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QUASI-CONFORMAL HARMONIC DIFFEOMORPHISM AND THE UNIVERSAL TEICHMÜLLER SPACE

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0. Introduction

The classical theory of Teichmüller spaces uses extensively the theory of quasi-conformal mappings. However, since the development of the basic results of Eells and Sampson [11], Hartman [15] (see also [3]), Schoen and Yau [27] and Sampson [25], one can now use harmonic maps to study Teichmüller spaces. See for example [10], [30] and [34]. Some important results in this direction were obtained by Wolf [34]. A starting point in the theory of [34] is the following. Using the fact that the Hopf differential of a harmonic map between two Riemann surfaces is holomorphic, it was proved [34] that the Teichmüller space T_g of a compact surface of genus $g > 1$ is homeomorphic to the space of holomorphic quadratic differentials of a fixed compact Riemann surface of the same genus. Then using the space of holomorphic quadratic differentials as the coordinate chart for the Teichmüller space T_g, one can study many important properties of T_g, see for example [34, 18]. Note that in the proof of the result of [34] mentioned above, the fact that the Teichmüller space T_g is of $(6g-6)$-dimensions was used. Later in [18], Jost gave another proof without using this fact.

The result in [34] can be put into a more general setting. In studying the relations between harmonic diffeomorphisms on the hyperbolic space $\mathbb{H}^2$ of dimension 2 and constant mean curvature cuts in the Minkowski 3-space, the second author [31, 32] was able to construct a map $\mathcal{B}$ (see §1 for definition) from the space $BQD(\mathbb{D})$ of holomor-

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phic quadratic differentials on the unit disc in $\mathbb{R}^2$, which are bounded with respect to the Poincaré metric, to the universal Teichmüller space T. In order to construct such a map, the second author proved that a harmonic diffeomorphism on $\mathbb{H}^2$ is quasi-conformal if and only if its Hopf differential is uniformly bounded. In [32], it was also showed that the map $\mathcal{B}$ is continuous with respect to the norm on $BQD(\mathbb{D})$ and the Teichmüller metric on T. However, in order to have applications, we have to know more about this map $\mathcal{B}$. One of the basic question, of course, is whether $\mathcal{B}$ is bijective. In fact, it was conjectured by Schoen [26] that the answer should be affirmative. Another basic question is whether $\mathcal{B}$ is a diffeomorphism. The injectivity part of Schoen's conjecture was proved by Li and the first author in [22] but the surjectivity part of the conjecture is still open.

In this paper, we will first prove the following results:

The map $\beta \circ \mathcal{B} : BQD(\mathbb{D}) \to T(1)$ is real analytic, where $T(1)$ is the image of the Bers imbedding β.

The image of $\beta \circ \mathcal{B}$ is open in $T(1)$ and $\beta \circ \mathcal{B}$ maps $BQD(\mathbb{D})$ analytically and diffeomorphically onto its image.

The well-known Bers imbedding β imbeds the universal Teichmüller space T as a bounded subset of the space of holomorphic quadratic differentials on $\mathbb{D}^*$ which are bounded with respect to the Poincaré metric, where $\mathbb{D}^*$ is the exterior of $\mathbb{D}$ in the Riemann sphere. Therefore, if $\mathcal{B}$ is surjective, then it will be a diffeomorphism onto the universal Teichmüller space. Later, we will say more on the image of $\mathcal{B}$.

Using our results and the results in [27] and [23], one can show that the map $\mathcal{B}$ descends to a map from the space of holomorphic quadratic differentials on a compact Riemann surface of genus $g > 1$ onto T_g. This gives another proof of the Teichmüller theorem which states that T_g is of finite dimension and is diffeomorphic to $\mathbb{R}^{6g-6}$. It is not hard to see that the descended map is in fact the inverse of the map introduced in [34]. Hence our result also gives another proof of Wolf's theorem without assuming the fact that T_g is of $(6g-6)$-dimension.

In order to prove that $\beta \circ \mathcal{B}$ is real analytic, first we have to study more carefully about the Bochner formula satisfied by the ∂-energy density of a harmonic map on $\mathbb{H}^2$. Then we apply the implicit function theorem to conclude the result. The method of proof is similar to that of [34]. However, since the Poincaré disk is noncompact, and since $BQD(\mathbb{D})$ is of infinite dimension, the argument is much more involved.

In order to prove that $\beta \circ \mathcal{B}$ is a diffeomorphism, we will compute the differential of $\beta \circ \mathcal{B}$. The differential is rather simple at the origin and looks almost like an identity map there. However, it is relatively more complicated elsewhere. It is rather difficult to see that the differential is surjective and has a bounded inverse. So, at a general point in $BQD(\mathbb{D})$, we will estimate the norm of the tension field of the composition of two quasi-conformal harmonic diffeomorphisms in terms of their Hopf differentials, and the norm of the difference of the Hopf differentials of two quasi-conformal harmonic diffeomorphisms in terms of their Teichmüller distance. Using these estimates, one can then prove that the image of $\mathcal{B}$ is open and the differential of $\beta \circ \mathcal{B}$ has a bounded inverse. The estimates themselves are also interesting in their own right. They may possibly be generalized to higher dimensions.

Using those estimates on the composition of two quasi-conformal harmonic diffeomorphisms, we will study more details on the structure of universal Teichmüller space via the map $\mathcal{B}$. Until now, the major works which have been done so far on the surjectivity of $\mathcal{B}$ are due to Li and the first author [21, 22, 23]. They proved, among other things, that if the normalized quasi-symmetric function on the ideal boundary is C^1 with non-vanishing energy density, then it has a unique quasi-conformal harmonic diffeomorphic extension to the Poincaré disc. Hence, those points in the universal Teichmüller space represented by this kind of normalized quasi-symmetric functions are in the image of $\mathcal{B}$. Akutagawa proved [2] independently the same result with stronger regularity (C^4) assumption of the boundary data. The examples of harmonic diffeomorphisms on $\mathbb{H}^2$ constructed in [9] turn out to be quasi-conformal. The boundary data of some of these examples are only Hölder at two isolated points. Using these examples and the method in [23], Wang [33] is able to obtain more quasi-conformal harmonic diffeomorphisms whose boundary data might not be smooth at finitely many points. We have some results in this direction. Let N be the subset of the universal Teichmüller space consisting of those equivalent classes such that the corresponding boundary data is a C^1 diffeomorphism on $\partial\mathbb{D}$. We will prove that:

There is a constant $\delta_0 > 0$ such that if $[\mu] \in N$, then $[\nu]$ lies in the image of $\mathcal{B}$ for all $[\nu]$ such that $\lambda([\nu], [\mu]) < \delta_0$, where λ is the Teichmüller metric. In particular, $\overline{N}$ is also in the image of $\mathcal{B}$.

The result is not a direct consequence of the inverse function theorem. In fact, using the estimates mentioned above, one can show that if a quasi-symmetric function f can be decomposed as $f_1 \circ f_2$ such that f_1 has a quasi-conformal harmonic (diffeomorphic) extension to the Poincaré disc and f_2 is C^1 with non-vanishing energy density, then f has a quasi-conformal harmonic (diffeomorphic) extension to Poincaré disc. This phenomenon may also be true for harmonic maps between hyperbolic spaces in higher dimensions. We will discuss this elsewhere.

In [21, 22, 23], the boundary behaviors of proper harmonic maps between hyperbolic spaces in any dimension have been studied. In this paper, we will use these results and the results on quasi-conformal mappings to study the relationship between the boundary regularity of a quasi-conformal harmonic diffeomorphism of the Poincaré disc onto itself and the decay rate of its Hopf differential in an explicit way. In fact, the second author first learned from R. Schoen that M. Wolf had told him that there should be some relation between the boundary regularity of harmonic self-map of the Poincaré disc and the decays of its Hopf differential. We will show that the norm of a holomorphic bounded quadratic differential decays faster at infinity if and only if it is more regular near the boundary when considered as a map from $\mathbb{D}$ onto $\mathbb{D}$. For example, we will also show that for a fixed $\alpha > 0$, the inverse image under $\mathcal{B}$ of the set in the universal Teichmüller space which corresponds to the set of $C^{1,\alpha}$ diffeomorphisms on $\partial\mathbb{D}$ is a linear subspace in $BQD(\mathbb{D})$. Using those results, it is easy to see that the set of C^1 diffeomorphisms on $\partial\mathbb{D}$ is nowhere dense in the universal Teichmüller space with respect to the Teichmüller metric.

The organization of this paper is as follows. We will set the notations, and state the preliminary results in §1. In §2, we will prove that $\beta \circ \mathcal{B}$ is real analytic. We will also compute the derivative of $\beta \circ \mathcal{B}$ and show that $\beta \circ \mathcal{B}$ is a diffeomorphism near the origin. The basic estimates on composition of two maps will be developed in §3 and will be used to prove the main results in §4. As an application, we will give another proof of the Teichmüller theorem. In §5, we will study the relationship between the boundary regularity and the decays of Hopf differential. More applications will also be given.

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1. Preliminary Results

Let B_1 be the open unit ball of the Banach space $L^\infty(\mathbb{D})$ of essentially bounded measurable (complex-valued) functions on the unit disc in $\mathbb{C}$ with sup-norm strictly less than one. In this paper, we will denote the universal Teichmüller space $B_1/\sim$ by T, where the equivalent relation is defined as follows: Let μ, $\nu \in B_1$ and f_μ, f_ν be the quasi-conformal homeomorphisms on $\mathbb{C}$ with complex dilatations equal μ and ν respectively in $\mathbb{D}$, conformal outside $\mathbb{D}$, and fixing the points 1, i, and -1. Then $\mu \sim \nu$ if and only if

$$f_\mu|_{\partial\mathbb{D}} = f_\nu|_{\partial\mathbb{D}},$$

or equivalently

$$f_\mu|_{\mathbb{D}^*} = f_\nu|_{\mathbb{D}^*},$$

where $\mathbb{D}^* = \{z \in \mathbb{C} \,|\, |z| > 1\}$.

