

## LEVEL SET APPROACH TO MEAN CURVATURE FLOW IN ARBITRARY CODIMENSION

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### Abstract

We develop a level set theory for the mean curvature evolution of surfaces with arbitrary co-dimension, thus generalizing the previous work [8, 15] on hypersurfaces. The main idea is to surround the evolving surface of codimension- $k$  in  $\mathbf{R}^d$  by a family of hypersurfaces (the level sets of a function) evolving with normal velocity equal to the sum of the  $(d - k)$  smallest principal curvatures. The existence and the uniqueness of a weak (level-set) solution is easily established by using mainly the results of [8] and the theory of viscosity solutions for second order nonlinear parabolic equations. The level set solutions coincide with the classical solutions whenever the latter exist. The proof of this connection uses a careful analysis of the squared distance from the surfaces. It is also shown that varifold solutions constructed by Brakke [7] are included in the level-set solutions. The idea of surrounding the evolving surface by a family of hypersurfaces with a certain property is related to the barriers of De Giorgi. An introduction to the theory of barriers and its connection to the level set solutions is also provided.

### 1. Introduction

Recently, Evans & Spruck [15] and, independently, Chen, Giga & Goto [8] developed a level set approach for hypersurfaces evolving by their mean curvature. We extend this approach to surfaces with arbitrary co-dimension.

In the classical setup, mean curvature flow is a geometric initial value problem. Starting from a smooth initial surface  $\Gamma_0$  in  $\mathbf{R}^d$ , the solution  $\Gamma_t$  evolves in time so that at each point its normal velocity vector is equal to its mean curvature vector. By parametric methods of differential geometry much has been obtained for convex or graph-like initial surfaces or for planar curves. See for instance Altschuler & Grayson [3], Ecker & Huisken [13], Gage & Hamilton [21], Grayson [23], and Huisken [25]. However for  $d \geq 3$ , initially smooth surfaces may develop geometric singularities. For example the dumbbell region in  $\mathbf{R}^3$  splits into two

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pieces in finite time (c.f. [2], [24]) or a “fat” enough torus closes its interior hole in finite time (c.f. [39]). Also it is easily seen that smooth curves in  $\mathbf{R}^3$  may self intersect in finite time.

Several weak solutions have been proposed. In his pioneering work, Brakke [7] uses geometric measure theory to construct a (generally nonunique) varifold solution with arbitrary co-dimension. Ilmanen’s monograph [26] provides an excellent account of this theory including the connections between different approaches and a partial regularity result. Also see Almgren, Taylor & Wang [1] for a related variational approach and the survey of Taylor, Cahn & Handwerker [40].

For codimension-one surfaces, a completely different approach, initially suggested in the physics literature by Ohta, Jasnow & Kawasaki [34], for numerical calculations by Sethian [35] and Osher & Sethian [33], represents the evolving surfaces as the level set of an auxiliary function solving an appropriate nonlinear differential equation. This “level-set” approach has been extensively developed by Chen, Giga & Goto [8] and, independently, by Evans & Spruck [15]. Their approach is this. Given an initial hypersurface  $\Gamma_0$ , select a function  $u_0 : \mathbf{R}^d \rightarrow \mathbf{R}$  so that

$$(1.1) \quad \Gamma_0 = \{x \in \mathbf{R}^d : u_0(x) = 0\}.$$

Consider then the Cauchy problem

$$(1.2) \quad u_t = |\nabla u| \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right), \quad \text{in } \mathbf{R}^d \times (0, \infty),$$

with initial data

$$(1.3) \quad u(x, 0) = u_0(x), \quad \forall x \in \mathbf{R}^d,$$

for the unknown scalar function  $u(x, t)$ . In the regions where  $u$  is smooth and  $\nabla u$  does not vanish,

$$\frac{u_t}{|\nabla u|}, \quad \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right)$$

are, respectively, the normal velocity and the scalar mean curvature of the level set of  $u$ . Hence (1.2) implies that each level set of  $u$  evolves according to its mean curvature, at least in the regions where  $u$  is smooth and  $\nabla u$  does not vanish. So it is reasonable to define

$$\Gamma_t = \{x \in \mathbf{R}^d : u(x, t) = 0\}.$$

Observe that (1.2) is degenerate, and it is not well defined when  $\nabla u$  is zero. Evans & Spruck and Chen, Giga & Goto overcame these difficulties

by using the theory of viscosity solutions ([11, 9, 10, 20]). In particular, in [8, 15] it is proved that under very general hypotheses, there is a unique viscosity solution  $u$  of (1.2), (1.3), and that  $\Gamma_t$  depends only on the geometric initial data  $\Gamma_0$ , but not on the auxiliary function  $u_0$  and whence  $\Gamma_t$  is a well defined evolution of  $\Gamma_0$ . Other interesting properties of  $\Gamma_t$ , including Hausdorff dimension estimates and local time existence of classical solutions, are obtained in a series of papers [16, 17, 18]. Also [8] demonstrates that the level set approach for hypersurfaces is robust enough to treat equations more general than the mean curvature flow.

More intrinsic definitions related to the level-set solutions have also been introduced. [37] recasts the definitions, constructions and uniqueness criteria into a different form using the (signed) distance function to the surface (also see [4]). In [27], Ilmanen uses smooth classical solutions as test functions to define set-theoretic subsolutions. These subsolutions were then used in [26] to prove a connection between the varifold solutions of Brakke and the level-set solutions. In [29], Ishii & Souganidis analyze general equations with arbitrary growth in the curvature term.

In [12], De Giorgi introduces the notion of barriers for very general equations, including the mean curvature flow with arbitrary codimension. For codimension-one surfaces, barriers are related to the level-set solutions of [8, 15] (see [5], [6]), and in higher co-dimensions De Giorgi's definition is the starting point of this paper. A discussion of the barriers and their connection to level-set solutions is given in §6 below. Finally, the singular limit of a reaction-diffusion equation with a cubic nonlinearity also provides an approximation and a possibly different definition for hypersurfaces moving by their curvature. However, this approach is shown to coincide with the previous definitions; see [19], [38] and the references therein. Katsoulakis & Souganidis [31] proves the convergence of a particle system to mean curvature flow.

Smooth surfaces with codimension- $k$  can be represented as the intersection of the level sets of  $k$  scalar functions with nonvanishing gradients on the surface. Then proceeding as in codimension-one case, we can obtain a *system* of partial differential equations generalizing (1.2). However, since this generalization is a degenerate system of equations, we can no longer employ (as was done in [15, 8]) the theory of viscosity solutions or any other existing theory to analyze the resulting equations. Therefore it is desirable to obtain an alternate approach using only one scalar function. We achieve this representation following the lectures of De Giorgi [12].

To explain the main idea, let  $\Gamma \subset \mathbf{R}^d$  be a smooth surface with co-

dimension  $k > 1$ ,  $v : \mathbf{R}^d \rightarrow [0, \infty)$  be an auxiliary function

$$\Gamma = \{x \in \mathbf{R}^d : v(x) = 0\}.$$

Assume that  $v$  is smooth near  $\Gamma$  and its spatial gradient does not vanish outside  $\Gamma$ . The key step is to express the curvature properties of  $\Gamma$  in terms of the derivatives of this auxiliary function  $v$ . To accomplish this, we consider the  $\epsilon$ -level set,  $\Gamma^\epsilon$ , of  $v$  for small  $\epsilon > 0$ . For  $x \notin \Gamma$  but near  $\Gamma$ , let  $J(x)$  be the symmetric,  $d \times d$  matrix

$$J(x) = \frac{1}{|\nabla v(x)|} P_{\nabla v(x)} \nabla^2 v(x) P_{\nabla v(x)},$$

where for a nonzero vector  $p \in \mathbf{R}^d$ ,

$$(1.4) \quad P_p = I - \frac{p \otimes p}{|p|^2}.$$

Further let

$$\lambda_1(J) \leq \lambda_2(J) \leq \dots \leq \lambda_{d-1}(J)$$

be the eigenvalues of  $J(x)$  corresponding to eigenvectors *orthogonal* to  $\nabla v(x)$  (note that  $J(x)\nabla v(x) = 0$ ). These eigenvalues are equal to the principal curvatures of the codimension-one surface  $\Gamma^\epsilon$ , oriented by  $\nabla v$  (see Remark 2.7).

Since  $\Gamma$  has codimension- $k$ , for small enough  $\epsilon$ , we expect  $\Gamma^\epsilon$  to have very large  $k - 1$  principal curvatures and the remaining  $d - k$  principal curvatures of  $\Gamma^\epsilon$  to be related to the geometry of  $\Gamma$ . Indeed, approaching to  $\Gamma$  from a normal direction  $p \in S^{d-1}$ , their sum converges to  $-H \cdot p$ , where  $H$  is the mean curvature vector of  $\Gamma$  (see Remark 3.3).

Preceding computations together with [8, 15] suggest the following level-set definition for the codimension- $k$  mean curvature flow. For a symmetric,  $d \times d$  matrix  $A$ , and  $p \in \mathbf{R}^d$  with  $p \neq 0$ , set  $X = P_p A P_p$ , and let

$$\lambda_1(X) \leq \lambda_2(X) \leq \dots \leq \lambda_{d-1}(X)$$

be the eigenvalues of  $X$  corresponding to eigenvectors *orthogonal* to  $p$  (observe that 0 is an eigenvalue of  $X$  corresponding to  $p$ ) and define

$$(1.5) \quad F(p, A) = \sum_{i=1}^{d-k} \lambda_i(X).$$

Given an initial data  $\Gamma_0$ , choose a scalar function  $u_0 : \mathbf{R}^d \rightarrow [0, \infty)$  satisfying (1.1). Then consider the equation

$$(1.6) \quad u_t = F(\nabla u, \nabla^2 u), \quad \text{in } \mathbf{R}^d \times (0, \infty),$$

with initial data (1.3), for the unknown *scalar* function  $u(x, t)$ . In §2, below we will show that (1.6) is degenerate parabolic and that the extension of the viscosity theory developed in [8] applies to (1.6). In particular, for a given uniformly continuous  $u_0$ , there is a unique viscosity solution  $u$  satisfying (1.6) and (1.3). Moreover

$$(1.7) \quad \Gamma_t = \{x \in \mathbf{R}^d : u(x, t) = 0\}$$

depends only on  $\Gamma_0$ , but not on  $u_0$ . Hence  $\Gamma_t$  is a well-defined evolution of  $\Gamma_0$ .

Clearly every weak theory has to be consistent with the classical solutions whenever the latter exist. In §3, we prove that if there is a classical solution  $\Gamma'_t$  of the geometric initial value problem, then it coincides with the level-set solution  $\Gamma_t$ . This is done by analyzing the properties of the distance function  $\delta(x, t)$  to  $\Gamma'_t$  and the square distance function  $\eta = \delta^2/2$ . We first show that  $\Gamma'_t$  is a classical solution of the mean curvature flow if and only if  $\eta$  is smooth in a neighborhood of  $\Gamma'_t$  and satisfies

$$\nabla \eta_t = \Delta \nabla \eta, \quad \text{on } \Gamma'_t;$$

see Lemma 3.7 below for the precise statement. Using this identity, we prove that  $\delta$  solves a parabolic equation in a tubular neighborhood of  $\Gamma'_t$ . Then it follows that for sufficiently large  $K$ ,  $e^{-Kt}\delta$  is a subsolution of (1.6) in a tubular neighborhood of  $\Gamma'_t$ . Thus by comparing  $\delta$  to a solution of (1.6) in this tubular region, we conclude that  $\Gamma'_t$  includes  $\Gamma_t$ . The reverse inclusion is proved after showing that  $\delta$  is a viscosity supersolution of (1.6).

This final property also suggests an intrinsic definition using the distance function as in [37]. Briefly, we say that  $\Gamma_t$  is a distance solution if its distance function  $\delta(x, t)$  is a viscosity supersolution of (1.6). Then as in codimension-one case, the zero level-set of  $u$  is the maximal distance solution; see Theorem 4.4 below.

In §5, we study the varifold solutions of Brakke. Following Ilmanen’s computations for hypersurfaces [26, §10], we show that the distance function to any Brakke solution is a viscosity supersolution of (1.6). Hence every Brakke solution is a distance solution. Since the level-set solution is the maximal distance solution, it includes the Brakke solutions. However, in general the level-set solution or a distance solution need not be a Brakke solution.

As mentioned at the beginning of this introduction, the starting point of our analysis is the notion of barriers defined by De Giorgi [12]. In §6, we give a brief introduction to De Giorgi’s barriers. Then we discuss the connection between the level set solutions and the barriers, in the

same spirit of the work of Bellettini & Paolini [6] in the codimension-1 case.

### 2. Level set solutions

We start with a brief review of several standard notation, definitions and results from the theory of viscosity solutions. An excellent introduction to this theory is the *User's Guide* [10].

For any function  $w$ , the upper semicontinuous envelope  $w^*$  of  $w$  is the smallest upper semicontinuous function that is greater than or equal to  $w$ . Similarly, the lower semicontinuous envelope  $w_*$  of  $w$  is the largest lower semicontinuous function that is less than or equal to  $w$ . Let  $F$  be as in (1.5). Then  $F^*(p, A) = F_*(p, A) = F(p, A)$  on  $p \neq 0$ , and for  $p = 0$ ,

$$F^*(0, A) = \max\{F(\nu, A) : |\nu| = 1\},$$

$$F_*(0, A) = \min\{F(\nu, A) : |\nu| = 1\}.$$

We continue with the definition of viscosity solutions. Although the unique viscosity solution of (1.6) is continuous, discontinuous sub- and supersolutions are often useful tools. So in the following definition we do not assume the continuity of  $u$ .

Let  $S^{d \times d}$  be the set of all symmetric,  $d \times d$  matrices.

**Definition 2.1** (Viscosity solutions). Let  $\Omega \subset \mathbf{R}^d$  be an open set, let  $u : \Omega \times [0, T) \rightarrow \mathbf{R}$  be a locally bounded function and let  $G : \mathbf{R} \times (\mathbf{R}^d \setminus \{0\}) \times S^{d \times d} \rightarrow \mathbf{R}$ .

a) We say that  $u$  is a *viscosity subsolution* of

$$(2.1) \quad u_t = G(u, \nabla u, \nabla^2 u)$$

in  $\Omega \times (0, T)$  if for any  $\phi \in C^2(\Omega \times (0, T))$

$$\phi_t(y, t) \leq G^*(u^*(y, t), u(\nabla\phi(y, t), \nabla^2\phi(y, t)))$$

at any local maximizer  $(y, t) \in \Omega \times (0, T)$  of the difference  $(u^* - \phi)$ . (If for a given  $\phi$  there are no local maximizers of the difference  $(u^* - \phi)$ , then there is nothing to check!)

b) Similarly, we say that  $u$  is a *viscosity supersolution* of (2.1) in  $\Omega \times (0, T)$  if for any  $\phi \in C^2(\Omega \times (0, T))$

$$\phi_t(y, t) \geq G_*(u_*(y, t), \nabla\phi(y, t), \nabla^2\phi(y, t))$$

at any local minimizer  $(y, t) \in \Omega \times (0, T)$  of the difference  $(u_* - \phi)$ . (Again if for a given  $\phi$  there are no local minimizers of the difference  $(u_* - \phi)$ , then there is nothing to check!)

c) Finally,  $u$  is a *viscosity solution* of (2.1) in  $\Omega \times (0, T)$  if it is both a viscosity subsolution and a viscosity supersolution of (2.1) in  $\Omega \times (0, T)$ .

We now state a comparison result which follows from Theorem 2.1 of [22].

