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FREE KLEINIAN GROUPS AND VOLUMES OF HYPERBOLIC 3-MANIFOLDS

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1. Introduction

The central result of this paper, Theorem 6.1, gives a constraint that must be satisfied by the generators of any free, topologically tame Kleinian group without parabolic elements. The following result is case (a) of Theorem 6.1.

Main Theorem. *Let $k \geq 2$ be an integer and let Φ be a purely loxodromic, topologically tame discrete subgroup of* $\mathrm{Isom}_+(\mathbf{H}^3)$ *which is freely generated by elements $\xi_1, \dots, \xi_k$. Let z be any point of $\mathbf{H}^3$ and set $d_i = \mathrm{dist}(z, \xi_i \cdot z)$ for $i = 1, \dots, k$. Then we have*

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \dots, k\}$ such that $d_i \geq \log(2k-1)$.

The last sentence of the Main Theorem, in the case $k = 2$, is equivalent to the main theorem of [14]. While most of the work in proving this generalization involves the extension from rank 2 to higher ranks, the main conclusion above is strictly stronger than the main theorem of [14] even in the case $k = 2$.

Like the main result of [14], Theorem 6.1 has applications to the study of large classes of hyperbolic 3-manifolds. This is because many subgroups of the fundamental groups of such manifolds can be shown to be free by topological arguments. The constraints on these free subgroups

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free by topological arguments. The constraints on these free subgroups impose quantitative geometric constraints on the shape of a hyperbolic manifold. As in [14] these can be applied to give volume estimates for hyperbolic 3-manifolds satisfying certain topological restrictions. The volume estimates obtained here, unlike those proved in [14], are strong enough to have qualitative consequences, as we shall explain below.

The following result is proved by combining the case $k = 3$ of the Main Theorem with the techniques of [15].

Corollary 9.2. *Let N be a closed orientable hyperbolic 3-manifold. Suppose that the first betti number $\beta_1(N)$ is at least 4, and that $\pi_1(N)$ has no subgroup isomorphic to the fundamental group of a surface of genus 2. Then N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$, and hence the volume of N is greater than 3.08.*

There is no reason to expect these estimates to be sharp. For instance, empirical evidence based on Weeks census [39] suggests that the conclusion of the corollary may hold under the hypothesis that $\beta_1(N)$ is at least 2, with no assumption on the surface subgroups of $\pi_1(N)$. However, the significance of our results lies elsewhere. The point is that these results imply that certain topological conditions on the manifold follow from an upper bound on the volume. More specifically, the volumes of hyperbolic 3-manifolds are known to form a well-ordered set of ordinal type ω^ω. If one lists the closed hyperbolic manifolds in ascending order of volume, the topological complexity of the manifolds tends to grow as one progresses through the list. We are interested in understanding this phenomenon in an explicit way.

The above result provides explicit information of this type. The volume of a cusped manifold is larger than that of any of its Dehn fillings, and is a limit point of the set of volumes of such fillings. There are 8 distinct volumes less than 3.08 among the volumes of orientable cusped manifolds in the Weeks census. Thus the result implies that each of the manifolds realizing the first 8ω volumes either has betti number at most 3 or has a fundamental group containing an isomorphic copy of a genus-2 surface group. (This conclusion is stated as Corollary 9.3.) It was not possible to deduce qualitative consequences of this sort in [14] because the lower bound of 0.92, obtained there for the volume of a closed hyperbolic 3-manifold of first betti number at least 3, is smaller than the least known volume of any hyperbolic 3-manifold.

Corollary 9.4 is similar to the above corollary but illustrates the applicability of our techniques to the geometric study of *infinite-volume* hyperbolic 3-manifolds. It asserts that a non-compact, topologically tame, orientable hyperbolic 3-manifold N without cusps always contains a hyperbolic ball of radius $\frac{1}{2}\log 5$ unless $\pi_1(N)$ either is a free group of rank 2 or contains an isomorphic copy of a genus-2 surface group.

Another application of Theorem 6.1 to non-compact finite-volume manifolds is the following result, which uses only the case $k = 2$ of the Main Theorem, but does not follow from the weaker form of the conclusion which appeared in [14].

Theorem 11.1. *Let $N = \mathbf{H}^3/\Gamma$ be a non-compact hyperbolic 3-manifold. If N has betti number at least 4, then N has volume at least π.*

Theorem 11.1 is deduced via Dehn surgery techniques from Proposition 10.1 and its Corollary 10.3, which are of independent interest. These results imply that if a hyperbolic 3-manifold satisfies certain topological restrictions, for example if its first betti number is at least 3, then there is a good lower bound for the radius of a tube about a short geodesic, from which one can deduce a lower bound for the volume of the manifold in terms of the length of a short geodesic. This lower bound approaches π as the length of the shortest geodesic tends to 0. Corollary 10.3 will be used in [13] as one ingredient in a proof of a new lower bound for the volume of a hyperbolic 3-manifold of betti number 3. This lower bound is greater than the smallest known volume of a hyperbolic 3-manifold, and therefore has the qualitative consequence that any smallest-volume hyperbolic 3-manifold has betti number at most 2.

The proof of the Main Theorem follows the same basic strategy as the proof of the main theorem of [14]. The Main Theorem is deduced from Theorem 6.1(d), which gives the same conclusion under somewhat different hypotheses. In 6.1(d), rather than assuming that the free Kleinian group Φ is topologically tame and has no parabolics, we assume that the manifold $\mathbf{H}^3/\Phi$ admits no non-constant positive superharmonic functions. As in [14], the estimate is proved in this case by using a Banach-Tarski-style decomposition of the area measure based on a Patterson construction. The deduction of the Main Theorem from 6.1(d) is based on Theorem 5.2, which asserts that, in the variety of representations of a free group F_k, the boundary of the set $CC(F_k)$ of convex-cocompact discrete faithful representations contains a dense G_δ consisting of representations whose images are "analytically tame" Kleinian groups without parabolics. This was proved in [14] in the case $k = 2$.

By definition, a rank-k free Kleinian group Γ without parabolics is analytically tame if the convex core of $\mathbf{H}^3/\Gamma$ can be exhausted by a sequence of geometrically well behaved compact submanifolds (a more exact definition is given in Section 5). The case $k = 2$ of Theorem 5.2

was established in [14] by combining a theorem of McMullen's [29] on the density of maximal cusps on the boundary of $\mathcal{CC}(F_k)$ with a special argument involving the canonical involution of a 2-generator Kleinian group. The arguments used in the proof of Theorem 5.2 make no use of the involution. This makes possible the generalization to arbitrary k, while also giving a new proof in the case $k = 2$. The ideas needed for the proof are developed in Sections 2 through 5, and will be sketched here.

In Section 3 we prove a general fact, Proposition 3.2, about a sequence (ρ_n) of discrete faithful representations of a finitely generated, torsion-free, non-abelian group G which converges to a maximal cusp ω. (For our purposes a maximal cusp is a discrete faithful representation ω of G into $\mathrm{PSL}_2(\mathbf{C})$ such that $\omega(G)$ is geometrically finite and every boundary component of the convex core of $\mathbf{H}^3/\omega(G)$ is a thrice-punctured sphere.) After passing to a subsequence one can assume that the Kleinian groups $\rho_n(G)$ converge geometrically to a Kleinian group $\widehat{\Gamma}$, which necessarily contains $\omega(G)$ as a subgroup. Proposition 3.2 then asserts that the convex core of $N = \mathbf{H}^3/\omega(G)$ embeds isometrically in $\mathbf{H}^3/\widehat{\Gamma}$. To prove this, we use Proposition 2.7, which combines an algebraic characterization of how conjugates of $\omega(G)$ can intersect in the geometric limit (Lemma 2.4), and a description of the intersection of the limit sets of two topologically tame subgroups of a Kleinian group (Theorem 2.5).

In Section 4 we construct a large submanifold D of the convex core of N which is geometrically well-behaved in the sense that ∂D has bounded area and the radius-2 neighborhood of ∂D has bounded volume. We use Proposition 3.2 to show that if ρ is a discrete faithful representation near enough to ω, then $\mathbf{H}^3/\rho(G)$ contains a nearly isometric copy of D. This copy is itself geometrically well-behaved in the same sense.

In Section 5 we specialize to the case $G = F_k$. We show that if a discrete faithful representation ρ is well-approximated by infinitely many maximal cusps, then its associated quotient manifold contains infinitely many geometrically well-behaved submanifolds. In fact, we show that the resulting submanifolds exhaust the convex core of the quotient manifold, and hence that the quotient manifold is analytically tame. We then apply McMullen's theorem to prove that there is a dense G_δ in the boundary of $\mathcal{CC}(F_k)$ consisting of representations which can be well approximated by maximal cusps.

In the argument given in [14], the involution of a 2-generator Kleinian group is used not only in the deformation argument, but also in the calculation based on the decomposition of the area measure in the case where $\mathbf{H}^3/\Phi$ supports no non-constant superharmonic functions. The absence of an involution in the k-generator case is compensated for by a new argument based on the elementary inequality established in Lemma 6.2. This leads to the stronger conclusion of the main theorem in the case $k = 2$.

Section 6 is devoted to the proof of Theorem 6.1.

We have mentioned that the application of Theorem 6.1 to the geometry of hyperbolic manifolds depends on a criterion for subgroups of fundamental groups of such manifolds to be free. The first such criterion in the case of a 2-generator subgroup was proved in [19] and independently in [37]. A partial generalization to k-generator subgroups, applying only when the given manifold is closed, was given in [3]. In Section 7 we give a criterion that includes the above results as special cases and is adapted to the applications in this paper.

In Section 8 we introduce a generalization of the notion of a Margulis number. We say that a positive number λ is a k-Margulis number for a Kleinian group Γ if the following condition holds: if $\xi_1, \ldots, \xi_k$ are elements of Γ, and there exists a point $z \in \mathbf{H}^3$ which is displaced less than λ by each ξ_i, then the group $\langle \xi_1, \ldots, \xi_k \rangle$ can be generated by $k-1$ abelian subgroups. In the case $k = 2$ the group $\langle \xi_1, \xi_2 \rangle$ would have to be abelian; thus the notion of a 2-Margulis number coincides with that of a Margulis number as defined in [14] and [33]. This notion and the related notion of a k-strong Margulis number prove useful for organizing the applications of the results of the earlier sections to the study of hyperbolic manifolds. The applications are presented in Sections 9, 10 and 11.

The pictures of limit sets of maximal cusps which appear in Section 3 were created by Yair Minsky, and were based on some earlier pictures by Chris Bishop. We are grateful to them for allowing us to use them here.

We close the introduction by mentioning a few notational conventions which are used throughout. We use $H \leq G$ to denote that H is a subgroup of G, and $H < G$ to denote that H is a proper subgroup of G. The translate of a set X by a group element γ is denoted $\gamma \cdot X$. Finally, we use $\operatorname{dist}(z, w)$ to denote the hyperbolic distance between points z and w in $\mathbf{H}^3$.

2. On the sphere at infinity

In this section we introduce the notion of the geometric limit of a sequence of Kleinian groups. We will consider a convergent sequence of discrete faithful representations into $\mathrm{PSL}_2(\mathbf{C})$ whose images converge geometrically. In general, the geometric limit of the images contains the image of the limit as a subgroup. The results in this section characterize the intersection of two conjugates of this subgroup and the intersection of their limit sets.

The group $\mathrm{PSL}_2(\mathbf{C})$ will be considered to act either by isometries on $\mathbf{H}^3$ or, via extension to the sphere at infinity, by Möbius transformations on the Riemann sphere $\overline{\mathbf{C}}$. The action of a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{C})$ partitions $\overline{\mathbf{C}}$ into two pieces, the *domain of discontinuity* $\Omega(\Gamma)$ and the *limit set* $\Lambda(\Gamma)$. The domain of discontinuity is the largest open subset of $\overline{\mathbf{C}}$ on which Γ acts properly discontinuously. If $\Lambda(\Gamma)$ contains two or fewer points, we say Γ is *elementary.* If Γ has an invariant circle in $\overline{\mathbf{C}}$ and preserves an orientation of the circle, then we say that Γ is *Fuchsian.*

By a *Kleinian group* we will mean a discrete non-elementary subgroup Γ of $\mathrm{PSL}_2(\mathbf{C})$. We will say that a Kleinian group Γ is *purely parabolic* if every non-trivial element is parabolic, or *purely loxodromic* if every non-trivial element is loxodromic.

Given a finitely generated group G, let $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ denote the variety of representations of G into $\mathrm{PSL}_2(\mathbf{C})$. A choice of k elements which generate G determines a bijection from $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ to an algebraic subset of $(\mathrm{PSL}_2(\mathbf{C}))^k$. We give $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ the topology that makes this bijection a homeomorphism onto the algebraic set with its complex topology. This topology on $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ is independent of the choice of the generators of G.

For the rest of this section, and throughout Section 3, we will assume that G is a finitely generated, non-abelian, torsion-free group.

Let $\mathcal{D}(G)$ denote the subspace of $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ consisting of those representations which are injective and have discrete image. It is a fundamental result of Jørgensen's [20] that $\mathcal{D}(G)$ is a closed subset of $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$. The proof of Jørgensen's result is based on an inequality for discrete subgroups of $\mathrm{PSL}_2(\mathbf{C})$. A second consequence of this inequality is the following lemma.

Lemma 2.1. (Lemma 3.6 in [21]) *Let (ρ_n) be a convergent sequence in $\mathcal{D}(G)$. If (g_n) is a sequence of elements of G such that $(\rho_n(g_n))$ converges to the identity, then there exists n_0 such that $g_n = 1$ for $n \geq n_0$.* q.e.d.

A sequence of discrete subgroups (Γ_n) is said to converge *geometrically* to a Kleinian group $\widehat{\Gamma}$ if and only if:

(1) for every $\gamma \in \widehat{\Gamma}$, there exist elements $\gamma_n \in \Gamma_n$ such that the sequence (γ_n) converges to γ, and

(2) whenever (Γ_{n_j}) is a subsequence of (Γ_n) and $\gamma_{n_j} \in \Gamma_{n_j}$ are elements such that the sequence (γ_{n_j}) converges to a Möbius transformation γ, we have $\gamma \in \widehat{\Gamma}$.

We call $\widehat{\Gamma}$ the *geometric limit* of (Γ_n).

The following basic fact is proved in Jørgensen-Marden [21].

Proposition 2.2. (Proposition 3.8 in [21]) *Let (ρ_n) be a sequence of elements of $\mathcal{D}(G)$ converging to ρ. Then $(\rho_n(G))$ has a geometrically convergent subsequence. If $\widehat{\Gamma}$ is the geometric limit of any such subsequence, then $\rho(G) \leq \widehat{\Gamma}$.*

The following fact will also be used.

Lemma 2.3. *Let (ρ_n) be a convergent sequence in $\mathcal{D}(G)$ such that $(\rho_n(G))$ converges geometrically to $\widehat{\Gamma}$. Then $\widehat{\Gamma}$ is torsion-free.*

Proof. Suppose that $\gamma \in \widehat{\Gamma}$ has finite order d. Let (g_n) be a sequence of elements of G such that $\rho_n(g_n)$ converges to γ. Then $\rho_n(g_n^d)$ converges to the identity. Hence by Lemma 2.1 we have $g_n^d = 1$ for large n. Since G is torsion-free, we have $g_n = 1$ for large n and therefore $\gamma = 1$. q.e.d.

The following lemma characterizes the intersection of two conjugates of $\rho(G)$ in the geometric limit $\widehat{\Gamma}$.