Let $BQD(\mathbb{D})$ be the vector space of holomorphic quadratic differentials on the unit disc which are bounded with respect to the Poincaré metric. The space $BQD(\mathbb{D})$ is a Banach space, known as Bers space, under the norm

$$\|\Phi\|_{QD} = \sup_{\mathbb{D}} \|\Phi\|,$$

for $\Phi = \phi(z)\mathrm{d}z^2 \in BQD(\mathbb{D})$, where $\|\Phi\|(z) = \rho^{-2}(z)|\phi(z)|$ and $\rho^2(z)|\mathrm{d}z|^2$ is the Poincaré metric on $\mathbb{D}$. Sometimes we write $\|\phi\|_{QD}$ and $\|\phi\|$ instead, if there will not cause any confusion.

In [32] the second author constructed a map $\mathcal{B}$ from $BQD(\mathbb{D})$ to T as follows: Let $\Phi = \phi(z)\mathrm{d}z^2 \in BQD(\mathbb{D})$. One solves uniquely a smooth function w satisfying

$$(1.1) \qquad \begin{cases} \Delta_p w = e^{2w} - \|\phi\|^2 e^{-2w} - 1, \\ e^{2w}\mathrm{d}s_p^2 \quad \text{complete on } \mathbb{D}, \\ \|\phi\| e^{-2w} \le 1, \end{cases}$$

where $\mathrm{d}s_p^2$ is the Poincaré metric on $\mathbb{D}$ and Δ_p is the corresponding Laplacian. Let us denote this correspondence by $\mathfrak{w}(\Phi)$. Then the mapping $\mathcal{B} : BQD(\mathbb{D}) \to T$ is given by

$$\mathcal{B}(\Phi) = \left[\rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}\right] \tag{1.2}$$

which is the equivalent class in T represented by

$$\mu = \rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}$$

for $\Phi = \phi(z)\mathrm{d}z^2 \in BQD(\mathbb{D})$. The existence of $\mathfrak{w}(\Phi)$ and the fact that $\|\mu\|_\infty < 1$ is proved in [32] using constant mean curvature spacelike surfaces in Minkowski 3-space; reproved and generalized in [29] recently using only techniques in partial differential equations and quasi-conformal maps. Let $0 < \alpha < 1$ be fixed and let Y be the Banach space of all functions u in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{D})$ such that $|u|^*_{2,\alpha,\mathbb{D}} < \infty$. Here for a bounded domain $\Omega \subset \mathbb{C}$, and $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$, we define

$$\begin{aligned} |u|^*_{2,\alpha,\Omega} = \sup\nolimits_\Omega |u| + \sup\nolimits_{x\in\Omega} d_x|\nabla_0 u|(x) + \sup\nolimits_{x\in\Omega} d_x^2|\nabla_0^2 u|(x) \\ + \sup_{x\neq y\in\Omega} d_{x,y}^{2+\alpha}\frac{|\nabla_0 u(x) - \nabla_0 u(y)|}{|x-y|^\alpha}, \end{aligned}$$

where d_x is the Euclidean distance from x to $\partial\Omega$, $d_{x,y} = \min\{d_x, d_y\}$, and ∇_0 is the Euclidean gradient of Ω. We define $|u|^*_{1,\alpha,\Omega}$ and $|u|^*_{0,\alpha,\Omega}$ similarly. We will also need the following norm

$$|u|^{(2)}_{0,\alpha,\mathbb{D}} = \sup_{x\in\Omega} d_x^2|u(x)| + \sup_{x\neq y\in\Omega} d_{x,y}^{2+\alpha}\frac{|u(x)-u(y)|}{|x-y|^\alpha}. \tag{1.3}$$

In Lemma 2.1, we will prove that $\mathfrak{w}(\Phi)$ is in fact in Y for $\Phi \in BQD(\mathbb{D})$.

There is yet another important description of the universal Teichmüller space T. Let $BQD(\mathbb{D}^*)$ be the space of bounded holomorphic quadratic differentials on $\mathbb{D}^*$ with respect to the Poincaré metric. Consider the map $S : B_1 \to BQD(\mathbb{D}^*)$ defined as follows: for $\mu \in B_1$, let f_μ be the quasi-conformal homeomorphism as before, then $S(\mu)$ is the Schwarzian derivative of f_μ restricted to $\mathbb{D}^*$ which is an element in $BQD(\mathbb{D}^*)$, see [19]. It is easy to see that S descends down to a map β from T into $BQD(\mathbb{D}^*)$. The mapping β is known as the Bers imbedding. It is well-known that the image of β is an open set and T is homeomorphic to its image with respect to the Teichmüller metric on T. Hence we may consider the image $T(1)$ of β to be the universal Teichmüller space. For more details of Bers imbedding, see [5, 19].

In this paper, we are interested in the map $\beta \circ \mathcal{B} : BQD(\mathbb{D}) \to T(1) \subset BQD(\mathbb{D}^*)$. Note that $\beta \circ \mathcal{B} = S \circ P \circ (\mathfrak{w}, id)$, where $\mathfrak{w}$, S are defined above, $(\mathfrak{w}, id) : BQD(\mathbb{D}) \to Y \times BQD(\mathbb{D})$ is defined as $(\mathfrak{w}, id)(\Phi) = (\mathfrak{w}(\Phi), \Phi)$, and $P : BQD(\mathbb{D}) \times Y \to L^\infty(\mathbb{D})$ is defined by $P(w, \Phi) = \rho^{-2}\overline{\phi}e^{-2w}$, if $\Phi = \phi \mathrm{d}z^2$.

We also need another formulation of the universal Teichmüller space to study effectively about the map $\mathcal{B}$, namely, the space of quasi-symmetric functions. For any $\mu \in B_1 \subset L^\infty$, by the Riemann mapping theorem, there is a biholomorphic map g which maps $f_\mu(\mathbb{D})$ onto $\mathbb{D}$ fixing 1, i and -1, where f_μ is the quasi-conformal map described at the beginning of this section. Let $F_\mu = g \circ f_\mu$. Then F_μ is a quasi-conformal map on $\mathbb{D}$ with complex dilatation μ. It is well-known that F_μ extends to a self-map of $\overline{\mathbb{D}}$ and by [6] the boundary map

$$h_\mu = F_\mu|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\mathbb{D}$$

is a quasi-symmetric self-map of $\partial\mathbb{D}$ fixing 1, i, -1. By [6] under this correspondence, T and the space $\tilde{T}$ of normalized, (that is, fixing 1, i, -1), quasi-symmetric self-maps of $\partial\mathbb{D}$ are homeomorphic with respect to the Teichmüller metrics [19]. Note that T is a group, but not a topological group, under composition. In particular, for μ and ν in B_1, $[\mu] \circ [\nu]$ is equal to the equivalent class of the complex dilatation of $F_\mu \circ F_\nu$. We have

Proposition 1.1. *Let $h \in \tilde{T}$ be a normalized quasi-symmetric self-map of $\partial\mathbb{D}$, and $[\mu]$ be the element in T corresponding to h under the correspondence described above. Then $[\mu]$ is in the image of $\mathcal{B}$ if and only if h has a normalized quasi-conformal harmonic (diffeomorphic) extension from $\mathbb{D}$ to $\mathbb{D}$ with respect to the Poincaré metric.*

Proof. If h has a normalized quasi-conformal harmonic extension $u : \mathbb{D} \to \mathbb{D}$, then it is proved in [32] that the Hopf differential Φ of u lies in $BQD(\mathbb{D})$ and that $\mathcal{B}(\Phi) = [\mu]$. Conversely, if $[\mu] \in \mathcal{B}(BQD(\mathbb{D}))$, then there exist $\Phi = \phi(z)\mathrm{d}z^2 \in BQD(\mathbb{D})$ such that

$$h = F_{\rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}}|_{\partial\mathbb{D}}.$$

It is proved in [32], see also [29], that there exists a normalized quasi-conformal harmonic diffeomorphism u from $\mathbb{D}$ onto $\mathbb{D}$ such that its complex dilatation is given by

$$\mu(\Phi) = \rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}.$$

Then by the uniqueness of normalized quasi-conformal self-maps of $\mathbb{D}$ with given complex dilatation, we see that

$$u \equiv F_{\rho^{-2}\overline{\phi}e^{-2w(\Phi)}}.$$

Hence, $u|_{\partial\mathbb{D}} = h$, which completes the proof of the proposition.

By the proposition, we see that $\mathcal{B}$ is surjective if and only if there exists a quasi-conformal harmonic (diffeomorphic) extension for any prescribed (normalized) quasi-symmetric self-maps of $\partial\mathbb{D}$ at the ideal boundary of the Poincaré disc. We also remark such an extension is unique for any given quasi-symmetric boundary data on $\partial\mathbb{D}$ by [23]. In fact, if there are two distinct normalized quasi-conformal harmonic (diffeomorphic) extensions for a given normalized quasi-symmetric boundary value, then the Hopf differentials of these two maps are distinct but have same image under $\mathcal{B}$.

2. Analyticity of $\beta \circ \mathcal{B}$

In this section, we are going to show that $\beta \circ \mathcal{B}$ defined in §1 is real analytic and it maps a neighborhood of 0 in $BQD(\mathbb{D})$ analytically and diffeomorphically onto a neighborhood of 0 in $BQD(\mathbb{D}^*)$. First of all, we need the following generalization of the result in [32]. Recall that Y is the Banach space of all complex-valued functions $u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{D})$ such that $|u|^*_{2,\alpha,\mathbb{D}}$ is finite, where $0 < \alpha < 1$ is a fixed constant.

Lemma 2.1. *Let $\Phi = \phi dz^2 \in BQD(\mathbb{D})$ and w be the unique real solution of*

$$\begin{cases} \Delta_p w = e^{2w} - \|\phi\|^2 e^{-2w} - 1, \\ e^{2w} ds_p^2 \quad \textit{complete on } \mathbb{D}, \\ \|\phi\| e^{-2w} \le 1. \end{cases} \tag{2.1}$$

Then $w \in Y$.

