)_y$. Now recall that the Clifford algebras satisfy

$$C\ell(T_x X_1) \otimes C\ell(T_y X_2) = C\ell(T_x X_1 \otimes T_y X_2).$$

This implies that E is a Clifford bundle over X. Let τ_i be the canonical involution of E_i. Then $\tau = \tau_1 \otimes \tau_2$ is the corresponding involution of E, and therefore the ± 1 eigenspaces $E^\pm$ of τ are given by

$$\begin{aligned} E^+ &= (E_1^+ \otimes E_2^+) \oplus (E_1^- \otimes E_2^-), \\ E^- &= (E_1^- \otimes E_2^+) \oplus (E_1^+ \otimes E_2^-). \end{aligned} \tag{9.2}$$

Let $D \colon C^\infty(X, E) \to C^\infty(X, E)$ be the Dirac operator of E and let $D^\pm$ be the restriction of D to $C^\infty(X, E^\pm)$. Then with respect to the decomposition (9.2), we have

$$\begin{aligned} D^+ &= \begin{pmatrix} D_1^+ \otimes \mathrm{Id} & -\mathrm{Id} \otimes D_2^- \\ \mathrm{Id} \otimes D_2^+ & D_1^- \otimes \mathrm{Id} \end{pmatrix}, \\ D^- &= \begin{pmatrix} D_1^- \otimes \mathrm{Id} & \mathrm{Id} \otimes D_2^- \\ -\mathrm{Id} \otimes D_2^+ & D_1^+ \otimes \mathrm{Id} \end{pmatrix}. \end{aligned} \tag{9.3}$$

Lemma 9.4. *The following equality holds:*

$$L^2\text{-}\operatorname{Ind} D^+ = (L^2\text{-}\operatorname{Ind} D_1^+) \cdot (L^2\text{-}\operatorname{Ind} D_2^+).$$

Proof. From (9.3) it follows that

$$
\begin{aligned}
D^-D^+ &= \begin{pmatrix} D_1^-D_1^+ \otimes \mathrm{Id} + \mathrm{Id} \otimes D_2^-D_2^+ & 0 \\ 0 & D_1^+D_1^- \otimes \mathrm{Id} + \mathrm{Id} \otimes D_2^+D_2^- \end{pmatrix}, \\
D^+D^- &= \begin{pmatrix} D_1^+D_1^- \otimes \mathrm{Id} + \mathrm{Id} \otimes D_2^-D_2^+ & 0 \\ 0 & D_1^-D_1^+ \otimes \mathrm{Id} + \mathrm{Id} \otimes D_2^+D_2^- \end{pmatrix}.
\end{aligned} \tag{9.5}
$$

Let $\Delta^\pm$ and $\Delta_i^\pm$ denote the closures of $D^\mp D^\pm$ and $D_i^\mp D_i^\pm$ in L^2, respectively. By Corollary 2.23 and Theorem 4.1 of [16], the kernels of these operators are all finite-dimensional, and (9.5) implies

$$
\begin{aligned}
\ker \Delta^+ &= (\ker \Delta_1^+ \otimes \ker \Delta_2^+) \oplus (\ker \Delta_1^- \otimes \ker \Delta_2^-), \\
\ker \Delta^- &= (\ker \Delta_1^- \otimes \ker \Delta_2^+) \oplus (\ker \Delta_1^+ \otimes \ker \Delta_2^-).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
L^2\text{-Ind } D^+ &= \dim\ker \Delta^+ - \dim\ker \Delta^- \\
&= (\dim\ker \Delta_1^+ - \dim\ker \Delta_1^-) \cdot (\dim\ker \Delta_2^+ - \dim\ker \Delta_2^-) \\
&= (L^2\text{-Ind } D_1^+) \cdot (L^2\text{-Ind } D_2^+).
\end{aligned}
$$

q.e.d.

Suppose that $\ker B_i = 0$. Then by Corollary 3.14 of [1], the L^2–index of D_i^+ is given by the index of the APS boundary problem, that is, it equals (0.3). Using Lemma 9.4, we get

$$
\begin{aligned}
L^2\text{-Ind } D^+ = &\int_{X_1} \omega_{D_1} \cdot \int_{X_2} \omega_{D_2} - \frac{1}{2}\eta(B_2) \int_{X_1} \omega_{D_1} \\
&- \frac{1}{2}\eta(B_1) \int_{X_2} \omega_{D_2} + \frac{1}{4}\eta(B_1)\eta(B_2).
\end{aligned} \tag{9.6}
$$