**Theorem 2.2** (Comparison). *Let  $u, v$  be respectively a viscosity subsolution and a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ . Suppose that  $u^*(\cdot, 0)$  or  $v_*(\cdot, 0)$  is uniformly continuous, and that there exists a constant  $K$  satisfying,*

$$|u(x, t)| + |v(x, t)| \leq K(1 + |x|).$$

Then

$$(u^* - v_*)(x, t) \leq \sup\{(u^* - v_*)(y, 0) : y \in \mathbf{R}^d\}, \quad \forall (x, t) \in \mathbf{R}^d \times [0, T].$$

Note that if  $u$  and  $v$  are uniformly continuous, then the assumption on the growth of  $u$  and  $v$  is automatically satisfied.

*Proof.* This theorem follows directly from Theorem 2.1 of [22]. In the following steps, we will show that  $F$  satisfies the hypotheses of [22, Theorem 2.1].

1. It is clear from the explicit forms of  $F^*$  and  $F_*$  that

$$F^*(0, O) = F_*(0, O) = 0.$$

Moreover, for every  $\rho > 0$ ,  $F$  is uniformly continuous on  $\{|p| \geq \rho\} \times S^{d \times d}$  and  $F(p, A)$  grows linearly in  $A$ .

2. Let  $H$  be a  $(d - 1)$ -dimensional space, let  $X$  be a symmetric bilinear form on  $H$  and let

$$\lambda_1(X) \leq \lambda_2(X) \leq \dots \leq \lambda_{d-1}(X)$$

be the eigenvalues of  $X$ . We claim that

$$(2.2) \quad \lambda_i(X) = \max\left\{\min_{\nu \in E} \frac{X\nu \cdot \nu}{|\nu|^2} : E \subset H, \text{ codim}(E) \leq i - 1\right\}.$$

The above identity is proved in [30, Theorem 6.44] and shows that  $\lambda_i(X)$  depend monotonically on  $X$ . For completeness, we give its elementary proof in the next step. Now the above formula and the definition of  $F$  imply that  $F$  is degenerate elliptic, i.e.,

$$F(p, A) \geq F(p, B), \quad \forall p \neq 0, A \geq B \in S^{d \times d},$$

because  $A \geq B$  implies  $P_p A P_p \geq P_p B P_p$  on the hyperplane  $H$  orthogonal to  $p$ . Hence  $F$  satisfies all the hypotheses of Theorem 2.1 of [22]. Set

$$V = v_* + \sup\{(u^* - v_*)(y, 0) : y \in \mathbf{R}^d\}.$$

Then  $V$  is a supersolution and  $u^*(x, 0) \leq V(x, 0)$  for all  $x$ . Hence, by Theorem 2.1 of [22],  $u^* \leq V$  in  $\mathbf{R}^d \times [0, T]$ .

**3.** In this step, we prove (2.2). Let  $L$  denote the right-hand side in (2.2). The inequality  $\lambda_i(X) \leq L$  easily follows by choosing  $E$  to be the vector space generated by the eigenvectors corresponding to  $\lambda_i, \dots, \lambda_{d-1}$ . To prove the reverse inequality, let  $E$  be any subspace with codimension at most  $(i - 1)$  and let  $E'$  be the vector space generated by the eigenvectors corresponding to  $\lambda_1(X), \dots, \lambda_i(X)$ . Since

$$\dim(E) + \dim(E') \geq (d - i) + i > d - 1,$$

there exists a unit vector  $\nu_0 \in E \cap E'$ . We thus have

$$\min_{\nu \in E} \frac{X\nu \cdot \nu}{|\nu|^2} \leq X\nu_0 \cdot \nu_0 \leq \lambda_i(X).$$

The final inequality follows from the fact that  $\nu_0$  belongs to  $E'$  and that  $E'$  is spanned by the first  $i$  eigenvectors.

Since  $F$  is geometric, i.e.,

$$(2.3) \quad F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X) \quad \forall \lambda > 0, \sigma \in \mathbf{R},$$

equation (1.6) is invariant under the relabelling of the level sets:

**Theorem 2.3.** (Invariance). *Let  $\theta : \mathbf{R} \rightarrow \mathbf{R}$  be a continuous nondecreasing function and let  $u$  be a viscosity subsolution (supersolution) of (1.6) in  $\Omega \times (0, T)$ . Then  $\theta(u)$  is still a viscosity subsolution (supersolution) of (1.6) in  $\Omega \times (0, T)$ .*

The above theorem follows from (2.3) and Theorem 5.6 in [8]. A formal proof can also be obtained by a direct computation.

The following existence theorem is an immediate corollary of [8, Theorem 6.8]

**Theorem 2.4** (Existence). *For any uniformly continuous function  $u_0$  there exists a unique, uniformly continuous viscosity solution  $u$  of (1.6) and (1.3).*

*Proof.* Since  $u_0$  is uniformly continuous, there is a constant  $K^* > 1$  satisfying

$$|u_0(x)| \leq K^*[1 + |x|].$$

1. For  $R > 0$ , let

$$u_0^R(x) = \min\{u_0(x) + h(|x| - R), R\},$$

where for  $r \leq 0$ ,  $h(r) = 0$  and for  $r \geq 0$ ,

$$h(r) = 2K^*[\sqrt{r^2 + 1} - 1].$$

Since  $h(|x| - R)$  grows faster than  $|u_0(x)|$  as  $|x| \rightarrow \infty$ ,  $u_0^R$  is equal to  $R$  outside a large ball. Hence Theorem 6.8 of [8] implies that there exists



a uniformly continuous viscosity solution  $u^R$  of (1.6) satisfying the initial condition  $u^R(x, 0) = u_0^R(x)$ . In the next several steps, we will show that  $u^R$  is equicontinuous in  $R$  and then we will let  $R \rightarrow \infty$ .

**2.** Since  $u_0$  is uniformly continuous and  $h$  is Lipschitz continuous, there is a modulus  $m$ , independent of  $R$ , satisfying

$$|u_0^R(x) - u_0^R(y)| \leq m(|x - y|), \quad x, y \in \mathbf{R}^d,$$

where a modulus  $m$  is a nondecreasing, continuous function on  $[0, \infty)$ , with  $m(0) = 0$ . Since (1.6) is translation invariant in space, the comparison result, Theorem 2.2, implies that

$$|u^R(x, t) - u^R(y, t)| \leq m(|x - y|), \quad \forall x, y \in \mathbf{R}^d, t \geq 0.$$

**3.** Fix  $x_0 \in \mathbf{R}^d$ . It is easy to verify that the function  $\phi(x, t) = |x - x_0|^2 + 2(d - k)t$  is a solution of (1.6). Now let

$$b(x, t) = u_0(x_0) + m(\sqrt{|x - x_0|^2 + 2(d - k)t}).$$

By Theorem 2.3,  $b$  is a viscosity solution of (1.6). Since by construction,  $u_0^R(x) \leq b(x, 0)$ , Theorem 2.2 implies that  $u^R \leq b$ . In particular,

$$u^R(x_0, t) - u_0^R(x_0) \leq m(2(d - k)t), \quad \forall t \geq 0.$$

An entirely similar argument using  $\hat{b} = u_0 - m(\dots)$  yields the opposite inequality. Then, for any  $t \geq 0$  we have,

$$|u^R(x_0, t) - u_0^R(x_0)| \leq m(2(d - k)t).$$

Hence the translation invariance in time of the equation implies that

$$|u^R(x_0, t) - u^R(x_0, s)| \leq m(2(d - k)|t - s|).$$

**4.** Previous steps show that the sequence  $u^R$  is equicontinuous. Also as  $R \rightarrow \infty$ ,  $u_0^R$  converges to  $u_0$  locally uniformly. Then the well known stability properties of the viscosity solutions (c.f. [8, Proposition 2.4]) together with the Ascoli-Arzelà Theorem enable us to let  $R \rightarrow \infty$  and construct a uniformly continuous solution  $u = \lim u^R$ , satisfying the initial data (1.3).

Next result yields that the zero level set of any viscosity solution at time  $t > 0$ , depends only on the zero level set of the initial data but not on the other level sets of the initial data. Similar results were already proved in [8, 15].

**Theorem 2.5.** *Assume that  $\Gamma_0$  is a closed subset of  $\mathbf{R}^d$ . Let  $u_0$  be any nonnegative, uniformly continuous function satisfying (1.1) and let*

$u$  be the viscosity solution of (1.6) satisfying the initial data (1.3). Then the zero level sets

$$\Gamma_t = \{x \in \mathbf{R}^d : u(x, t) = 0\}$$

are independent of the choice of  $u_0$ .

*Proof.* Let  $u'(x, t)$  be the unique uniformly continuous, viscosity solution of (1.6) satisfying

$$u'_0(x) := u'(x, 0) = \text{dist}(x, \Gamma_0).$$

We will prove that the sets

$$\Gamma_t := \{x \in \mathbf{R}^d : u(x, t) = 0\}, \quad \Gamma'_t := \{x \in \mathbf{R}^d : u'(x, t) = 0\},$$

coincide for all  $t \geq 0$ .

1. Set

$$\omega(t) := \sup \left\{ u_0(x) : \text{dist}(x, \Gamma_0) \leq t \right\}.$$

Since  $u_0$  is uniformly continuous,  $\omega$  is nondecreasing and uniformly continuous. Moreover,

$$0 \leq u_0(x) \leq \omega(\text{dist}(x, \Gamma_0)), \quad \forall x \in \mathbf{R}^d.$$

By Theorem 2.3,  $u(x, t)$  and  $\omega(u'(x, t))$  are solutions of (1.6), and by Theorem 2.2 we conclude that

$$0 \leq u(x, t) \leq \omega(u'(x, t)), \quad \forall x \in \mathbf{R}^d, t \in [0, +\infty).$$

Hence  $\Gamma'_t \subset \Gamma_t$  for any  $t \geq 0$ .

2. Let  $\chi(x, t)$  be the indicator of the zero level set of  $u$  and  $w = 1 - \chi$ , i.e.,  $w(x, t) = 0$  if  $u(x, t) = 0$  and otherwise  $w(x, t) = 1$ . Observe that

$$w(x, t) = \liminf_{\epsilon \rightarrow 0, (y, s) \rightarrow (x, t)} h^\epsilon(u(y, s)), \quad x \in \mathbf{R}^d, t \geq 0,$$

where

$$h^\epsilon(r) = 0, \text{ for } r \leq 0, \quad h^\epsilon(r) = 1, \text{ for } r \geq \epsilon, \quad h^\epsilon(r) = \frac{r}{\epsilon}, \text{ for } 0 \leq r \leq \epsilon.$$

Then by Theorem 2.3 and the stability theorem [8, Proposition 2.4],  $w$  is a viscosity supersolution of (1.6). Since  $u$  is continuous and  $u_0 = u(\cdot, 0)$  satisfies (1.1), we have

$$u'_0(x) \wedge 1 = \text{dist}(x, \Gamma_0) \wedge 1 \leq w(x, 0).$$

Hence by the comparison result, Theorem 2.2, we conclude that

$$0 \leq u'(x, t) \wedge 1 \leq w(x, t).$$

Consequently,  $\Gamma_t$  is included in  $\Gamma'_t$ .

**Definition 2.6.** For a given closed set  $\Gamma_0$ , let  $u$  be as in the statement of the previous theorem. Then the zero level sets  $\{\Gamma_t\}_{t \geq 0}$  of  $u(\cdot, t)$  are called the  $(d - k)$ -level set flow of  $\Gamma_0$ .

Using Theorem 2.5 and the same argument of [15], it is easy to see that the  $(d - k)$ -level set flow has the semigroup property, i.e.,  $\Gamma_{t+s}$  coincides with the evolution at time  $s$  of  $\Gamma_t$ . When dealing with unbounded sets, the restriction to uniformly continuous functions is necessary in view of a counterexample constructed by Ilmanen in [28].

*Remark 2.7.* Let us assume that  $u$  is a classical solution of (1.6) in  $\Omega \times (0, T)$ , i.e.,  $u$  is  $C^2$  and its spatial gradient does not vanish. For a real number  $\tau$ , consider the sets

$$E_t := \{x \in \mathbf{R}^d : u(x, t) = \tau\}$$

with the orientation induced by  $\nu := \nabla u / |\nabla u|$ . Let

$$B(\xi, \eta) := -(\xi \cdot d_\eta^{E_t} \nu)$$

be the second fundamental form of  $E_t$  (see [32, p.13]). Then the principal curvatures of  $E_t$  are equal to the eigenvalues,  $\kappa_1, \dots, \kappa_{d-1}$ , of the symmetric bilinear form

$$B(\xi, \eta) := -B(\xi, \eta) \cdot \nu = \xi \cdot d_\eta^{E_t} \nu$$

on the tangent space to  $E_t$ . With this sign convention (opposite to the one adopted in [32, p.30-32]), the mean curvature vector of  $E_t$  is given by

$$H = -(\kappa_1 + \dots + \kappa_{d-1})\nu$$

and the convex sets have nonnegative principal curvatures when oriented by the outer normal. A simple computation shows that  $B$  coincides with  $P_\nu \nabla^2 u P_\nu / |\nabla u|$  in the tangent space to  $E_t$ , hence

$$\frac{F(\nabla u(x, t), \nabla^2 u(x, t))}{|\nabla u(x, t)|}$$

represents the sum of the smallest  $(d - k)$  principal curvatures of  $E_t$ .

Arguing as in [15] we obtain that each level set of  $u$  flows in the direction  $-\nu$  with velocity equal to the sum of the smallest  $(d - k)$  principal curvatures.

### 3. Agreement with smooth flows

In this section, we will show that the level set solutions and the classical solutions agree whenever the latter exist. Our analysis is based on the properties of the distance and the square distance functions. We

start by proving several elementary properties of these functions. Let  $\Gamma$  be a compact subset of  $\mathbf{R}^d$  and define

$$\delta(x) := \text{dist}(x, \Gamma), \quad \eta(x) := \frac{1}{2}\delta^2(x),$$

$$I_\rho(\Gamma) := \{x \in \mathbf{R}^d : \delta(x) \leq \rho\}.$$

**Theorem 3.1.** *Let  $\Gamma$  be a smooth embedded manifold of codimension- $k$  without boundary. Then there is  $\sigma > 0$  such that  $\eta$  is smooth in  $I_\sigma(\Gamma)$ . Moreover for any  $x \in \Gamma$ , the matrix  $\nabla^2\eta(x)$  represents the orthogonal projection on the normal space to  $\Gamma$  at  $x$  and*

$$(3.1) \quad \delta(x+p) = |p|,$$

for any  $p$  orthogonal to  $\Gamma$  at  $x$  and  $|p| \leq \sigma$ .

*Proof.* Fix  $x_0 \in \Gamma$ . By the smoothness of  $\Gamma$ , there is a constant  $s > 0$  and a smooth orthonormal vector field

$$(\nu^1, \dots, \nu^k) : B_s(x_0) \cap \Gamma \rightarrow \mathbf{R}^{dk}$$

spanning the normal space to  $\Gamma$ . Set

$$\Phi(x, \alpha) = x + \sum_{j=1}^k \alpha_j \nu^j(x), \quad x \in B_s(x_0) \cap \Gamma, \alpha \in \mathbf{R}^k.$$

Using local coordinates, we compute that the Jacobian  $J\Phi(x_0, 0)$  is equal to the identity matrix. Hence by the implicit function theorem, there is  $r \in (0, s)$  satisfying,

(1) In  $(B_r(x_0) \cap \Gamma) \times B_r^k(0)$ ,  $\Phi$  is one to one and its Jacobian is nowhere singular;

(2)  $V = \Phi((B_r(x_0) \cap \Gamma) \times B_r^k(0))$  is an open set containing  $x_0$ .