Proof. The fact that $w \in C^\infty(\mathbb{D})$ and that $\sup_{\mathbb{D}} |w| < \infty$ are proved in [32]. Note that $\Delta_0 = (4/(1-|z|^2)^2)\,\Delta_p$, where Δ_0 is the Euclidean Laplacian. Since $\sup_{\mathbb{D}} \|\phi\| < \infty$, we have

$$\Delta_0 w(z) = O\left(d_z^{-2}\right),$$

where d_z is the Euclidean distance from z to $\partial\mathbb{D}$. So, by (4.45) in p.70 of [12] and the fact that w is bounded,

$$\sup_{\mathbb{D}} d_z |\nabla_0 w|(z) < \infty.$$

Then, the interpolation inequalities in [12] implies that

$$|w|^*_{0,\alpha,\mathbb{D}} < \infty. \tag{2.2}$$

Also, it is not hard to prove that $h(z) = (1 - |z|^2)^{-2}$ satisfies

$$|h|^{(2)}_{0,\alpha,\mathbb{D}} < \infty, \tag{2.3}$$

where the norm is defined as in (1.3). Hence by (6.11) in [12], the functions $f_\pm(z) = 4(1-|z|^2)^{-2}e^{\pm 2w(z)}$ satisfy

$$|f_\pm|^{(2)}_{0,\alpha,\mathbb{D}} < \infty. \tag{2.4}$$

Let $g(z) = (1-|z|^2)^2\phi(z)$. Using the fact that $\sup_{\mathbb{D}} |(1-|z|^2)^2\phi(z)| < \infty$ and that ϕ is holomorphic, together with the gradient estimates for harmonic functions [12], it is easy to see that $\sup_{z\in\mathbb{D}} d_z|\nabla_0 g|(z) < \infty$. As before, we have $|g|^*_{0,\alpha,\mathbb{D}} < \infty$. Therefore, combining this with (2.2), (2.4) and using (6.11) in [12] again, we have

$$|\Delta_0 w|^{(2)}_{0,\alpha,\mathbb{D}} < \infty,$$

where the norm is defined as in (1.3). Finally, by Theorem 4.8 in [12], we see that $w \in Y$.

By the lemma, we see that $\mathfrak{w}$ maps $BQD(\mathbb{D})$ into Y. Therefore $S \circ P \circ (\mathfrak{w}, id)$ in §1 is well-defined. We note that S is analytic on $B_1 \subset L^\infty(\mathbb{D})$ [13] and it is easy to see that P is real analytic. Hence, recalling that $\beta \circ \mathcal{B} = S \circ P \circ (\mathfrak{w}, id)$, to show that $\beta \circ \mathcal{B}$ is real analytic, it is sufficient to show that $\mathfrak{w}$ is real analytic. That is, we need to show that $\mathfrak{w} : BQD(\mathbb{D}) \to Y$ has Fréchet's derivatives $\mathfrak{w}^{(n)}$ of all orders and that for any $\Phi \in BQD(\mathbb{D})$ there is a $\delta > 0$ such that

$$\mathfrak{w}(\Phi + \Psi) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathfrak{w}^{(n)}(\Phi)(\Psi) \quad \forall\, \|\Psi\|_{QD} \le \delta.$$

It turns out much easier to see the real-analyticity of $\mathfrak{w}$ by extending it to a complex analytic map between complex Banach spaces. Let

$$X = \{(\Phi, \Psi) \mid \Phi, \Psi \in BQD(\mathbb{D})\},$$

be the complex Banach space with complex scalar multiplication defined by

$$\tau(\Phi,\Psi) = (\tau\Phi, \overline{\tau}\Psi), \quad \tau \in \mathbb{C}$$

and with norm

$$\|(\Phi,\Psi)\|_X = \frac{1}{2}\left(\|\Phi\|_{QD} + \|\Psi\|_{QD}\right).$$

Let

$$Z = \{f \in C^{\alpha}_{\mathrm{loc}}(\mathbb{D}) \mid f \text{ complex-valued, } |f|^*_{0,\alpha,\mathbb{D}} < \infty\}.$$

Then Z is also a complex Banach space.

Define $F : X \times Y \to Z$ by

$$F(\mathfrak{p}, w) = \Delta_p w - e^{2w} + \langle \Phi, \Psi\rangle_p e^{-2w} + 1,$$

for $\mathfrak{p} = (\Phi,\Psi) \in X$, $w \in Y$, where Δ_p and $\langle\cdot,\cdot\rangle_p$ are the Laplacian and pointwise inner product with respect to the Poincaré metric respectively. That is $\langle \Phi,\Psi\rangle_p = \rho^{-4}\phi\overline{\psi}$, if $\Phi = \phi \mathrm{d}z^2$ and $\Psi = \psi \mathrm{d}z^2$. Recall that $\rho^2|\mathrm{d}z|^2$ is the Poincaré metric on $\mathbb{D}$.

Lemma 2.2. *$F : X \times Y \to Z$ is well-defined. That is, $F(\mathfrak{p}, w) \in Z$ for any $\mathfrak{p} = (\Phi,\Psi) \in X$ and $w \in Y$.*

Proof. Since $|w|^*_{2,\alpha,\mathbb{D}} < \infty$, $\Delta_p w$ is bounded. To estimate $|\Delta_p w|^*_{0,\alpha,\mathbb{D}}$, we take any x, $y \in \mathbb{D}$. Without lost of generality, we may assume that $d_x = 1 - |x| \le 1 - |y| = d_y$. Let $d_{x,y} = \min(d_x, d_y)$. Then

$$\begin{aligned}
&\left|(1-|x|^2)^2\Delta_0 w(x) - (1-|y|^2)^2\Delta_0 w(y)\right| \\
&\le \left|(1-|x|^2)^2 - (1-|y|^2)^2\right| |\Delta_0 w(y)| + (1-|x|^2)^2\,|\Delta_0 w(x) - \Delta_0 w(y)| \\
&\le 4\left(\big||x|-|y|\big| \cdot \left|2 - |x|^2 - |y|^2\right| |\Delta_0 w(y)| + d^2_{x,y}\,|\Delta_0 w(x) - \Delta_0 w(y)|\right) \\
&\le 16|x-y|^{\alpha}\left(d_y^{2-\alpha}|\Delta_0 w(y)| + d_{x,y}^{-\alpha} \cdot d_{x,y}^{2+\alpha}\frac{|\Delta_0 w(x) - \Delta_0 w(y)|}{|x-y|^{\alpha}}\right) \\
&\le 16|x-y|^{\alpha}\left(d_y^{-\alpha} + d_{x,y}^{-\alpha}\right)|w|^*_{2,\alpha,\mathbb{D}}.
\end{aligned}$$

Using the assumption that $d_x \le d_y$, i.e. $d_{x,y} = d_x$, we have

$$\begin{aligned}
d^{\alpha}_{x,y}\frac{|(1-|x|^2)^2\Delta_0 w(x) - (1-|y|^2)^2\Delta_0 w(y)|}{|x-y|^{\alpha}} &\le 16(d_x^{\alpha}d_y^{-\alpha} + 1)|w|^*_{2,\alpha,\mathbb{D}} \\
&\le 32|w|^*_{2,\alpha,\mathbb{D}}.
\end{aligned}$$

Therefore

$$|\Delta_p w|^*_{0,\alpha,\mathbb{D}} < \infty.$$

By interpolation inequalities in [12], we see that $|e^{2w}|^*_{0,\alpha,\mathbb{D}}$ and $|e^{-2w}|^*_{0,\alpha,\mathbb{D}}$ are also finite. As in the proof of Lemma 2.1 and using the fact that

$$|fg|^*_{0,\alpha,\mathbb{D}} \le |f|^*_{0,\alpha,\mathbb{D}}|g|^*_{0,\alpha,\mathbb{D}}.$$

for $f,\ g\ \in\ Z$, one can prove that $|\langle\Phi,\Psi\rangle_p|^*_{0,\alpha,\mathbb{D}}\ <\ \infty$ and hence $|\langle\Phi,\Psi\rangle_p e^{-2w}|^*_{0,\alpha,\mathbb{D}} < \infty$. Therefore, $F(\mathfrak{p},w) \in Z$, i.e., F is well-defined.

Lemma 2.3. *$F : X \times Y \to Z$ is complex analytic with derivative given by*

$$\begin{aligned} DF(\mathfrak{p}_0,w_0)(\mathfrak{p},w) = \Delta_p w\ &-2\left(e^{2w_0} + \langle\Phi_0,\Psi_0\rangle_p e^{-2w_0}\right) w \\ &+ \left(\langle\Phi_0,\Psi\rangle_p + \langle\Phi,\Psi_0\rangle_p\right) e^{-2w_0}, \end{aligned}$$

for any $(\mathfrak{p}_0,w_0)$, $(\mathfrak{p},w) \in X \times Y$ with $\mathfrak{p}_0 = (\Phi_0,\Psi_0)$ and $\mathfrak{p} = (\Phi,\Psi)$.

Proof. By Theorem 2.3.3 in [4], it is sufficient to show that F is Gateaux differentiable. Let $\mathfrak{p}_0 = (\Phi_0,\Psi_0)$, $\mathfrak{p} = (\Phi,\Psi) \in X$ and w, $w_0 \in Y$. Then for any $\tau \in \mathbb{C}$, by noticing that $\mathfrak{p}_0 + \tau\mathfrak{p} = (\Phi_0 + \tau\Phi, \Psi_0 + \overline{\tau}\Psi)$, we have

$$\begin{aligned} &F(\mathfrak{p}_0 + \tau\mathfrak{p}, w_0 + \tau w) - F(\mathfrak{p}_0,w_0) \\ &= \tau\Delta_p w - e^{2w_0}\left(e^{2\tau w} - 1\right) + \langle\Phi_0,\Psi_0\rangle_p e^{-2w_0}\left(e^{-2\tau w} - 1\right) \\ &\quad + \tau\left(\langle\Phi_0,\Psi\rangle_p + \langle\Phi,\Psi_0\rangle_p + \tau\langle\Phi,\Psi\rangle_p\right) e^{-2(w_0+\tau w)} \end{aligned}$$

Since

$$e^{\pm 2\tau w} - 1 = \sum_{n=1}^{\infty} \tau^n \frac{(\pm 2w)^n}{n!} = \pm 2w\tau + \tau^2 \sum_{n=2}^{\infty} \tau^{n-2} \frac{(\pm 2w)^n}{n!},$$

$$|w^n|^*_{0,\alpha,\mathbb{D}} \le \left(|w|^*_{0,\alpha,\mathbb{D}}\right)^n,$$

and

$$|w|^*_{0,\alpha,\mathbb{D}} \le C|w|^*_{2,\alpha,\mathbb{D}},$$

for some absolute constant C, we have

$$\lim_{\tau\to 0} \frac{e^{\pm 2\tau w} - 1}{\tau} = \pm 2w,$$

in Z. Similarly,

$$\lim_{\tau\to 0} e^{-2(w_0+\tau w)} = e^{-2w_0},$$

in Z. From these, it is easy to see that F is Gateaux differentiable with the desired derivative. This completes the proof of the lemma.