We shall now compare (9.6) with the result obtained by applying our index formla (0.10). Firstly, it follows from (9.5) that

$$
\begin{aligned}
&\operatorname{tr} e^{-t\Delta^+}(x,x) - \operatorname{tr} e^{-t\Delta^-}(x,x) \\
&\quad = \left(\operatorname{tr} e^{-t\Delta_1^+}(x,x) - \operatorname{tr} e^{-t\Delta_1^-}(x,x)\right) \left(\operatorname{tr} e^{-t\Delta_2^+}(x,x) - \operatorname{tr} e^{-t\Delta_2^-}(x,x)\right).
\end{aligned}
$$

By Theorem 5 of [10], this equality implies that the local index densities are related by $\omega_D = \omega_{D_1} \wedge \omega_{D_2}$, and therefore we obtain

$$
\int_X \omega_D = \int_{X_1} \omega_{D_1} \cdot \int_{X_2} \omega_{D_2}.
$$

It remains to compute the eta invariants. If we use the terminology of §1, then the hypersurfaces Z_i are given by $Z_1 = Y_1 \times X_2$ and $Z_2 = X_1 \times Y_2$. We may use γ_i to identify $E_i^+|_{Y_i}$ with $E_i^-|_{Y_i}$ which we call S_i. From (9.3) it follows that on $\mathbb{R}^+ \times Z_i$, D^+ can be written as

$$D^+ = \rho_i \left(\frac{\partial}{\partial u_i} + A_i \right),$$

where

$$A_i = \begin{pmatrix} B_i \otimes \mathrm{Id} & \mathrm{Id} \otimes D_j^- \\ \mathrm{Id} \otimes D_j^+ & -B_i \otimes \mathrm{Id} \end{pmatrix}, \quad i,j = 1,2, \quad i \neq j, \tag{9.7}$$

and ρ_i denotes Clifford multiplication by the inward unit mormal vector field. Let $\mathcal{A}_i$ be the unique self-adjoint extension of A_i in L^2. The eta-invariant $\eta(\mathcal{A}_i)$ of $\mathcal{A}_i$ is defined as in (0.9). To compute $\eta(\mathcal{A}_i)$, we introduce the following function defined in terms of the spectrum $\mathrm{Spec}(B_i)$ of B_i. Let

$$\theta_i(t) = \sum_{\lambda \in \mathrm{Spec}(B_i)} \frac{\operatorname{sign} \lambda}{2} \operatorname{erfc}(|\lambda|\sqrt{t}), \quad t > 0,\ i = 1,2, \tag{9.8}$$

where $\operatorname{erfc}(x)$ is the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du.$$

The series (9.8) is absolutely convergent, and as $t \to \infty$ we have

$$|\theta_i(t)| \leq Ce^{-ct}, \tag{9.9}$$

for some constants $C, c > 0$. The small time behaviour of $\theta_i(t)$ is described by

Lemma 9.10. *As $t \to 0$, we have*

$$\theta_i(t) = -\frac{1}{2}\eta(B_i) + O(t), \quad i = 1,2.$$

Proof. Differentiating (9.8) yields

$$\frac{\partial \theta_i}{\partial t}(t) = -\frac{1}{\sqrt{4\pi t}} \mathrm{Tr}(B_i e^{-tB_i^2}). \tag{9.11}$$

Since Y_i is odd-dimensional, it follows from Theorem 2.4 of [3] that

$$\mathrm{Tr}\ (B_i e^{-tB_i^2}) = b_i t^{1/2} + O(t^{3/2})$$

as $t \to 0$. Hence, we get

$$\theta_i(t) = a_i + c_i t + O(t^2)$$

as $t \to 0$. The constant term of this expansion can be computed in the same way as in [1, p.53], which gives $a_i = -1/2\,\eta(B_i)$. q.e.d.

Lemma 9.12. *The eta invariants of $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by*

$$\eta(\mathcal{A}_1) = \eta(B_1)\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_1)\eta(B_2) - 2\int_0^\infty \frac{\partial\theta_1}{\partial t}(t)\theta_2(t)\,dt,$$

and

$$\eta(\mathcal{A}_2) = \eta(B_2)\int_{X_1}\omega_{D_1} - \frac{1}{2}\eta(B_1)\eta(B_2) - 2\int_0^\infty \theta_1(t)\frac{\partial\theta_2}{\partial t}(t)\,dt.$$

Proof. First, observe that by (9.9) and Lemma 9.10, the infinite integrals are absolutely convergent. We consider $\mathcal{A}_1$. From (9.7) it follows that

$$\begin{aligned}&\operatorname{tr}\Big(A_1 e^{-t\mathcal{A}_1^2}((y,x),(y,x))\Big)\\ (9.13)\qquad &= \operatorname{tr}\Big(B_1 e^{-tB_1^2}(y,y)\Big)\Big\{\operatorname{tr}(e^{-t\Delta_2^+}(x,x)) - \operatorname{tr}(e^{-t\Delta_2^-}(x,x))\Big\},\end{aligned}$$

for $y \in Y_1$ and $x \in X_2$. Integration over Y_1 gives $\operatorname{Tr}(B_1 e^{-tB_1^2})$ on the right-hand side. It remains to investigate the integral over X_2.