For  $y \in V$ , let

$$\Psi(y) = (x(y), \alpha(y)) \in (B_r(x_0) \cap \Gamma) \times B_r^k(0)$$

be the smooth inverse of  $\Phi$ . Choose  $\sigma \in (0, r/2)$  such that  $B_\sigma(x_0) \subset V$ . We wish to relate the functions  $x(y)$ ,  $\alpha(y)$  to the distance function. So for  $y \in B_\sigma(x_0)$ , let  $x \in \Gamma$  be the minimizer of the distance, i.e.,  $\delta(y) = |x-y|$ . Then it is clear that the minimizer  $x$  belongs to  $B_r(x_0) \cap \Gamma$  and is equal to  $x(y)$ . Moreover,  $\delta(y) = |\alpha(y)|$  and consequently

$$2\eta(y) = |\alpha(y)|^2 = \sum_{j=1}^k \alpha_j^2(y), \quad y \in B_\sigma(x_0).$$

Hence  $\eta$  is smooth and (3.1) holds by construction. Since  $\Gamma$  is compact, we use a covering argument to extend these properties to a tubular neighborhood  $I_\sigma(\Gamma)$ .

Finally, let  $N = (N_{ij})$  be the orthogonal projection on the normal space to  $\Gamma$  at  $x_0$ . Then for  $z \in \mathbf{R}^d$ ,

$$\eta(x_0 + z) = \frac{|Nz|^2}{2} + o(|z|^2) = \frac{1}{2}Nz \cdot z + o(|z|^2),$$

where as usual  $o(r)$  is any function satisfying  $|o(r)|/r \rightarrow 0$  as  $r \downarrow 0$ . By differentiating twice with respect to  $z$  and evaluating at  $z = 0$ , we find  $\eta_{ij}(x_0) = N_{ij}$ .

Next, we will show that the eigenvalues and the eigenvectors of  $\nabla^2\eta$  propagate along the characteristics of the distance function, and  $k$  eigenvalues of  $\nabla^2\eta$  are exactly equal to 1 as long as  $\eta$  is smooth. Then using the properties of the eigenvalues of  $\nabla^2\eta$ , we will establish a relation between the mean curvature of  $\Gamma$  and  $\nabla^3\eta$ .

Let  $x_0 \in \Gamma$  and  $p$  be a unit vector orthogonal to  $\Gamma$  at  $x_0$ . Let  $\Omega$  be the maximal open set on which  $\eta$  is smooth, and define

$$t^* := t^*(x_0) = \sup\{\tau > 0 : x_0 + tp \in \Omega, \forall t \in [0, \tau]\}.$$

Then  $t^* \geq \sigma$ , where  $\sigma$  is as in the previous theorem. For  $t \in [0, t^*]$ , let

$$B(t) := \nabla^2\eta(x_0 + tp),$$

and

$$\lambda_1(t) \leq \lambda_2(t) \leq \dots \leq \lambda_d(t)$$

be the eigenvalues of  $B(t)$ .

**Theorem 3.2.** *For  $t \in [0, t^*]$ , the eigenvectors of  $B(t)$  are independent of  $t$ ,  $B(t)$  has exactly  $k$  eigenvalues equal to one and the remaining  $(d - k)$  eigenvalues are strictly less than one. Moreover, for any  $\sigma$  satisfying  $I_\sigma(\Gamma) \subset \Omega$ , there is a constant  $C = C(\sigma)$ , independent of  $x_0$  and  $p$ , such that*

$$(3.2) \quad |\lambda_i(t)| \leq C\delta(x_0 + tp) = Ct, \quad \forall t \in [0, \sigma], i = 1, \dots, d - k.$$

Finally, the map

$$t \in (0, t^*) \mapsto F(\nabla\delta(x_0 + tp), \nabla^2\delta(x_0 + tp))$$

is nonincreasing in  $(0, t^*)$ .

*Proof.* 1. Since  $\delta$  is smooth in  $\Omega \setminus \Gamma$ ,  $|\nabla\delta| = 1$  on this set. Then, using the summation convention, we compute that,

$$(3.3) \quad \delta_j\delta_j = 1, \quad \delta_{ij}\delta_j = 0, \quad \delta_{ijk}\delta_j + \delta_{ij}\delta_{jk} = 0,$$

in  $\Omega \setminus \Gamma$ , and in  $\Omega$ ,

$$(3.4) \quad \eta_j \eta_j = 2\eta, \quad \eta_{ij} \eta_j = \eta_i, \quad \eta_{ijk} \eta_j + \eta_{ij} \eta_{jk} = \eta_{ik},$$

where a subscript denotes differentiation with respect to that variable.

**2.** Since  $\delta(x_0 + tp) = t$  and  $\nabla \eta(x_0 + tp) = p\delta$ , by the third identity in (3.4),

$$(3.5) \quad \begin{aligned} \frac{d}{dt} B_{ij}(t) &= \eta_{ijk}(x_0 + tp)p_k = \eta_{ijk}(x_0 + tp) \frac{\eta_k(x_0 + tp)}{\delta(x_0 + tp)} \\ &= \frac{B_{ij}(t) - B_{ik}(t)B_{kj}(t)}{t}, \quad t \in (0, t^*). \end{aligned}$$

Let  $z_1, \dots, z_d$  be any basis such that  $B(\sigma)$  is diagonal, and for  $i = 1, \dots, d$ , let  $\mu_i(t)$  be the unique solution of

$$\frac{d}{dt} \mu_i(t) = \frac{\mu_i(t)(1 - \mu_i(t))}{t}, \quad t \in (0, t^*),$$

satisfying  $\mu_i(\sigma) = \lambda_i(\sigma)$ . Recall that  $\lambda_i(\sigma)$ 's are the eigenvalues of  $B(\sigma)$ . Then the matrices

$$\hat{B}(t) = \sum_{i=1}^d \mu_i(t) z_i \otimes z_i$$

solve the differential equation (3.5) and satisfy  $\hat{B}(\sigma) = B(\sigma)$ . By the uniqueness we have  $B = \hat{B}$ . Therefore the eigenvectors of  $B(t)$  are equal to  $z_i$ , and the eigenvalues  $\lambda_i(t)$ 's solve

$$(3.6) \quad \frac{d}{dt} \lambda_i(t) = \frac{\lambda_i(t)(1 - \lambda_i(t))}{t}, \quad t \in (0, t^*).$$

**3.** In view of Theorem 3.1,  $B(0)$  is the orthogonal projection on the normal space to  $\Gamma$  at  $x_0$ . Hence  $k$  eigenvalues of  $B(0)$  are equal to one and the remaining  $(d - k)$  of them are equal to zero. By the differential equation (3.6), we conclude that if for some  $i$  we have  $\lambda_i(0) = 1$ , then  $\lambda_i(t) = 1$  for all  $t \in [0, t^*)$ . Moreover if  $\lambda_i(0) = 0$  for some  $i$ , then  $\lambda_i(t) < 1$  for all  $t \in [0, t^*)$ . Hence for any  $t \in [0, t^*)$ ,  $B_{ij}(t)$  has exactly  $k$  eigenvalues equal to 1, and its restriction to the normal space of  $\Gamma$  at  $x_0$  is equal to the identity. The remaining eigenvalues are less than one, and the corresponding eigenvectors span the tangent space of  $\Gamma$  at  $x_0$ . Moreover, the differential equation (3.6) yields

$$\frac{\lambda_i(t)}{t} = \frac{\lambda_i(\sigma)}{\sigma + (t - \sigma)\lambda_i(\sigma)}, \quad t \in (0, \sigma].$$

Therefore if  $\lambda_i(\sigma) < 0$ , then  $\lambda_i(t) < 0$  for all  $t$  and

$$\left| \frac{\lambda_i(t)}{t} \right| \leq \left| \frac{\lambda_i(\sigma)}{\sigma} \right|, \quad t \in [0, \sigma].$$

If, however,  $\lambda_i(\sigma) \in (0, 1)$ , then  $\lambda_i(t) \in [0, 1)$  for all  $t$ . Moreover,

$$\left| \frac{\lambda_i(t)}{t} \right| \leq \frac{\lambda_i(\sigma)}{\sigma(1 - \lambda_i(\sigma))}, \quad t \in [0, \sigma].$$

In summary, for all  $t \in (0, \sigma]$  and  $i = 1, \dots, d - k$ , we have,

$$\left| \frac{\lambda_i(t)}{t} \right| \leq C := \max \left\{ \frac{|\lambda|}{\sigma[1 \wedge (1 - \lambda)]} : \lambda < 1 \text{ eigenvalue of } \nabla^2 \eta(x), \delta(x) = \sigma \right\}.$$

4. To prove the final statement of the theorem, we differentiate the identity  $\eta_i = \delta \delta_i$  to obtain

$$\delta_{ij} = \frac{\eta_{ij} - \delta_i \delta_j}{\delta}.$$

Since  $\nabla \delta(x_0 + tp) = p$ , for all  $t \in (0, t^*)$ ,

$$\nabla^2 \delta(x_0 + tp) = \frac{B(t) - p \otimes p}{\delta}, \quad \forall t \in (0, t^*).$$

Therefore  $\nabla^2 \delta(x_0 + tp)$  has  $(k - 1)$  eigenvalues equal to  $1/\delta(x_0 + tp)$ , one eigenvalue (corresponding to  $p$ ) equal to 0, and the remaining  $(d - k)$  eigenvalues less than  $1/\delta(x_0 + tp)$ . Let  $\beta_1(t) \leq \beta_2(t) \dots \leq \beta_{d-k}(t)$  be these eigenvalues. Since  $\beta_i(t) = \lambda_i(t)/t$ , by (3.6),  $\beta_i(t)' = -\beta_i^2(t)$ . Hence  $\beta_i$ 's are nonincreasing and therefore

$$F(\nabla \delta(y + tp), \nabla^2 \delta(y + tp)) = \sum_{i=1}^{d-k} \beta_i(t)$$

is also nonincreasing.

*Remark 3.3.* Since  $\beta_i(t)' = -\beta_i^2(t)$ , for  $i = 1, \dots, d - k$ , the eigenvalues  $\beta_i(t)$  of  $\nabla^2 \delta(x_0 + tp)$ , converge, as  $t \downarrow 0$ , to real numbers  $\beta_i$ , depending on  $p$ . Clearly these numbers are related to the geometry of  $\Gamma$ . We conjecture the following: let (c.f. [32, p.13])

$$B : T_{x_0}(\Gamma) \times T_{x_0}(\Gamma) \rightarrow N_{x_0}(\Gamma)$$

be the second fundamental form of  $\Gamma$ , where  $T_{x_0}(\Gamma)$ ,  $N_{x_0}(\Gamma)$  are, respectively, the tangent and the normal spaces of  $\Gamma$  at  $x_0$ . Then  $\beta_i$ 's are equal to the eigenvalues of the symmetric bilinear form,

$$h(v, w) := -B(v, w) \cdot p, \quad v, w \in T_{x_0}(\Gamma).$$

Since the above conjecture is only tangentially related to this paper, its analysis will be pursued elsewhere. However the proof of Theorem 3.5 can be used to prove that the sum of  $\beta_i$  is equal to  $-H \cdot p$ ; see (3.16) below for the stationary flow.

*Remark 3.4.* For  $A \in S^{d \times d}$  and  $p \in \mathbf{R}^d, p \neq 0$ , let  $P_p$  and  $X = P_p A P_p$  as in (1.4). Let

$$\tilde{\lambda}_1(X) \leq \tilde{\lambda}_2(X) \leq \dots \leq \tilde{\lambda}_d(X)$$

be the eigenvalues of  $X$  and define

$$\tilde{F}(p, A) := \sum_{i=1}^{d-k+1} \tilde{\lambda}_i(X).$$

Recall that in the definition of  $F$ , we only used the eigenvalues of  $X$  that are orthogonal to  $p$ . Therefore, it is easy to check that  $\tilde{F} \leq F$ , and they are equal if and only if at least  $k$  eigenvalues of  $X$  are nonnegative. In particular,  $F$  coincides with  $\tilde{F}$  in the codimension-1 case (in this case  $F$  also coincides with the function  $\hat{F}(p, A) = \text{trace}(P_p A)$  considered in [15], [8]). Moreover, step 4 of Theorem 3.2 shows that

$$(3.7) \quad F(\nabla\delta, \nabla^2\delta) = \tilde{F}(\nabla\delta, \nabla^2\delta)$$

in the region where  $\delta$  is smooth.

We are now ready to express the mean curvature vector in terms of  $\eta$ .

**Theorem 3.5** *Let  $H(x)$  be the mean curvature vector of  $\Gamma$  at  $x$ . Then*

$$(3.8) \quad H(x) = -\Delta \nabla \eta(x), \quad x \in \Gamma.$$

*Proof.* The mean curvature vector  $H$  of  $\Gamma$  is characterized by the property

$$(3.9) \quad \int_{\Gamma} \text{div}^{\Gamma} \phi \, d\mathcal{H}^{d-k} = - \int_{\Gamma} H \cdot \phi \, d\mathcal{H}^{d-k},$$

$$\forall \phi = (\phi_1, \dots, \phi_d) \in C^1(\mathbf{R}^d, \mathbf{R}^d),$$

where, using the summation convention,  $\text{div}^{\Gamma} \phi = d_i^{\Gamma} \phi_i$  is the tangential divergence of  $\phi$ , and for a scalar function  $\varphi$

$$d^{\Gamma} \varphi = (d_1^{\Gamma} \varphi, \dots, d_d^{\Gamma} \varphi)$$

denote the tangential gradient of  $\varphi$ , i.e., the projection of  $\nabla \varphi$  on the tangent space to  $\Gamma$ . The integration by parts formula, (3.9), is also related to the first variation of area (see [36]), and motivates the study of flow by the mean curvature.



We claim that  $H = -\Delta \nabla \eta$  satisfies (3.9). Indeed, by the divergence formula on manifolds (see [36]),

$$\int_{\Gamma} \operatorname{div}^{\Gamma} X \, d\mathcal{H}^{d-k} = 0,$$

for any tangent vectorfield  $X$ . Given a smooth vector field  $\phi$ , let  $X$  be the tangential component of  $\phi$ , and  $P_{ij}$  be the projection on the tangent space. The divergence formula yields

$$\begin{aligned} 0 &= \int_{\Gamma} \operatorname{div}^{\Gamma} X \, d\mathcal{H}^{d-k} = \int_{\Gamma} d_j^{\Gamma} (P_{ji} \phi_i) \, d\mathcal{H}^{d-k} \\ &= \int_{\Gamma} P_{ij} d_j^{\Gamma} \phi_i + \phi_i d_j^{\Gamma} P_{ji} \, d\mathcal{H}^{d-k} = \int_{\Gamma} d_i^{\Gamma} \phi_i + \phi_i d_j^{\Gamma} P_{ji} \, d\mathcal{H}^{d-k}. \end{aligned}$$

Hence,  $H_i = d_j^{\Gamma} P_{ji}$ . Since by Theorem 3.1,  $\nabla^2 \eta$  is the projection onto the normal space,  $P = I - \nabla^2 \eta$  and therefore,

$$\begin{aligned} H_i &= d_j^{\Gamma} P_{ji} = -d_j^{\Gamma} \eta_{ji} = -P_{lj} \eta_{lj} \\ &= -\eta_{jji} + \eta_{lj} \eta_{lji}, \end{aligned}$$

where as before, a subscript of  $\eta$  denotes differentiation with respect to that variable, and all derivatives of  $\eta$  are evaluated on the surface  $\Gamma$ . We now claim that  $\eta_{lj} \eta_{lji}$  is equal to zero. Indeed,

$$2\eta_{lj} \eta_{lji} = (\eta_{lj}^2)_i = \left( \sum_{k=1}^d \lambda_k^2 \right)_i,$$

where  $\lambda_1, \dots, \lambda_d$  are the eigenvalues of  $\nabla^2 \eta$ . By the previous theorem, the sum of the squares of the eigenvalues is equal to  $k + o(\delta)$  near  $\Gamma$ , and therefore it has zero derivative on  $\Gamma$ . Hence  $\eta_{lj} \eta_{lji} = 0$  and  $H_i = -\eta_{jji}$ .

Next we give a definition of classical solutions.