Lemma 2.4. *Let* $\mathfrak{p}_0 = (\Phi_0, \Phi_0)$ *and* $w_0 = \mathfrak{w}(\Phi_0)$ *be as in Lemma 2.1. Then there exists a neighborhood* N *of* $\mathfrak{p}_0$ *in* X *and a complex analytic mapping* $w = \tilde{\mathfrak{w}}(\mathfrak{p})$ *from* N *to* Y *with* $w_0 = \tilde{\mathfrak{w}}(\mathfrak{p}_0)$ *such that*

$$F(\mathfrak{p}, \tilde{\mathfrak{w}}(\mathfrak{p})) = 0.$$

Moreover, $w = \tilde{\mathfrak{w}}(\mathfrak{p})$ *is the unique solution of* $F(\mathfrak{p}, w) = 0$ *near* $(\mathfrak{p}_0, w_0)$.

Proof. By Lemma 2.1, $w_0 \in Y$ and $F(\mathfrak{p}_0, w_0) = 0$. Lemmas 2.2 and 2.3 show that F is a complex analytic map from $X \times Y$ to Z. Moreover, the derivative of F with respect to the variable $w \in Y$ at $(\mathfrak{p}_0, w_0)$ is given by

$$DF_w(\mathfrak{p}_0, w_0)(v) = \Delta_p v - 2\left(e^{2w_0} + \|\Phi_0\|^2 e^{-2w_0}\right) v, \quad \forall\, v \in Y.$$

We claim that $DF_w(\mathfrak{p}_0, w_0)$ has a bounded inverse from Z to Y. Firstly, let $v \in Y$ such that

$$\Delta_p v - 2\left(e^{2w_0} + \|\Phi_0\|^2 e^{-2w_0}\right) v = 0.$$

Since the coefficients are real-valued, v is bounded as a consequence of $v \in Y$, and w_0 is bounded by [32], we can apply the generalized maximum principle of Cheng and Yau [8] to conclude that $v \equiv 0$. Therefore $DF_w(\mathfrak{p}_0, w_0)$ is one-to-one.

Next we would like to show that $DF_w(\mathfrak{p}_0, w_0)$ is onto. Let $f \in Z$ and consider the linear equation

$$\Delta_p v - 2\left(e^{2w_0} + \|\Phi_0\|^2 e^{-2w_0}\right) v = f. \tag{2.5}$$

Since $f \in Z$, f is bounded. Note that $\|\Phi_0\|$ and w_0 are also bounded. Hence, one can use the method of sub- and super-solutions [32] to find a bounded solution v of (2.5) and a constant C_1 depending only on $\mathfrak{p}_0$ such that $|v|_{0,\mathbb{D}} \le C_1 |f|_{0,\mathbb{D}}$. As in the proof of Lemma 2.1, by using the fact that $f \in Z$ and the definition of $|\cdot|^{(2)}_{0,\alpha,\mathbb{D}}$ as in (1.3), the $|\cdot|^{(2)}_{0,\alpha,\mathbb{D}}$ norms of the functions $(1-|z|^2)^{-2}\left(e^{2w_0} + \|\Phi_0\|^2 e^{-2w_0}\right)(z)$ and

$(1-|z|^2)^{-2}f(z)$ are bounded. Then, the interior Schauder estimate, Theorem 6.2 in [12], implies that

$$\begin{aligned}|v|^*_{2,\alpha,\mathbb{D}} &\le C_2\left(|v|_{0,\mathbb{D}} + |f|^*_{0,\alpha,\mathbb{D}}\right)\\ &\le C_2\left(C_1|f|_{0,\mathbb{D}} + |f|^*_{0,\alpha,\mathbb{D}}\right)\\ &\le C_3|f|^*_{0,\alpha,\mathbb{D}},\end{aligned}$$

for some constants C_2 and C_3 depending only on $\mathfrak{p}_0$, w_0, and α. Therefore, $v \in Y$ and $DF_w(\mathfrak{p}_0, w_0)$ is onto. The above arguments also show that the inverse of $DF_w(\mathfrak{p}_0, w_0)$ is bounded. Now the conclusion of the lemma follows immediately from the analytic implicit function theorem (3.3.2) in [4].

Theorem 2.5. *The map $\beta \circ \mathcal{B} : BQD(\mathbb{D}) \to T(1)$ is real analytic, where $T(1)$ is the image of the Bers imbedding β.*

Proof. We have observed that, in order to prove the theorem, it is sufficient to prove that $\mathfrak{w}$ is real analytic. Let $\Phi_0 \in BQD(\mathbb{D})$, $\mathfrak{p}_0 = (\Phi_0, \Phi_0) \in X$, and $w_0 = \mathfrak{w}(\mathfrak{p}_0)$. Then, by Lemma 2.4, there is a neighborhood N of $\mathfrak{p}_0$ in X and a complex analytic map $\tilde{\mathfrak{w}} : N \to Y$, such that $F(\mathfrak{p}, \tilde{\mathfrak{w}}(\mathfrak{p})) = 0$, for $\mathfrak{p} \in N$, where F is the map defined in Lemma 2.3. Moreover, $\tilde{w}(\mathfrak{p})$ is the only solution of $F(\mathfrak{p}, w) = 0$ near $(\mathfrak{p}_0, w_0)$. Let $\Phi \in BQD(\mathbb{D})$ and $w = \mathfrak{w}(\Phi)$. We would like to show that if $\|\Phi - \Phi_0\|_{QD}$ is small then $|w - w_0|^*_{2,\alpha,\mathbb{D}}$ is also small. It is proved in [32] that

$$\sup_{\mathbb{D}} |w - w_0| \to 0 \quad \text{as} \quad \|\Phi - \Phi_0\|_{QD} \to 0.$$

Letting $\eta = w - w_0$, we have

$$\begin{aligned}e^{-2w_0}\Delta_p \eta = e^{2\eta} - |\mu_0|^2 e^{-2\eta} - 1 + &|\mu_0|^2\\ &+ (\|\Phi\| - \|\Phi_0\|)\left(|\mu| e^{-2w_0} + |\mu_0| e^{-2w}\right),\end{aligned}$$

where $|\mu| = \|\Phi\| e^{-2w}$ and $|\mu_0| = \|\Phi_0\| e^{-2w_0}$. Since w_0 is bounded and $\eta \to 0$ as $\|\Phi - \Phi_0\|_{QD} \to 0$, (4.45) in [12] implies that

$$\sup_{\mathbb{D}} d_z |\nabla_0 \eta|(z) \to 0 \quad \text{as} \quad \|\Phi - \Phi_0\|_{QD} \to 0.$$

and hence $|\eta|^*_{0,\alpha,\mathbb{D}} \to 0$. Putting it back to the equation, as in the proof of Lemma 2.1, we conclude that $|w - w_0|^*_{2,\alpha,\mathbb{D}} \to 0$ as $\|\Phi - \Phi_0\|_{QD} \to 0$. Therefore, if Φ is close to Φ_0, then $w = \mathfrak{w}(\Phi)$ is close to w_0 in Y. So

$(\mathfrak{p}, w)$ is close to $(\mathfrak{p}_0, w_0)$ in $X \times Y$. Since $F(\mathfrak{p}, w) = 0$ by the definition of w, where $\mathfrak{p} = (\Phi, \Phi)$, we have $\mathfrak{w}(\Phi) = \tilde{\mathfrak{w}}(\mathfrak{p})$ provided Φ is close to Φ_0 in $BQD(\mathbb{D})$. Therefore, by the definition of X, $\mathfrak{w}$ is real analytic near Φ_0. As Φ_0 is an arbitrary element in $BQD(\mathbb{D})$, the proof of the theorem is completed.

Theorem 2.6. *The differential of the analytic map $\beta \circ \mathcal{B} : BQD(\mathbb{D}) \to T(1)$ at 0 is given by*

$$D(\beta \circ \mathcal{B})(0)(\Psi) = -\frac{1}{2} z^{-4} \overline{\psi(\overline{z}^{-1})} dz^2 \quad \in BQD(\mathbb{D}^*), \tag{2.6}$$

where $\Psi = \psi(z)dz^2 \in BQD(\mathbb{D})$. Therefore, $\beta \circ \mathcal{B}$ is an analytic diffeomorphism in a neighborhood of $0 \in BQD(\mathbb{D})$. In particular, there exists a unique quasi-conformal harmonic (diffeomorphic) extension with respect to the Poincaré metric of any quasi-symmetric boundary data at the ideal boundary of $\mathbb{H}^2$ provided its dilatation is small enough.

Proof. Let us denote $\mu(\Phi) = P(\mathfrak{w}(\Phi), \Phi) \in B_1$. Then $\beta \circ \mathcal{B} = S \circ \mu$ as we have seen in §1. We also note that $\mathcal{B}(\Phi) = [\mu(\Phi)]$. Now for $\Phi = \phi(z)\mathrm{d}z^2$ and $\Psi = \psi(z)\mathrm{d}z^2$ in $BQD(\mathbb{D})$,

$$D\mu(\Phi)(\Psi) = \rho^{-2}\overline{\psi}e^{-2\mathfrak{w}(\Phi)} - 2\rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)} D\mathfrak{w}_{\Phi}(\Psi).$$

Taking $\Phi = 0$, we have $\mathfrak{w}(0) \equiv 0$ and

$$D\mu(0)(\Psi) = \rho^{-2}\overline{\psi}\mathrm{d}z^2.$$

By the variational formula of S in [1], see also [5], we have

$$D(\beta \circ \mathcal{B})(0)(\Psi)(z)\mathrm{d}z^2 = \left(-\frac{6}{\pi} \int_{\mathbb{D}} \frac{\rho^{-2}(\zeta)\overline{\psi(\zeta)}}{(\zeta - z)^4} |\mathrm{d}\zeta|^2 \right) \mathrm{d}z^2 \quad \forall z \in \mathbb{D}^*.$$

Therefore,

$$-2z^{-4}\overline{D(\beta \circ \mathcal{B})(0)(\Psi)(\overline{z}^{-1})} = \int_{\mathbb{D}} \rho^{-2}(\zeta)\psi(\zeta) \overline{\left[\frac{12}{\pi(1 - \overline{z}\zeta)^4} \right]} |\mathrm{d}\zeta|^2, \ \forall z \in \mathbb{D}.$$

Then using the reproducing formula [5], see also [16], we conclude that

$$-2z^{-4}\overline{D(\beta \circ \mathcal{B})(0)(\Psi)(\overline{z}^{-1})} = \psi(z) \quad \forall z \in \mathbb{D},$$

which is equivalent to (2.6).