Lemma 2.4. *Let (ρ_n) be a sequence of elements of $\mathcal{D}(G)$ converging to ρ. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Then $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$ is a (possibly trivial) purely parabolic group for each $\gamma \in \widehat{\Gamma} - \rho(G)$.*

Proof. Suppose that $\rho(a)$ is a nontrivial element of $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$ for some $\gamma \in \widehat{\Gamma} - \rho(G)$. We will show that $\rho(a)$ is parabolic.

Assume to the contrary that $\rho(a)$ is loxodromic. We may write $\rho(a) = \gamma\rho(b)\gamma^{-1}$ for some $b \in G$. Choose $g_n \in G$ such that $(\rho_n(g_n))$ converges to γ. Then $(\rho_n(g_n b g_n^{-1}))$ converges to $\rho(a)$, and hence $(\rho_n(a^{-1} g_n b g_n^{-1}))$ converges to 1. It follows from Lemma 2.1 that there exists an integer n_0 such that $a = g_n b g_n^{-1}$ for all $n \geq n_0$. Hence $g_{n_0}^{-1} g_n$ is contained in the centralizer of b for all $n \geq n_0$. Applying ρ_n and passing to the limit, we have that $\rho(g_{n_0}^{-1})\gamma$ commutes with $\rho(b)$.

Since the Kleinian group $\widehat{\Gamma}$ is torsion-free by Lemma 2.3 and the element $\rho(b) \in \widehat{\Gamma}$ is loxodromic, the centralizer of $\rho(b)$ in $\widehat{\Gamma}$ is cyclic. Thus there are integers j and k such that $(\rho(g_{n_0}^{-1})\gamma)^j = \rho(b)^k$. A second application of Lemma 2.1 shows that for some $n_1 \geq n_0$ we have $(g_{n_0}^{-1} g_n)^j = b^k$ for all $n \geq n_1$. But since G is isomorphic to a torsion-free Kleinian group, each element of G has at most one j^{th} root. Hence b^k has a unique j^{th} root c, and $g_{n_0}^{-1} g_n = c$ for $n \geq M$. Thus $g_n = g_{n_0} c$ for large n, so the sequence (g_n) is eventually constant. Since $(\rho_n(g_n))$ converges to γ, this implies that γ is an element of $\rho(G)$, which contradicts our hypothesis that $\gamma \in \widehat{\Gamma} - \rho(G)$. q.e.d.

Next we consider the intersection of the limit set of $\rho(G)$ with its image under an element of the geometric limit $\widehat{\Gamma}$.

The following definition will be useful. Let Γ_1 and Γ_2 be subgroups of the Kleinian group Γ. We will say that a point $p \in \Lambda(\Gamma_1) \cap \Lambda(\Gamma_2)$ is in $\mathrm{P}(\Gamma_1, \Gamma_2)$ if and only if $\mathrm{Stab}_{\Gamma_1}(p) \cong \mathbf{Z}$, $\mathrm{Stab}_{\Gamma_2}(p) \cong \mathbf{Z}$, and $\langle \mathrm{Stab}_{\Gamma_1}(p), \mathrm{Stab}_{\Gamma_2}(p) \rangle \cong \mathbf{Z} \oplus \mathbf{Z}$. In particular, p must be a parabolic fixed point of both Γ_1 and Γ_2.

We will make use of the following result, due to Soma [34] and Anderson [1], which provides the link between the intersection of the limit sets of a pair of subgroups and the limit set of the intersection of the subgroups.

Recall that a Kleinian group Γ is *topologically tame* if $\mathbf{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold.

Theorem 2.5. *Let Γ_1 and Γ_2 be nonelementary, topologically tame subgroups of the Kleinian group Γ. Then*

$$\Lambda(\Gamma_1) \cap \Lambda(\Gamma_2) = \Lambda(\Gamma_1 \cap \Gamma_2) \cup \mathrm{P}(\Gamma_1, \Gamma_2).$$

The next lemma shows that the term $\mathrm{P}(\Gamma_1, \Gamma_2)$ may be ignored in the case where Γ_1 and Γ_2 are distinct conjugates of $\rho(G)$ by elements of the geometric limit $\widehat{\Gamma}$.

Lemma 2.6. *Let (ρ_n) be a sequence in $\mathcal{D}(G)$ converging to ρ. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Then for each $\gamma \in \widehat{\Gamma} - \rho(G)$, the set $\mathrm{P}(\rho(G), \gamma\rho(G)\gamma^{-1})$ is empty.*

Proof. The argument runs along much the same line as the proof of Proposition 2.4. Suppose that $p \in \mathrm{P}(\rho(G), \gamma\rho(G)\gamma^{-1})$, that $\mathrm{Stab}_{\rho(G)}(p) \cong \mathbf{Z}$ is generated by $\rho(a)$ and that $\mathrm{Stab}_{\gamma\rho(G)\gamma^{-1}}(p) \cong \mathbf{Z}$ is generated by $\gamma\rho(b)\gamma^{-1}$. Note that each of the elements a and b generates its own centralizer.

Choose $g_n \in G$ so that $(\rho_n(g_n))$ converges to γ. Since $\rho(a)$ commutes with $\gamma\rho(b)\gamma^{-1}$, we conclude as in the proof of Lemma 2.4 that there exists an integer n_0 such that a commutes with $g_n b g_n^{-1}$ for all $n \geq n_0$. Since a generates its centralizer in G, each of the elements $g_n b g_n^{-1}$ for $n \geq n_0$ must be a power of a. But these are all conjugate elements, while distinct powers of a are not conjugate. Therefore we must have that $g_n b g_n^{-1} = g_{n_0} b g_{n_0}^{-1}$ for all $n \geq n_0$.

Thus $g_{n_0}^{-1} g_n$ commutes with b for all $n \geq n_0$. Since the centralizer of b is cyclic, we may now argue exactly as in the proof of 2.4 that the sequence (g_n) must be constant for $n \geq n_1 \geq n_0$, a contradiction to our hypothesis that $\gamma \in \widehat{\Gamma} - \rho(G)$. q.e.d.

As an immediate consequence of Lemma 2.4, Theorem 2.5, and Lemma 2.6, we have the following proposition.

Proposition 2.7. *Let (ρ_n) be a sequence in $\mathcal{D}(G)$ converging to ρ. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$ and that $\rho(G)$ is topologically tame. Then for any $\gamma \in \widehat{\Gamma} - \rho(G)$ the group $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$ is purely parabolic and*

$$\Lambda(\rho(G)) \cap \gamma \cdot \Lambda(\rho(G)) = \Lambda(\rho(G) \cap \gamma\rho(G)\gamma^{-1}).$$

Hence if $\Lambda(\rho(G)) \cap \gamma \cdot \Lambda(\rho(G))$ is non-empty then it must contain only the fixed point of $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$.

3. In the convex core

We continue to assume that G is a finitely generated nonabelian torsion-free group. We consider a sequence (ρ_n) in $\mathcal{D}(G)$ converging to a representation ω which is a maximal cusp (defined below). If the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$, then the hyperbolic manifold $N = \mathbf{H}^3/\omega(G)$ is a covering space of $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$. The main result of this section states that in this situation the restriction of the covering projection gives an embedding of the convex core of N into $\widehat{N}$.

It may be helpful to recall the most basic situation in which the convex core of a manifold does not embed in a manifold which it covers. Let $\widehat{N}$ be a hyperbolic 3-manifold, and let $f : S \to \widehat{N}$ be a totally geodesic isometric immersion of a finite area surface S. Let N be the cover of $\widehat{N}$ associated to $\pi_1(S)$, so that f lifts to a totally geodesic embedding $\widetilde{f} : S \to N$. Since $\widetilde{f}(S)$ is totally geodesic, $\Lambda(\pi_1(S))$ is a circle and $\widetilde{f}(S)$ is the convex core of N. The convex core of N embeds in $\widehat{N}$ if and only if f is an embedding. Notice that f is not an embedding if and only if there exists an element γ of $\pi_1(\widehat{N})$ such that $\gamma(\Lambda(\pi_1(S)))$ intersects $\Lambda(\pi_1(S))$ transversely, hence in at least two points.

We will be dealing with the case where the algebraic limit is a maximal cusp, and hence each boundary component of the convex core of our algebraic limit is a totally geodesic thrice-punctured sphere (Lemma 3.1). We will see, as in the example above, that if the convex core of the algebraic limit does not embed, then there must be an element γ of $\widehat{\Gamma} - \omega(G)$ such that $\gamma(\Lambda(\omega(G)))$ intersects $\Lambda(\omega(G))$ in at least two points. An application of Proposition 2.7 will complete the proof.

Given a Kleinian group Γ, define its *convex hull* $\mathrm{CH}(\Gamma)$ in $\mathbf{H}^3$ to be the smallest non-empty convex set in $\mathbf{H}^3$ which is invariant under the action of Γ. Thus $\mathrm{CH}(\Gamma)$ is the intersection of all half-spaces in $\mathbf{H}^3$, whose closures in the compactification $\mathbf{H}^3 \cup \overline{\mathbf{C}}$ contain $\Lambda(\Gamma)$. (Recall that a Kleinian group is, by definition, non-elementary so that its limit set has more than two points.)

The *convex core* of $N = \mathbf{H}^3/\Gamma$ is $\mathrm{C}(N) = \mathrm{CH}(\Gamma)/\Gamma$. We say that N, or equivalently Γ, is *geometrically finite* if Γ is finitely generated and $\mathrm{C}(N)$ has finite volume.

The *injectivity radius* $\mathrm{inj}_N(x)$ of N at the point x is half the length of the shortest homotopically non-trivial closed loop passing through x. Note that injectivity radius increases under lifting to a covering space. That is, if N is a cover of $\widehat{N}$ with covering map $\pi : N \to \widehat{N}$, and x is any point of N, then $\mathrm{inj}_N(x) \geq \mathrm{inj}_{\widehat{N}}(\pi(x))$.

Given a hyperbolic 3-manifold N, define the ϵ-*thick part* of N as

$$N_{\mathrm{thick}(\epsilon)} = \{x \in N \,|\, \mathrm{inj}_N(x) \geq \frac{\epsilon}{2}\},$$

and the ϵ-*thin part* of N as

$$N_{\mathrm{thin}(\epsilon)} = \{x \in N \,|\, \mathrm{inj}_N(x) \leq \frac{\epsilon}{2}\}.$$

We recall that $\mathrm{C}(N) \cap N_{\mathrm{thick}(\epsilon)}$ is compact for all $\epsilon > 0$ if and only if N is geometrically finite (see Bowditch [7]). Hence, for a geometrically finite hyperbolic 3-manifold N, the sets $\mathrm{C}(N) \cap N_{\mathrm{thick}(1/m)}$ for $m \geq 1$ form an exhaustion of $\mathrm{C}(N)$ by compact subsets.

We say that a representation ω in $\mathcal{D}(G)$ is a *maximal cusp* if $N = \mathbf{H}^3/\omega(G)$ is geometrically finite and every component of the boundary $\partial\,\mathrm{C}(N)$ of its convex core is a thrice-punctured sphere. We will further require that $\omega(G)$ not be a Fuchsian group. (This rules out only the case that $\omega(G)$ is in the (unique) conjugacy class of finite co-area Fuchsian groups uniformizing the thrice-punctured sphere.) Maximal cusps are discussed at length by Keen, Maskit and Series in [22], where the image groups are termed maximally parabolic.

A proof of the following lemma appears in [22]; since we will be using the lemma heavily, we include a sketch of the proof here.

Lemma 3.1. *Let $\omega \in \mathcal{D}(G)$ be a maximal cusp, and let $N = \mathbf{H}^3/\omega(G)$. Then each component of $\partial\,\mathrm{C}(N)$ is totally geodesic.*

Proof. Since the universal cover of $\mathrm{C}(N)$ is $\mathrm{CH}(\omega(G))$, it suffices to show that each component of $\partial\,\mathrm{CH}(\omega(G))$ is a totally geodesic hyperplane or, equivalently, that each component of $\Omega(\omega(G))$ is a disk bounded by a circle on the sphere at infinity.

Recall, for example from Epstein-Marden [17], that $\partial\,\mathrm{C}(N)$ is homeomorphic to $\Omega(\omega(G))/\omega(G)$. Moreover, by the Ahlfor's Finiteness theorem $\Omega(\omega(G))/\omega(G)$ has finite area. Thus each component S of $\Omega(\omega(G))/\omega(G)$ must be a thrice-punctured sphere. Write $S = \Delta/\Gamma_\Delta$,

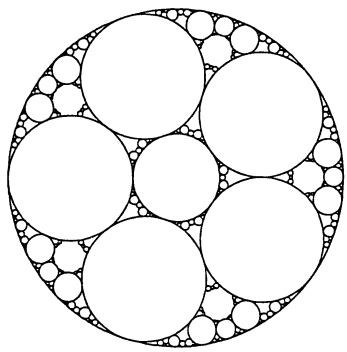

FIGURE 1. The domain of discontinuity of a maximal cusp is a union of round disks

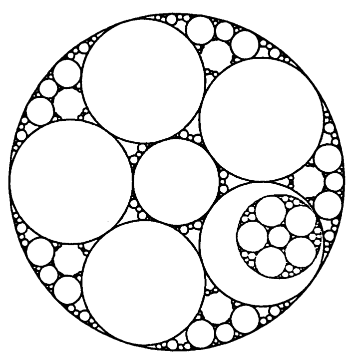

FIGURE 2. The limit set of a maximal cusp may intersect its translate in at most one point.

where Δ is a component of $\Omega(\omega(G))$, and Γ_Δ is the subgroup of $\omega(G)$ stabilizing Δ.

Since Δ/Γ_Δ is a thrice-punctured sphere, the group Γ_Δ must be a Fuchsian group, and Δ must be a disk bounded by a circle on the sphere at infinity. (For a proof, see Chapter IX.C of Maskit's book [26]; for a picture see Figure 1.) q.e.d.

We are now ready to prove the main result of this section. A map between locally compact spaces will be called an *embedding* if it is proper and one-to-one.

Proposition 3.2. *Let* (ρ_n) *be a sequence of elements of* $\mathcal{D}(G)$ *converging to a maximal cusp* ω. *Suppose that the groups* $\rho_n(G)$ *converge geometrically to* $\widehat{\Gamma}$. *Let* $N = \mathbf{H}^3/\omega(G)$, $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$, *and let* $\pi : N \to \widehat{N}$ *be the covering map. Then* $\pi|_{C(N)}$ *is an embedding.*

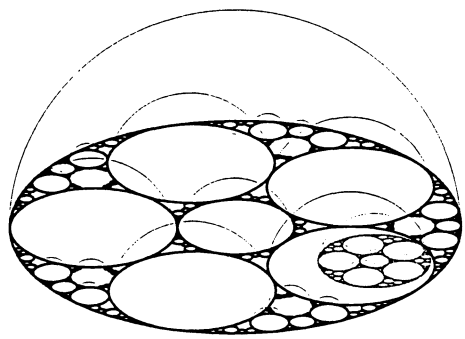

FIGURE 3. The convex hull of the limit set of a maximal cusp cannot intersect its translate

Proof. We first note that if (x_i) is a sequence in $\mathrm{C}(N)$ leaving every compact set, then $\lim_{i\mapsto\infty} \mathrm{inj}_N(x_i) = 0$. Hence, $\lim_{i\mapsto\infty} \mathrm{inj}_{\widehat{N}}(\pi(x_i)) = 0$, which implies that $(\pi(x_i))$ leaves every compact subset of $\widehat{N}$. Thus, $\pi|_{\mathrm{C}(N)}$ is a proper mapping.

It remains to show that π is injective. The universal cover of $\mathrm{C}(N)$ is $\mathrm{CH}(\omega(G))$. So it suffices to show that $\mathrm{CH}(\omega(G))\cap\gamma\cdot\mathrm{CH}(\omega(G))$ is empty for each $\gamma \in \widehat{\Gamma} - \omega(G)$. For notational convenience, set $X = \mathrm{CH}(\omega(G))$.