**Definition 3.6.** Let  $(\Gamma_t)_{t \in [0, T]}$  be a family of smooth embedded  $(d - k)$ -submanifolds of  $\mathbf{R}^d$  without boundary. We say that  $(\Gamma_t)_{t \in [0, T]}$  is a *smooth  $(d - k)$ -dimensional, mean curvature flow* if there exists a smooth deformation map

$$\phi : \Gamma_0 \times [0, T] \rightarrow \mathbf{R}^d$$

satisfying the following:

- (1) for every  $t \leq T$ ,  $\phi(\cdot, t)$  is one to one, and on  $\Gamma_0$  the tangential Jacobian of  $\phi(\cdot, t)$  has full rank  $(d - k)$ ;
- (2)  $\phi(\Gamma_0, t) = \Gamma_t$ , for any  $t \in [0, T]$ ;

$$(3) \quad \phi(x, 0) = x \text{ and}$$

$$\phi_t(x, t) = H(\phi(x, t), t), \quad \forall x \in \Gamma_0, t \in [0, T],$$

where  $H(\phi(x, t), t)$  is the mean curvature vector of  $\Gamma_t$  at  $\phi(x, t)$ .

For future use we make one more definition. We say that  $(\Gamma_t)_{t \in [0, T]}$  is a *smooth flow* if there is a deformation map  $\phi$  satisfying the first two conditions in Definition 3.6 and that for every  $x \in \Gamma_0$ ,  $\phi_t(x, t)$  is orthogonal to  $\Gamma_t$  at  $\phi(x, t)$ . Note that since the mean curvature vector is orthogonal to the surface (see for instance [36]), smooth mean curvature flow is also a smooth flow.

The following characterization of the mean curvature flow in terms of the square distance function  $\eta$  (c.f. (3.11) below) was first stated in [12].

**Lemma 3.7.** *Let  $(\Gamma_t)_{t \in [0, T]}$  be a smooth flow. Then, there exists  $\sigma > 0$  such that the function*

$$\eta(x, t) := \frac{1}{2} \text{dist}^2(x, \Gamma_t)$$

is smooth in  $\{(x, t) \in \mathbf{R}^d \times [0, T] : \eta \leq \sigma\}$ . Moreover, the displacement of the flow is given by

$$(3.10) \quad \phi_t(x, t) = -\nabla \eta_t(\phi(x, t), t) \quad \forall t \in [0, T], x \in \Gamma_0.$$

In particular,  $(\Gamma_t)_{t \in [0, T]}$  is a smooth mean curvature flow if and only if

$$(3.11) \quad \nabla \eta_t = \Delta \nabla \eta, \quad \text{on } \Gamma_t.$$

*Proof.* Since the Jacobian of  $\phi(\cdot, t)$  has full rank on  $\Gamma_0$ , the smoothness of  $\eta$  can be proved as in Theorem 3.1.

1. Fix  $y_0 \in \Gamma_t$ , and let  $x_0 \in \Gamma_0$  be the unique point satisfying

$$y_0 = \phi(x_0, t).$$

Since  $\phi(x_0, t+h) \in \Gamma_{t+h}$ ,

$$\eta(\phi(x_0, t+h), t+h) = 0.$$

We differentiate the above identity with respect to  $h$  twice, and then evaluate it at  $h = 0$ . Since  $\phi(x_0, t) = y_0$  and  $\nabla \eta(y_0, t) = 0$ , we have

$$\eta_{tt}(y_0, t) + \nabla^2 \eta(y_0, t) \phi_t(x_0, t) \cdot \phi_t(x_0, t) + 2 \nabla \eta_t(y_0, t) \cdot \phi_t(x_0, t) = 0.$$

2. By the definition of a smooth flow,  $\phi_t(x_0, t)$  is orthogonal to  $\Gamma_t$  at  $y_0$  and, by (3.1),

$$\eta(\phi(x_0, t+h) - h \phi_t(x_0, t+h), t+h) = \frac{1}{2} h^2 |\phi_t(x_0, t+h)|^2$$

for  $h$  small enough. Observe that

$$|\phi(x_0, t + h) - h\phi_t(x_0, t + h) - y_0| \leq Ch^2,$$

for some constant  $C$ . Since  $\nabla\eta$  vanishes at  $(y_0, t)$ ,

$$\eta(y_0, t + h) = \frac{1}{2}h^2|\phi_t(x_0, t)|^2 + o(h^2).$$

Therefore

$$\eta_{tt}(y_0, t) = |\phi_t(x_0, t)|^2.$$

**3.** Combine the two previous steps. The result is:

$$|\phi_t(x_0, t)|^2 + \nabla^2\eta(y_0, t)\phi_t(x_0, t) \cdot \phi_t(x_0, t) + 2\nabla\eta_t(y_0, t) \cdot \phi_t(x_0, t) = 0.$$

Since  $\phi_t(x_0, t)$  is orthogonal to  $\Gamma_t$  at  $y_0$  and  $\nabla^2\eta(x_0, t)$  is the orthogonal projection on the normal space of  $\Gamma_t$  at  $y_0$ ,

$$\nabla^2\eta(y_0, t)\phi_t(x_0, t) \cdot \phi_t(x_0, t) = |\phi_t(x_0, t)|^2,$$

and

$$2|\phi_t(x_0, t)|^2 + 2\nabla\eta_t(y_0, t) \cdot \phi_t(x_0, t) = 0.$$

**4.** Let  $\epsilon > 0$  and  $y \in \mathbf{R}^d$ . Then, if  $\phi(z, t) \in \Gamma_t$  is the point of least distance of  $y_0 + y$  from  $\Gamma_t$ , then we can find  $\rho > 0$  so small that  $|y| < \rho$  implies  $|z - x_0| < \epsilon$ . For any  $y \in B_\rho(0)$  we have

$$\delta(y_0 + y, t + \tau) \leq |y_0 + y - \phi(z, t)| + \tau|\phi_t(z, t)| + o(\tau)$$

hence

$$\delta(y_0 + y, t + \tau) \leq \delta(y_0 + y, t) + \tau|\phi_t(z, t)| + o(\tau)$$

and

$$\eta(y_0 + y, t + \tau) \leq \eta(y_0 + y, t) + \tau\delta(y_0 + y, t)|\phi_t(z, t)| + o(\tau).$$

Let  $\tau \downarrow 0$  and use our choice of  $\rho$ :

$$\eta_t(y_0 + y, t) \leq |y| \sup_{z \in B_\epsilon(x_0)} |\phi_t(z, t)|,$$

so that

$$|\nabla\eta_t(y_0, t)| \leq \sup_{z \in B_\epsilon(x_0)} |\phi_t(z, t)|.$$

By letting  $\epsilon \downarrow 0$ , we conclude that  $|\nabla\eta_t(y_0, t)| \leq |\phi_t(x_0, t)|$  and, by Step 3,  $\phi_t(x_0, t) = -\nabla\eta_t(y_0, t)$ .

The smooth mean curvature flow is a system of partial differential equations in  $\eta$ . But quite surprisingly, it turns out to be equivalent to a differential inequality in  $\delta$ . This observation was first made in [37] for codimension-one flows.

**Theorem 3.8.** *Let  $(\Gamma_t)_{t \in [0, T]}$  be a smooth flow and let  $\Omega \subset \mathbf{R}^d \times (0, T)$  be the maximal open set on which  $\eta$  is smooth. Then  $\Gamma_t$  is a smooth  $(d - k)$  dimensional mean curvature flow if and only if*

$$(3.12) \quad \delta_t(x, t) \geq F(\nabla\delta(x, t), \nabla^2\delta(x, t)), \quad \forall (x, t) \in \Omega, \quad x \notin \Gamma_t.$$

*Proof.* 1. Suppose that  $\Gamma_t$  is a smooth mean curvature flow and let

$$\Omega' := \{(x, t) \in \Omega : x \notin \Gamma_t\}.$$

We compute that on  $\Omega'$ ,

$$\eta_{it} = (\delta\delta_i)_t = \delta_t\delta_i + \delta\delta_{it},$$

and

$$(3.13) \quad \delta_i\eta_{it} = \delta_t + \delta\delta_{it}\delta_i = \delta_t + \frac{1}{2}\delta(\delta_i\delta_i)_t = \delta_t.$$

Similarly, using (3.3) we get

$$\Delta\eta_i = \delta_i\Delta\delta + 2\delta_j\delta_{ij} + \delta\Delta\delta_i = \delta_i\Delta\delta + \delta\Delta\delta_i,$$

and therefore,

$$(3.14) \quad \delta_i\Delta\eta_i = \Delta\delta + \delta\delta_i\Delta\delta_i,$$

in  $\Omega'$ . Set  $\alpha_i = \eta_{it} - \Delta\eta_i$ . Then (3.13) and (3.14) imply

$$\alpha_i\delta_i = \delta - \Delta\delta - \delta\delta_i\Delta\delta_i.$$

By means of the last identity in (3.3), we conclude that,

$$(3.15) \quad \alpha_i\delta_i = \delta_t - \Delta\delta + \delta\|\nabla^2\delta\|^2,$$

where  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbf{R}^{d^2}$ . Since  $\nabla^2\delta$  is a symmetric matrix,  $\|\nabla^2\delta\|^2$  is equal to the sum of the squares of the eigenvalues. Hence,

$$\alpha_i\delta_i = \delta_t - \sum_{i=1}^d \beta_i + \delta \sum_{i=1}^d \beta_i^2, \quad \text{in } \Omega',$$

where  $\beta_1 \leq \dots \leq \beta_d$  are the eigenvalues of  $\nabla^2\delta$ . By step 4 of Theorem 3.2, the last  $k - 1$  eigenvalues are equal to  $\delta^{-1}$ , and by (3.2) the remaining  $(d - k + 1)$  of them are bounded in a tubular neighborhood

$$I_\sigma = \{(x, t) \in \mathbf{R}^d \times [0, T] : \delta(x, t) \leq \sigma\}.$$

Therefore, using (3.7) we obtain

$$\begin{aligned}
 \alpha_i \delta_i &= \delta_t - \sum_{i=1}^{d-k+1} \beta_i + \delta \sum_{i=1}^{d-k+1} \beta_i^2 + \delta \sum_{i=d-k+2}^d \beta_i (\delta \beta_i - 1) \\
 (3.16) \qquad &= \delta_t - F(\nabla \delta, \nabla^2 \delta) + \delta C(x, t), \qquad \text{in } I_\sigma \cap \Omega',
 \end{aligned}$$

for some bounded function  $C(x, t)$ . Since  $\Gamma_t$  is a smooth mean curvature flow for each  $i$ ,  $\alpha_i = 0$  on  $\Gamma_t$ . So we have

$$\begin{aligned}
 (3.17) \qquad \lim_{x \rightarrow x_0} \delta_t(x, t) - F(\nabla \delta(x, t), \nabla^2 \delta(x, t)) &= 0, \\
 \forall x_0 \in \Gamma_t, t \in (0, T).
 \end{aligned}$$

**2.** Now, fix  $x \in \Omega'$  and  $t \in [0, T]$ . Let  $x_0 \in \Gamma_t$  be the unique point satisfying  $\delta(x, t) = |x_0 - x|$ . Then with  $p = (x - x_0)/|x - x_0|$  we have

$$\begin{aligned}
 \frac{d}{ds}(\delta_t(x_0 + sp, t)) &= \nabla \delta_t(x_0 + sp, t) \cdot p \\
 &= \nabla \delta_t(x_0 + sp, t) \cdot \nabla \delta(x_0 + sp, t) \\
 &= \frac{1}{2} \frac{d}{dt} (|\nabla \delta(x_0 + sp, t)|^2) = 0,
 \end{aligned}$$

for any  $s \in (0, \delta(x, t))$ . In view of Theorem 3.2 and the above calculation, the map

$$s \mapsto \delta_t(x_0 + sp, t) - F(\nabla \delta(x_0 + sp, t), \nabla^2 \delta(x_0 + sp, t))$$

is nondecreasing in  $(0, \delta(x, t)]$ . Now we obtain (3.12) from (3.17), by letting  $s \downarrow 0$ .

**3.** Conversely, suppose that (3.12) holds in  $\Omega'$ . Let

$$(3.18) \qquad \gamma(x, t) = \nabla \eta_t(x, t) - \Delta \nabla \eta(x, t).$$

By continuity, we need only to show that  $\gamma(y, t) = 0$  for any  $t \in (0, T)$ ,  $y \in \Gamma_t$ . Since  $(\Gamma_t)_{t \in [0, T]}$  is a smooth flow, by Theorem 3.7,  $\nabla \eta_t(y, t)$  is orthogonal to  $\Gamma_t$  at  $y$ . Also the mean curvature vector  $H = -\nabla \Delta \eta(y, t)$  is normal to the surface  $\Gamma_t$  at  $y$ , (see for example [36]). Therefore, it suffices to show that  $p \cdot \gamma(y, t) \geq 0$  for any unit vector  $p$  normal to  $\Gamma_t$  at  $y$ .

Let  $\sigma > 0$  be as in Theorem 3.1 (with  $\Gamma = \Gamma_t$ ), and for  $s \in (0, \sigma]$  let  $x_s = y + sp$ . By Theorem 3.1 and Theorem 3.2, we have  $\delta(x_s, t) = s$

and  $\nabla\delta(x_s, t) = p$  for any  $s \in (0, \sigma]$ . Multiply (3.18) by  $p$  and proceed as in step one to obtain

$$\begin{aligned} p \cdot \gamma(x_s, t) &= \delta_t(x_s, t) - \Delta\delta(x_s, t) + \delta(x_s, t)\|\nabla^2\delta(x_s, t)\|^2 \\ &= \delta_t(x_s, t) - F(\nabla\delta(x_s, t), \nabla^2\delta(x_s, t)) + O(\delta(x_s, t)). \end{aligned}$$

Since by hypothesis  $\delta$  is a classical supersolution of (1.6) in  $\Omega'$ , we let  $s \downarrow 0$  to obtain  $p \cdot \gamma(y, t) \geq 0$ .

**Corollary 3.9.** *Let  $(\Gamma_t)_{t \in [0, T]}$  be a smooth  $(d - k)$ -dimensional mean curvature flow and let  $u(x, t)$  be the unique viscosity solution of (1.6) with initial data  $u(x, 0) = \text{dist}(x, \Gamma_0)$ . Then*

$$\Gamma_t = \{x \in \mathbf{R}^d : u(t, x) = 0\}$$

for any  $t \in [0, T]$ . Moreover,  $\delta(x, t) := \text{dist}(x, \Gamma_t)$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ .

*Proof.*

1. Choose  $\sigma > 0$  so that  $\eta$  is smooth on

$$Q_\sigma := \{(x, t) : 0 \leq t \leq T, \delta(x, t) \leq \sigma\}.$$

Fix  $(x, t) \in Q_\sigma$ ,  $x \notin \Gamma_t$  and choose  $y \in \Gamma_t$  such that  $\delta(x, t) = |y - x|$ . For  $s \in [0, \delta(t, x)]$ , set

$$x_s := y + s \frac{x - y}{|x - y|},$$

and let

$$\beta_1(s) \leq \beta_2(s) \leq \dots \leq \beta_d(s)$$

be the eigenvalues of  $\nabla^2\delta(x_s, t)$ . The following are proved in Theorem 3.2:

$$\beta_{d-k+2}(s) = \dots = \beta_d(s) = \frac{1}{\delta(x_s, t)} = \frac{1}{s},$$

(this condition is empty for  $k = 1$ ), and for  $i = 1, \dots, d - k + 1$ ,

$$(3.19) \quad \frac{d}{ds} \beta_i(s) = -(\beta_i(s))^2 \Rightarrow \beta_i(s) = \frac{\beta_i}{1 + s\beta_i} \quad s \in (0, \sigma],$$

for suitable real constants  $\beta_i$ . Moreover in view of (3.2),  $|\beta_i|$ 's are uniformly bounded by some constant  $C$ , independent of  $x$  and  $t \in [0, T]$ . By step 2 of Theorem 3.8,  $\delta_t(x_s, t)$  is constant and by (3.17),

$$\lim_{s \downarrow 0} \delta_t(x_s, t) - F(\nabla\delta(x_s, t), \nabla^2\delta(x_s, t)) = 0.$$

Reducing  $\sigma$ , if necessary, we may assume that  $\sigma C < 1/2$  (recall that  $C$  is an upper bound for  $\beta_i$ 's in (3.19) ). For  $0 < s' < s \leq \sigma$ , we use (3.7) to compute  $F$ . The result is:

$$\begin{aligned} &\delta_t(x_s, t) - F(\nabla\delta(x_s, t), \nabla^2\delta(x_s, t)) \\ &= \delta_t(x_{s'}, t) - F(\nabla\delta(x_{s'}, t), \nabla^2\delta(x_{s'}, t)) \\ &\quad + [F(\nabla\delta(x_{s'}, t), \nabla^2\delta(x_{s'}, t)) - F(\nabla\delta(x_s, t), \nabla^2\delta(x_s, t))] \\ &= O(s') + \sum_{i=1}^{d-k+1} \frac{\beta_i}{1+s'\beta_i} - \frac{\beta_i}{1+s\beta_i}. \end{aligned}$$

Let  $s' \downarrow 0$ . Then

$$\begin{aligned} \delta_t(x_s, t) - F(\nabla\delta(x_s, t), \nabla^2\delta(x_s, t)) &\leq (d-k+1) \frac{sC^2}{1-sC} \\ &\leq 2C^2(d-k+1)\delta(x_s, t). \end{aligned}$$

Set  $C^* := 2C^2(d-k+1)$ . We have proved that  $\delta$  is a subsolution of

$$(3.20) \quad \delta_t \leq F(\nabla\delta, \nabla^2\delta) + C^*\delta$$

in

$$Q'_\sigma := \{(x, t) \in \mathbf{R}^d \times (0, T) : 0 < \delta(x, t) < \sigma\}.$$

2. Set  $W := e^{-C^*t}(\delta \wedge \sigma/2)$ . We claim that  $W$  is a viscosity subsolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ . Indeed, let

$$w := H_\sigma(\delta), \quad H_\sigma(r) := r \wedge \sigma/2, \quad r \geq 0.$$

We only need to show that  $w$  is a viscosity subsolution of (3.20) in  $\mathbf{R}^d \times (0, T)$ . Suppose that for some test function  $\psi$ ,  $w - \psi$  attains its maximum at  $(x_0, t_0) \in \mathbf{R}^d \times (0, T)$ . Since  $\Gamma_t$  is smooth we conclude that  $x_0 \notin \Gamma_{t_0}$ .

3. Suppose that  $\delta(x_0, t_0) > \sigma/2$  (the opposite case will be considered in the next step). Then  $\delta > \sigma/2$  near  $(x_0, t_0)$  and  $H \equiv \sigma/2$  near  $(x_0, t_0)$ . So, at  $(x_0, t_0)$ ,  $\psi_t = 0$ ,  $\nabla\psi = 0$ ,  $\nabla^2\psi \geq 0$ . Hence

$$(3.21) \quad 0 = \psi_t(x_0, t_0) \leq F^*(\nabla\psi(x_0, t_0), \nabla^2\psi(x_0, t_0)) + C^*\delta(x_0, t_0).$$

4. Suppose that  $\delta(x_0, t_0) \leq \sigma/2$ . Recall that  $x_0 \notin \Gamma_{t_0}$ . So  $(x_0, t_0) \in Q'_\sigma$ , and the first step yields

$$\delta_t(x_0, t_0) \leq F(\nabla\delta(x_0, t_0), \nabla^2\delta(x_0, t_0)) + C^*\delta(x_0, t_0).$$

Now the smoothness of  $\delta$  near  $(x_0, t_0)$  implies that  $\psi$  satisfies (3.21).

5. By Theorem 2.2,  $0 \leq W \leq u$  in  $\mathbf{R}^d \times [0, T]$ . Therefore the zero set of  $u$  is contained in the zero set of  $W$  for all  $t \in [0, T]$ . Observe that the zero level set of  $W$  is equal to  $\Gamma_t$ .

6. To prove the opposite inclusion, set  $V = H_\sigma(\delta)$ , where  $H_\sigma$  is as in the second step. Using Theorem 3.8 and Theorem 2.3, it is easy to show that  $V$  is a classical supersolution of (1.6) in  $Q'_\sigma$  and a viscosity supersolution of (1.6) in  $\{(x, t) \in \mathbf{R}^d \times (0, T) : V(x, t) > 0\}$ . Therefore Lemma 3.11 below implies that  $V$  is a supersolution in all of  $\mathbf{R}^d \times (0, T)$ . By the comparison result, Theorem 2.2, we have

$$0 \leq u \wedge \frac{\sigma}{2} \leq V = \delta \wedge \frac{\sigma}{2}.$$

Hence  $\Gamma_t$  is included in the zero level set of  $u$ .

7. In this step, we show that  $\delta$  is a viscosity supersolution in  $\mathbf{R}^d \times (0, T)$ . Let  $\phi$  be a smooth function, and  $(x_0, t_0) \in \mathbf{R}^d \times (0, T)$  be a minimizer of  $(\delta - \phi)$ . Choose  $y_0 \in \Gamma_{t_0}$  satisfying  $\delta(x_0, t_0) = |x_0 - y_0|$ , and set

$$\varphi(y, t) = \phi(y + x_0 - y_0, t).$$

Then by the subadditivity of  $\delta$ , it is easy to show that  $(y_0, t_0)$  is a minimizer of  $(\delta - \varphi)$ . Let  $V$  be as in the previous step. Since  $y_0 \in \Gamma_{t_0}$ ,  $(y_0, t_0)$  is also a minimizer of  $(V - \varphi)$ . Since  $V$  is a viscosity supersolution of (1.6), we have

$$\varphi_t \geq F_*(\nabla\varphi, \nabla^2\varphi),$$

at  $(y_0, t_0)$ . Hence  $\phi$  satisfies the above inequality at  $(x_0, t_0)$ , and therefore  $\delta$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, \infty)$ .

*Remark 3.10.* Let  $\tilde{F}$  be the function defined in Remark 3.4. It is easy to check that the results of §2 apply to  $\tilde{F}$  as well. In particular, given a uniformly continuous viscosity solution of

$$u_t = \tilde{F}(\nabla u, \nabla^2 u), \quad \{u(\cdot, 0) = 0\} = \Gamma_0,$$

the sets  $\tilde{\Gamma}(t) := \{u(\cdot, t) = 0\}$  are well defined and depend only on  $\Gamma_0$ .

Moreover, by (3.7), Theorem 3.8 trivially holds with  $\tilde{F}$  in place of  $F$ . Also Theorem 3.9 holds with  $\tilde{F}$  in place of  $F$ , hence the weak solution  $\tilde{\Gamma}_t$  is consistent with the classical solution. However, the inequality  $\tilde{F} \leq F$  and Theorem 2.2 easily imply that  $\Gamma_t \subset \tilde{\Gamma}_t$  in general. By an explicit computation it can be seen that the inclusion is strict if  $\Gamma_0 = \{x \in \mathbf{R}^d : 1 \leq |x| \leq 2\}$ .

We continue with the proof of the lemma already used in step 6. For future reference, we will prove a slightly more general version than needed in that step.

**Lemma 3.11** *Let  $w : \mathbf{R}^d \times [0, T] \rightarrow [0, \infty)$  be a lower semicontinuous function satisfying the following conditions:*



- (i) for every  $(x, t) \in \mathbf{R}^d \times (0, T)$  with  $w(x, t) = 0$ , there is a sequence  $(x_n, t_n) \rightarrow (x, t)$  such that  $w(x_n, t_n) = 0$  and  $t_n < t$ ;
- (ii)  $w$  is a viscosity supersolution of (1.6) on  $\{(x, t) : w(x, t) > 0, t \in (0, T)\}$ ;
- (iii)  $|w(t, x) - w(t, y)| \leq L|x - y|$  for some suitable constant  $L$ .

Then  $w$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ .

*Proof.* For  $\epsilon > 0$ , let  $h_\epsilon(r) = (r - \epsilon)^+$ . We claim that  $h_\epsilon(w)$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ . Suppose that for some test function  $\Psi$ ,  $h_\epsilon(w) - \Psi$  has a minimum at  $(x_0, t_0) \in \mathbf{R}^d \times (0, T)$ . Adding a fourth order perturbation to  $\Psi$ , we may assume that the minimum is strict.

1. Suppose that  $w(x_0, t_0) > 0$ , the opposite case will be discussed in the next step. Choose a sequence of strictly increasing functions  $\{f_n\}$  uniformly converging to  $h_\epsilon$  on  $\mathbf{R}$ . Since  $(x_0, t_0)$  is a strict minimum it is easy to see that there are local minimizers  $(x_n, t_n)$  of  $f_n(w) - \Psi$  converging to  $(x_0, t_0)$ . By the lower semicontinuity of  $w$ ,  $w(x_n, t_n) > 0$  for  $n$  large enough. Since by Theorem 2.3,  $f_n(w)$  is a viscosity supersolution in  $\{w > 0\}$ , we have

$$\Psi_t(x_n, t_n) \geq F_*(\nabla\Psi(x_n, t_n), \nabla^2\Psi(x_n, t_n)).$$

By send  $n \rightarrow +\infty$  we obtain the above inequality at  $(x_0, t_0)$ .

2. Suppose that  $w(x_0, t_0) = 0$ . By (i) there is a sequence  $(x_n, t_n) \rightarrow (x_0, t_0)$  with  $w(x_n, t_n) = 0$  and  $t_n < t_0$ . Recall that by hypothesis,  $w$  is Lipschitz continuous in the  $x$  variable. Therefore for sufficiently large  $n$ ,  $h_\epsilon(w(x_0, t_n)) = 0$ . Since  $t_n \uparrow t$ ,  $\Psi_t(x_0, t_0) \geq 0$ . Also  $h_\epsilon(w(x, t_0)) \equiv 0$  for all  $x$  near  $x_0$ . Hence  $\nabla\Psi(x_0, t_0) = 0$ ,  $\nabla^2\Psi(t_0, x_0) \leq 0$ , and at  $(t_0, x_0)$  we have  $\Psi_t = 0 \geq F_*(\nabla\Psi, \nabla^2\Psi)$ . Let  $\epsilon \downarrow 0$ , and use the stability property of viscosity supersolutions (see for instance [8], Proposition 2.4) to conclude that  $w$  is a viscosity supersolution in  $\mathbf{R}^d \times (0, T)$ .

#### 4. Distance solutions

In the previous section, it is shown that the distance function of a smooth mean curvature flow is a viscosity supersolution of (1.6) in all of  $\mathbf{R}^d \times (0, \infty)$ . This suggests the following definition of a weak solution of the mean curvature motion.

**Definition 4.1.** We say that  $\{\Gamma_t\}_{t \in (0, T)}$  is a *distance solution* of the  $(d - k)$ -dimensional mean curvature flow, if the distance function

$$\delta(x, t) = \text{dist}(x, \Gamma_t), \quad x \in \mathbf{R}^d, t \in (0, T),$$

is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ .

For codimension-one flow, a similar definition was first given in [37]. The above is an *intrinsic* definition using the distance function to the surface instead of an auxiliary function  $u(x, t)$  used to define the level set solutions. However, as opposed to the level-set solutions, for a given  $\Gamma_0$ , there may be more than one distance solution. Nonuniqueness of distance solutions is related to the “fattenning” of the unique level set solution. We will show in Theorem 4.4 below, that distance solutions and the level set solutions are very closely related.

We study the properties of distance solutions satisfying an initial condition. However, as set valued maps we do not expect continuity at time zero. For instance, consider the planar mean curvature flow with initial data

$$\Gamma_0 = \{(x_1, x_2) \in \mathbf{R}^2 : (x_1)^2 + (x_2)^2 = 1\} \cup \{(0, x_2) \in \mathbf{R}^2 : x_2 \in [1, 2]\}.$$

Then at time  $t$ , the only solution (level set, distance, etc) with the above initial data is the circle with radius  $\sqrt{1 - 2t}$ . Hence the line segment  $\{(0, x_2) \in \mathbf{R}^2 : x_2 \in [1, 2]\}$  disappears instantaneously. So we need to make precise how the initial data is achieved.

**Definition 4.2.** For a given closed set  $\Gamma^*$ , we say that  $\{\Gamma_t\}_{t \in (0, T)}$  satisfies the the initial inclusion

$$(4.1) \quad \Gamma_0 \subset \Gamma^*,$$

if

$$\delta_*(x, 0) := \liminf_{t \downarrow 0} \delta(x, t) \geq \text{dist}(x, \Gamma^*).$$

Geometrically the above condition is equivalent to

$$(4.2) \quad \liminf_{t \downarrow 0} \Gamma_t := \bigcap_{t \in (0, T)} \overline{\bigcup_{s \in (0, t)} \Gamma_s} \subset \Gamma^*.$$

Note that for level set solutions, the geometric initial data is imposed by requiring (1.1) and the continuity of the viscosity solution  $u$ .

In this section, we will show that the level set solution is the maximal distance solution satisfying the initial inclusion (4.2). We start our analysis by proving an equivalent formulation for the distance solutions. Let  $\chi_\Gamma(x, t)$  be the indicator of the set  $\Gamma_t$  evaluated at  $x$ .

**Lemma 4.3**  $\{\Gamma_t\}_{t \in [0, T]}$  is a distance solution of the  $(d-k)$ -dimensional mean curvature flow if and only if  $1 - \chi_\Gamma$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ .

*Proof.* Let  $\{\Gamma_t\}_{t \in [0, T]}$  be a distance solution, and  $h^\epsilon$  be as in the second step of Theorem 2.5, i.e.,

$$h^\epsilon(r) = 0, \text{ for } r \leq 0, \quad h^\epsilon(r) = 1, \text{ for } r \geq \epsilon, \quad h^\epsilon(r) = \frac{r}{\epsilon}, \text{ for } 0 \leq r \leq \epsilon.$$

Then by the stability of viscosity solutions (c.f. [8, Proposition 2.4]),

$$w(x, t) = \liminf_{\epsilon \rightarrow 0, (y, s) \rightarrow (x, t)} h^\epsilon(\delta(y, s)), \quad x \in \mathbf{R}^d, t \in [0, T],$$

is a viscosity supersolution of (1.6). Also it is easy to check that  $w$  is the lower semicontinuous envelope of  $1 - \chi_\Gamma$ . Hence  $1 - \chi_\Gamma$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ .

To prove the sufficiency, suppose that  $1 - \chi_\Gamma$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, T)$ . For  $K > 0$ , set

$$v^K(x, t) = \inf\{K(1 - \chi_\Gamma)(y, t) + |x - y| : y \in \mathbf{R}^d\}.$$

Then it is easy to prove that  $v^K$  is a viscosity supersolution of (1.6); see for instance [20, Section V.7]. Note that

$$v^K(x, t) = \delta(x, t) \wedge \inf\{K + |x - y| : y \notin \Gamma_t\}.$$

Now let  $K \rightarrow \infty$ . Then we conclude that  $\delta = \lim v^K$  is a viscosity supersolution.

Let  $\Gamma^*$  be a closed set and let  $u$  be the unique, uniformly continuous viscosity solution of (1.6) satisfying  $u(x, 0) = \text{dist}(x, \Gamma^*)$ . Recall that the level set solution is the zero level set:

$$\Gamma'_t = \{x \in \mathbf{R}^d : u(x, t) = 0\}.$$

**Theorem 4.4.** *The level set solution  $\Gamma'_t$  is the maximal distance solution satisfying (4.1).*

*Proof.* Let  $h^\epsilon$  be as in the previous proof. Set

$$w(x, t) = \liminf_{\epsilon \rightarrow 0, (y, s) \rightarrow (x, t)} h^\epsilon(u(y, s)), \quad x \in \mathbf{R}^d, t \in [0, \infty).$$

Then  $w$  is a viscosity supersolution of (1.6), and  $w$  is equal to the lower semicontinuous envelope of  $1 - \chi_{\Gamma'}$ .