It is easy to see from (2.6) that $D(\beta \circ \mathcal{B})(0)$ is an isomorphism. Therefore, by the inverse function theorem, $\beta \circ \mathcal{B}$ is an analytic diffeomorphism in a neighborhood of $0 \in BQD(\mathbb{D})$. By the definition of the map, we see that if f is a normalized quasi-symmetric function on the ideal boundary of $\mathbb{H}^2$ with small dilatation, then f can be extended to a quasi-conformal harmonic diffeomorphism. This map is not only unique in a neighborhood of the class of the identity map in the universal Teichmüller space, but it is also globally unique by the result of Li-Tam [23]. This completes the proof of the theorem.

Remark 2.1. Similar to the proof of the theorem, one can use the formula for the variation of the Bers embedding at an arbitrary point [16] to find $D(\beta \circ \mathcal{B})(\Phi)$ for arbitrary $\Phi \in BQD(\mathbb{D})$. In fact, for any variation $\Psi \in BQD(\mathbb{D})$, we have

$$D(\beta \circ \mathcal{B})(\Phi)(\Psi) = \left[-\frac{6}{\pi}\int_{\mathbb{D}} \frac{D\mu_\Phi(\Psi)(\zeta)((f_\mu)_\zeta(\zeta))^2}{(f_\mu(\zeta) - f_\mu(z))^4}|\mathrm{d}\zeta|^2\right](f_\mu'(z))^2\mathrm{d}z^2,$$

where $\mathfrak{w}(\Phi)$ as in §1, $\mu = \mu(\Phi) = \rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}$, f_μ is the quasi-conformal homeomorphism on $\mathbb{C}$ corresponding to μ defined in §1 ,

$$D\mu(\Phi)(\Psi) = \rho^{-2}\overline{\psi}e^{-2\mathfrak{w}(\Phi)} - 2\rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}D\mathfrak{w}_\Phi(\Psi),$$

and $D\mathfrak{w}(\Phi)(\Psi)$ is the unique bounded solution of

$$\Delta_p v = 2\left(e^{2\mathfrak{w}(\Phi)} + \|\Phi\|^2 e^{-2\mathfrak{w}(\Phi)}\right)v - 2\mathrm{Re}\langle\Phi,\Psi\rangle_p e^{-2\mathfrak{w}(\Phi)}.$$

We will see later that $D(\beta \circ \mathcal{B})_\Phi$ is bijective for any $\Phi \in BQD(\mathbb{D})$ and hence $\beta \circ \mathcal{B}$ is an analytic diffeomorphism from $BQD(\mathbb{D})$ onto it image.

3. Estimates on Composition of Two Maps

In this section, we will give some estimates on the composition of two maps on $\mathbb{H}^2$ which will be used later. Let w be a quasi-conformal harmonic diffeomorphism from $\mathbb{H}^2$ onto itself. Then its Hopf differential $\Phi = \phi \mathrm{d}z^2$ is holomorphic and $\|\Phi\|_{QD}$ is finite by the work of [32]. We begin with the following lemma:

Lemma 3.1. *Let w be a quasi-conformal harmonic diffeomorphism on $\mathbb{H}^2$. Let $\Phi = \phi dz^2$ be the Hopf differential of w. Then*

$$|\nabla(\log\|\partial w\|)| \leq C_1\|\Phi\|_{QD}$$

for some absolute constant C_1, where ∇ is the gradient in $\mathbb{H}^2$ and $\|\partial w\|$ is the ∂-energy density of w.

Proof. Let $\psi = \log\|\partial w\|$ and $a = \|\Phi\|_{QD}$. We use the upper half space model for $\mathbb{H}^2$ and write $z = x + iy$ in the domain, $u + iv$ in the target, with $y > 0$ and $v > 0$. Since w is a quasi-conformal diffeomorphism, $\|\partial w\| > 0$ everywhere and hence ψ is well-defined and smooth in $\mathbb{H}^2$. By [32]

$$1 \le e^{2\psi} \le \frac{1+\sqrt{1+4a^2}}{2}. \tag{3.1}$$

Using the fact that w is quasi-conformal and that Φ is the Hopf differential of w, we have

$$\sup_{\mathbb{D}} e^{-2\psi}\|\phi\| < 1. \tag{3.2}$$

The Bochner formula in [27], see also [25], implies that

$$y^2\Delta_0\psi = e^{2\psi} - \|\phi\|^2 e^{-2\psi} - 1,$$

where Δ_0 is the Euclidean Laplacian. Hence, by (3.1) and (3.2), we have

$$y^2\,|\Delta_0\psi| \le a. \tag{3.3}$$

Let $z_0 = x_0 + iy_0 \in \mathbb{H}^2$. Applying the estimate in p.70 in [12] to $D_1 = D_{z_0}(\frac{1}{4}y_0)$ and $D_2 = D_{z_0}(\frac{1}{2}y_0)$, where $D_{z_0}(r)$ is the Euclidean disc of radius r with center z_0, and using (3.3), we see that there is an absolute constant C_2 such that

$$y_0|\nabla_0\psi|(z_0) \le C_2(\sup_{D_2}|\psi| + a).$$

Combining this with (3.1) and using the fact that $\log(1+t) \le t$ for $t \ge 0$, the lemma follows. Recall that, given a C^2 map w from the hyperbolic plane into itself, and if we use the upper half space models, then the norm of the tension field of w is given by

$$\|\tau(w)\|(z) = \frac{4y^2}{g}\left|w_{z\bar{z}} + \frac{i}{g}w_z w_{\bar{z}}\right|,$$

where $z = x + iy$ and $w = f + ig$.

Lemma 3.2. *Let h and w be quasi-conformal harmonic diffeomorphisms on $\mathbb{H}^2$. Then the norm of the tension field $\|\tau(h \circ w)\|$ of the composition map $h \circ w$ satisfies*

$$\|\tau(h \circ w)\|(z) \le C_3 \|\Psi\|_{QD} \left(1 + \|\Psi\|_{QD}\right)^{\frac{1}{2}} \|\Phi\|(z)$$

for some absolute constant C_3, where $\Phi = \phi dz^2$ and $\Psi = \psi dw^2$ are the Hopf differentials of w and h respectively.

Proof. We use again the upper half space model for $\mathbb{H}^2$. Let $z = x + iy$, $w = u + iv$, and $h = f + ig$. The norm of the tension field of $h \circ w$ at a point $z = x + iy$ is given by

$$\|\tau(h \circ w)\| = \frac{4y^2}{g} |A|, \tag{3.4}$$

where

$$A = (h \circ w)_{z\bar{z}} + \frac{i}{g}(h \circ w)_z (h \circ w)_{\bar{z}}.$$

Since

$$\begin{aligned} (h \circ w)_z &= h_w w_z + h_{\bar{w}} \bar{w}_z, \\ (h \circ w)_{\bar{z}} &= h_w w_{\bar{z}} + h_{\bar{w}} \bar{w}_{\bar{z}}, \end{aligned}$$

and

$$(h \circ w)_{z\bar{z}} = h_{ww} w_{\bar{z}} w_z + h_{w\bar{w}} \bar{w}_{\bar{z}} w_z + h_w w_{z\bar{z}} + h_{\bar{w}w} w_{\bar{z}} \bar{w}_z + h_{\bar{w}\bar{w}} \bar{w}_{\bar{z}} \bar{w}_z + h_{\bar{w}} \bar{w}_{z\bar{z}},$$

we have

$$\begin{aligned} A &= (h \circ w)_{z\bar{z}} + \frac{i}{g} (h \circ w)_z (h \circ w)_{\bar{z}} \\ &= (h \circ w)_{z\bar{z}} + \frac{i}{g} \left(h_w w_z + h_{\bar{w}} \bar{w}_z\right) \left(h_w w_{\bar{z}} + h_{\bar{w}} \bar{w}_{\bar{z}}\right) \\ &= h_{ww} w_{\bar{z}} w_z + h_{w\bar{w}} \bar{w}_{\bar{z}} w_z + h_w w_{z\bar{z}} \\ &\quad + h_{\bar{w}w} w_{\bar{z}} \bar{w}_z + h_{\bar{w}\bar{w}} \bar{w}_{\bar{z}} \bar{w}_z + h_{\bar{w}} \bar{w}_{z\bar{z}} \\ &\quad + \frac{i}{g} \left(h_w^2 w_z w_{\bar{z}} + h_w h_{\bar{w}} \left(w_z \bar{w}_{\bar{z}} + \bar{w}_z w_{\bar{z}}\right) + h_{\bar{w}}^2 \bar{w}_z \bar{w}_{\bar{z}}\right). \end{aligned} \tag{3.5}$$

As w and h are harmonic,

$$w_{z\bar{z}} + \frac{i}{v} w_z w_{\bar{z}} = 0, \tag{3.6}$$

$$h_{w\bar{w}} + \frac{i}{g} h_w h_{\bar{w}} = 0. \tag{3.7}$$

By (3.5)–(3.7), we have

$$\begin{aligned} A &= h_w \left(w_{z\bar{z}} + \frac{i}{g} h_w w_z w_{\bar{z}} \right) + h_{\bar{w}} \left(\bar{w}_{z\bar{z}} + \frac{i}{g} h_{\bar{w}} w_z \bar{w}_{\bar{z}} \right) \\ &\quad + h_{ww} w_{\bar{z}} w_z + h_{\bar{w}\bar{w}} \bar{w}_z \bar{w}_{\bar{z}} \\ &= w_{\bar{z}} w_z \left(h_{ww} + i h_w \left(-\frac{1}{v} + \frac{h_w}{g} \right) \right) \\ &\quad + \bar{w}_z \bar{w}_{\bar{z}} \left(h_{\bar{w}\bar{w}} + i h_{\bar{w}} \left(\frac{1}{v} + \frac{h_{\bar{w}}}{g} \right) \right). \end{aligned} \tag{3.8}$$