Since ω is assumed to be a maximal cusp, each component of $\partial\,\mathrm{C}(N)$ is totally geodesic. Hence each component of ∂X is a plane H in $\mathbf{H}^3$ whose boundary at infinity is a circle C which lies in $\Lambda(\omega(G))$.

If $X \cap \gamma \cdot X$ is not empty, there are two possibilities: Either there is a point in $\partial X \cap \gamma \cdot \partial X$, or a component of ∂X lies entirely within $\gamma \cdot X$ (or vice versa).

We begin with the case that there is a point x in $\partial X \cap \gamma \cdot \partial X$. There then exist a plane H in ∂X and a plane H' in $\gamma \cdot \partial X$ with $x \in H \cap H'$. If two planes in $\mathbf{H}^3$ meet, either their intersection is a line or they are equal. If we let C denote the boundary at infinity of H and C' the boundary at infinity of H', then either $C\cap C'$ contains exactly two points or $C = C'$. However, $C\cap C'$ is contained in $\Lambda(\omega(G))\cap\gamma\cdot\Lambda(\omega(G))$, so that $\Lambda(\omega(G)) \cap \gamma \cdot \Lambda(\omega(G))$ contains at least two points, which contradicts Proposition 2.7.

The second possibility is that a component H of ∂X lies entirely within $\gamma \cdot X$. In this case, the boundary at infinity C of H lies in the boundary at infinity of $\gamma \cdot X$, which is exactly $\gamma \cdot \Lambda(\omega(G))$. However, C also lies in $\Lambda(\omega(G))$; hence $\Lambda(\omega(G)) \cap \gamma \cdot (\Lambda(\omega(G)))$ contains C. This also contradicts Proposition 2.7. See Figures 2 and 3 for an illustration of how $\Lambda(\omega(G))$ can meet its translate by γ. q.e.d.

Remark 3.3. The conclusion of Proposition 3.2 holds, by the same argument, whenever N is topologically tame and $\partial\,\mathrm{C}(N)$ is totally geodesic.

In general, the convex core of the algebraic limit need not embed in the geometric limit. However, one can define the visual hull of a Kleinian group Γ to be the set of all points in $\mathbf{H}^3$ such that the visual area of every component of $\Omega(\Gamma)$ is at most $\frac{1}{2}$. The visual core is then defined to be the quotient of the visual hull. Notice that if Γ is a maximal cusp, then its visual core and its convex core coincide.

Anderson and Canary [2] use techniques similar to those developed in this section to prove that the visual core embeds whenever the algebraic limit has connected limit set and no accidental parabolics. They also show that, under the same assumptions, there is a compact core for the algebraic limit which embeds in the geometric limit. It has recently been discovered that the visual core of the algebraic limit need not embed in the algebraic limit even if the algebraic limit has connected limit set.

4. Near a maximal cusp

In this section, we will prove that if a representation in $\mathcal{D}(G)$ is near enough to a maximal cusp, then its associated hyperbolic 3-manifold contains a nearly isometric copy of an ϵ-truncated convex core of the maximal cusp. We first define this ϵ-truncated object and describe some of its useful attributes.

We recall that it follows from the Margulis lemma that there exists a constant λ_0, such that if $\epsilon < \lambda_0$ and N is a hyperbolic 3-manifold, then every component P of $N_{\mathrm{thin}(\epsilon)}$ is either a solid torus neighborhood of a closed geodesic, or the quotient of a horoball H by a group Θ of parabolic elements fixing H (see for example [4]). In the second case, Θ is isomorphic to either $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$. Moreover, H is precisely invariant under $\Theta < \Gamma$, by which we mean that if $\gamma \in \Gamma$ and $\gamma \cdot H \cap H \neq \emptyset$, then $\gamma \in \Theta$ and $\gamma \cdot H = H$. If $\Theta \cong \mathbf{Z}$, we call P a *rank-one cusp*, and if $\Theta \cong \mathbf{Z} \oplus \mathbf{Z}$, we call P a *rank-two cusp.* Recall also that there exists $L(\epsilon) > 0$, such that any two components of $N_{\mathrm{thin}(\epsilon)}$ are separated by a distance of at least $L(\epsilon)$.

The next lemma gives the structure of $N_{\mathrm{thin}(\epsilon)}$ for sufficiently small ϵ. For a proof see Section 6 of Morgan [31].

Lemma 4.1. *Let N be a geometrically finite hyperbolic 3-manifold. There exists $\delta(N) < \lambda_0$, such that if $\epsilon \leq \delta(N)$ and P is a component of $N_{\mathrm{thin}(\epsilon)}$, then the following hold:*

(i) *P is non-compact,*

(ii) *∂P meets* $\mathrm{C}(N)$ *orthogonally along each component of their intersection.*

(iii) $E = \partial P \cap \mathrm{C}(N)$ *is a Euclidean surface with geodesic boundary, and* $\operatorname{diam} E \leq 1$.

(iv) *If P is a rank-one cusp, then E is an annulus, and if P is a rank-two cusp then E is a torus; and*

(v) $\mathrm{C}(N) \cap P$ *is homeomorphic to* $E \times [0, \infty)$.

In particular, for any $\epsilon \leq \delta(N)$ *the set* $N_{\text{thick}(\epsilon)} \cap \mathrm{C}(N)$ *is a compact 3-manifold with piecewise smooth boundary.*

Given a geometrically finite hyperbolic 3-manifold N we may define its ϵ-truncated convex core $D_\epsilon(N)$ to be the intersection of its convex core $\mathrm{C}(N)$ with the ϵ-thick part $N_{thick(\epsilon)}$ of N. The above lemma completely characterizes $\mathrm{C}(N) - D_\epsilon(N)$ when $\epsilon < \delta(N)$.

Recall that a compact submanifold of N is said to be a *compact core* for N if the inclusion map is a homotopy equivalence.

Lemma 4.2. *Suppose that N is a geometrically finite hyperbolic 3-manifold, and* $\delta(N) > \epsilon > 0$. *Then $D_\epsilon(N)$ is a compact core for N.*

Proof. First recall that the inclusion of $\mathrm{C}(N)$ into N is a homotopy equivalence. Moreover, each component of $\mathrm{C}(N) - D_\epsilon(N)$ is homeomorphic to $E \times (0, \infty)$ for some Euclidean surface E. Thus, the inclusion of $D_\epsilon(N)$ into $\mathrm{C}(N)$ is a homotopy equivalence, and the result follows. q.e.d.

For any subset X of a hyperbolic manifold N we will denote by $\mathcal{N}_r(X)$ the closed neighborhood of radius r of X. In the case $N = \mathbf{H}^3/\omega(G)$, where ω is a maximal cusp, Proposition 4.4 will provide bounds for both the area of $\partial D_\epsilon(N)$ and the volume of $\mathcal{N}_2(\partial D_\epsilon(N))$. These bounds will depend only on the topological type of N and not on ϵ. We first recall the following special case of Lemma 8.2 in [8] (see also Proposition 8.12.1 of Thurston [36]).

Lemma 4.3. *There is a constant $\kappa > 0$, such that for any maximal cusp $\omega \in \mathcal{D}(G)$ and any collection S of components of the boundary of the convex core of $N = \mathbf{H}^3/\omega(G)$, the neighborhood $\mathcal{N}_3(S)$ has volume less than* $\kappa \operatorname{area} S$. q.e.d.

If $N = \mathbf{H}^3/\omega(G)$, where $\omega(G)$ is a maximal cusp, we will denote by $\sigma(N)$ the number of components of $\partial\,\mathrm{C}(N)$, and by $\tau(N)$ the number of rank-two cusps of N. We set

$$\alpha(N) = \frac{7}{2}\pi\sigma(N) + 2\pi\tau(N)$$

and

$$\beta(N) = 2\pi\kappa\sigma(N) + \pi e^4\tau(N),$$

where κ is the constant given by Lemma 4.3.

Lemma 4.4. *Let ω be a maximal cusp and let $N = \mathbf{H}^3/\omega(G)$. If $\delta(N) > \epsilon > 0$, then*

(i) $\operatorname{area} \partial D_\epsilon(N) \leq \alpha(N)$, *and*

(ii) $\operatorname{vol}(\mathcal{N}_2(\partial D_\epsilon(N))) \leq \beta(N)$.

Proof. Notice that $\partial D_\epsilon(N) = S \cup E$ where $S = \partial\,\mathrm{C}(N) \cap N_{\mathrm{thick}(\epsilon)}$ and $E = \mathrm{C}(N) \cap \partial N_{\mathrm{thick}(\epsilon)}$. Since $S \subset \partial\,\mathrm{C}(N)$ and each component of $\partial\,\mathrm{C}(N)$ is a thrice-punctured sphere, we have $\operatorname{area} S \leq \operatorname{area} \partial\,\mathrm{C}(N) = 2\pi\sigma(N)$. By Lemma 4.1, each component of E is a Euclidean manifold of diameter at most 1, and therefore has area at most π. Since each component of $\partial\,\mathrm{C}(N)$ contains three components of ∂E, there are $\frac{3}{2}\sigma(N)$ annular components of E. Moreover, there are $\tau(N)$ toroidal components of E. Hence the first assertion follows.

Let $\widehat{S}$ denote the union of S with the annular components of E. Since each annular component of E has diameter less than 1,

$$\mathcal{N}_2(\widehat{S}) \subset \mathcal{N}_3(S) \subset \mathcal{N}_3(\partial\,\mathrm{C}(N)).$$

Thus Lemma 4.3 guarantees that

$$\operatorname{vol} \mathcal{N}_2(\widehat{S}) \leq \kappa \operatorname{area}(\partial\,\mathrm{C}(N)) = 2\pi\kappa\sigma(N).$$

Now, if T is a toroidal component of E, then $\mathcal{N}_2(T)$ is (the quotient of) a region isometric to $T \times (-2, 2)$ with the metric $ds^2 = e^{-2t} ds_T^2 + dt^2$, which has volume less than $2\pi e^4$. Hence the second assertion is proved. q.e.d.

Our next result, Proposition 4.5, asserts (among other things) that the hyperbolic manifold associated to a representation which is near enough to a maximal cusp contains a biLipschitz copy of the ϵ-truncated convex core of the manifold associated to the maximal cusp.

We first outline the argument. Suppose that (ρ_n) is a sequence of representations in $\mathcal{D}(G)$ which converges to a maximal cusp ω, and that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Let $N = \mathbf{H}^3/\omega(G)$ and $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$. If $\pi : N \to \widehat{N}$ is the covering map, Proposition 3.2 implies that $\pi|_{D_\epsilon(N)}$ is an embedding. Since $(\rho_n(G))$ converges geometrically to $\widehat{\Gamma}$, larger and larger chunks of $\widehat{N}$ are nearly isometric to larger and larger chunks of $N_n = \mathbf{H}^3/\rho_n(G)$. In particular, for all large enough n there exists a 2-biLipschitz embedding $f_n : V_n \to N_n$ where $\pi(D_\epsilon(N)) \subset V_n \subset \widehat{N}$. The desired biLipschitz copy of $D_\epsilon(N)$ is $f_n(\pi(D_\epsilon(N)))$.

In order to carry out the program outlined above, it will be necessary to make consistent choices of base points in different hyperbolic 3-manifolds. We will use the following convention. If z is a point in $\mathbf{H}^3$ and Γ is a Kleinian group, we will let z_Γ denote the image of z in the hyperbolic manifold $\mathbf{H}^3/\Gamma$. In the case that $\Gamma = \rho(G)$ for some representation $\rho \in \mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ we will write $z_\rho = z_{\rho(G)}$.

If a codimension-0 submanifold X of a hyperbolic manifold N is connected and has piecewise smooth boundary, then it has two natural distance functions. In the *extrinsic metric* the distance between two points of X is equal to their distance in N, while in the *intrinsic metric* the distance is the infimum of the lengths of rectifiable paths in X joining the two points. Observe that if X and Y are submanifolds of hyperbolic manifolds, and $f\colon X \to Y$ is a K-biLipschitz map with respect to the extrinsic metrics, then f is also K-biLipschitz with respect to the intrinsic metrics.

Proposition 4.5. *Suppose that $\omega \in \mathcal{D}(G)$ is a maximal cusp, and set $N = \mathbf{H}^3/\omega(G)$. Let $\epsilon > 0$ be given with the property that $\epsilon < \delta(N)$, and let z be a point of $\mathbf{H}^3$ such that z_ω lies in the interior of $D_\epsilon(N)$. Then there is a neighborhood $U(\epsilon, z, \omega)$ of ω in $\mathcal{D}(G)$ such that for each $\rho \in U(\epsilon, z, \omega)$, there exists a map $\phi : D_\epsilon(N) \to N' = \mathbf{H}^3/\rho(G)$, with the following properties:*

(1) *ϕ maps $D_\epsilon(N)$ homeomorphically onto a manifold with piecewise smooth boundary, and is 2-biLipschitz with respect to the intrinsic metrics on $D_\epsilon(N)$ and $\phi(D_\epsilon(N))$,*
(2) *$\phi(z_\omega) = z_\rho$,*
(3) *$\mathrm{vol}\,\mathcal{N}_1(\partial(\phi(D_\epsilon(N)))) \le 8\beta(N)$, and*
(4) *$\phi(N_{\mathrm{thin}(\delta)} \cap D_\epsilon(N)) \subset N'_{\mathrm{thin}(2\delta)}$ for any $\delta < \frac{\lambda_0}{2}$, where λ_0 is the Margulis constant.*

Proof of 4.5. Let (ρ_n) be a sequence in $\mathcal{D}(G)$ that converges to ω, and set $N_n = \mathbf{H}^3/\rho_n(G)$. It suffices to prove that (ρ_n) has a subsequence (ρ_{n_i}) such that there exist maps $\phi_i : D_\epsilon(N) \to N_{n_i}$ with properties (1)—(4).

Given any sequence (ρ_n) in $\mathcal{D}(G)$ converging to ω, Proposition 2.2 guarantees that there exists a subsequence $(\rho_{n_i}(G))$ of $(\rho_n(G))$ which converges geometrically to a group $\widehat{\Gamma}$ such that $\omega(G) \subset \widehat{\Gamma}$. Let $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$, and let $\pi : N \to \widehat{N}$ be the associated covering map. From Proposition 3.2 it follows that $\pi|_{C(N)}$ is an embedding, so that $\pi|_{D_\epsilon(N)}$ is an embedding. Let $D = \pi(D_\epsilon(N))$. Since π is a local isometry, we use Lemma 4.4 to find that

$$\mathrm{vol}\,\mathcal{N}_2(\partial D) \le \mathrm{vol}\,\mathcal{N}_2(\partial D_\epsilon(N)) \le \beta(N).$$

Since $(\rho_{n_i}(G))$ converges geometrically to $\widehat{\Gamma}$, from Corollary 3.2.11 in [10] or Theorem E.1.13 in [4] it follows that there exist smooth submanifolds $V_i \subset \widehat{N}$, numbers r_i and α_i, and maps $f_i : V_i \to N_{n_i}$ such that:

(i) V_i contains $B(r_i, z_{\widehat{\Gamma}})$, the closed radius-$r_i$ neighborhood of $z_{\widehat{\Gamma}}$,
(ii) $f_i(z_{\widehat{\Gamma}}) = z_{\rho_{n_i}}$,
(iii) r_i converges to ∞, and α_i converges to 1,
(iv) f_i maps V_i diffeomorphically onto $f(V_i)$ and is α_i-biLipschitz with respect to the extrinsic metrics on V_i and $f(V_i)$.