By the previous lemma,  $\{\Gamma'_t\}_{t > 0}$  is a distance solution. Also the continuity of  $u$  and the initial data  $u(x, 0) = \text{dist}(x, \Gamma^*)$  imply that  $\{\Gamma'_t\}_{t > 0}$  satisfies (4.1) in the sense of Definition 4.2.

Let  $\{\Gamma_t\}_{t \in (0, T)}$  be another distance solution satisfying (4.1), and let  $\delta$  be its distance function. Then by definition,  $\delta$  is a viscosity supersolution of (1.6). Moreover, (4.1) yields that

$$\delta_*(x, 0) \geq \text{dist}(x, \Gamma^*) = u(x, 0), \quad x \in \mathbf{R}^d.$$

Hence by Theorem 2.2, we have

$$\delta \geq \delta_* \geq u, \quad \text{in } \mathbf{R}^d \times [0, T].$$

Therefore,  $\Gamma'_t$  contains any other distance solution satisfying (4.1).

## 5. Varifold solutions of Brakke

In this section, we compare the varifold solutions of Brakke [7] with level set solutions defined in §2. The definition given by Brakke involves varifolds. Here we use the formulation of Ilmanen [26], using the Radon measures on  $\mathbf{R}^d$ . We will show that the support of these Radon measures is a distance solution in the sense defined in the previous section. Hence by Theorem 4.4, they are included in the level set solution. We start by recalling Ilmanen's definition, which implies the Brakke's original definition [26, §6].

Let  $(\mu_t)_{t \geq 0}$  be a family of Radon measures on  $\mathbf{R}^d$ . Following [26, §6], we call  $(\mu_t)_{t \geq 0}$  a *Brakke motion* provided that for all  $t \geq 0$  and all  $\phi \in C_c^2(\mathbf{R}^d \rightarrow [0, \infty))$ ,

$$\bar{D}_t \mu_t(\phi) \leq \mathcal{B}(\mu_t, \phi), \quad \mu_t(\phi) = \int_{\mathbf{R}^d} \phi d\mu_t,$$

where for any real valued function  $f$ ,  $\bar{D}_t f(t)$  is the upper derivative,

$$\bar{D}_t f(t) = \limsup_{s \rightarrow t} \frac{f(s) - f(t)}{s - t},$$

and for any Radon measure  $\mu$ ,  $\mathcal{B}(\mu, \phi)$  is defined as follows. Suppose the following:

- (i)  $\mu[\{\phi > 0\}]$  is a  $(d - k)$  rectifiable Radon measure,
- (ii)  $|\delta V|[\{\phi > 0\}]$  is a Radon measure, where  $V$  is the varifold corresponding to the rectifiable measure  $\mu[\{\phi > 0\}]$ , and  $\delta V$  is its first variation (see for example [36], or [26, §1]),
- (iii)  $|\delta V|[\{\phi > 0\}]$  is absolutely continuous with respect to  $\mu[\{\phi > 0\}]$ , with a Radon-Nikodym derivative

$$H \in L^2(\mathbf{R}^d \rightarrow \mathbf{R}^d, d\mu[\{\phi > 0\}]).$$

Then

$$\mathcal{B}(\mu, \phi) = \int -\phi |H|^2 + \nabla \phi \cdot S^\perp \cdot H d\mu,$$

where  $S(x)$  is the projection onto the tangent space  $T_x \mu$ . If however,  $\mu$  and  $\phi$  do not satisfy any one of the above conditions (i),(ii),(iii), then we set  $\mathcal{B}(\mu, \phi) = -\infty$ .

For a Brakke motion  $(\mu_t)_{t \geq 0}$ ,  $d\mu := d\mu_t dt$  is a Radon measure on  $\mathbf{R}^d \times [0, \infty)$  (see (5.2) below). Let  $\Gamma$  be the support of  $\mu$ , and for  $t \geq 0$  let  $\hat{\Gamma}_t$  be the support of  $\mu_t$ .

**Lemma 5.1.** *Let  $(\mu_t)_{t \geq 0}$  be a Brakke motion. Then*

$$(5.1) \quad \Gamma = \overline{\bigcup_{t>0} \hat{\Gamma}_t \times \{t\}}.$$

Moreover, for any  $(x_0, t_0) \in \Gamma$  with  $t_0 > 0$ , there exists a sequence  $(x_n, t_n)$  converging to  $(x_0, t_0)$  such that  $x_n \in \hat{\Gamma}_{t_n}$  and  $t_n < t_0$ . Finally, if  $(x, 0) \in \Gamma$ , then  $x \in \hat{\Gamma}_0$ .

*Proof.* Let

$$\mathcal{C} := \bigcup_{t>0} \hat{\Gamma}_t \times \{t\}.$$

1. The inclusion  $\Gamma \subset \bar{\mathcal{C}}$  is immediate. Suppose that  $t_0 > 0$  and  $(x_0, t_0) \notin \Gamma$ . Then there is  $0 < \epsilon < t_0$  and a smooth function  $\xi : \mathbf{R}^d \rightarrow \mathbf{R}^+$  with compact support such that  $\xi(x_0) > 0$  and

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \mu_t(\xi) dt = 0.$$

Hence for almost every  $t \in (t_0 - \epsilon, t_0 + \epsilon)$ , we have  $\mu_t(\xi) = 0$ . According to [26, 7.2(ii)], the following limits exist and satisfy

$$(5.2) \quad \lim_{t \uparrow t_0} \mu_t(\xi) \geq \mu_{t_0}(\xi) \geq \lim_{t \downarrow t_0} \mu_t(\xi).$$

In particular  $\mu_{t_0}(\xi) = 0$  and  $x_0 \notin \hat{\Gamma}_{t_0}$ . So we have proved that

$$\mathcal{C} \subset \Gamma \subset \bar{\mathcal{C}}.$$

Since  $\Gamma$  is closed, (5.1) follows.

2. Suppose that  $x_0 \in \hat{\Gamma}_{t_0}$  for some  $t_0 > 0$ . For any  $\epsilon > 0$ , choose a smooth cut-off function  $\xi : \mathbf{R}^d \rightarrow [0, 1]$  which is equal to one on  $B_\epsilon(x_0)$  and zero on  $\mathbf{R}^d \setminus B_{2\epsilon}(x_0)$ . By the first inequality in (5.2), we have

$$\limsup_{t \uparrow t_0} \mu_t(B_{2\epsilon}(x_0)) \geq \lim_{t \uparrow t_0} \mu_t(\xi) \geq \mu_{t_0}(\xi) > 0.$$

Since  $\epsilon > 0$  is arbitrary, we can now easily create the desired sequence  $(x_n, t_n)$ .

3. Now suppose that  $(x_0, t_0) \in \Gamma$ . In view of the previous step, we may assume that  $x_0 \notin \hat{\Gamma}_{t_0}$ . Then there is  $\rho > 0$  such that  $\mu_{t_0}(B_\rho(x_0)) = 0$ . By Brakke's clearing out lemma, [7, p.164], there is  $\tau > t_0$  satisfying,

$$\mu_t(B_{\rho/2}(x_0)) = 0, \quad \forall t \in [t_0, \tau].$$

Hence  $B_{\rho/2}(x_0)$  and  $\hat{\Gamma}_t$  are disjoint for all  $t \in [t_0, \tau]$ . Since  $(x_0, t_0) \in \Gamma$ , (5.1) implies that

$$(x_0, t_0) \in \overline{\bigcup_{0 < t < t_0} \hat{\Gamma}_t \times \{t\}}.$$

We now use this inclusion and the previous step in a diagonal argument to construct the desired sequence.

4. If  $x \notin \hat{\Gamma}_0$ , there is  $\rho > 0$  satisfying  $\mu_0(B_\rho(x)) = 0$ . Then by Brakke’s clearing out lemma, there is  $\tau > 0$  such that  $\mu_t(B_{\rho/2}(x)) = 0$  for any  $t \in [0, \tau]$ . Hence  $(x, 0)$  is not in  $\Gamma$ .

**Lemma 5.2.** *Let  $X \in S^{d \times d}$  and  $Xu = 0$  for some unit vector  $u$ . Then*

$$F(u, X) \leq X : S + \|X\| \|Su\|^2,$$

for any  $(d - k)$ -plane  $S$ . Here  $S$  is identified with its projection matrix,  $X : S$  is the scalar product of the matrices  $X$  and  $S$ , and  $\|X\|$  is the operator norm of  $X$ .

*Proof.* Since the statement is rotationally invariant, we assume that  $u = e_d$  and  $X$  is diagonal, i.e.,  $X_{ij} = \lambda_i \delta_{ij}$ . Then the equation  $Xu = 0$  implies that  $\lambda_d = 0$ . We may also assume that

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{d-1},$$

so that

$$\|X\| = \max\{|\lambda_1|, \dots, |\lambda_{d-1}|\} = |\lambda_{d-1}| \geq |\lambda_{d-k}|.$$

Since  $u = e_d$  and  $S$  is a projection matrix,  $S_{dd} = Su \cdot u = |Su|^2$ . Moreover  $X = P_u X P_u$  and therefore

$$F(u, X) = \sum_{i=1}^{d-k} \lambda_i.$$

So it suffices to prove that

$$\sum_{i=1}^{d-k} \lambda_i \leq \sum_{i=1}^{d-1} \lambda_i S_{ii} + |\lambda_{d-k}| S_{dd},$$

an inequality which is equivalent to

$$\sum_{i=1}^{d-k} \lambda_i (1 - S_{ii}) \leq \sum_{i=d-k+1}^{d-1} \lambda_i S_{ii} + |\lambda_{d-k}| S_{dd}.$$

Since  $0 \leq S_{ij} \leq 1$ , and the trace of  $S$  is equal to  $(d - k)$ , we have

$$\begin{aligned} \sum_{i=1}^{d-k} \lambda_i(1 - S_{ii}) &\leq \lambda_{d-k} \sum_{i=1}^{d-k} (1 - S_{ii}) = \lambda_{d-k} \left( d - k - \sum_{i=1}^{d-k} S_{ii} \right) \\ &= \lambda_{d-k} \sum_{i=d-k+1}^d S_{ii} \leq \sum_{i=d-k+1}^{d-1} \lambda_i S_{ii} + |\lambda_{d-k}| S_{dd}. \end{aligned}$$

**Proposition 5.3.** *Let  $(\mu_t)_{t \geq 0}$  be a Brakke motion. Set*

$$\delta(x, t) := \text{dist}(x, \{y : (y, t) \in \Gamma\}).$$

*Then  $\delta$  is a viscosity supersolution of  $\delta_t \geq F_*(\nabla\delta, \nabla^2\delta)$  in  $\mathbf{R}^d \times (0, \infty)$ .*

*Proof.* Suppose that for some test function  $\psi$  the function  $\delta - \psi$  attains a local minimum at  $(x_0, t_0) \notin \Gamma$  with  $t_0 > 0$ . We wish to show that at  $(x_0, t_0)$  we have  $\psi_t \geq F_*(\nabla\psi, \nabla^2\psi)$ . We argue by contradiction. So we assume that

$$\beta := -[\psi_t(x_0, t_0) - F_*(\nabla\psi(x_0, t_0), \nabla^2\psi(x_0, t_0))] > 0,$$

and then obtain a contradiction in several steps.

Without loss of generality we may assume that  $\psi(x_0, t_0) = \delta(x_0, t_0)$ ,  $\psi$  is globally Lipschitz continuous, and the infimum of  $\delta - \psi$  is strictly positive on the complement of any ball containing  $(x_0, t_0)$ .

1. Since  $\delta(x_0, t_0) > 0$  and any distance function is semi-concave on its positive set, we conclude that  $\delta$  is differentiable with respect to  $x$  at  $(x_0, t_0)$ . Therefore  $|\nabla\psi(x_0, t_0)| = |\nabla\delta(x_0, t_0)| = 1$ , and

$$F_*(\nabla\psi(x_0, t_0), \nabla^2\psi(x_0, t_0)) = F(\nabla\psi(x_0, t_0), \nabla^2\psi(x_0, t_0)).$$

Choose  $0 < \epsilon < t_0$  such that for all  $|t - t_0| \leq 2\epsilon$  and  $|x - x_0| \leq 2\epsilon$ ,

$$(5.3) \quad \psi_t(x, t) - F(\nabla\psi(x, t), \nabla^2\psi(x, t)) \leq -\beta/2,$$

and

$$(5.4) \quad 2 \geq |\nabla\psi(x, t)| \geq 1/2.$$

2. Set

$$\alpha_0 := \delta(x_0, t_0) + \inf\{(\delta - \psi)(t, x) : |x - x_0| \geq \epsilon \text{ or } |t - t_0| \geq \epsilon \text{ or both}\}.$$

By our assumptions on  $\psi$  and  $\delta$ ,  $\alpha_0 > \delta(x_0, t_0) > 0$ . Choose  $y_0 \in \mathbf{R}^d$  so that

$$\alpha := \delta(x_0, t_0) = |x_0 - y_0|, \quad (y_0, t_0) \in \Gamma,$$

and define

$$\Omega(t) := \{x \in \mathbf{R}^d : \psi(x + x_0 - y_0, t) \leq \alpha\}.$$

Note that  $y_0 \in \partial\Omega(t_0)$ .

3. We claim that

$$\{x \in \mathbf{R}^d : (x, t) \in \Gamma\} \subset \Omega(t), \quad \forall t \geq 0.$$

Indeed, suppose that  $\delta(x, t) = 0$ . Then

$$(5.5) \quad \begin{aligned} \psi(x + x_0 - y_0, t) &\leq \delta(x + x_0 - y_0, t) \\ &\leq \delta(x, t) + |x_0 - y_0| = \alpha. \end{aligned}$$

4. In this step, we will show that, for any  $|t - t_0| \geq \epsilon$ ,

$$\{x : \text{dist}(x, \partial\Omega(t)) \leq \frac{\alpha_0 - \alpha}{2}, \delta(x, t) = 0\} = \emptyset.$$

Suppose that  $\text{dist}(x, \partial\Omega(t)) \leq (\alpha_0 - \alpha)/2$ . Since  $|t - t_0| \geq \epsilon$ , the definition of  $\alpha_0$  yields

$$\psi(y, t) \leq \delta(y, t) + \alpha - \alpha_0, \quad \forall y \in \mathbf{R}^d.$$

Choose  $y \in \partial\Omega(t)$  satisfying

$$|x - y| = \text{dist}(x, \partial\Omega(t)) \leq \frac{\alpha_0 - \alpha}{2}.$$

Since  $\psi(y + x_0 - y_0, t) = \alpha$ ,

$$\begin{aligned} \delta(x, t) &\geq \delta(x + x_0 - y_0, t) - |x_0 - y_0| \\ &\geq \delta(y + x_0 - y_0, t) - |x - y| - |x_0 - y_0| \\ &\geq \psi(y + x_0 - y_0, t) - (\alpha - \alpha_0) - \frac{\alpha_0 - \alpha}{2} - \alpha \\ &= \frac{\alpha_0 - \alpha}{2} > 0. \end{aligned}$$

5. Finally we claim that for any  $t \geq 0$ ,

$$\{x \in \mathbf{R}^d : \text{dist}(x, \partial\Omega(t)) \leq \frac{\alpha_0 - \alpha}{2[\|\nabla\psi\|_\infty + 1]}, \delta(x, t) = 0\} \subset B_\epsilon(y_0).$$

Let  $x$  be an element of the set on the left. In view of step 4, we may assume that  $|t - t_0| < \epsilon$ . Choose  $y \in \partial\Omega(t)$  such that

$$|y - x| = \text{dist}(x, \partial\Omega(t)) \leq \frac{\alpha_0 - \alpha}{2[\|\nabla\psi\|_\infty + 1]}.$$



Since  $\psi(y + x_0 - y_0, t) = \alpha$ ,

$$\begin{aligned} (\delta - \psi)(x + x_0 - y_0, t) &\leq \delta(x, t) + \alpha - \psi(y + x_0 - y_0, t) \\ &\quad + \|\nabla\psi\|_\infty |x - y| \\ &= \|\nabla\psi\|_\infty \frac{\alpha_0 - \alpha}{2[\|\nabla\psi\|_\infty + 1]} \\ &< \alpha_0 - \alpha. \end{aligned}$$

Since  $|t - t_0| < \epsilon$ , by the definition of  $\alpha_0$ ,  $x + x_0 - y_0 \in B_\epsilon(x_0)$  and therefore,  $x \in B_\epsilon(y_0)$ .