Let $Q_2^\pm$ be a parametrix for $e^{-t\Delta_2^\pm}$ defined as by (3.3) in [16]. From (9.1) it follows that $\operatorname{tr} Q_2^+(x,x,t) = \operatorname{tr} Q_2^-(x,x,t)$ for $x \in [2,\infty)\times Y_2$. Together with (3.5) in [16], this implies that

$$\operatorname{tr}(e^{-t\Delta_2^+}(x,x)) - \operatorname{tr}(e^{-t\Delta_2^-}(x,x))$$

is absolutely integrable on X_2, and with respect to t we can differentiate under the integral sign. Using Lemma 7.6 together with these observations, it is easy to prove that

$$\frac{\partial}{\partial t}\int_{X_2}\Big\{\operatorname{tr}(e^{-t\Delta_2^+}(x,x)) - \operatorname{tr}(e^{-t\Delta_2^-}(x,x))\Big\}\,dx = \frac{1}{\sqrt{4\pi t}}\operatorname{Tr}\Big(B_2 e^{-tB_2^2}\Big).$$

Hence, by (9.13) and (9.11) we get

$$\begin{aligned}&\int_{X_2}\left\{\operatorname{tr}(e^{-t\Delta_2^+}(x,x)) - \operatorname{tr}(e^{-t\Delta_2^-}(x,x))\right\}dx\\ &= L^2\text{-}\operatorname{Ind} D_2^+ - \frac{1}{\sqrt{4\pi}}\int_t^\infty u^{-1/2}\operatorname{Tr}\left(B_2 e^{-uB_2^2}\right)du \\ &= L^2\text{-Ind } D_2^+ + \theta_2(t).\end{aligned} \tag{9.14}$$

As observed above, the L^2–index of D_2^+ can be computed by the index formula (0.3). Hence, the right-hand side of (9.14) equals

$$\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_2) + \theta_2(t),$$

and we obtain in consequence of (9.13) and (9.11),

$$\begin{aligned}\eta(\mathcal{A}_1) &= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\int_{Y_1}\int_{X_2}\operatorname{tr}\left(A_1 e^{-t\mathcal{A}_1^2}((y,x),(y,x))\right)dy\,dx\,dt\\ &= \left(\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_2)\right)\eta(B_1)\\ &\quad + \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\operatorname{Tr}\left(B_1 e^{-tB_1^2}\right)\theta_2(t)\,dt\\ &= \eta(B_1)\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_1)\eta(B_2) - 2\int_0^\infty \frac{\partial\theta_1}{\partial t}(t)\theta_2(t)\,dt.\end{aligned}$$

The computation of $\eta(\mathcal{A}_2)$ is similar. q.e.d.

Applying our index formula together with Lemma 9.12 yields

$$\begin{aligned}L^2\text{-Ind } D^+ &= \int_{X_1}\omega_{D_1}\cdot\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_1)\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_2)\int_{X_1}\omega_{D_1}\\ &\quad + \frac{1}{2}\eta(B_1)\eta(B_2) + \int_0^\infty \frac{\partial}{\partial t}(\theta_1(t)\theta_2(t))\,dt.\end{aligned}$$

By Lemma 9.10, the last integral equals $-\frac{1}{4}\eta(B_1)\eta(B_2)$ and, after replacing the integral by this term, our index formula coincides with (9.6), as it should.

Remark. We observe that by our assumption $\ker B_i = 0$, the continuous spectrum of $\Delta^\pm$ has a positive lower bound. Therefore, if we modify the manifold X and the Clifford bundle E on a compact set, the boundary contribution in the index formula for the corresponding Dirac operator will be the same. This follows from a relative index formula as proved in [11].

The boundary contribution in the index formula (9.6) has a natural decomposition where each term is associated with a particular stratum of the boundary at infinity. As the formula suggests, one may regard $\eta(B_i)\int_{X_j}\omega_{D_j}$, $i \neq j$, as being associated with $Y_i \times X_j$, and $\frac{1}{4}\eta(B_1)\eta(B_2)$ as being attached to the corner $Y_1 \times Y_2$. We do not know if such a natural decomposition exists in general. If it exists, it must be related to a natural splitting of the eta invariants $\eta(\mathcal{A}_1^+)$ and $\eta(\mathcal{A}_2^+)$ occurring in (0.10).