Choose d so that $D \subset B(d, z_{\widehat{\Gamma}})$. Set $\mu = \max\{1, \lambda_0/2\}$. We may assume that the subsequence (ρ_{n_i}) has been chosen so that $\alpha_i < 2$ and $r_i > d + 2\mu$ for all i. This condition on r_i implies that $\mathcal{N}_{2\mu}(D)$ is contained in the interior of V_i.

We claim that $\mathcal{N}_\mu(f_i(D)) \subset f_i(V_i)$. To prove the claim, we consider the frontier X of $\mathcal{N}_{2\mu}(D)$ in $\widehat{N}$ and the frontier Y_i of $f_i(\mathcal{N}_{2\mu}(D))$ in N_{n_i}. Since f_i is a homeomorphism onto its image, from the invariance of domain it follows that $f_i(X) = Y_i$. Since f_i^{-1} is extrinsically α_i-Lipschitz with $\alpha_i < 2$, and every point of X has distance 2μ from D, every point of $f_i(X) = Y_i$ must be a distance greater than μ from $f_i(D)$. Thus Y_i is disjoint from $\mathcal{N}_\mu(f_i(D))$. Since $f_i(\mathcal{N}_{2\mu}(D))$ contains $f_i(D)$ and is disjoint from the frontier Y_i of $\mathcal{N}_\mu(f_i(D))$, we have $f_i(\mathcal{N}_{2\mu}(D)) \supset \mathcal{N}_\mu(f_i(D))$, and the claim thus follows.

In particular, $\mathcal{N}_1(f_i(\partial D)) \subset f_i(V_i)$. Again using that f_i^{-1} is extrinsically 2-Lipschitz we conclude that $\mathcal{N}_1(f_i(\partial D)) \subset f_i(\mathcal{N}_2(\partial D))$. On the other hand, since f_i is extrinsically 2-Lipschitz, it is intrinsically 2-Lipschitz and can therefore increase the volume by at most a factor of 8. Hence

$$\text{vol}\,\mathcal{N}_1(\partial f_i(D)) \leq 8\,\text{vol}\,\mathcal{N}_2(\partial D) \leq 8\beta(N). \tag{1}$$

We now define $\phi_i \colon D_\epsilon(N) \to N_{n_i}$ to be $f_i \circ \pi$, and will complete the proof of the proposition by showing that ϕ_i has properties (1)—(4).

Since π is a local isometry and $\pi|_{D_\epsilon(N)}$ is an embedding, the map $\pi|_{D_\epsilon(N)}$ is an isometry between $D_\epsilon(N)$ and $\pi(D_\epsilon(N))$ with respect to their intrinsic metrics. Since $f_i \colon D \to f_i(D)$ is an extrinsically (and hence intrinsically) 2-biLipschitz homeomorphism, it follows that ϕ_i has property (1).

We have $\phi_i(z_\omega) = f_i(\pi(z_\omega)) = f_i(z_{\widehat{\Gamma}}) = z_{\rho_{n_i}}$, which is property (2). Property (3) follows from equation (1) since $\phi_i(D_\epsilon(N)) = f_i(D)$.

It remains to check property (4). Suppose that $x \in N_{\text{thin}(\delta)} \cap D_\epsilon(N)$ and $\delta < \frac{\lambda_0}{2}$. Then there exists a homotopically non-trivial loop C (in N) based at x and having length at most δ. Notice that $\phi_i(C)$ has length at most 2δ. Hence $\phi_i(x)$ must lie in the 2δ-thin part of N_{n_i} unless $\phi_i(C)$ is homotopically trivial. But since $\phi_i(C)$ has length at most 2δ, it is contained in the closed δ-neighborhood of $\phi_i(x)$ in N_{n_i}. Thus if $\phi_i(C)$ were homotopically trivial in N, it would lift to a loop in a ball of radius δ in $\mathbf{H}^3$ whose center projects to $\phi_i(x)$. Hence $\phi_i(C)$ would be null-homotopic in $B(\delta, \phi_i(x)) \subset \mathcal{N}_\delta(\phi_i(D)) \subset \mathcal{N}_\mu(\phi_i(D)) \subset f_i(V_i)$. This would imply that $f_i^{-1}(\phi_i(C)) = \pi(C)$ is homotopically trivial in $\widehat{N}$, in contradiction to the fact that π is a covering map. Thus, ϕ_i has property (4), and the proof of Proposition 4.5 is complete. q.e.d.

Remark 4.6. Lemmas 4.3 and 4.4 have analogues for general geometrically finite hyperbolic 3-manifolds, but the constants would also depend on the minimal length of a compressible curve in $\partial\, \mathrm{C}(N)$. Proposition 4.5 remains true, by similar arguments, whenever N is geometrically finite and every component of $\partial\, \mathrm{C}(N)$ is totally geodesic. One may use arguments similar to those in Section 3 of [11] to prove that if ρ is sufficiently near to a maximal cusp ω, then $\phi(D_\epsilon(N_\omega))$ is a compact core for N_ρ.

5. All over the boundary of Schottky space

We now restrict to the case where G is the free group F_k on k generators, where $k \geq 2$. We set $\mathcal{D}_k = \mathcal{D}(F_k)$. Recall that $\mathcal{D}_k$ is a closed subset of $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$. Let $\mathcal{CC}_k$ denote the subset of $\mathcal{D}_k$ consisting of representations which are *convex-cocompact*, i.e., are geometrically finite and have purely loxodromic image. Moreover, let $\mathcal{B}_k = \overline{\mathcal{CC}_k} - \mathcal{CC}_k \subset \mathcal{D}_k$. It is known (see Marden [24]) that $\mathcal{CC}_k$ is an open subset of $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$. (The quotient of $\mathcal{CC}_k$ under the action of $\mathrm{PSL}_2(\mathbf{C})$ is often called Schottky space.)

Let $\mathcal{M}_k$ denote the set of maximal cusps in $\mathcal{D}_k$. It is a theorem of Maskit's [28] that $\mathcal{M}_k \subset \mathcal{B}_k$. McMullen has further proved that $\mathcal{M}_k$ is a dense subset of $\mathcal{B}_k$. This result, though not written down, is in the spirit of McMullen's earlier result [29] that maximal cusps are dense in the boundary of any Bers slice of quasi-Fuchsian space.

The main result of this section, Theorem 5.2, asserts that there is a dense G_δ-set of purely loxodromic, analytically tame representations in $\mathcal{B}_k$. This theorem generalizes and provides an alternate proof of Theorem 8.2 in [14].

The proof of Theorem 5.2 makes use of Proposition 4.5 and McMullen's theorem. We use Proposition 4.5 to show that if $\rho \in \mathcal{B}_k$ is well-approximated by an infinite sequence of maximal cusps, then its convex core can be exhausted by nearly isometric copies of the truncated convex cores of the maximal cusps; this implies that ρ is analytically tame. McMullen's theorem guarantees that the G_δ-set of points in $\mathcal{B}_k$ which are well-approximated by an infinite sequence of maximal cusps is dense.

We now recall the definition of an analytically tame hyperbolic 3-manifold.

Definition. A hyperbolic 3-manifold N with finitely generated fundamental group is *analytically tame* if $\mathrm{C}(N)$ may be exhausted by a sequence of compact submanifolds $\{M_i\}$ with piecewise smooth boundary such that

(1) $M_i \subset \overset{\circ}{M}_j$ if $i < j$, where $\overset{\circ}{M}_j$ denotes the interior of M_j considered as a subset of $\mathrm{C}(N)$,
(2) $\cup \overset{\circ}{M}_i = \mathrm{C}(N)$,
(3) there exists a number $K > 0$ such that the boundary ∂M_i of M_i has area at most K for all i, and
(4) there exists a number $L > 0$ such that $\mathcal{N}_1(\partial M_i)$ has volume at most L for every i.

While the definition of analytic tameness is geometric in nature, it does have important analytic consequences. In particular, for an analytically tame group Γ one can control the behavior of positive Γ-invariant superharmonic functions on $\mathbf{H}^3$. Specifically, we will make extensive use of the following result, which is contained in Corollary 9.2 of [8].

Proposition 5.1. *If $N = \mathbf{H}^3/\Gamma$ is analytically tame and $\Lambda(\Gamma) = \overline{\mathbf{C}}$, then all positive superharmonic functions on N are constant.* q.e.d.

We are now in a position to state Theorem 5.2.

Theorem 5.2. *For all $k \geq 2$, there exists a dense G_δ-set $\mathcal{C}_k$ in $\mathcal{B}_k$, which consists entirely of analytically tame Kleinian groups whose limit set is the entire sphere at infinity.*

The proof of Theorem 5.2 involves the following three lemmas. The first lemma is contained in Chuckrow [12].

Lemma 5.3. *The set $\mathcal{U}_k$ of purely loxodromic representations in $\mathcal{B}_k$ is a dense G_δ-set in $\mathcal{B}_k$. Moreover, if $\rho \in \mathcal{U}_k$, then $\Lambda(\rho(F_k)) = \overline{\mathbf{C}}$.* q.e.d.

The second lemma is an adaptation of Bonahon's bounded diameter lemma [5].

Lemma 5.4. *For every $\delta > 0$, there is a number $c_k(\delta)$ with the following property. Let $\epsilon > 0$ be given, let ω be any maximal cusp in $\mathcal{D}_k$, and set $N = \mathbf{H}^3/\omega(F_k)$. If $\delta(N) > \epsilon$, then any two points in $\partial D_\epsilon(N)$ may be joined by a path β in $\partial D_\epsilon(N)$ such that $\beta \cap N_{\mathrm{thick}(\delta)}$ has length at most $c_k(\delta)$.*

Proof. Let $\delta > 0$ be given. In order to define $c_k(\delta)$ we consider a hyperbolic 2-manifold P which is homeomorphic to a thrice-punctured sphere; there is only one such hyperbolic 2-manifold up to isometry. Since $P_{\mathrm{thick}(\delta)}$ is a compact subset of the metric space P, it has a finite diameter $d(\delta)$. It is clear that any two points in P may be joined by a path β such that $\beta \cap P_{\mathrm{thick}(\delta)}$ has length at most $d(\delta)$. We set $c_k(\delta) = (2k-2)d(\delta) + 3k - 3$.

Now let ω be any maximal cusp in $\mathcal{D}_k$, and set $N = \mathbf{H}^3/\omega(F_k)$. We consider an arbitrary component S of $\partial\,\mathrm{C}(N)$. Then S is a totally geodesic thrice-punctured sphere. Hence S, with its intrinsic metric, is isometric to P. Furthermore, the inclusion homomorphism $\pi_1(S) \to \pi_1(N)$ is injective, and hence $S_{\mathrm{thin}(\delta)} \subset N_{\mathrm{thin}(\delta)}$. It follows that any two points in S may be joined by a path β in S such that $\beta \cap N_{\mathrm{thick}(\delta)}$ has length at most $d(\delta)$.

Now, since $D_\epsilon(N)$ is a compact core for N and $\pi_1(N)$ is a free group of rank k, we see that $D_\epsilon(N)$ is a handlebody of genus k. In particular, there are exactly $2k-2$ components of $\partial\,\mathrm{C}(N)$ and exactly $3k-3$ annular components of $\partial D_\epsilon(N) - \partial\,\mathrm{C}(N)$. Also recall, from Lemma 4.1, that each component of $\partial D_\epsilon(N) - \partial\,\mathrm{C}(N)$ has diameter at most 1. Thus, since $\partial D_\epsilon(N)$ is connected, any two points in $\partial D_\epsilon(N)$ may be joined by a path β such that $\beta \cap N_{\mathrm{thick}(\delta)}$ has length at most $(2k-2)d(\delta) + 3k - 3 = c_k(\delta)$. q.e.d.

In the following lemma and the rest of this section, we arbitrarily fix a base point $z^0 \in \mathbf{H}^3$, and let $\mathcal{X}_k$ denote the set of all representations $\rho \in \mathcal{B}_k$ such that z^0 lies in the interior of $\mathrm{CH}(\rho(F_k))$ relative to $\mathbf{H}^3$.

Lemma 5.5. *The set $\mathcal{X}_k$ is an open dense subset of $\mathcal{B}_k$.*

Proof. Given a representation $\rho \in \mathcal{X}_k$, we have that z^0 lies in the interior of $\mathrm{CH}(\rho(F_k))$. Hence z^0 lies in the interior of some ideal tetrahedron T with vertices in $\Lambda(\rho(F_k))$. Since the fixed points of the elements of $\rho(F_k)$ are dense in $\Lambda(\rho(F_k))$, we may assume that the vertices of T are attracting fixed points of elements $\rho(g_1), \ldots, \rho(g_4)$ of $\rho(F_k)$. It follows that for any $\rho' \in \mathcal{B}_k$ sufficiently close to ρ, the attracting fixed points of $\rho'(g_1), \ldots, \rho'(g_4)$ span a tetrahedron having z^0 as an interior point. Thus z^0 lies in the interior of $\mathrm{CH}(\rho'(F_k))$. This shows that $\mathcal{X}_k$ is an open subset of $\mathcal{B}_k$.

If $\rho \in \mathcal{U}_k$, then $\rho \in \mathcal{X}_k$ since $\Lambda(\rho(F_k)) = \overline{\mathbf{C}}$. Hence, Lemma 5.3 implies that $\mathcal{X}_k$ is dense. q.e.d.

Proof of 5.2. By the result of McMullen's discussed at the beginning of this section, $\mathcal{M}_k$ is a dense subset of $\mathcal{B}_k$. In view of Lemma 5.5 it follows that $\mathcal{M}_k \cap \mathcal{X}_k$ is also dense in $\mathcal{B}_k$. For each $\omega \in \mathcal{M}_k \cap \mathcal{X}_k$ and each $\epsilon > 0$ we will define a neighborhood $V(\epsilon, \omega)$ of ω in $\mathcal{X}_k \subset \mathcal{B}_k$.

Set $N_\omega = \mathbf{H}^3/\omega(F_k)$. Since $\omega \in \mathcal{X}_k$, $z^0 \in \mathrm{CH}(\omega(F_k))$; in the basepoint convention given in Section 4 we have $z^0_\omega \in \mathrm{C}(N_\omega)$. Hence either $z^0_\omega \in D_\epsilon(N_\omega)$, or z^0_ω lies in the interior of the ϵ-thin part of N_ω. Let $\eta(\epsilon, \omega) = \min\{\epsilon, \delta(N_\omega)\}$. If $z^0_\omega \in D_\epsilon(N_\omega)$, then we set $V(\epsilon, \omega) = U(\eta(\epsilon, \omega), z^0, \omega) \cap \mathcal{X}_k$, where $U(\eta(\epsilon, \omega), z^0, \omega)$ is the open set given by Proposition 4.5. If

z^0_ω lies in the ϵ-thin part of N_ω, we take $V(\epsilon, \omega)$ to be a neighborhood of ω in $\mathcal{X}_k$ such that for every $\rho \in V$ the point z^0_ρ lies in the interior of the ϵ-thin part of $\mathbf{H}^3/\rho(F_k)$. (Such a neighborhood exists because there is an element $g \in F_k$ such that $\operatorname{dist}(z^0, \omega(g) \cdot z^0) < \epsilon$. For any ρ sufficiently close to ω we have $\operatorname{dist}(z^0, \rho(g) \cdot z^0) < \epsilon$.)

We now set $W_k(\epsilon) = \cup_{\omega \in \mathcal{M}_k \cap \mathcal{X}_k} V(\epsilon, \omega)$. Since $\mathcal{M}_k \cap \mathcal{X}_k$ is dense in $\mathcal{B}_k$, the set $W_k(\epsilon)$ is an open dense subset of $\mathcal{X}_k$ for every $\epsilon > 0$. Since $\mathcal{U}_k$ is a dense G_δ-set in $\mathcal{B}_k$, it follows that

$$\mathcal{C}_k = \mathcal{U}_k \cap \bigcap_{m \in \mathbf{Z}_+} W_k(\frac{1}{m})$$

is a dense G_δ-set in $\mathcal{X}_k$. In order to complete the proof, we need only to show that each element of $\mathcal{C}_k$ is analytically tame and has the entire sphere as its limit set.