6. Choose  $\gamma \in (0, \epsilon/2)$  such that

$$(5.6) \quad \gamma \leq \frac{\alpha_0 - \alpha}{4[\|\nabla\psi\|_\infty + 1]},$$

and the signed distance  $r(x, t)$  to  $\Omega(t)$ ,

$$r(x, t) := \begin{cases} -\text{dist}(x, \partial\Omega(t)), & \text{in } \Omega(t), \\ \text{dist}(x, \partial\Omega(t)), & \text{in } \mathbf{R}^d \setminus \Omega(t), \end{cases}$$

is smooth in the region

$$\{(x, t) : |t - t_0| \leq \epsilon, x \in B_\epsilon(y_0), \text{dist}(x, \partial\Omega(t)) \leq 2\gamma\}.$$

Observe that the above choice of  $\gamma$  is possible, because of (5.4) and the smoothness of  $\psi$ .

7. Let  $h(s) := ([\gamma - |s|]^+)^p$  with  $p > 2$  to be chosen later. Set

$$\Phi(x, t) := h(r(x, t) - \frac{\gamma}{2}), \quad f(t) := \mu_t(\Phi).$$

Since  $y_0 \in \partial\Omega(t_0)$ ,  $\Phi(y_0, t_0) > 0$ . Moreover,  $(y_0, t_0) \in \Gamma$ . Therefore by (5.1),  $f$  is not identically equal to zero in  $(t_0 - \epsilon, t_0 + \epsilon)$ . On the other hand, step 4 implies that for  $|t - t_0| \geq \epsilon$ , we have  $|r| \geq (\alpha_0 - \alpha)/2 \geq 2\gamma$  on  $\Gamma$ . Hence  $f$  is identically equal to zero for  $|t - t_0| \geq \epsilon$ . We will obtain a contradiction by showing that for suitably chosen  $p$ , we have  $\bar{D}_t f(t) \leq 0$  for any  $t \geq 0$ .

By steps 3, 4, 5 and 6,  $\Gamma \cap \text{spt } \Phi \subset U$ , where

$$(5.7) \quad U := \{(x, t) : |t - t_0| \leq \epsilon, x \in B_\epsilon(y_0), -\frac{1}{2}\gamma \leq r(x, t) < 0\}.$$

Hence our choice of  $\gamma$  in step 6 implies that  $\Phi$  is twice differentiable on  $\Gamma$ .

8. By the previous step,  $\Phi$  is a time varying admissible function in Brakke's definition. Hence we conclude that for any  $t > 0$ , either we have  $\bar{D}_t f(t) = -\infty$ , or  $\text{spt } \mu_t \cap \{x : (x, t) \in \text{spt } \Phi\}$  is  $(d - k)$ -rectifiable, and at such a time point  $t$ , we follow [26, p.60, step 3]. The result is:

$$\begin{aligned} \bar{D}_t f(t) &\leq \int [-\Phi|H|^2 + \nabla\Phi \cdot S^\perp \cdot H + \Phi_t] d\mu_t \\ &= \int [-\Phi|H|^2 + \nabla\Phi \cdot S \cdot H + \nabla\Phi \cdot H + \Phi_t] d\mu_t \\ &= \int [-|\sqrt{\Phi}H + \frac{S\nabla\Phi}{2\sqrt{\Phi}}|^2 + \frac{|S\nabla\Phi|^2}{4\Phi} - S : \nabla^2\Phi + \Phi_t] d\mu_t \\ &\leq \int [\frac{|S\nabla\Phi|^2}{4|\Phi|} - S : \nabla^2\Phi + \Phi_t] d\mu_t, \end{aligned}$$

where  $S = S(x)$  is the projection matrix onto  $T_x\mu_t$ ,  $S\nabla\Phi$  is the tangential gradient of  $\Phi$ , and we have used the first variation formula (3.9) in the third step. Proceeding as in [26, p.60], we compute the derivatives of  $\Phi$  in terms of the derivatives of  $h$  evaluated at  $r - \frac{\gamma}{2}$ :

$$\bar{D}_t f(t) \leq \int [(\frac{(h')^2}{4h} - h'')|S\nabla r|^2 + h'(-S : \nabla^2 r + r_t)] d\mu_t.$$

By Lemma 5.2.,

$$F(\nabla r, \nabla^2 r) \leq S : \nabla^2 r + \|\nabla^2 r\| \|S\nabla r\|^2.$$

In view of (5.7),  $h'(r - \gamma/2) \geq 0$  on  $\Gamma \cap \text{spt } \Phi$ . Therefore,

$$\begin{aligned} \bar{D}_t f(t) &\leq \int [\frac{1}{4} \frac{(h')^2}{h} - h'' + \Lambda|h'|] |S\nabla r|^2 d\mu_t \\ &\quad + \int h'(r_t - F(\nabla r, \nabla^2 r)) d\mu_t, \end{aligned}$$

where  $\Lambda$  is a constant satisfying  $\|\nabla^2 r\| \leq \Lambda$  on  $U$ . Now choose  $p = p(\Lambda)$  in the definition of  $h$  so that the first integral in the above expression is nonpositive. Hence

$$(5.8) \quad \bar{D}_t f(t) \leq \int h'[r_t - F(\nabla r, \nabla^2 r)] d\mu_t.$$

9. By (5.4) we obtain  $|\nabla\psi(x, t)| \geq 1/2$  for any  $|t - t_0| \leq 2\epsilon$ ,  $x \in B_{2\epsilon}(x_0)$ . Also note that  $\nabla r$  and  $\nabla\psi$  induce the same orientation of

$\partial\Omega(t)$ . Therefore for any  $t \in [t_0 - \epsilon, t_0 + \epsilon]$  and any  $x \in B_{2\epsilon}(y_0) \cap \partial\Omega(t)$ , we have  $r_t = \psi_t/|\nabla\psi|$  and

$$F(\nabla r, \nabla^2 r) = \frac{F(\nabla\psi, \nabla^2\psi)}{|\nabla\psi|},$$

where all derivatives of  $\psi$  are evaluated at  $(x + x_0 - y_0, t)$ . From (5.3) and (5.4) it follows that

$$r_t(y, t) - F(\nabla r(y, t), \nabla^2 r(y, t)) \leq -\beta/4,$$

$$(5.9) \quad \forall |t - t_0| \leq \epsilon, y \in B_{2\epsilon}(y_0) \cap \partial\Omega(t).$$

Let  $(x, t) \in \Gamma \cap \text{spt } \Phi$ . Since  $2\gamma < \epsilon$ , by (5.7) and the smoothness of  $r$  (c.f. step 6), there is a unique  $y \in \partial\Omega(t) \cap B_{2\epsilon}(y_0)$  satisfying  $r(x, t) = -|y - x|$ . Since the eigenvalues of  $\nabla^2 \delta$  decrease moving away from  $\partial\Omega(t)$  along characteristics of the distance (see step 4 of Theorem 3.2), by (5.9),

$$\begin{aligned} r_t(x, t) - F(\nabla r(x, t), \nabla^2 r(x, t)) &\leq r_t(y, t) - F(\nabla r(y, t), \nabla^2 r(y, t)) \\ &\leq -\beta/4, \end{aligned}$$

which together with (5.8) shows that  $\bar{D}_t f(t) \leq 0$  for all  $t > 0$ . This, however, contradicts with the fact that  $f(t) = 0$  for all  $|t - t_0| > \epsilon$  and  $f(t) > 0$  for some  $|t - t_0| \leq \epsilon$ . Hence  $\delta$  is a viscosity supersolution of (1.6) in  $\{\delta > 0\}$ .

**10.** In view of the previous step and Lemma 5.1, all hypotheses of Lemma 3.11 are satisfied. Hence  $\delta$  is a viscosity supersolution of (1.6) in  $\mathbf{R}^d \times (0, \infty)$ .

**Theorem 5.4.** *Let  $(\mu_t)_{t \geq 0}$  be a Brakke motion and let  $u$  be a non-negative, uniformly continuous, viscosity solution of (1.6) satisfying*

$$\text{spt } \mu_0 \subset \{x \in \mathbf{R}^d : u(x, 0) = 0\}.$$

*Then for all  $t \geq 0$ ,*

$$\text{spt } \mu_t \subset \{x \in \mathbf{R}^d : u(x, t) = 0\}.$$

*Proof.* Since  $\{u(\cdot, 0) = 0\}$  contains  $\text{spt } \mu_0$ , Lemma 5.1 implies that  $\{u(\cdot, 0) = 0\}$  contains  $\{\delta(\cdot, 0) = 0\}$ . Then, arguing exactly as in Theorem 2.5 we construct a nondecreasing uniformly continuous function  $\omega(t)$  such that  $\omega(0) = 0$  and

$$0 \leq u(x, 0) \leq \omega(\delta(x, 0)), \quad \forall x \in \mathbf{R}^d.$$

By Theorem 2.2,  $0 \leq u \leq \omega(\delta)$  on  $\mathbf{R}^d \times (0, +\infty)$  and this gives the desired inclusion, because  $\{\delta(\cdot, t) = 0\}$  contains  $\text{spt } \mu_t$ .

*Remark 5.5.* Since Ilmanen's solutions of the mean curvature flow, obtained by elliptic regularization in [26], satisfy Brakke's condition, our result applies also to them. Theorem 5.4 can be used, in some situations, to show the occurrence of the *fattening phenomenon*, i.e.,  $\mathcal{H}^{d-k+1}(\Gamma_t) > 0$  for some  $t > 0$  even though the initial set  $\Gamma_0$  has dimension  $(d-k)$ . It suffices to find a family  $\{\mu_t^i\}_{i \in I}$  of Brakke varifolds such that  $\mu_0^i$  are supported in  $\Gamma_0$  and

$$\mathcal{H}^{d-k+1}\left(\bigcup_{i \in I} \text{spt } \mu_t^i\right) > 0$$

for some  $t > 0$ .

## 6. Geometric supersolutions of De Giorgi

In this section, we compare the level set approach with a purely geometric approach based on the notion of barriers, recently introduced by De Giorgi in [12]. In addition to the general definition of barriers, [12] also contains the characterization of the smooth mean curvature flow as a system of equations for  $\eta$  (c.f. (3.10)) and the idea to evolve hypersurfaces by the sum of the smallest  $(d-k)$  principal curvatures. Both of these observations were crucial in the development of this paper.

We start with De Giorgi's general definition of barriers.

**Definition 6.1 (De Giorgi).** Let  $(S, \leq)$  be a partially ordered set and let  $\mathcal{F}$  be a class of functions defined on intervals  $[a, b] \subset [0, +\infty)$ , with values in  $S$ . We say that  $\phi : [0, +\infty) \rightarrow S$  is a *barrier* relative to  $\mathcal{F}$ , and we write  $\phi \in \text{Barr}(\mathcal{F})$ , provided that the following implication holds for any  $f \in \mathcal{F}$ ,

$$f : [a, b] \rightarrow S, \quad f(a) \leq \phi(a) \quad \Rightarrow \quad f(t) \leq \phi(t), \quad \forall t \in [a, b].$$

If  $S$  is a complete lattice, then the infimum any family of barriers is still a barrier. For any  $s \in S$ , this suggests the following definition of the least barrier,  $\text{barr}(\mathcal{F}, s)$ , that is greater than  $s$  at time 0,

$$\text{barr}(\mathcal{F}, s)(t) := \inf \left\{ \phi(t) : \phi \in \text{Barr}(\mathcal{F}), \quad s \leq \phi(0) \right\}.$$

Heuristically, we think of  $\mathcal{F}$  as the set of all classical solutions and  $\text{Barr}(\mathcal{F})$  as the set of all supersolutions. Then in analogy with the Perron's method,  $\text{barr}(\mathcal{F}, s)$  is a *weak* solution with initial data  $s$ .

In this section, we apply the above definition of barriers to the codimension- $k$  mean curvature flow, assuming  $k > 1$  (the codimension-1 case is studied in [6]). Following [12], we take  $S$  to be the collection of all

subsets of  $\mathbf{R}^d$ , ordered by inclusion. Then there are two choices for  $\mathcal{F}$ . First one, denoted by  $\mathcal{F}$ , is the class of smooth codimension- $k$  mean curvature flows  $\{\Gamma_t\}_{t \in [a,b]}$ , given in Definition 3.6 up to a translation in time. The second choice  $\mathcal{F}^*$  is the collection of all maps with values in compact sets  $\{\Omega_t\}_{t \in [a,b]}$  such that  $\{\partial\Omega_t\}_{t \in [a,b]}$  is a smooth codimension-1 flow (see Definition 3.6) and the signed distance,  $r(x, t)$  from  $\partial\Omega_t$ ,

$$r(x, t) := \text{dist}(x, \Omega_t) - \text{dist}(x, \mathbf{R}^d \setminus \Omega_t),$$

satisfies

$$\frac{\partial r}{\partial t} \geq F(\nabla r, \nabla^2 r) \quad \text{on} \quad \{(x, t) \in \mathbf{R}^d \times [a, b] : r(x, t) = 0\}.$$

Given a compact set  $\Gamma_0$ , in [12] De Giorgi uses the choice  $\mathcal{F}$  to define a weak solution of the codimension- $k$  mean curvature flow starting from  $\Gamma_0$ . His definition is,

$$(6.1) \quad \Lambda_t := \bigcap_{\rho > 0} \text{barr}(\mathcal{F}, N_\rho(\Gamma_0))(t),$$

where  $N_\rho(\Gamma_0)$  is the open  $\rho$ -neighbourhood of  $\Gamma_0$ . Using  $\mathcal{F}^*$  instead of  $\mathcal{F}$  one obtains the following definition

$$(6.2) \quad \Gamma_t := \bigcap_{\rho > 0} \text{barr}(\mathcal{F}^*, N_\rho(\Gamma_0))(t).$$

We will show in Theorem 6.4 below, that the latter definition is the same as the level set flow. In the codimension-1 case, the connections between the level set approach and the barrier approach are investigated by Bellettini and Paolini in [5], [6].

**Conjecture [De Giorgi].** *For any open set  $A \subset \mathbf{R}^d$  and  $t \geq 0$ , we have*

$$(6.3) \quad \text{barr}(\mathcal{F}, A)(t) = \text{barr}(\mathcal{F}^*, A)(t).$$

*If the above conjecture holds, then  $\Gamma_t$  and  $\Lambda_t$  defined above agree and both are equal to the level set solution defined in §2.*

*Remark 6.2.* The inclusion  $\subset$  in (6.3) is not very hard to prove. Let  $\phi(t)$  be the family of sets on the right-hand side. We will prove that  $\phi \in \text{Barr}(\mathcal{F})$ . Let  $\{\hat{\Gamma}_t\}_{t \in [a,b]}$  be any function in  $\mathcal{F}$  satisfying  $\Gamma_a \subset \phi(a)$ . Choose  $\sigma > 0$  such that  $\eta(x, t) = \text{dist}^2(x, \Gamma_t)$  is smooth on

$$Q := \{(x, t) : a \leq t \leq b, \eta(x, t) \leq 4\sigma^2\}.$$

For  $t \in [a, b]$ , set  $\Omega_t = \{\eta(\cdot, t) < \sigma^2\}$ . Then using Theorem 3.8, we can show that the family  $\{\Omega_t\}_{t \in [a,b]}$  belongs to  $\mathcal{F}^*$ . Indeed, the signed

distance  $r(x, t)$  from  $\Omega_t$  is equal to  $\text{dist}(x, \Gamma_t) - \sigma$  near  $\partial\Omega_t$  and outside  $\Omega_t$ .