One possible approach of obtaining such a splitting is to use the decomposition of the spectrum of $\mathcal{A}_i^+$ into the point spectrum and the continuous spectrum. Let Z be an odd-dimensional Riemannian manifold with cylindrical ends. Let $A : C^\infty(Z,F) \to C^\infty(Z,F)$ be a Dirac operator on Z, and let $\mathcal{A}$ be the unique self-adjoint extension of A in L^2. By Theorem 4.1 of [16], the point spectrum of $\mathcal{A}$ consists of real eigenvalues of finite multiplicity and the number of eigenvalues, counted with multiplicity, in $(-\mu,\mu)$, $\mu > 0$, is bounded by $C(1+\mu^{2n})$. Therefore, the series

$$\eta_d(\mathcal{A},s) = \sum_{\lambda\neq 0} \frac{\text{sign } \lambda}{|\lambda|^s}, \tag{9.15}$$

where λ runs over the nonzero eigenvalues of $\mathcal{A}$, is absolutely convergent in the half-plane $\mathrm{Re}(s) > 2n$. We do not know if, for general A, this series admits a meromorphic continuation to $\mathbb{C}$. Now consider the special case where, for example, $Z = Y_1 \times X_2$ and $A = A_1$ as defined by (9.7). Let $\mathcal{A}_{1,d}$ be the restriction of $\mathcal{A}_1$ to the subspace of $L^2(Y_1 \times X_2, S_1 \otimes E_2)$ which is spanned by all L^2 eigensections of $\mathcal{A}_1$. Furthermore, let $\Delta_{2,d}^{\pm}$ be the restriction of $\Delta_2^{\pm}$ to the subspace of $L^2(X_2, E_2^{\pm})$ which corresponds to the point spectrum of $\Delta_2^{\pm}$. Then from (9.7) it follows that

$$\begin{aligned}
&\operatorname{tr}\Big(A_1 e^{-t\mathcal{A}_{1,d}^2}((y,x),(y,x))\Big)\\
&\quad = \operatorname{tr}\Big(B_1 e^{-tB_1^2}(y,y)\Big)\Big\{\operatorname{tr}\big(e^{-t\Delta_{2,d}^+}(x,x)\big) - \operatorname{tr}\big(e^{-t\Delta_{2,d}^-}(x,x)\big)\Big\},
\end{aligned}$$

for $y \in Y_1$ and $x \in X_2$. The right-hand side is absolutely integrable and we have

$$\int_{Y_1}\int_{X_2} \operatorname{tr}\Big(A_1 e^{-t\mathcal{A}_{1,d}^2}((y,x),(y,x))\Big)\,dy\,dx = \operatorname{Tr}\Big(B_1 e^{-tB_1^2}\Big)\, L^2\text{-Ind } D_2^+.$$

This implies that $\eta_d(\mathcal{A}_1,s)$ has a meromorphic continuation which is regular at $s=0$, and the value at zero, which we denote by $\eta_d(\mathcal{A}_1)$, is given by

$$\eta_d(\mathcal{A}_1) = \eta(B_1)\, L^2\text{-Ind } P^+ = \eta(B_1)\int_{X_2}\omega_{D_2} - \frac{1}{2}\eta(B_1)\eta(B_2). \tag{9.16}$$

Set

$$\eta_c(\mathcal{A}, s) = \eta(\mathcal{A}, s) - \eta_d(\mathcal{A}, s), \ \operatorname{Re}(s) > 2n.$$

This is the contribution of the continuous spectrum to the eta function of $\mathcal{A}$. Again we do not know if, in general, $\eta_c(\mathcal{A}, s)$ has an analytic continuation to $\mathbb{C}$. For $A = A_1$, however, by the above results, $\eta_c(\mathcal{A}_1, s)$ has an analytic continuation which is regular at $s = 0$. Let $\eta_c(\mathcal{A}_1) = \eta_c(\mathcal{A}_1, 0)$ be the corresponding eta invariant. From Lemma 9.12 and (9.16) it follows that

$$\eta_c(\mathcal{A}_1) = -2\int_0^\infty \frac{\partial \theta_1}{\partial t}(t)\, \theta_2(t)\, dt.$$

A similar formula holds for $\mathcal{A}_2$ and we have

$$\eta_c(\mathcal{A}_1) + \eta_c(\mathcal{A}_2) = \frac{1}{2}\eta(B_1)\eta(B_2).$$

Thus the decomposition of the spectrum of $\mathcal{A}_i$ leads to the splitting

$$\eta(\mathcal{A}_i) = \eta_d(\mathcal{A}_i) + \eta_c(\mathcal{A}_i)$$

of the eta invariants which in turn induces a natural decomposition of the boundary term in (9.6). It has to be seen if this approach can be generalized.

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