Let $\rho : F_k \to \mathrm{PSL}_2(\mathbf{C})$ be a representation in $\mathcal{C}_k$. Set $N = \mathbf{H}^3/\rho(F_k)$. Lemma 5.3 guarantees that $\mathrm{C}(N) = N$.

By the definition of $\mathcal{C}_k$, for every $m \in \mathbf{Z}_+$ there exists a maximal cusp $\omega_m : F_k \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho \in V(\frac{1}{m}, \omega_m)$. Let $N_m = \mathbf{H}^3/\omega_m(F_k)$ and let m_0 be a positive integer such that $\frac{1}{2m_0}$ is less than the injectivity radius of N at z^0_ρ. In what follows we consider an arbitrary integer $m \geq m_0$.

By the definition of m_0 the point z^0_ρ lies in the $\frac{1}{m}$-thick part of N. From the definition of the sets $V(\frac{1}{m}, \omega_m)$, it follows that $\rho \in U(\eta(\frac{1}{m}, \delta(N_m)), z^0, \omega_m)$. Let $D_m = D_{\eta(\frac{1}{m}, \delta(N_m))}(N_m)$. Proposition 4.5 guarantees that there is a map $\phi_m : D_m \to N$, such that:

(1) ϕ_m maps D_m homeomorphically onto a manifold with piecewise smooth boundary, and is 2-biLipschitz with respect to the intrinsic metrics on D_m and $\phi_m(D_m)$,
(2) $\phi_m(z^0_{\omega_m}) = z^0_\rho$,
(3) $\operatorname{vol} \mathcal{N}_1(\partial(\phi_m(D_m))) \leq 8\beta(N_m) = 32\pi\kappa(k-1)$, and
(4) $\phi_m((N_m)_{\mathrm{thin}(\delta)} \cap D_m) \subset N_{\mathrm{thin}(2\delta)}$ for any $\delta < \frac{\lambda_0}{2}$,

where κ is the constant given by Lemma 4.3. Notice that since $\pi_1(N_m)$ is a free group we have $\tau(N_m) = 0$ and hence $\beta(N_m) = 2\pi\kappa\sigma(N_m) = 4\pi\kappa(k-1)$.

We set $\delta_0 = \lambda_0/3$ and $M_m = \phi_m(D_m)$. Then by (4) we have

$$M_m \cap \phi_m(D_m \cap (N_m)_{\mathrm{thin}(\delta_0)}) \subset N_{\mathrm{thin}(2\delta_0)}.$$

Hence, by Lemma 5.4, any two points in ∂M_m may be joined by a path β in ∂M_m such that $\beta \cap N_{\mathrm{thin}(2\delta_0)}$ has length at most $2c_k(\delta_0)$.

Let $r > 0$, and let $X(r)$ denote the set of points $x \in N$ for which there exists a path β beginning in $B(r, z^0_\rho)$ and ending at x such that

$\beta \cap N_{\mathrm{thick}(2\delta_0)}$ has length at most $2c_k(\delta_0)$. Since $\rho(F_k)$ is purely loxodromic, each component of $N_{\mathrm{thin}(2\delta_0)}$ is compact. Moreover, the components of $N_{\mathrm{thin}(2\delta_0)}$ are separated by a distance of at least $L(2\delta_0)$, so there exists only a finite number of components of $N_{\mathrm{thin}(2\delta_0)}$ contained in $X(r)$. Therefore $X(r)$ is compact. Set $\zeta(r) = \min_{x\in X(r)} \mathrm{inj}_N(x)$, and $m_1 = \max(m_0, 2/\zeta(r))$.

If $m > m_1$ then $\partial M_m \cap B(r, z_\rho^0) = \emptyset$, since any point in ∂M_m may be joined by a path of length at most $2c_k(\delta_0)$ to a point of injectivity radius $\leq \frac{2}{m}$; namely any point in $\phi_m(\partial D_m - \partial\,\mathrm{C}(N_m))$. Since $z_\rho^0 \in M_m$ by (2), and since $\partial M_m \cap B(r, z_\rho^0) = \emptyset$, we see that $B(r, z_\rho^0) \subset \mathring{M}_m$ for every $m > m_1$.

We therefore have $\cup_{m>m_1} \mathring{M}_m = N = \mathrm{C}(N)$. We may pass to a subsequence M_{m_j} such that $M_{m_j} \subset \mathring{M}_{m_{j+1}}$ for all j and $\cup_{j\in \mathbf{Z}_+} \mathring{M}_{m_j} = N = \mathrm{C}(N)$.

By (3) we have

$$\mathrm{vol}\,\mathcal{N}_1(\partial M_m) \leq 32\pi\kappa(k-1)$$

for all $m > m_1$. Moreover, using (1) and Lemma 4.4 we obtain

$$\mathrm{area}\,\partial M_m \leq 4\alpha(N_m) = 14\pi\sigma(N_m) = 28\pi(k-1).$$

Thus N is analytically tame. q.e.d.

6. Free groups and displacements

This section is devoted to the proof of the following theorem, which includes the Main Theorem stated in the introduction.

Theorem 6.1. *Let $k \geq 2$ be an integer and let Φ be a Kleinian group which is freely generated by elements $\xi_1, \ldots, \xi_k$. Suppose that one of the following holds:*

(a) *Φ is purely loxodromic and topologically tame,*
(b) *Φ is geometrically finite,*
(c) *Φ is analytically tame and $\Lambda(\Phi) = \overline{\mathbf{C}}$,*
(d) *the hyperbolic 3-manifold $\mathbf{H}^3/\Phi$ admits no non-constant positive superharmonic functions.*

Let z be any point of $\mathbf{H}^3$ and set $d_i = \mathrm{dist}(z, \xi_i \cdot z)$ for $i = 1, \ldots, k$. Then we have

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \ldots, k\}$ such that $d_i \geq \log(2k-1)$.

Note that if we had $d_i < \log(2k-1)$ for $i = 1, \dots, k$ it would follow that

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} > k \cdot \frac{1}{2k} = \frac{1}{2}.$$

Thus the last sentence of Theorem 6.1 does indeed follow from the preceding sentence.

Conditions (a)–(d) of Theorem 6.1 are by no means mutually exclusive. In particular, according to Proposition 5.1, condition (c) implies condition (d).

The following elementary inequality will be needed for the proof of Theorem 6.1.

Lemma 6.2. *Let x and y be non-negative real numbers with $x+y \leq 1$. Set $p = \frac{1}{2}(x+y)$. Then we have*

$$\left(\frac{1-x}{x}\right)\left(\frac{1-y}{y}\right) \geq \left(\frac{1-p}{p}\right)^2.$$

Proof. We can write $x = p + \alpha$ and $y = p - \alpha$ for some $\alpha \in \mathbf{R}$. We find by direct calculation that $p^2(1-x)(1-y) - (1-p)^2 xy = (1-2p)\alpha^2$. But $p \leq 1/2$ since $x + y \leq 1$. Hence $p^2(1-x)(1-y) \geq (1-p)^2 xy$, and the assertion follows. q.e.d.

A basic step in the proof of Theorem 6.1 is the observation that if the inequality in conclusion holds on the boundary $\mathcal{B}_k$ of $\mathcal{CC}_k$, then it also holds on $\mathcal{CC}_k$. This observation is contained in the following lemma.

Lemma 6.3. *For $k > 1$ let F_k denote the free group on the k generators $x_1 \dots x_k$, and let ρ: $F_k \to \mathrm{PSL}_2(\mathbf{C})$ be any representation in $\mathcal{CC}_k$. Then given any point z in $\mathbf{H}^3$ there exists a representation $\rho_z \in \mathcal{B}_k$ such that*

$$\operatorname{dist}(z, \rho(x_i)\cdot z) = \operatorname{dist}(z, \rho_z(x_i)\cdot z)$$

for $i = 1, \dots, k$.

Proof. We consider the set

$$R_z = \{\sigma \in \operatorname{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C})) \,|\, \operatorname{dist}(z, \sigma(x_i)\cdot z) = \operatorname{dist}(z, \rho(x_i)\cdot z)\}.$$

It suffices to show that R_z is connected and contains a point of $\operatorname{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C})) - \mathcal{CC}_k$. The connectedness follows immediately from the easy observation that for any positive number d the set $\{\xi \in \operatorname{Isom}_+(\mathbf{H}^3) \,|\, \operatorname{dist}(z, \xi \cdot z) = d\}$ is path-connected. Since R_z contains a representation with an invariant line passing through z, $R_z - \mathcal{CC}_k$ is clearly non-empty. q.e.d.

Proof of Theorem 6.1. We first prove that condition (d) implies the conclusion of the theorem.

We use the terminology of [14]. We set $\Psi = \{\xi_1, \xi_1^{-1}, \dots, \xi_k, \xi_k^{-1}\} \subset \Phi$. According to Lemma 5.3 of [14], there exists a number $D \in [0,2]$ such that, for any $z_0 \in \mathbf{H}^3$, there exist a Φ-invariant D-conformal density $\mathfrak{M} = (\mu_z)$ for $\mathbf{H}^3$ and a family $(\nu_\psi)_{\psi\in\Psi}$ of Borel measures on S_∞ such that:

(i) $\mu_{z_0}(S_\infty) = 1$;
(ii) $\mu_{z_0} = \sum_{\psi\in\Psi} \nu_\psi$; and
(iii) for each $\psi \in \Psi$ we have

$$\int (\lambda_{\psi,z_0})^D d\nu_{\psi^{-1}} = 1 - \int d\nu_\psi.$$

If condition (d) of the theorem holds, it follows from Proposition 3.9 of [14] that any Φ-invariant conformal density for $\mathbf{H}^3$ is a constant multiple of the area density $\mathfrak{A}$. In view of condition (i) above we must in fact have $\mathfrak{M} = \mathfrak{A}$. In particular $D = 2$.

For $i = 1, \dots, k$ we set $\nu_i = \nu_{\xi_i}$ and $\nu_i' = \nu_{\xi_i^{-1}}$. We denote by α_i and β_i the total masses of the measures ν_i and ν_i' respectively. After possibly interchanging the roles of ξ_i and ξ_i^{-1} we may assume that $\alpha_i \le \beta_i$. (Interchanging the roles of ξ_i and ξ_i^{-1} does not affect the truth of the conclusion of the lemma, since $d_i = \operatorname{dist}(z_0, \xi_i \cdot z_0) = \operatorname{dist}(z_0, \xi_i^{-1} \cdot z_0)$.)

By conditions (i) and (ii) above we have $\sum_{i=1}^k (\alpha_i + \beta_i) = 1$. In particular for each i we have $0 \le \beta_i \le 1$, and since $\alpha_i \le \beta_i$ we have $0 \le \alpha_i \le 1/2$. Since $\mathfrak{M} = \mathfrak{A}$, condition (ii) also implies that $\nu_i \le A_{z_0}$, where A_{z_0} denotes the area measure on S_∞ centered at z_0. By applying condition (iii) above to $\psi = \xi_i^{-1}$ we get that $\int_{S_\infty} \lambda^2_{\xi_i^{-1}, z_0} d\nu_i = 1 - \beta_i$. And by definition we have $\nu_i(S_\infty) = \alpha_i$. Thus the hypotheses of Lemma 5.5 of [14] hold with $\nu = \nu_i$, $a = \alpha_i$ and $b = 1 - \beta_i$. Hence by Lemma 5.5 of [14] we have

$$d_i = \operatorname{dist}(z_0, \xi_i^{-1} \cdot z_0) \ge \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}.$$

(This is a corrected version of the conclusion of Lemma 5.5 of [14]. In the published version of [14] the inequality appeared with the roles of a and b reversed.)

Thus

$$d_i \ge \frac{1}{2} \log \frac{(1-\beta_i)(1-\alpha_i)}{\alpha_i \beta_i}.$$

Since $\alpha_i + \beta_i \le \sum_{i=1}^k (\alpha_i + \beta_i) = 1$, it follows from Lemma 6.2 that

$$\frac{(1-\alpha_i)(1-\beta_i)}{\alpha_i \beta_i} \ge (\frac{1-p_i}{p_i})^2,$$

where $p_i = (\alpha_i + \beta_i)/2$. Thus

$$d_i \geq \log \frac{1-p_i}{p_i},$$

or equivalently

$$\frac{1}{1+e^{d_i}} \leq p_i.$$

Hence

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \sum_{i=1}^{k} p_i = \frac{1}{2}\sum_{i=1}^{k}(\alpha_i + \beta_i) = \frac{1}{2}.$$

This completes the proof that condition (d) implies the conclusion of the theorem. In view of Proposition 5.1, condition (c) also implies the conclusion of the theorem.

Next we assume that condition (a) holds and deduce the conclusion of the theorem.

Since Φ is purely loxodromic and free of finite rank, either $\Lambda(\Phi) = \overline{\mathbf{C}}$ or Φ is a Schottky group (see Maskit [27]). In the case that $\Lambda(\Phi) = \overline{\mathbf{C}}$, we use Theorem 8.1 in [8], which states that a topologically tame hyperbolic 3-manifold is analytically tame.

Thus in this case condition (c) holds, and hence the conclusion of the theorem is true.

Now suppose that Φ is a Schottky group. Let F_k denote the abstract free group generated by $\{x_1, \ldots, x_k\}$. Let $\rho_0 : F_k \to \Phi$ denote the unique isomorphism that takes x_i to ξ_i for $i = 1, \ldots, k$. We may regard ρ_0 as a representation of F_k in $PSL_2(\mathbf{C})$. Since Φ is a Schottky group we have $\rho_0 \in \mathcal{CC}_k$.

We define a continuous, non-negative real-valued function f_z on the representation space $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$ by setting

$$f_z(\rho) = \sum_{i=1}^{k} \frac{1}{1 + \exp \operatorname{dist}(z, \rho(x_i) \cdot z)} \quad .$$

We must show that for any point z in $\mathbf{H}^3$ and any representation ρ in $\mathcal{CC}_k$ we have $f_z(\rho) \leq \frac{1}{2}$. Let z and ρ be given. By Lemma 6.3 there exists a representation $\rho_z \in \mathcal{B}_k$ such that the point z is displaced the same distance by $\rho_0(x_i)$ as by $\rho_z(x_i)$ for $i = 1, \ldots, k$. Thus $f_z(\rho_z) = f_z(\rho_0)$, so it suffices to show that $f_z(\rho) \leq \frac{1}{2}$ for any representation $\rho \in \mathcal{B}_k$. To see this we consider the dense G_δ-set $\mathcal{C}_k \subset \mathcal{B}_k$ given by Theorem 5.2. Recall that every representation in $\mathcal{C}_k$ maps F_k isomorphically onto an analytically tame Kleinian group whose limit set is the entire sphere at infinity. Thus for any $\rho \in \mathcal{C}_k$ the group $\rho(F_k)$ satisfies condition (c) of the present theorem, and we therefore have $f_z(\rho) \leq \frac{1}{2}$. Since $\mathcal{C}_k$ is dense in $\mathcal{B}_k$ and f_z is continuous, we have $f_z(\rho) \leq \frac{1}{2}$ for every $\rho \in \mathcal{B}_k$.

Finally, we verify the conclusion of the theorem under the assumption that condition (b) holds.