Since $\|\partial h\| > 0$ everywhere, direct computations show that

$$h_{ww} + i h_w \left(-\frac{1}{v} + \frac{h_w}{g} \right) = h_w \left(\log \|\partial h\|^2 \right)_w. \tag{3.9}$$

Similarly, at the points where $\|\bar{\partial} h\| > 0$, we have

$$h_{\bar{w}\bar{w}} + i h_{\bar{w}} \left(\frac{1}{v} + \frac{h_{\bar{w}}}{g} \right) = h_{\bar{w}} \left(\log \|\bar{\partial} h\|^2 \right)_{\bar{w}}. \tag{3.10}$$

Using the fact that $\psi = g^{-2} h_w \bar{h}_w$ is holomorphic, we have, at those points where $\|\bar{\partial} h\| > 0$,

$$\begin{aligned} h_{\bar{w}} \left(\log \|\bar{\partial} h\|^2 + \log \|\partial h\|^2 \right)_{\bar{w}} &= h_{\bar{w}} \left(\log \|\psi\|^2 \right)_{\bar{w}} \\ &= h_{\bar{w}} \left(\frac{4 v_{\bar{w}}}{v} + \frac{\left(\overline{\psi}\right)_{\bar{w}}}{\overline{\psi}} \right) \\ &= h_{\bar{w}} \left(\frac{2i}{v} + \frac{\left(\overline{\psi}\right)_{\bar{w}}}{\overline{\psi}} \right). \end{aligned}$$

Combining this with (3.10), we then have

$$\begin{aligned} h_{\bar{w}\bar{w}} + i h_{\bar{w}} \left(\frac{1}{v} + \frac{h_{\bar{w}}}{g} \right) &= h_{\bar{w}} \left(\frac{2i}{v} + \frac{\left(\overline{\psi}\right)_{\bar{w}}}{\overline{\psi}} - \left(\log \|\partial h\|^2 \right)_{\bar{w}} \right) \\ &= \frac{2i h_{\bar{w}}}{v} + \frac{g^2 \left(\overline{\psi}\right)_{\bar{w}}}{\bar{h}_{\bar{w}}} - h_{\bar{w}} \left(\log \|\partial h\|^2 \right)_{\bar{w}}, \end{aligned} \tag{3.11}$$

at those points where $\|\bar{\partial} h\| > 0$. Since h_w is never 0, ψ and hence $\bar{\partial} h$ is either identically zero or having isolated zeros. Therefore, it is easy to see that (3.11) is true at all points. Since ψ is holomorphic, gradient estimate for harmonic functions implies that

$$\begin{aligned} |\nabla_0 \psi|(w) &\leq \frac{C_4}{v} \sup_{D_w(\frac{1}{2}v)} |\psi| \\ &\leq \frac{C_5}{v^3} \|\Psi\|_{QD}, \end{aligned} \tag{3.12}$$

at a point $w = u + iv$, where ∇_0 is the Euclidean gradient and C_4, C_5 are absolute constants. By the results in [32],

$$1 \leq \|\partial h\|^2 \leq \frac{1 + \sqrt{1 + 4\|\Psi\|_{QD}^2}}{2}. \tag{3.13}$$

Combining (3.4), (3.8), (3.9), (3.11), (3.12), (3.13), Lemma 3.1, and the fact that $\phi = v^{-2} w_z \bar{w}_z$, we have

$$\begin{aligned} &\|\tau(h \circ w)\| \\ &= \frac{4y^2}{g} \Big| w_z w_{\bar{z}} h_w \left(\log \|\partial h\|^2\right)_w \\ &\quad + \bar{w}_z \bar{w}_{\bar{z}} \left(\frac{2ih_{\bar{w}}}{v} + \frac{g^2 \left(\overline{\psi}\right)_{\bar{w}}}{\overline{h}_{\bar{w}}} - h_{\bar{w}} \left(\log \|\partial h\|^2\right)_{\bar{w}} \right) \Big| \\ &\leq 4 \left| y^2 \phi \cdot \frac{v}{g} h_w \cdot v \left(\log \|\partial h\|^2\right)_w + y^2 \overline{\phi} \left(2i \cdot \frac{v}{g} h_{\bar{w}} + \frac{v^2 g}{\overline{h}_{\bar{w}}} \cdot \overline{\psi}_{\bar{w}} \right) \right| \\ &\leq C \|\Psi\|_{QD} \left(1 + \|\Psi\|_{QD}\right)^{\frac{1}{2}} \|\Phi\|, \end{aligned}$$

for some absolute constant C. This completes the proof of the lemma.

Remark 3.1. In Lemma 3.1, we may relax the condition that w is quasi-conformal. If we assume that w has bounded energy density, then the gradient of $\log \|\partial w\|$ is still bounded by a constant depending on the upper bound of the energy density provided $\|\partial w\| \geq C > 0$ for some constant C. Also the estimate is local in nature. Similarly, in Lemma 3.2, $\|\Psi\|_{QD}$ can be replaced by a constant depending only on the lower and upper bounds of the energy density of h, as long as the lower bound is large than 0. The esitmate is also local.

Lemma 3.3. *Let h and w be two quasi-conformal harmonic diffeomorphism on $\mathbb{H}^2$. Let Φ and Ψ be the Hopf differentials of w and h respectively. Then, in the upper space model for $\mathbb{H}^2$ with $z = x + iy$, $w = u + iv$, and $h = f + ig$, we have, at any point $z \in \mathbb{H}^2$,*

$$\begin{aligned}
&\left|\frac{y^2}{(g\circ w)^2}\,(h\circ w)_z\left(\overline{h\circ w}\right)_z - \frac{y^2}{v^2}w_z\bar{w}_z\right| \\
&\le \|\Phi\|(z)\left(\|\partial h\|^2(w(z)) + \|\bar{\partial} h\|^2(w(z)) - 1\right) \\
&\quad + \|\Psi\|(w(z))\left(\|\partial w\|^2(z) + \|\bar{\partial} w\|^2(z)\right) \\
&\le 5\|\Psi\|_{QD}\left(1 + \|\Phi\|_{QD}\right).
\end{aligned}$$

Proof. Using the fact that $\|\partial w\| > \|\bar{\partial} w\|$ and the result in [32] that

$$1 \le \|\partial h\|^2 \le \frac{1+\sqrt{1+4\|\Psi\|_{QD}^2}}{2} \le 1 + \|\Psi\|_{QD}, \tag{3.14}$$

and

$$1 \le \|\partial w\|^2 \le \frac{1+\sqrt{1+4\|\Phi\|_{QD}^2}}{2} \le 1 + \|\Phi\|_{QD}, \tag{3.15}$$

at the point z

$$\begin{aligned}
&\left|\frac{y^2}{(g\circ w)^2}\,(h\circ w)_z\left(\overline{h\circ w}\right)_z - \frac{y^2}{v^2}w_z\bar{w}_z\right| \\
= &\quad \left|\frac{y^2}{g^2}\left(w_z\bar{w}_z\left(h_w\bar{h}_{\bar{w}} + \bar{h}_w h_{\bar{w}}\right) + h_w\bar{h}_w(w_z)^2\right.\right. \\
&\quad \left.\left. + h_{\bar{w}}\bar{h}_{\bar{w}}(\bar{w}_z)^2\right) - \frac{y^2}{v^2}\cdot w_z\bar{w}_z\right| \\
= &\quad \left|\frac{y^2 w_z\bar{w}_z}{v^2}\left(\frac{v^2}{g^2}\cdot h_w\bar{h}_{\bar{w}}\right.\right. \\
&\quad \left.\left. + \frac{v^2}{g^2}\bar{h}_w h_{\bar{w}} - 1\right) + \frac{y^2}{g^2}\left(h_w\bar{h}_w(w_z)^2 + h_{\bar{w}}\bar{h}_{\bar{w}}(\bar{w}_z)^2\right)\right| \qquad (3.16) \\
\le &\quad \|\Phi\|(z)\left(\|\partial h\|^2(w(z)) + \|\bar{\partial} h\|^2(w(z)) - 1\right) \\
&\quad + \|\Psi\|(w(z))\left(\|\partial w\|^2(z) + \|\bar{\partial} w\|^2(z)\right).
\end{aligned}$$

If $||\Psi||_{QD} \le 1$, then by (3.14)

$$||\bar{\partial} h||^2 \le \frac{||\Psi||_{QD}^2}{||\partial h||^2} \le ||\Psi||_{QD}$$

and

$$||\partial h||^2 + ||\bar{\partial} h||^2 - 1 \le 2||\Psi||_{QD}.$$

If $||\Psi||_{QD} \ge 1$, then by (3.14)

$$||\bar{\partial} h||^2 \le ||\partial h||^2 \le 2||\Psi||_{QD}.$$

Hence

$$||\partial h||^2 + ||\bar{\partial} h||^2 - 1 \le 3||\Psi||_{QD}.$$

Combining these with (3.16) and (3.15), the lemma follows.

In the following lemma, we still use the upper space model for $\mathbb{H}^2$ in the domian and the target. We write $z = x + iy$ in the domain, and a map from $\mathbb{H}^2$ into itself is given by w with $Im(w) > 0$.

Lemma 3.4. *Let $w = u + iv$ and $\tilde{w} = \tilde{u} + i\tilde{v}$ be two quasi-confomal diffeomorphisms of $\mathbb{H}^2$. Suppose w is harmonic and $\tilde{w}$ has uniformly bounded energy density. Let $a = \sup_{\mathbb{H}^2} ||\tau(\tilde{w})|| < \infty$, where $||\tau(\tilde{w})||$ is the norm of the tension field of $\tilde{w}$. Suppose $|\tilde{w} - w| \le b \min\{\tilde{v},\ v\}$ for some constant $0 < b \le 1$, where $|\tilde{w} - w|$ is the Euclidean distance between $\tilde{w}$ and w. Then*

$$\sup_{z \in \mathbb{H}^2} \left| \frac{y}{\tilde{v}} \nabla_0 \tilde{w} - \frac{y}{v} \nabla_0 w \right| \le a + C_5 b,$$

for some constant C_5 depending only on the upper bounds of the energy density of w and $\tilde{w}$, where ∇_0 is the Euclidean gradient.