By Lemma 6.3, the sets  $\phi(t)$  are open for any  $t$ . Reducing  $\sigma$ , if necessary, we may assume that  $\Omega_a \subset \phi(a)$ . Since  $\phi$  is a barrier relative to  $\mathcal{F}^*$ ,  $\Omega_t \subset \phi(t)$  and therefore

$$\Gamma_t \subset \Omega_t \subset \phi(t)$$

for any  $t \in [a, b]$ . Since  $\phi \in \text{Barr}(\mathcal{F})$ , the inclusion  $\subset$  in (6.3) follows.

In the following lemma we prove some elementary topological properties of least barriers.

**Lemma 6.3.** *Let  $A \subset \mathbf{R}^d$  be an open set and let  $\phi(t) = \text{barr}(\mathcal{F}^*, A)(t)$ ,  $K(t) = \mathbf{R}^d \setminus \phi(t)$ . Then  $\phi(t)$  is open for any  $t$ , and the map  $t \mapsto K(t)$  is upper semicontinuous from the left, i.e.,  $(x_h, t_h) \rightarrow (x, t)$ ,  $x_h \in K(t_h)$  and  $t_h < t$  implies  $x \in K(t)$ .*

*Proof.* The translation invariance of the class  $\mathcal{F}^*$  easily implies that the interior of a barrier is still a barrier. Hence the minimality of  $\phi$  forces  $\phi(t)$  to coincide with its interior for any  $t$ .

To check the upper semicontinuity property we define

$$\tilde{K}(t) := \bigcap_{0 < \tau < t} \overline{\bigcup_{s \in (t-\tau, t]} K(s)}, \quad \tilde{K}(0) = K(0),$$

and verify that  $\tilde{K}(t) = K(t)$  for any  $t$ . Since  $\tilde{K}(t)$  contains  $K(t)$  and  $\phi(t)$  is the least barrier, we will prove that  $\tilde{\phi}(t) := \mathbf{R}^d \setminus \tilde{K}(t)$  is a barrier.

Let  $\{\Omega_t\}_{t \in [a, b]} \in \mathcal{F}^*$  such that  $\Omega(a) \subset \phi(a)$ . For  $\tau$  small enough we can assume that  $y + \Omega_{a+\epsilon} \subset \phi(a)$  for any  $\epsilon \in [0, \tau)$  and any  $y$  such that  $|y| < \tau$ . The barrier property implies

$$N_\tau(\Omega_t) \subset \phi(s), \quad \forall s \in (t - \tau, t]$$

for any  $t \geq a$ . Hence,

$$\Omega_t \subset \text{Interior} \left( \bigcap_{s \in (t-\tau, t]} \phi(s) \right).$$

In particular, since the set on the right is contained in  $\tilde{\phi}(t)$ , we obtain  $\Omega_t \subset \tilde{\phi}(t)$ .

The proof of the inclusion of the level set flow in the sets  $\Gamma_t$  defined in (6.2) crucially depends on the following approximation property. Similar properties for codimension-1  $\mathcal{F}$ -flows have been proved in [16], [25], [27].

**Approximation property.** Assume that  $U$  is a bounded open set with  $C^{1,1}$  boundary  $\Sigma$ . Let  $r(x, t)$  be the unique viscosity solution of (1.6) with the initial condition  $r(\cdot, 0)$  equal to the signed distance from

$\Sigma$ . Then, there exist  $T > 0$ , an open set  $A$  containing  $\Sigma$ , and a family of functions  $r^\epsilon$  such that with  $D = \bar{A} \times [0, T]$  we have the following:

- (1)  $r^\epsilon \in C^\infty(D)$ ,  $|\nabla r^\epsilon| > 0$  on  $D$ , and  $r^\epsilon$  are supersolutions of (1.6) in  $A \times (0, T)$ ;
- (2)  $r^\epsilon$  uniformly converges to  $r$  on  $D$ .

We remark that in the approximation property there is no hope in general to look for smooth solutions of (1.6) because of the lack of smoothness of  $F(p, X)$  on  $(\mathbf{R}^d \setminus \{0\}) \times S^{d \times d}$ . This is also the main motivation for the definition of  $\mathcal{F}^*$ . We can now prove the following equivalence result.

**Theorem 6.4.** *Assume that the approximation property holds. Let  $\{\Gamma_t\}_{t \in [0, +\infty)}$  be as in (6.2). Then,  $\Gamma_t$  is equal to the level set solution defined in §2.*

*Proof.* Let  $u(x, t)$  be the unique viscosity solution of (1.6) with the initial condition  $u(x, 0) = \text{dist}(x, \Gamma_0)$ , and set  $\Gamma'_t := \{u(\cdot, t) = 0\}$ .

1. To prove the inclusion  $\Gamma'_t \supset \Gamma_t$ , it suffices to show that  $\phi_\rho(t) := \{u(\cdot, t) \leq \rho\}$  is a barrier relative to  $\mathcal{F}^*$ , for any  $\rho > 0$ . Because if  $\phi_\rho$  is a barrier, then

$$\text{barr}(\mathcal{F}^*, N_\rho(\Gamma))(t) \subset \phi_\rho(t), \quad \forall \rho > 0,$$

and we obtain the inclusion  $\Gamma'_t \supset \Gamma_t$  by letting  $\rho \downarrow 0$ . The fact that  $\phi_\rho$  is a barrier follows from Theorem 2.2.

Indeed, let  $\{\Omega_t\}_{t \in [a, b]}$  be a function in  $\mathcal{F}^*$ , such that  $\Omega_a \subset \phi_\rho(a)$ . Set  $\delta(x, t) := \text{dist}(x, \Omega(t))$ . Arguing as in Corollary 3.9, we can prove that  $\delta$  is a supersolution of (1.6) in  $\mathbf{R}^d \times (a, b)$ . On the other hand, since  $\Omega_a \subset \phi_\rho(a)$ , there exists a nondecreasing uniformly continuous function  $\omega(t)$  such that  $\omega(0) \leq 0$  and  $\omega(\delta(\cdot, a)) \geq u(\cdot, a) - \rho$  (see Theorem 2.5 for the construction of  $\omega$ ). By Theorem 2.2, Theorem 2.3 and the continuity of  $u, \delta$ , we get  $\omega(\delta(\cdot, t)) \geq u(\cdot, t) - \rho$  for  $t \in [a, b]$ . Hence

$$x \in \Omega_t, \Rightarrow \delta(x, t) = 0, \Rightarrow u(x, t) \leq \rho, \Rightarrow x \in \phi_\rho(t)$$

for any  $t \in [a, b]$ . This shows that  $\phi_\rho$  is a barrier.

2. To prove the opposite inclusion  $\subset$ , we define  $K(t) := \mathbf{R}^d \setminus \phi(t)$  and show that the function  $f(t) := \text{dist}(\Gamma_t, K(t))$  is nondecreasing. It is easy to see that the monotonicity of  $f$  follows by the following two properties:

$$(6.4) \quad \liminf_{s \uparrow t} f(s) \geq f(t), \quad \forall t > 0,$$

and

$$(6.5) \quad \forall t \geq 0 \text{ s. t. } f(t) > 0 \quad \exists T > t \text{ s. t. } f(s) \geq f(t), \quad \forall s \in [t, T].$$

Inequality (6.4) follows by the upper semicontinuity of the map  $t \mapsto K(t)$  (see Lemma 6.3): if  $(x_h, y_h, s_h)$  is a sequence converging to  $(x, y, t)$  such that  $s_h < t, x_h \in \Gamma_{s_h}, y_h \in K(s_h)$  and

$$\lim_{h \rightarrow +\infty} |x_h - y_h| = \liminf_{s \uparrow t} f(s),$$

we have  $x \in \Gamma_t, y \in K(t)$  and the inequality follows.

**3.** In this step we begin the proof of (6.5). By the  $C^{1,1}$  interpolation lemma of Ilmanen [27] there exists a bounded open set  $U$  containing  $\Gamma_t$  whose boundary  $\Sigma$  is  $C^{1,1}$  and satisfies

$$(6.6) \quad \text{dist}(\Gamma_t, \Sigma) + \text{dist}(\Sigma, K(t)) = \text{dist}(\Gamma_t, K(t)).$$

Let  $r_t$  be the signed distance function from  $\Sigma$ , and let  $r(x, s)$  be the unique viscosity solution of 1.6 with the initial condition  $r(\cdot, t) = r_t$ . By setting  $U_s := \{r(\cdot, s) < 0\}$ , Theorem 2.2 yields the inclusion

$$y + \Gamma_s \subset U_s \quad \forall s \geq t$$

for any  $y \in \mathbf{R}^d$  such that  $|y| < \text{dist}(\Gamma_t, U_t)$ . In particular

$$(6.7) \quad \text{dist}(\Gamma_s, \partial U_s) \geq \text{dist}(\Gamma_t, \partial U_t).$$

Taking into account that

$$\text{dist}(\Gamma_s, \partial U_s) + \text{dist}(\partial U_s, K(s)) \leq \text{dist}(\Gamma_s, K(s)), \quad \forall s \geq 0$$

and (6.6), (6.7), in order to prove (6.5) we need only to prove in the next step the existence of  $T > t$  such that

$$(6.8) \quad \text{dist}(\partial U_s, K(s)) \geq \text{dist}(\partial U_t, K(t)), \quad \forall s \in [t, T].$$

**4.** Let  $A, T > t, r^\epsilon$  be given by the approximation property up to a translation in time. Possibly reducing  $T$  we can assume that  $\{r^\epsilon(\cdot, s) = \delta\} \subset A$  for any  $s \in [t, T]$ , any  $\epsilon \in (0, \epsilon_0)$ , and  $\delta \in (0, \delta_0)$ .

Since  $r^\epsilon$  are supersolutions of (1.6) and smooth, the sets  $U_s^{\epsilon, \delta} := \{r^\epsilon(\cdot, s) \leq \delta\}$  belong to  $\mathcal{F}^*$  (the functions  $r_\epsilon$  need not be the distance functions). Since  $\phi(t)$  is a barrier, we get

$$(6.9) \quad \text{dist}(\partial U_s^{\epsilon, \delta}, K(s)) \geq \text{dist}(\partial U_t^{\epsilon, \delta}, K(t))$$

for any  $s \in [t, T], \epsilon \in (0, \epsilon_0), \delta \in (0, \delta_0)$ . Now, it is easy so see that the uniform convergence of  $r^\epsilon$  to  $r$  implies

$$\liminf_{\epsilon \rightarrow 0^+} \text{dist}(\partial U_t^{\epsilon, \delta}, K(t)) \geq \text{dist}(\{r(\cdot, t) = \delta\}, K(t))$$

and



$$\limsup_{\epsilon \rightarrow 0^+} \text{dist}(\partial U_s^{\epsilon, \delta}, K(s)) \leq \text{dist}(\partial U_s, K(s)).$$

This yields

$$\text{dist}(\partial U_s, K(s)) \geq \text{dist}(\{r(\cdot, t) = \delta\}, K(t))$$

for any  $s \in [t, T]$ . Letting  $\delta \downarrow 0$  we obtain (6.8).

### 7. Extensions and examples

As we have already seen several important properties of the codimension-one flow generalize to flows with arbitrary codimension. However, there are differences. The level set solution  $\Gamma_t$  of the arbitrary codimension flow is the zero level set of a *nonnegative* auxiliary function  $u(\cdot, t)$ . Alternatively,  $\Gamma_t$  is the set of minimizers of  $u(\cdot, t)$ . On the other hand, in the codimension-one case, any level set can be used to define the level set flow. This simple observation implies that certain properties of the codimension-one flow cannot be generalized. Most importantly, consider the codimension-one flow of two disjoint, compact subsets  $U_0, V_0$  of  $\mathbf{R}^d$ . It is easy to construct a Lipschitz continuous  $u_0 : \mathbf{R}^d \rightarrow \mathbf{R}$  such that,

$$U_0 = \{x : u_0(x) = 0\}, \quad V_0 = \{x : u_0(x) = 1\}.$$

Let  $u(x, t)$  be the unique viscosity solution of (1.2) with initial data  $u_0$ . Then the codimension-one flows

$$U_t = \{x : u(x, t) = 0\}, \quad V_t = \{x : u(x, t) = 1\},$$

are disjoint for all time  $t \geq 0$ . In fact, since the Lipschitz constant of  $u(\cdot, t)$  is nonincreasing in time, the distance between  $U_t$  and  $V_t$  is nondecreasing in time. This proof uses the flexibility, in codimension-one, to use any level set of  $u$  to define the level set flow, and therefore it can not be generalized to an arbitrary codimension. Indeed, this “disjointness” property is false in higher codimensions. Consider the codimension-two flow in  $\mathbf{R}^3$  with initial sets,

$$U_0 = \{(x, y, z) : x^2 + y^2 = 4, z = 0\},$$

$$V_0 = \{(x, y, z) : (x - 2)^2 + z^2 = 4, y = 0\}.$$

Then the level set solutions are given by

$$U_t = \{(x, y, z) : x^2 + y^2 = 4 - 2t, z = 0\},$$

$$V_t = \{(x, y, z) : (x - 2)^2 + z^2 = 4 - 2t, y = 0\}.$$

Observe that the distance between two solutions is decreasing, and at  $t = 1.5$  they have a nonempty intersection at  $(1, 0, 0)$ . Interestingly, the level set flow  $\Gamma_t$  of the initial data  $\Gamma_0 = U_0 \cup V_0$  should become “fat” for  $t > 1.5$ .

Alternatively, we summarize the above observation as follows. Let  $U_t, V_t$  be the level set flows of two disjoint compact sets  $U_0$  and  $V_0$ , respectively. Then, in the codimension-one case the level set flow starting from  $U_0 \cup V_0$  is equal to the union of  $U_t$  and  $V_t$ , but this property is not true for higher codimension flows. However, the ellipticity of the function  $F(p, A)$  in (1.5) shows that there is an inclusion principle for hypersurfaces flowing by the sum of the smallest  $(d - k)$ -principal curvatures, hence the level set approach can be used to describe this motion. Moreover, Corollary 3.9 shows that these hypersurfaces are “barriers” against codimension- $k$  surfaces flowing by the mean curvature, giving a connection between the two flows.

*Extensions.* Consider the geometric equation

$$(7.1) \quad V = H + \pi g(x, t),$$

where  $g(x, t) \in \mathbf{R}^d$  is a given Lipschitz vector field,  $\pi$  is the projection onto the normal space while, as before,  $V$  and  $H$  are, respectively, the normal velocity and the mean curvature vectors. Then the theory developed in the previous sections extends to the above equation with only minor modifications. For example,  $\Gamma_t$  is a classical solution of (7.1) if and only if

$$\nabla \eta_t = \nabla \Delta \eta - \nabla^2 \eta g, \quad \text{on } \Gamma_t,$$

where  $2\eta = \delta^2$ , and  $\delta$  is the distance function to  $\Gamma_t$ . Moreover, the above equation holds if and only if there are positive constants  $C, \sigma$  satisfying

$$-C\delta \leq \delta_t - F(\nabla \delta, \nabla^2 \delta) + \nabla \delta \cdot g \leq C\delta,$$

on  $\{\delta < \sigma\}$ . These observations lead to a result, analogous to Corollary 3.9; namely, whenever a classical solution of (7.1) exists, then it is equal to the zero level set of the unique viscosity solution  $u$  of

$$u_t = F(\nabla u, \nabla^2 u) - \nabla u \cdot g.$$

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