We continue to denote by F_k the abstract free group on k generators. We fix an isomorphism $\rho : F_k \to \Phi$, which we regard as a discrete, faithful representation of F_k in $\mathrm{Isom}_+(\mathbf{H}^3)$. In view of the geometric finiteness of $\Phi = \rho(F_k)$, a theorem of Maskit's [28] guarantees that there exists a sequence of discrete faithful representations $\{\rho_j : F \to \mathrm{Isom}_+(\mathbf{H}^3)\}$ such that (for all j) $\rho_j(F)$ is geometrically finite and purely loxodromic, and ρ_j converges (as a sequence of representations) to ρ. Given $z \in \mathbf{H}^3$, we set

$$m_j = \sum_{i=1}^{k} \frac{1}{1 + e^{\mathrm{dist}(\rho_j(x_i)\cdot z, z)}} \qquad \textit{and} \qquad m = \sum_{i=1}^{k} \frac{1}{1 + e^{d_i}}.$$

Since each ρ_j satisfies (a), we have $m_j \leq \frac{1}{2}$. But clearly $\{m_j\}$ converges to m, so $m \leq \frac{1}{2}$. This is the conclusion of the theorem. q.e.d.

7. Topology and free subgroups

The results of the last section can be used to study the geometry of a hyperbolic manifold N. One writes N in the form $\mathbf{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group, and applies the estimate given by Theorem 6.1 to suitable free subgroups Φ of Γ to deduce geometric information about N. Of course, such applications require that one be able to produce free subgroups of Γ. In this section we concentrate on the problem of giving sufficient conditions for a subgroup of $\Gamma \cong \pi_1(N)$ to be free.

This problem can be attacked with purely topological techniques. Results which use 3-manifold theory to deduce, from topological hypotheses, that certain subgroups of $\pi_1(N)$ are free have appeared in [37], in Section VI.4 of [19] and in the appendix to [3]. The first two sources consider the case of a 2-generator subgroup, while the results of the third apply to higher rank subgroups, but require that N be closed. In this section we give a systematic treatment of this topic. We prove a result that includes the results of [3] as special cases and is suitable for the applications in this paper.

We shall follow a couple of conventions that are widely used in low-dimensional topology. Unlabeled homomorphisms between fundamental groups are understood to be induced by inclusion maps. Base points will be suppressed whenever it is clear from the context how to choose consistent base points.

Recall that an orientable piecewise linear 3-manifold N is said to be *irreducible* if every PL 2-sphere in N bounds a PL ball. We shall say that N is *simple* if N is irreducible and if for every rank-2 free abelian subgroup A of $\pi_1(N)$, there is a closed PL subspace E of N, piecewise linearly homeomorphic to $T^2 \times [0, \infty)$, such that A is contained in a conjugate of $\mathrm{im}(\pi_1(E) \to \pi_1(N))$. (The subgroup $\mathrm{im}(\pi_1(E) \to \pi_1(N))$ of $\pi_1(N)$ is itself well-defined up to conjugacy.)

We shall say that an orientable PL 3-manifold N without boundary has *cusp-like ends* if it is PL homeomorphic to the interior of a compact manifold-with-boundary M such that (i) every component of ∂M is a torus and (ii) for every component B of ∂M, the inclusion homomorphism $\pi_1(B) \to \pi_1(M)$ is injective. In particular, note that if N is closed, then it has cusp-like ends.

Recall that the *rank* of a group Γ is the minimal cardinality of a generating set for Γ.

A group Γ is termed *freely indecomposable* if it is non-trivial and is not a free product of two non-trivial subgroups.

For any non-negative integer g we denote by S_g the closed orientable surface of genus g.

Theorem 7.1. *Let N be a simple orientable PL 3-manifold without boundary. Suppose that $k = \mathrm{rank}\, \pi_1(N) < \infty$, and that $\pi_1(N)$ is freely indecomposable and has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$. Then either $\pi_1(N)$ is a free abelian group, or N has cusp-like ends.*

Proof. According to [32], there is a compact PL manifold-with-boundary $M \subset N$ such that $\pi_1(M) \to \pi_1(N)$ is an isomorphism. Among all compact PL manifolds-with-boundary with this property we may suppose M to have been chosen so as to minimize the number of components $r = r_M$ of ∂M. We may assume if we like that $r > 0$, for if $r = 0$, then $N = M$ is closed, and in particular it has cusp-like ends. Let $B_1, \ldots, B_r$ denote the components of ∂M, and g_i the genus of B_i for $i = 1, \ldots, r$. Since $\pi_1(M) \cong \pi_1(N)$ has rank k, the first betti number of M is at most k.

From Poincaré-Lefschetz duality and the exact homology sequence of $(M, \partial M)$ it follows that the total genus $\sum g_i$ of ∂M is at most the first betti number of M. Thus $\sum g_i \le k$.

We must have $g_i > 0$ for $i = 1, \ldots, r$. Indeed, if B_i is a 2-sphere for some i, then B_i bounds a PL ball $K \subset N$ since N is irreducible. We have either $K \supset M$ or $K \cap M = B_i$. If $K \supset M$, then since $\pi_1(M) \to \pi_1(N)$

is an isomorphism we have $\pi_1(N) = 1$, in contradiction to the free indecomposability of $\pi_1(N)$. If $K \cap M = B_i$, then $M' = M \cup K$ has fewer boundary components than M, and $\pi_1(M) \to \pi_1(N)$ is an isomorphism; this contradicts our choice of M.

Let us consider the case in which $\pi_1(B_i) \to \pi_1(M)$ has a non-trivial kernel for some $i \le r$. According to the Loop Theorem [35], M contains a properly embedded disk D such that ∂D is homotopically non-trivial in ∂M. If D separates M, both components of $M - D$ have boundary components of positive genus and are therefore non-simply connected. This contradicts the free indecomposability of $\pi_1(M)$. Hence D does not separate M, and M is a free product of an infinite cyclic group with a group isomorphic to $\pi_1(M-D)$. The latter group must be trivial in view of the free indecomposability of $\pi_1(M) \cong \pi_1(N)$. Thus $\pi_1(N) \cong \pi_1(M)$ is infinite cyclic in this case, and in particular free abelian.

From this point on we assume that $\pi_1(B_i) \to \pi_1(M)$ is injective for $i = 1, \dots, r$. Since by the hypothesis of the theorem, $\pi_1(M) \cong \pi_1(N)$ has no subgroup isomorphic to $\pi_1(S_{g_i})$ for $2 \le g_i \le k-1$, each g_i is either ≤ 1 or $\ge k$. We have seen that the g_i are all strictly positive and that their sum is at most k. Hence we must have either (i) $r = 1$ and $g_1 = k$, or (ii) $g_i = 1$ for $i = 1, \dots, r$.

Suppose that (i) holds. Then ∂M is a connected surface of genus k. Hence the Euler characteristic $\chi(\partial M)$ is equal to $2 - 2k$. We have $\chi(M) = \frac{1}{2}\chi(\partial M) = 1 - k$. Now as a compact PL 3-manifold with non-empty boundary, M admits a simplicial collapse to a 2-complex L. In particular M is homotopy-equivalent to L, and hence to the CW-complex L' obtained from L by identifying a maximal tree in the 1-skeleton of L to a point. If c_i denotes the number of i-cells in L', we have $c_0 = 1$ and $1 - c_1 + c_2 = \chi(L') = \chi(M) = 1 - k$, so that $c_1 - c_2 = k$. But $\pi_1(L') \cong \pi_1(M) \cong \pi_1(N)$ has a presentation with c_1 generators and c_2 relations, and $k = c_1 - c_2$ is by definition the deficiency of the presentation. On the other hand, k is by hypothesis the rank of $\pi_1(N)$. It is a theorem due to Magnus [23] that if a group Γ has rank k and admits a presentation of deficiency k, then Γ is free of rank k. Since $\pi_1(N)$ is freely indecomposable, we must have $k = 1$, and $\pi_1(N)$ must be infinite cyclic in this case as well.

Now suppose that (ii) holds. Then $B_1, \dots, B_r$ are tori. Since the inclusion homomorphisms $\pi_1(B_i) \to \pi_1(M)$ are injective, the groups $A_i = \mathrm{im}(\pi_1(B_i) \to \pi_1(M))$ are free abelian groups of rank 2. Since N is simple, there are closed PL subspaces $E_1, \dots, E_r$ of N, each piecewise linearly homeomorphic to $T^2 \times [0, \infty)$, such that A_i is contained in a conjugate of $\mathrm{im}(\pi_1(E_i) \to \pi_1(N))$ for $i = 1, \dots, r$. It follows from

Proposition 5.4 of [38] that B_i is isotopic to ∂E_i for $i = 1, \ldots, r$. Hence we may suppose the E_i to have been chosen so that $\partial E_i = B_i$. For each $i \leq r$ we have either $M \subset E_i$ or $M \cap E_i = B_i$.

If $M \cap E_i = B_i$ for $i = 1, \ldots, r$, we have $N = M \cup E_1 \cdots \cup E_r$. It follows that in this case N is PL homeomorphic to the interior of M, and hence that N has cusp-like ends.

There remains the case in which $M \subset E_i$ for some i. In the sequence of inclusion homomorphisms

$$\pi_1(B_i) \to \pi_1(M) \to \pi_1(E_i) \to \pi_1(N)$$

the composition of the first two arrows (from the left) is the isomorphism $\pi_1(B_i) \to \pi_1(E_i)$, and the composition of the last two arrows is the isomorphism $\pi_1(M) \to \pi_1(N)$. Thus the entire sequence consists of isomorphisms, and hence $\pi_1(N)$ is a rank-2 free abelian group in this case. q.e.d.

We shall say that a group is *semifree* if it is a free product of abelian groups.

Corollary 7.2. *Let N be an orientable hyperbolic 3-manifold of infinite volume. Suppose that $k = \operatorname{rank} \pi_1(N) < \infty$, and that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then $\pi_1(N)$ is semifree.*

Proof. Let us write $N = \mathbf{H}^3/\Gamma$, where $\Gamma \cong \pi_1(N)$ is a discrete torsion-free subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. Since Γ is finitely generated, it can be written as a free product $\Gamma_1 * \cdots * \Gamma_n$ of freely indecomposable subgroups. Since N has infinite covolume, so does the manifold H^3/Γ_i for $i = 1, \ldots, n$. The rank k_i of $\Gamma_i \cong \pi_1(N_i)$ is at most k. Hence the hypothesis of the corollary implies that Γ_i has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Applying Theorem 7.1 with N_i in place of N, we conclude that for each $i \leq n$, either Γ_i is free abelian or N_i has cusp-like ends. But the latter alternative is impossible because a hyperbolic manifold with cusp-like ends has finite volume (see [4, D.3.18]). Thus all the Γ_i are free abelian and hence Γ is semifree. q.e.d.

Recall that a group Γ is termed *k-free,* where k is a cardinal number, if every subgroup of Γ whose rank is at most k is free. We shall say that Γ is *k-semifree* if every subgroup of Γ whose rank is at most k is semifree.

Corollary 7.3. *Let N be an orientable hyperbolic 3-manifold, and let k be a non-negative integer. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. In addition suppose that either*

(i) *N has infinite volume, or*

(ii) *every subgroup of $\pi_1(N)$ whose rank is at most k is of infinite index in $\pi_1(N)$.*

Then $\pi_1(N)$ is k-semifree.

Proof. Let Δ be any subgroup of $\pi_1(N)$ whose rank is at most k. Let $\tilde{N}$ denote the covering space of N associated to H. If either (i) or (ii) holds, $\tilde{N}$ has infinite volume. Hence by Corollary 7.2, Δ is semifree. q.e.d.

Hypothesis (ii) of 7.3 clearly holds if the first betti number of N is at least $k+1$. According to Proposition 1.1 of [33], it also holds if $H_1(N, \mathbf{Z}/p)$ has rank at least $k+2$ for some prime p. Thus we have:

Corollary 7.4. *Let N be an orientable hyperbolic 3-manifold, and let k be a non-negative integer. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. In addition suppose that either*

(i) *the first betti number of N is at least $k+1$, or*

(ii) *$H_1(N, \mathbf{Z}/p)$ has rank at least $k+2$ for some prime p.*

Then $\pi_1(N)$ is k-semifree.

Remark 7.5. If the orientable hyperbolic 3-manifold N has no cusps, then every abelian subgroup of $\pi_1(N)$ is infinite cyclic; thus $\pi_1(N)$ is k-semifree for a given k if and only if it is k-free. Hence if N has no cusps we may replace "semifree" by "free" in the conclusions of Corollaries 7.3 and 7.4.

8. Strong Margulis numbers and k-Margulis numbers

In order to unify the different applications of the results of the last two sections it is useful to introduce a little formalism. Let Γ be a discrete torsion-free subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. Recall from [33] and [14] that a positive number λ is termed a *Margulis number* for the group Γ, or the orientable hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$, if whenever ξ and η are non-commuting elements of Γ, and $z \in \mathbf{H}^3$, we have

$$\max\{\mathrm{dist}(\xi \cdot z, z), \mathrm{dist}(\eta \cdot z, z)\} \geq \lambda.$$

We shall say that λ is a *strong Margulis number* for Γ or N, if whenever ξ and η are non-commuting elements of Γ, we have

$$\frac{1}{1+e^{\mathrm{dist}(\xi \cdot z, z)}} + \frac{1}{1+e^{\mathrm{dist}(\eta \cdot z, z)}} \leq \frac{2}{1+e^{\lambda}}\ .$$

Notice that if λ is a strong Margulis number for Γ, then λ is also a Margulis number for Γ.

More generally, let $k \geq 2$ be an integer, and let λ be a positive real number. We shall say that λ is a *k-Margulis number* for the discrete torsion-free group $\Gamma \leq \mathrm{Isom}_+(\mathbf{H}^3)$ or $N = \mathbf{H}^3/\Gamma$ if any k elements $\xi_1, \ldots, \xi_k \in \Gamma$ and for any $z \in \mathbf{H}^3$, we have that either

(i) $\max_{i=1}^{k} \mathrm{dist}(\xi_i \cdot z, z) \geq \lambda$, or
(ii) the group $\langle \xi_1, \ldots, \xi_k \rangle$ is generated by at most $k-1$ abelian subgroups.

We say that λ is a *strong k-Margulis number* for Γ or N, if for any k elements $\xi_1, \ldots, \xi_k \in \Gamma$ and any $z \in \mathbf{H}^3$, we have that either

(i)
$$\sum \frac{1}{1 + e^{\mathrm{dist}(\xi_i \cdot z, z)}} \leq \frac{k}{1 + e^{\lambda}},$$
or
(ii) the group $\langle \xi_1, \ldots, \xi_k \rangle$ is generated by at most $k-1$ abelian subgroups.

Note that λ is a (respectively, strong) 2-Margulis number for Γ if and only if it is a (respectively, strong) Margulis number for Γ. Note also that if λ is a strong k-Margulis number for Γ, then λ is also a k-Margulis number for Γ.

In this section we will use Theorem 6.1 and the corollaries of Theorem 7.1 to prove that under various conditions a hyperbolic 3-manifold has $\log(2k-1)$ as a strong k-Margulis number. In the following three sections these results will be used to obtain lower bounds for the volume of various classes of hyperbolic 3-manifolds.

Our first result is an easy consequence of Theorem 6.1(a). We shall say that a Kleinian group Γ is *k-tame*, where k is a positive integer, if every subgroup of Γ having rank at most k is topologically tame.