Proof. Let $k = \sup_{z \in \mathbb{H}^2} \left| \frac{y}{\tilde{v}} \nabla_0 \tilde{w} - \frac{y}{v} \nabla_0 w \right|$. By [32], w has uniformly bounded energy density. Therefore, together with the assumption on $\tilde{w}$, we see that k is finite.

Let $m = \sup_{\mathbb{H}^2} (e(w) + e(\tilde{w}))$. Note that $|\tilde{w} - w| \le b \min\{\tilde{v},\ v\}$, which implies that $\left|\frac{v}{\tilde{v}} - 1\right| \le b$, we have

$$\begin{aligned} \left| \left(\frac{y}{v} \nabla_0 (\tilde{w} - w) \right) - \left(\frac{y}{\tilde{v}} \nabla_0 \tilde{w} - \frac{y}{v} \nabla_0 w \right) \right| &= \left| \left(\frac{y}{v} - \frac{y}{\tilde{v}} \right) \nabla_0 \tilde{w} \right| \\ &\le \frac{y}{v} |\nabla_0 \tilde{w}| \cdot \left| \frac{v}{\tilde{v}} - 1 \right| \\ &\le C_6 b, \end{aligned} \tag{3.17}$$

for some constant C_6 depending only on m. Since w is a harmonic map,

$$\begin{aligned}\left|\frac{y^2}{\tilde v}\Delta_0\tilde w - \frac{y^2}{v}\Delta_0 w\right| &\le \|\tau(\tilde w)\| + \left|\frac{y^2}{\tilde v^2}\tilde w_z\tilde w_{\bar z} - \frac{y^2}{v^2}w_z w_{\bar z}\right| \\ &\le a + \frac{y}{\tilde v}\cdot|\tilde w_z|\cdot\left|\frac{y}{\tilde v}\tilde w_{\bar z} - \frac{y}{v}w_{\bar z}\right| + \frac{y}{v}\cdot|w_{\bar z}|\left|\frac{y}{\tilde v}\tilde w_z - \frac{y}{v}w_z\right| \\ &\le C_7\left(a + (e(\tilde w)+e(w))^{\frac12}k\right) \\ &\le C_8\,(a+k)\,,\end{aligned}$$

for some absolute constant C_7 and some constant C_8 depending only on m. Hence, using $|w-\tilde w| \le b\min\{v,\ \tilde v\}$ and the harmonicity of w again, we have

$$\begin{aligned}\left|\frac{y^2}{\tilde v}\left(\Delta_0\tilde w - \Delta_0 w\right)\right| &\le \left|\frac{y^2}{\tilde v}\Delta_0\tilde w - \frac{y^2}{v}\Delta_0 w\right| + \left|\left(\frac{y^2}{\tilde v} - \frac{y^2}{v}\right)\Delta_0 w\right| \\ &\le C_8(a+k) + \left|\left(\frac{v}{\tilde v} - 1\right)\cdot\frac{y^2}{v^2}w_z w_{\bar z}\right| \\ &\le C_9(a+b+k),\end{aligned}$$

for some constant C_9 depending only on m. Therefore, (3.17) and the assumption that $|\tilde w - w| \le b\min\{\tilde v,\ v\}$ for some $b \le 1$ imply that

$$\begin{aligned}\left|\Delta_0\left(y^2(\tilde w - w)\right)\right| &= |y^2\Delta_0(\tilde w - w) + 2\langle\nabla_0(y^2), \nabla_0(\tilde w - w)\rangle \\ &\quad + (\tilde w - w)\Delta_0(y^2)| \\ &\le C_9(a+b+k)\tilde v + 4(k + C_6 b)v + 2bv \\ &\le C_{10}(a+b+k)v,\end{aligned}$$

for some constant C_{10} depending only on m. Using (3.17) again and the fact that w is harmonic, we have

$$\begin{aligned}
&\left|\Delta_0\left(\frac{y^2}{v}(\tilde{w}-w)\right)\right| \\
&= \left|\frac{1}{v}\Delta_0\left(y^2(\tilde{w}-w)\right)+2\langle\nabla_0(\frac{1}{v}),\nabla_0\left(y^2(\tilde{w}-w)\right)\rangle\right. \\
&\quad \left.+\left(y^2(\tilde{w}-w)\right)\Delta_0(\frac{1}{v})\right| \\
&\le C_{10}(a+b+k)+\left|\frac{4y(\tilde{w}-w)}{v^2}\langle\nabla_0 v,\nabla_0 y\rangle\right| \\
&\quad +\left|\frac{2y^2}{v^2}\langle\nabla_0 v,\nabla_0(\tilde{w}-w)\rangle\right| \\
&\quad +\left|\left(\frac{y^2(\tilde{w}-w)}{v^2}\right)\Delta_0 v-\left(\frac{2y^2(\tilde{w}-w)}{v^3}\right)|\nabla_0 v|^2\right| \qquad (3.18) \\
&\le C_{11}(a+b+k),
\end{aligned}$$

for some constant C_{11} depending only on m. Now for any $z_0 = x_0 + iy_0 \in \mathbb{H}^2$, let $D_1 = D_{z_0}(\frac{1}{4}y_0)$ and $D_2 = D_{z_0}(\frac{1}{2}y_0)$, where $D_{z_0}(r)$ is the Euclidean disc of radius r with center z_0. By [12 (p.70, (3.18))] and the assumption that $|\tilde{w}-w| \le b\min\{\tilde{v},\ v\}$, for a fixed $0<\alpha<1$,

$$\begin{aligned}
\left[\frac{y^2}{v}(\tilde{w}-w)\right]^*_{1,\alpha,D_1} &\le C_{12}\left(\sup_{D_2}\left|\frac{y^2}{v}(\tilde{w}-w)\right|+C_{11}(a+b+k)(y_0)^2\right) \\
&\le C_{13}(a+b+k)y_0^2,
\end{aligned}$$

where C_{12} depends only on α and C_{13} depends only on α and m. Therefore, the interpolation inequality Lemma 6.32 in [12] implies that, for any $\epsilon > 0$,

$$\begin{aligned}
&\sup\nolimits_{z=x+iy\in D_1}\left|d_z\nabla_0\left(\tfrac{y^2}{v}(\tilde{w}-w)\right)(z)\right| \\
&\le C_{14}\sup\nolimits_{D_1}\left|\tfrac{y^2}{v}(\tilde{w}-w)\right|+\epsilon\left[\tfrac{y^2}{v^2}(\tilde{w}-w)\right]^*_{1,\alpha,D_1} \\
&\le C_{14}b(y_0)^2+C_{13}\cdot\epsilon(a+b+k)(y_0)^2,
\end{aligned}$$

where d_z is the Euclidean distance from z to ∂D_1 and C_{14} is a constant depending only on α and ϵ. Evaluate at z_0, we have

$$\left|\nabla_0\left(\frac{y^2}{v}(\tilde{w}-w)\right)\right| \le y\left(C_{15}b+C_{16}\cdot\epsilon(a+b+k)\right),$$

at every point $z = x + iy \in \mathbb{H}^2$, where C_{15} depends only on α and ϵ and C_{16} depends only on m and α. Hence, by $|\tilde{w} - w| \leq b\min\{\tilde{v},\ v\}$ again, we have

$$\left|\frac{y}{v}\nabla_0(\tilde{w} - w)\right| \leq C_{17}b + C_{16}\cdot\epsilon(a + b + k),$$

for some constant C_{17} depending only on m, α and ϵ. Combine this with (3.17), we have

$$\left|\frac{y}{\tilde{v}}\nabla_0\tilde{w} - \frac{y}{v}\nabla_0 w\right| \leq C_{18}b + C_{16}\cdot\epsilon(a + b + k), \tag{3.19}$$

for some constant C_{18} depending only on m, α, and ϵ. Taking supremum over $\mathbb{H}^2$, we have

$$k \leq C_{18}b + C_{16}\cdot\epsilon(a + b + k).$$

Fix α, say $\alpha = \frac{1}{2}$, then one can choose ϵ small enough so that $C_{16}\cdot\epsilon = \frac{1}{2}$. Noticing that C_{16} depends only on m and α, the lemma follows.

4. Local Diffeomorphic Property of $\beta \circ \mathcal{B}$

In this section, we will prove that $\beta \circ \mathcal{B}$ has an open image and is an analytic diffeomorphism onto its image. Note that by the result of [23], $\mathcal{B}$ is injective. It is well-known that β is also injective, therefore $\beta \circ \mathcal{B}$ is injective. We will continue to use the upper half-plane model for the Hyperbolic 2-space $\mathbb{H}^2$ in order to simplify calculations.

Theorem 4.1. *The image of $\beta \circ \mathcal{B}$ is open in $T(1)$ and $\beta \circ \mathcal{B}$ maps $BQD(\mathbb{D})$ analytically and diffeomorphically onto its image.*

Proof. First, let us prove that the image of $\beta \circ \mathcal{B}$ is open. Let Ψ_0 be an element in the image of $\beta \circ \mathcal{B}$ and $\Phi_0 \in BQD(\mathbb{D})$ such that $(\beta \circ \mathcal{B})(\Phi_0) = \Psi_0$. Let w_0 be the quasi-conformal harmonic diffeomorphism of $\mathbb{H}^2$ fixing 0, 1, and ∞, such that its Hopf differential is Φ_0. The existence and uniqueness of w_0 is proved in [32]. Let μ_0 be the complex dilatation of w_0, then $[\mu_0] = \mathcal{B}(\Phi_0)$ and $\beta([\mu_0]) = \Psi_0$. By [19], there exists $\delta_0 > 0$ and a constant $C_0 > 0$ depending only on Ψ_0, such that if $\|\Psi - \Psi_0\|_{QD} \leq \delta_0$, then Ψ lies in the image of β; and if $\beta([\mu]) = \Psi$, then

$$C_0^{-1}\|\Psi - \Psi_0\|_{QD} \leq \lambda([\mu], [\mu_0]) \leq C_0\|\Psi - \Psi_0\|_{QD}, \tag{4.1}$$

where $[\nu] \in T$ is the equivalent class represented by $\nu \in B_1$ and λ is the Teichmüller metric. Since $\mathcal{B}(0) = 0$ and $\beta(0) = 0$, by Theorem 2.6 and (4.1) applying to the case that $\Phi_0 = 0$, there are constants $\delta > 0$ and $C_1 > 0$, such that if $\lambda([\nu], 0) \leq \delta$, then there exists a unique $\Phi \in BQD(\mathbb{D})$ such that $\mathcal{B}(\Phi) = [\nu]$ and

$$C_1^{-1}\|\Phi\|_{QD} \leq \lambda([\nu], 0) \leq C_1\|\Phi\|_{QD}. \tag{4.2}$$