Proposition 8.1. *Let $k \geq 2$ be an integer and let Γ be a discrete subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. Suppose that Γ is k-free, k-tame and purely loxodromic. Then $\log(2k-1)$ is a strong k-Margulis number for Γ.*

Proof. If $\xi_1, \ldots, \xi_k \in \Gamma$ are elements of Γ, then the group $\langle \xi_1, \ldots, \xi_k \rangle$ is topologically tame, purely loxodromic and free of some rank $\leq k$. If its rank is k, then it is freely generated by $\xi_1, \ldots, \xi_k$; hence for any $z \in \mathbf{H}^3$, we have

$$\sum \frac{1}{1 + e^{\mathrm{dist}(\xi_i \cdot z, z)}} \leq \frac{1}{2} = \frac{k}{1 + e^{\log(2k-1)}}$$

by Theorem 6.1(a). If $\langle \xi_1, \ldots, \xi_k \rangle$ has rank $\leq k-1$, then in particular it is generated by at most $k-1$ abelian subgroups. q.e.d.

Corollary 8.2. *Let $k \geq 2$ be an integer, and let N be a non-compact, topologically tame orientable hyperbolic 3-manifold without cusps. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then $\log(2k-1)$ is a strong k-Margulis number for N.*

Proof. Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a discrete, non-cocompact, purely loxodromic subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. According to Proposition 3.2 in [8], every finitely generated subgroup of Γ is topologically tame. In particular Γ is k-tame. On the other hand, since N has infinite volume and $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$, Corollary 7.3 and Remark 7.5 guarantee that $\Gamma \cong \pi_1(N)$ is k-free. The desired conclusion therefore follows from Proposition 8.1. q.e.d.

It is worth pointing out that the following corollary can be deduced from Proposition 8.1, although a more general result, Corollary 8.7, will be proved below by a slightly different argument.

Corollary 8.3. *Let $k \geq 2$ be an integer and let N be a closed orientable hyperbolic 3-manifold. Suppose that the first betti number of N is at least $k+1$ and that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then $\log(2k-1)$ is a strong k-Margulis number for N.*

Proof. Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a discrete, cocompact, purely loxodromic subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. By Corollary 7.4 and Remark 7.5, $\Gamma \cong \pi_1(N)$ is k-free. On the other hand, since N has first betti number at least $k+1$, any subgroup Γ' of Γ having rank at most k is contained in the kernel of a surjective homomorphism $\beta : \Gamma \to \mathbf{Z}$. According to Proposition 8.4 of [9], it follows that Γ' is topologically tame. Thus Γ is k-tame and the desired conclusion follows from Proposition 8.1. q.e.d.

The following result gives information not contained in Proposition 8.1 because the group Γ is allowed to have parabolic elements.

Proposition 8.4. *Let $k \geq 2$ be an integer and let Γ be a discrete subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. Suppose that Γ is k-semifree. Suppose in addition that for every subgroup Γ' of Γ having rank at most k, either*

(i) *Γ' is geometrically finite, or*
(ii) *$N' = \mathbf{H}^3/\Gamma'$ admits no non-constant positive superharmonic functions.*

Then $\log(2k-1)$ is a strong k-Margulis number for Γ.

Proof. If $\xi_1, \dots, \xi_k \in \Gamma$ are elements of Γ, the group $\langle \xi_1, \dots, \xi_k \rangle$ is semifree. Thus we may write it as a free product $A_1 * \cdots * A_r$, where r is an integer $\leq k$ and $A_1, \dots, A_r$ are free abelian groups. The sum of the ranks of the A_i is at most k. If $r < k$, or if some A_i has rank > 1, then $\langle \xi_1, \dots, \xi_k \rangle$ is generated by at most $k-1$ abelian subgroups. Now suppose that $r = k$ and that the A_i are all cyclic. Then $\langle \xi_1, \dots, \xi_k \rangle$ is free of rank k and is therefore freely generated by $\xi_1, \dots, \xi_k$. If condition (i) of the hypothesis of the proposition holds, it follows from Theorem 6.1(b) that for any $z \in \mathbf{H}^3$ we have

$$\sum \frac{1}{1 + e^{\operatorname{dist}(\xi_i \cdot z, z)}} \le \frac{1}{2} = \frac{k}{1 + e^{\log(2k-1)}}.$$

If condition (ii) holds, the same conclusion follows from Theorem 6.1(d). q.e.d.

If a torsion-free Kleinian group Γ is geometrically finite and has infinite covolume, then a theorem of Thurston's (see Proposition 7.1 in Morgan [31]) guarantees that every finitely generated subgroup of Γ is geometrically finite. This yields the following corollary to Proposition 8.4.

Corollary 8.5. *Let $k \ge 2$ be an integer, and let Γ be a discrete subgroup of* $\operatorname{Isom}_+(\mathbf{H}^3)$ *which is geometrically finite and k-semifree and has infinite covolume. Then* $\log(2k-1)$ *is a strong k-Margulis number for* Γ.

This result can also be combined with the results from Section 7 as in the following corollary.

Corollary 8.6. *Let $k \ge 2$ be an integer, and let N be a geometrically finite orientable hyperbolic 3-manifold of infinite volume. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$. Then* $\log(2k-1)$ *is a strong k-Margulis number for N.* q.e.d.

Proof. We write $N = \mathbf{H}^3/\Gamma$, where Γ is a geometrically finite Kleinian group. Since N has infinite volume and $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$, Corollary 7.3 guarantees that $\Gamma \cong \pi_1(N)$ is k-semifree. The assertion now follows from Corollary 8.5. q.e.d.

The next corollary generalizes Corollary 8.3.

Corollary 8.7. *Let $k \ge 2$ be an integer, and let N be an orientable hyperbolic 3-manifold of finite volume. Suppose that the first betti number of N is at least $k+1$, and $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$. Then* $\log(2k-1)$ *is a strong k-Margulis number for N.*

Proof. We write $N = \mathbf{H}^3/\Gamma$, where Γ is a Kleinian group of finite covolume. It follows from Corollary 7.4 that $\Gamma \cong \pi_1(N)$ is k-semifree. To complete the proof it suffices to show that for every subgroup Γ'

of Γ whose rank is at most k, one of the hypotheses (i) and (ii) of Proposition 8.4 holds.

Since N has first betti number at least $k+1$, the subgroup Γ' is contained in the kernel of a surjective homomorphism $\beta : \pi_1(N) \to \mathbf{Z}$. Therefore, by Corollary E in [9], Γ' is either geometrically finite, or N has a finite cover $\widehat{N}$ which fibers over the circle and Γ' is topologically tame and contains the fiber subgroup Γ'' of $\widehat{N}$. In the latter case we have $\Lambda(\Gamma') = \overline{\mathbf{C}}$, since $\Lambda(\Gamma'') = \overline{\mathbf{C}}$. Corollary 9.2 in [8] then guarantees that $N' = \mathbf{H}^3/\Gamma'$ admits no non-constant positive superharmonic functions. q.e.d.

Finally, by specializing some of the results stated above to the case $k=2$ we obtain some sufficient conditions for $\log 3$ to be a Margulis number for a hyperbolic 3-manifold.

Corollary 8.8. *Let $N = \mathbf{H}^3/\Gamma$ be an orientable hyperbolic 3-manifold, such that either*

(i) *N is geometrically finite and has infinite volume,*
(ii) *N is topologically tame, purely loxodromic, and has infinite volume, or*
(iii) *N has finite volume and its first betti number is at least 3.*

Then $\log 3$ *is a strong Margulis number for* Γ.

Proof. As we observed at the beginning of this section, a Margulis number is the same thing as a 2-Margulis number. Under the hypothesis (i), (ii) or (iii), the assertion follows respectively from Corollary 8.6, Corollary 8.2, or Corollary 8.7. The general version of each of these corollaries includes the assumption that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$. For $k=2$ this condition is vacuously true. q.e.d.

Remark 8.9. Given Corollary 8.8 it seems reasonable to conjecture that log 3 is a strong Margulis number for any infinite volume hyperbolic 3-manifold. We notice that our conjecture would follow from the conjecture that every free 2-generator Kleinian group is a limit of Schottky groups. There appear to exist closed hyperbolic 3-manifolds for which $\log 3$ is not even a Margulis number; computations by Hodgson and Weeks give strong evidence that the Weeks manifold [40] does not contain a ball of radius $(\log 3)/2$.

9. Geometric estimates for closed manifolds

In this section we will prove the results promised in the introduction concerning the balls of radius $\frac{1}{2}\log 5$ and the volume estimates for closed manifolds of first betti number at least 4. This will be done by combining the results of the last section with the following result, which illustrates the use of the notion of a k-Margulis number for $k > 2$.

Theorem 9.1. *Let N be an orientable hyperbolic 3-manifold without cusps. Suppose that $\pi_1(N)$ is 3-free. Let λ be a 3-Margulis number for N. Then either N contains a hyperbolic ball of radius $\lambda/2$, or $\pi_1(N)$ is a free group of rank 2.*

Before giving the proof of Theorem 9.1 we shall point out how to use it to prove the corollaries stated in the introduction.

Corollary 9.2. *Let N be a closed orientable hyperbolic 3-manifold. Suppose that the first betti number $\beta_1(N)$ is at least 4, and that $\pi_1(N)$ has no subgroup isomorphic to $\pi_1(S_2)$. Then N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$. Hence the volume of N is greater than 3.08.*

Proof. According to Corollary 7.4 and Remark 7.5, the group $\pi_1(N)$ is 3-free. By Corollary 8.3, $\log 5$ is a strong 3-Margulis number, and *a fortiori* a 3-Margulis number, for N. It therefore follows from Theorem 9.1 that either N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$ or $\pi_1(N)$ is a free group of rank 2. The latter alternative is impossible, because Γ, as the fundamental group of a closed hyperbolic 3-manifold, must have cohomological dimension 3, whereas a free group has cohomological dimension 1. Thus N must contain a hyperbolic ball of radius $\frac{1}{2}\log 5$.

The lower bound on the volume now follows by applying Böröczky's density estimate for hyperbolic sphere-packings as in [14]. q.e.d.

Let $\mathcal{W}$ denote the set of all finite volumes of orientable hyperbolic 3-manifolds. Then $\mathcal{W}$ is a set of positive real numbers, and by restricting the usual ordering of the real numbers we can regard $\mathcal{W}$ as an ordered set. It is a theorem of Thurston's, based on work due to Jorgensen and Gromov, that $\mathcal{W}$ is a well-ordered set having ordinal type ω^ω and that there are at most a finitely many of isometry types of hyperbolic 3-manifolds with a given volume. (See [4, E.1]) Thus there is a unique order-preserving bijection between $\mathcal{W}$ and the set of ordinal numbers less than ω^ω. Let us denote by v_c the element of $\mathcal{W}$ corresponding to the ordinal number c.

Corollary 9.3. *Let c be any ordinal number less than 8ω, and let N be any orientable hyperbolic 3-manifold with $\operatorname{vol} N = v_c$. Then either the first betti number of N is at most 3, or $\pi_1(N)$ contains an isomorphic copy of $\pi_1(S_2)$.*

Proof. Assume that N has first betti number at least 4 and contains no isomorphic copy of $\pi_1(S_2)$. Then by Corollary 9.2 we have $v_c = \operatorname{vol} N > 3.08$. On the other hand, Weeks census (see [18] and [39]) lists 8 distinct volumes less than 3.08 for orientable manifolds with one cusp.

The volume of such a cusped manifold is the limit, from below, of the volumes of its Dehn fillings (see Theorem E.7.2 in [4]). Hence the result follows. q.e.d.

Corollary 9.4. *Let N be a non-compact, topologically tame, orientable hyperbolic 3-manifold without cusps. Suppose that: (i) $\pi_1(N)$ is not a free group of rank 2, and (ii) $\pi_1(N)$ has no subgroup isomorphic to $\pi_1(S_2)$. Then N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$.*

Proof. According to Corollary 8.2 and Remark 7.5, the group $\pi_1(N)$ is 3-free. By Corollary 8.3, $\log 5$ is a strong 3-Margulis number, and *a fortiori* a 3-Margulis number, for N. It therefore follows from Theorem 9.1 (and hypothesis (ii)) that N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$. q.e.d.

The rest of this section is devoted to the proof of Theorem 9.1. The essential ideas of the proof appear in the proof of Theorem B in [15]. We begin by reviewing and extending a few notions from [15].

As in [15], we shall say that elements $z_1, \ldots, z_r$ of a group Γ are *independent* if they freely generate a (free, rank-r) subgroup of Γ. Recall that the *rank* of a finitely generated group G to be the minimal cardinality of a generating set for G.

As in [15], a Γ-*labeled complex*, where Γ is a group, is defined to be an ordered pair $(K, (X_v)_v)$, where K is a simplicial complex, and $(X_v)_v$ is a family of cyclic subgroups of Γ indexed by the vertices of K. If $(K, (X_v)_v)$ is a Γ-labeled complex, then for any subcomplex L of K we denote by $\Theta(L)$ the subgroup of Γ generated by all the groups X_v, where v ranges over the vertices of L.

In this paper we shall use one notion which appeared only implicitly in [15]. Let Γ be a group and let $(K, (X_v)_v)$ be a Γ-labeled complex. By a *natural action of* Γ *on* $(K, (X_v)_v)$ we shall mean a simplicial action of Γ on K such that for each vertex v of K we have $X_{\gamma \cdot v} = \gamma X_v \gamma^{-1}$. The following result could have been stated and used in [15].

Proposition 9.5. *Let Γ be a finitely generated 3-free group in which every non-trivial element has a cyclic centralizer. Let $(K, (X_v)_v)$ be a Γ-labeled complex which admits a natural Γ-action. Suppose the following: X_v is a maximal cyclic subgroup of Γ for every vertex v of K, K is connected and has more than one vertex, the link of every vertex of K is connected, for every 1-simplex e of K the group $\Theta(|e|)$ is non-abelian, and there is no 2-simplex σ of K such that $\Theta(|\sigma|)$ is free of rank 3. Then $\Theta(K)$ is a free group of rank 2.*

Proof. The hypotheses of the above proposition include those of Proposition 4.3 of [15]. According to the latter result, $\Theta(K)$ has *local rank* 2; according to the definitions given in [15], this means that every finitely generated subgroup of $\Theta(K)$ is contained in a subgroup of rank ≤ 2, but that not every finitely generated subgroup of $\Theta(K)$ is contained in a subgroup of rank ≤ 1. On the other hand, the existence of a natural action of Γ on $(K, (X_v)_v)$ clearly implies that $\Theta(K)$ is a normal subgroup of Γ.

Now choose any vertex v_0 of K, and let x_0 denote a generator of $X_0 = X_{v_0}$. Since x_0 has a cyclic centralizer and X_0 is a maximal cyclic subgroup of Γ, the element x_0 generates its own centralizer in Γ. Now it is a special case of Proposition 4.4 of [15] that if Θ is a normal subgroup of a finitely generated 3-free group Γ, Γ is 3-free over some finitely generated subgroup of Θ, and Θ has local rank 2 and contains an element x_0 which generates its own centralizer in Γ, then Γ is a free group of rank 2.

This completes the proof. q.e.d.

Proof of Theorem 9.1. As in [15], for any infinite cyclic group X of isometries of $\mathbf{H}^3$, generated by a loxodromic isometry, and for any $\lambda > 0$, we denote by $Z_\lambda(X)$ the set of points $z \in \mathbf{H}^3$ such that $\operatorname{dist}(z, \xi \cdot z) < \lambda$ for some non-trivial element ξ of X.

Suppose that N satisfies the hypotheses of Theorem 9.1 but contains no ball of radius $\lambda/2$. We shall prove the theorem by showing that $\pi_1(N)$ is a free group of rank 2. Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a purely loxodromic Kleinian group. Then according to the discussion in subsection 3.4 of [15], the indexed family $(Z_\lambda(X))_{X \in \mathcal{X}}$, where $\mathcal{X} = \mathcal{X}_\lambda(N)$ denotes the set of all maximal cyclic subgroups X of Γ such that $Z_\lambda(X) \neq \emptyset$, is an open covering of $\mathbf{H}^3$, and the nerve $K = K_\lambda(N)$ of this covering is a simplicial complex. By definition the vertices of K are in natural one-one correspondence with the maximal cyclic subgroups in the set $\mathcal{X}$. If we denote by $X_v \in \mathcal{X}$ the maximal cyclic subgroup corresponding to a vertex v, then $(K, (X_v)_v)$ is a Γ-labeled complex.

We shall show that the group Γ and the Γ-labeled complex $(K, (X_v)_v)$ satisfy the hypotheses of Proposition 9.5. By the hypothesis of the theorem, Γ is 3-free. Since Γ is a purely loxodromic Kleinian group, it has the property that each of its non-trivial elements has a cyclic centralizer.

In order to construct a natural action of Γ on $(K, (X_v)_v)$, we first define an action of Γ on the set of vertices of K by $X_{\gamma \cdot v} = \gamma X_v \gamma^{-1}$. If $v_0, \ldots, v_m$ are the vertices of an m-simplex of K, then we have

$$\bigcap_{0 \leq i \leq m} Z_\lambda(\gamma X \gamma^{-1}) = \bigcap_{0 \leq i \leq m} \gamma \cdot Z_\lambda(X) = \gamma \cdot \bigcap_{0 \leq i \leq m} Z_\lambda(X) \neq \emptyset,$$

so that $\gamma \cdot v_0, \ldots, \gamma \cdot v_m$ are the vertices of an m-simplex of K. Thus the action of Γ on the vertex set extends to a simplicial action on K.

It is immediate from the definitions that this is a natural action on $(K, (X_v)_v)$.

By Proposition 3.4 of [15], K is a connected simplicial complex with more than one vertex, and the link of every vertex of K is connected. Now let e be any 1-simplex of K, and let v and w denote its vertices. Let x_v and x_w be generators of X_v and X_w. We have $v \neq w$ and hence $X_v \neq X_w$; that is, the elements x_v and x_w generate distinct maximal cyclic subgroups of Γ. Since the abelian subgroups of Γ are cyclic, it follows that $\Theta(|e|) = \langle x_v, x_w \rangle$ is non-abelian.

Finally, we claim that if σ is a 2-simplex of K, the group $\Theta(|\sigma|)$ cannot be free of rank 3. To prove this, let u, v and w denote the vertices of σ, and let ξ_u, ξ_v and ξ_w be generators of X_u, X_v and X_w. From the definition of the nerve K it follows that $Z_\lambda(X_u) \cap Z_\lambda(X_v) \cap Z_\lambda(X_w) \neq \emptyset$. Let z be any point of $Z_\lambda(X_u) \cap Z_\lambda(X_v) \cap Z_\lambda(X_w)$. By definition there are non-trivial elements of X_u, X_v and X_w, say $\eta_u = \xi_u^{n_u}$, $\eta_v = \xi_v^{n_v}$ and $\eta_w = \xi_w^{n_w}$, such that $\mathrm{dist}(z, \eta_u \cdot z)$, $\mathrm{dist}(z, \eta_v \cdot z)$ and $\mathrm{dist}(z, \eta_w \cdot z)$ are less than λ. Since λ is a 3-Margulis number for Γ, $\langle \eta_u, \eta_v, \eta_w \rangle$ is generated by at most two abelian subgroups. Now if $\Theta(|\sigma|)$ is free of rank 3, then ξ_u, ξ_v and ξ_w are independent, and so are η_u, η_v and η_w. This means that $\langle \eta_u, \eta_v, \eta_w \rangle$ is free of rank 3, and thus cannot be generated by two abelian subgroups. The claim is proved.

Thus Γ and $(K, (X_v)_v)$ satisfy all the hypotheses of Proposition 9.5. Hence Γ is a free group of rank 2, as required. q.e.d.

Remark 9.6. It is possible to drop the hypothesis that N has no cusps in Theorem 9.1. Because $\pi_1(N)$ is 3-free, N can only have rank 1 cusps. The construction of the Γ-labeled complex in the proof of 9.1 can still be carried out, although the arguments in [15] must be extended to account for the fact that some of the sets $Z_\lambda(X)$ will be horoballs instead of cylinders.

10. Volumes and short geodesics

Let C be a non-trivial closed geodesic in a closed hyperbolic 3-manifold N. Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a cocompact, torsion-free, discrete group of isometries of $\mathbf{H}^3$. Then C is the image in N of the axis A_γ of some non-trivial (and hence loxodromic) element $\gamma \in \Gamma$ which is uniquely determined up to conjugacy. Let us set

$$R = \frac{1}{2} \min_{\delta} \mathrm{dist}(A_\gamma, \delta \cdot A_\gamma),$$

where δ ranges over all elements of Γ which do not commute with γ. If we denote by Z the set of all points in $\mathbf{H}^3$ whose distance from A_γ is less than R, from the definition of R it follows that $Z \cap \delta \cdot Z = \emptyset$ for every $\delta \in \Gamma$ not commuting with γ; hence the quotient $Z/\langle\gamma\rangle$ embeds in N. The resulting isometric copy of $Z/\langle\gamma\rangle$ in N is called the *maximal embedded tube* about the geodesic C, and the number R is called the *radius* of the tube. If the geodesic C has length l, then the volume of the maximal embedded tube about C is given by the formula

$$\pi l \sinh^2 R, \tag{2}$$

which is therefore a lower bound for the volume of N.

In this section we prove the following result.

Proposition 10.1. *Let N be an orientable hyperbolic 3-manifold having* $\log 3$ *as a strong Margulis number. Let C be a closed geodesic in N, and let l denote its length. If R denotes the radius of the maximal embedded tube about C, then*

$$\cosh 2R \geq \frac{e^{2l} + 2e^l + 5}{(\cosh \frac{l}{2})(e^l - 1)(e^l + 3)}.$$

Combining this with Corollary 8.8 immediately yields

Corollary 10.2. *Let N be an orientable hyperbolic 3-manifold of finite volume whose first betti number is at least* 3. *Let C be a closed geodesic in N, and let l denote its length. If R is the radius of the maximal embedded tube about C, then*

$$\cosh 2R \geq \frac{e^{2l} + 2e^l + 5}{(\cosh \frac{l}{2})(e^l - 1)(e^l + 3)}.$$

The above results will also be used to give volume estimates for hyperbolic 3-manifolds containing short geodesics (see 10.3, 10.5 and 10.6 below).

Proof of 10.1. By the definition of R there is an element δ of Γ, not commuting with γ, such that the distance from A_γ to $\delta \cdot A_\gamma = A_{\delta\gamma\delta^{-1}}$ is $2R$. Let B denote the common perpendicular to the lines A_γ and $\delta \cdot A_\gamma$, and let z and w denote the points of intersection of B with A_γ and $\delta \cdot A_\gamma$ respectively. Then $\mathrm{dist}(z, w) = 2R$. Let us write $w = \delta \cdot u$ where u is a point of A_γ. Since γ acts on A_γ as a translation of length l, there is an integer m such that $\mathrm{dist}(u, \gamma^m \cdot z) \leq l/2$. Hence $\mathrm{dist}(w, \delta\gamma^m \cdot z) \leq l/2$. The triangle with vertices z, w and $\delta\gamma^m \cdot z$ has a right angle at w. Writing $\alpha = \mathrm{dist}(z, \delta\gamma^m \cdot z)$ for the hypotenuse of this right triangle and applying the Hyperbolic Pythagorean Theorem, lead to

$$\cosh \alpha = \cosh 2R \cosh \mathrm{dist}(w, \delta\gamma^m \cdot z) \leq \cosh \frac{l}{2} \cosh 2R. \tag{3}$$

Since γ and δ do not commute, neither do the elements γ and $\delta\gamma^m$ of Γ. Applying the definition of a strong Margulis number with $\xi = \gamma$ and $\eta = \delta\gamma^m$, and using $\alpha = \operatorname{dist}(z, \delta\gamma^m \cdot z)$ and $l = \operatorname{dist}(z, \gamma \cdot z)$, we obtain

$$\frac{1}{1+e^\alpha} + \frac{1}{1+e^l} \leq \frac{1}{2},$$

which can be rewritten in the form

$$e^\alpha \geq \frac{e^l+3}{e^l-1}. \tag{4}$$

On the other hand, using (2) we find that

$$\begin{aligned} e^\alpha &= \cosh\alpha + \sinh\alpha \\ &= \cosh\alpha + \sqrt{\cosh^2\alpha - 1} \\ &\leq \cosh 2R \cosh\frac{l}{2} + \sqrt{\cosh^2 2R \cosh^2\frac{l}{2} - 1}. \end{aligned}$$

Combining this with (3) gives

$$\cosh 2R \cosh\frac{l}{2} + \sqrt{\cosh^2 2R \cosh^2\frac{l}{2} - 1} \geq \frac{e^l+3}{e^l-1}. \tag{5}$$

The equation

$$x\cosh\frac{l}{2} + \sqrt{x^2\cosh^2\frac{l}{2} - 1} = \frac{e^l+3}{e^l-1}$$

has the solution

$$x_0 = \frac{e^{2l}+2e^l+5}{(\cosh\frac{l}{2})(e^l-1)(e^l+3)}.$$

Since the function $x\cosh\frac{l}{2} + \sqrt{x^2\cosh^2\frac{l}{2} - 1}$ is monotone increasing for $x \geq 1$, it follows from (4) that

$$\cosh 2R \geq x_0,$$

which is the conclusion of Proposition 10.1. q.e.d.

Let us define a function $V(x)$ for $x > 0$ by

$$V(x) = \frac{\pi x}{e^x - 1}\left(\frac{e^{2x}+2e^x+5}{2(\cosh\frac{x}{2})(e^x+3)}\right) - \frac{\pi x}{2},$$

and note that

$$\lim_{x\to 0} V(x) = \pi.$$

Since $\sinh^2 R = \frac{1}{2}(\cosh 2R - 1)$, Proposition 10.1 and the above formula (1) for the volume of a maximal tube now imply:

Lemma 10.3. *Let N be an orientable hyperbolic 3-manifold having $\log 3$ as a strong Margulis number. Let C be a closed geodesic in N, and let l denote its length. Then the maximal embedded tube about C has volume at least $V(l)$.*

The following result will permit us to put the information given by the above lemma in a more useful form.

Proposition 10.4. *The function $V(x)$ is monotonically decreasing for $x > 0$.*

Proof. For $x \geq 0$ we set

$$f(x) = \frac{V(2x)}{\pi}.$$

Since

$$\begin{aligned}
&(\cosh^2 x)(e^{4x} + 2e^{2x} - 3)^2 f'(x) \\
&\quad = (\cosh x - x\sinh x)(e^{4x} + 2e^{2x} - 3)(e^{4x} + 2e^{2x} + 5) \\
&\qquad -(\cosh^2 x)(e^{4x} + 2e^{2x} - 3)^2 \\
&\qquad -32x(\cosh x)(e^{4x} + e^{2x}),
\end{aligned}$$

$$\begin{aligned}
&(\cosh x)(e^{4x} + 2e^{2x} - 3)^2 f'(x) \\
&\quad \leq (e^{4x} + 2e^{2x} - 3)(e^{4x} + 2e^{2x} + 5) \\
&\qquad -(e^{4x} + 2e^{2x} - 3)^2 - 32x(e^{4x} + e^{2x}) \\
&\quad = 8(e^{4x}(1 - 4x) + 2e^{2x}(1 - 2x) - 3).
\end{aligned}$$

But the function $e^{4x}(1 - 4x) + 2e^{2x}(1 - 2x) - 3$ is negative-valued for $x > 0$, because it vanishes at 0, and its derivative $-xe^{4x} - 4xe^{2x}$ is negative for $x > 0$. Thus $f'(x) < 0$ for $x > 0$. q.e.d.

Combining Lemma 10.3 with Proposition 10.4, we immediately obtain:

Corollary 10.5. *Let N be an orientable hyperbolic 3-manifold having $\log 3$ as a strong Margulis number. Let λ be a positive number, and suppose that N contains a closed geodesic of length at most λ. Then the maximal embedded tube about C has volume at least $V(\lambda)$. In particular the volume of N is at least $V(\lambda)$.*

Corollary 10.6. *Let N be an orientable hyperbolic 3-manifold which has first betti number at least 3. Let λ be a positive number, and suppose that N contains a closed geodesic of length at most λ. Then the volume of N is at least $V(\lambda)$.*

Proof. We may assume that N has finite volume, as otherwise the assertion is trivial. It then follows from Corollary 8.8 that $\log 3$ is a Margulis number for N. The assertion now follows from Corollary 10.5. q.e.d.

We observe above that $\lim_{x\to 0} V(x) = \pi$. Thus Corollary 10.6 implies that if an orientable hyperbolic 3-manifold N has betti number at least 3 and contains a very short geodesic, the volume of N cannot be much less than π. Explicitly, we can say for example that if N contains a geodesic of length at most 0.1, then the volume of N is at least $V(0.1) = 2.906\ldots$. We already get non-trivial information from 10.3 and 10.4 if N contains a closed geodesic of length at most 1; in this case the results imply that N has volume at least $V(1) = 0.956\ldots$. This is greater than the smallest *known* volume $0.943\ldots$ of a closed orientable hyperbolic 3-manifold, which is in turn greater than the lower bound 0.92 established in [14] for the volume of an arbitrary closed orientable hyperbolic 3-manifold of first betti number at least 3.

In [13], Corollary 10.6 will be used as one ingredient in proving that any orientable hyperbolic 3-manifold with first betti number at least 3 has a volume exceeding that of the smallest known example, and hence that any smallest-volume orientable hyperbolic 3-manifold has first betti number at most 2.

11. A volume bound for non-compact manifolds

Theorem 11.1. *Let $N = \mathbf{H}^3/\Gamma$ be a non-compact hyperbolic 3-manifold. If N has first betti number at least 4, then N has volume at least π.*

Proof. We may assume that N has finite volume. In this case N is homeomorphic to the interior of a compact 3-manifold M with non-empty boundary ∂M which consists of a finite collection of tori. Let T_1 be a torus in ∂M and let M_n be the result of the $(1, n)$ Dehn filling of M along T_1, in terms of some fixed system of coordinates on T_1. Notice that M_n has first betti number at least 3, since N had betti number at least 4.

Thurston's Hyperbolic Dehn Surgery Theorem (see [36]) guarantees that the interior of M_n admits a hyperbolic structure for all large enough n (see also Theorem E.5.1 in [4].) Let $N_n = \mathbf{H}^3/\Gamma_n$ be a hyperbolic manifold homeomorphic to the interior of M_n. Then $\operatorname{vol} N_n < \operatorname{vol} N$ for all n and $\operatorname{vol} N_n$ converges to $\operatorname{vol} N$ (see Theorem E.7.2 in [4]). Moreover, we may assume that Γ_n converges geometrically to Γ (see Theorem E.6.2 in [4]).

Let γ_n denote an element of Γ_n representing the shortest closed geodesic in N_n. Since Γ_n converges geometrically to Γ, N has k cusps and N_n has $k-1$ cusps (for every n), we see that $l_n = l(\gamma_n)$ converges to 0 (see Theorem E.2.4 in [4]). By Corollary 10.3 we have

$$\operatorname{vol} N_n \geq V(l_n) = \frac{\pi l}{e^{l_n}-1}\left(\frac{e^{2l_n}+2e^{l_n}+5}{2(\cosh\frac{l_n}{2})(e^{l_n}+3)}\right) - \frac{\pi l_n}{2}.$$

Recall that $V(l_n)$ converges to π, since l_n converges to 0. Therefore $\operatorname{vol} N \geq \pi$. q.e.d.

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