Now given Ψ_0 in the image of $\beta \circ \mathcal{B}$ and $\Phi_0 \in BQD(\mathbb{D})$ such that $(\beta \circ \mathcal{B})(\Phi_0) = \Psi_0$. We may choose δ_0 in (4.1) small enough, such that $C_0\delta_0 \leq \delta$. Hence, by (4.1), if $\|\Psi - \Psi_0\| \leq \delta_0$, then there exists a unique $[\mu] \in T$ such that $\beta([\mu]) = \Psi$, and

$$\begin{aligned} C_0^{-1}\|\Psi - \Psi_0\|_{QD} &\leq \lambda([\mu], [\mu_0]) \\ &\leq C_0\|\Psi - \Psi_0\|_{QD} \\ &\leq \delta. \end{aligned} \tag{4.3}$$

Fix a Ψ which satisfies (4.3) and let $\mu \in B_1$ such that $\beta([\mu]) = \Psi$. We would like to show that $[\mu]$ can be represented by a complex dilatation of a quasi-conformal harmonic diffeomorphism. Let us denote by $[\mu] \circ [\mu_0]^{-1}$ the class in T represented by $f_\mu|_{\partial\mathbb{H}^2} \circ f_{\mu_0}|_{\partial\mathbb{H}^2}^{-1}$, where f_μ and f_{μ_0} are the quasi-conformal homeomorphisms defined in §1 corresponding to μ and μ_0 respectively. Then, $\lambda([\mu] \circ [\mu_0]^{-1}, 0) = \lambda([\mu], [\mu_0]) \leq \delta$. Therefore, by (4.3) and the choice of δ, there exists a unique $\Phi_1 \in BQD(\mathbb{D})$ such that $\mathcal{B}(\Phi_1) = [\mu] \circ [\mu_0]^{-1}$. By (4.2) and (4.3) again, we have

$$\begin{aligned} \|\Phi_1\|_{QD} &\leq C_1 \cdot \lambda([\mu] \circ [\mu_0]^{-1}, 0) \\ &= C_1 \cdot \lambda([\mu], [\mu_0]) \\ &\leq C_0C_1\|\Psi - \Psi_0\|_{QD} \\ &\leq C_0C_1\delta_0. \end{aligned} \tag{4.4}$$

Using the definition of $\mathcal{B}$, we can find a unique quasi-conformal harmonic diffeomorphism h on $\mathbb{H}^2$ fixing 0, 1, ∞, such that its Hopf differential is Φ_1. Let μ_h be the complex dilatation of h. Then the fact that $1 \leq \|\partial h\|$ (see [32]), and (4.4), we have

$$\begin{aligned} |\mu_h| &= \frac{\|\bar\partial h\|}{\|\partial h\|} \\ &= \frac{\|\Phi_1\|}{\|\partial h\|^2} \\ &\leq \|\Phi_1\|_{QD} \\ &\leq C_0 C_1 \delta_0. \end{aligned} \tag{4.5}$$

If we choose δ_0 even smaller, we have $\|\mu_h\|_\infty \leq \frac{1}{2}$. Note that the choice of δ_0 depends only on Ψ_0, Φ_0, C_0, C_1, and δ. Since w_0 is a quasi-conformal diffeomorphism, we have

$$\|\mu_{h\circ w_0}\|_\infty \leq \epsilon, \tag{4.6}$$

for some $1 > \epsilon > 0$, depending only on the complex dilatation of w_0 which in turns depending only on Φ_0 by the result of [32], where $\mu_{h\circ w_0}$ is the complex dilatation of $h \circ w_0$. Using again the result of [32],

$$1 \leq \|\partial h\|^2 \leq \frac{1+\sqrt{1+4\|\Phi_1\|_{QD}}}{2},$$

and

$$1 \leq \|\partial w_0\|^2 \leq \frac{1+\sqrt{1+4\|\Phi_0\|_{QD}}}{2}.$$

Let $e(h \circ w_0)$ be the energy density of the map $h \circ w_0$. Then, together with (4.4) and (4.6), there is a constant $C_2 > 0$ depending only on Φ_0, C_0, C_1, and δ_0, such that

$$C_2^{-1} \leq e(h \circ w_0) \leq C_2, \tag{4.7}$$

provided that $\|\Psi - \Psi_0\|_{QD} \leq \delta_0$.

We claim that there exists a quasi-conformal harmonic diffeomorphism with the same boundary data as $h \circ w_0$. In fact, let q be a fixed point in target $\mathbb{H}^2$ and $B_q(R)$ be the geodesic ball of radius R centered at q. Let $\Omega_R = (h \circ w_0)^{-1}(B_p(R))$. Then $\{\Omega_R\}$ is a compact exhaustion of $\mathbb{H}^2$ with smooth boundary. For each R, by [14], there is a harmonic map w_R on Ω_R, such that $w_R = h \circ w_0$ on $\partial\Omega_R$. Let $d_R(z)$

be the hyperbolic distance between $w_R(z)$ and $h \circ w_0(z)$, then using the computations in [22], (4.6) and (4.7), we have

$$(4.8)\ \Delta d_R \geq -\|\tau(h \circ w_0)\| + 2C_2^{-1}\left(1 + \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\right)^{-1} \cdot \frac{\cosh d_R - 1}{\sinh d_R},$$

in $B_p(R)$ in the sense of distribution, where $\|\tau(h \circ w_0)\|$ is the norm of the tension field of $h \circ w_0$. By Lemma 3.2 and (4.4),

$$\begin{aligned} \|\tau(h \circ w_0)\| &\leq C_3\|\Phi_1\|_{QD}\left(1 + \|\Phi_1\|_{QD}\right)^{\frac{1}{2}}\|\Phi_0\|_{QD} \\ &\leq C_4\|\Psi - \Psi_0\|_{QD}, \end{aligned} \tag{4.9}$$

where C_3 is an absolute constant and C_4 is a constant depending only on Φ_0, C_0, C_1 and δ_0. Hence there exists $0 < \delta_1 \leq \delta_0$, depending only on Φ_0, C_0, C_1, and δ_0, such that if $\|\Psi - \Psi_0\|_{QD} \leq \delta_1$, then

$$\sup_{\mathbb{H}^2}\|\tau(h \circ w_0)\| \leq C_2^{-1}\left(1 + \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\right)^{-1}.$$

By (4.8), (4.9) and the maximum principle, if we let $d_R^* = \sup_{\Omega_R} d_R$, we have

$$\frac{\cosh d_R^* - 1}{\sinh d_R^*} \leq \frac{1}{2}.$$

In particular,

$$d_R^* \leq C_5, \tag{4.10}$$

for some absolute constant C_5, for all R. We also have

$$d_R^* \leq C_6\|\Psi - \Psi_0\|_{QD}, \tag{4.11}$$

for some constant C_6 depending only on Φ_0, C_0, C_1 and δ_0, provided $\|\Psi - \Psi_0\|_{QD} \leq \delta_1$, where we have used the fact that $t \leq C(\cosh t - 1)/\sinh t$ for some absolute constant C, for $0 \leq t \leq C_5$. Let $x \in \mathbb{H}^2$, $R > 0$ large enough so that $B_x(1) \subset \Omega_R$, for all $y \in B_x(1)$,

$$\begin{aligned} d_{\mathbb{H}^2}(w_R(x), w_R(y)) &\leq d_{\mathbb{H}^2}(w_R(x), (h \circ w_0)(x)) \\ &\quad + d_{\mathbb{H}^2}((h \circ w_0)(x), (h \circ w_0)(y)) \\ &\quad + d_{\mathbb{H}^2}(w_R(y), (h \circ w_0)(y)) \\ &\leq 2C_5 + \sqrt{C_2}, \end{aligned}$$

where we have used (4.7) and (4.10). Using the energy density estimate of Cheng [7], there exists a constant C_7 depending only on Φ_0, C_0, C_1, and δ_0, such that

$$e(w_R)(x) \leq C_7 \tag{4.12}$$

provided $B_x(1) \subset \Omega_R$. Arguing as in [22] and [23], after passing to a subsequence, w_R converges uniformly on compact subsets to a proper harmonic map w of $\mathbb{H}^2$. By (4.10), it is easy to see that $\sup_{\mathbb{H}^2} d_{\mathbb{H}}(w, h \circ w_0) \leq C_5$. Hence w is continuous as a map from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$. Since $w|_{\partial\mathbb{D}} = h \circ w_0|_{\partial\mathbb{D}}$ which is a homeomorphisms, w is globally one-to-one by the degree theory of maps and the fact that $J(w) > 0$ everywhere. By (4.12), the energy density of w is uniformly bounded. In particular, w is quasi-conformal by the results in [32], which completes the proof of the claim. Note that, by (4.11), we have

$$d^* = \sup_{\mathbb{H}^2} d_{\mathbb{H}^2}(w, h \circ w_0) \leq C_6 \|\Psi - \Psi_0\|_{QD}. \tag{4.13}$$

To conclude the first half of the theorem, let Φ be the Hopf differential of w, then $\Phi \in BQD(\mathbb{D})$ as w has uniformly bounded energy density. Since w and $h \circ w_0$ have the same boundary data, the complex dilatation of w is equivalent to that of $h \circ w_0$. Moreover, the complex dilatation of h is in $[\mu] \circ [\mu_0]^{-1}$ and the complex dilatation of w_0 is in $[\mu_0]$, therefore, by the definition of $\mathcal{B}$, we have $\mathcal{B}(\Phi) = ([\mu] \circ [\mu_0]^{-1}) \circ [\mu_0]$ which is $[\mu]$. Since $\beta([\mu]) = \Psi$, $\beta \circ \mathcal{B}(\Phi) = \Psi$. So Ψ is in the image of $\beta \circ \mathcal{B}$ provided $\|\Psi - \Psi_0\|_{QD} \leq \delta_1$. This proves that the image of $\beta \circ \mathcal{B}$ is open.

To prove the second half of the theorem, we observe that, by Theorem 2.5, $\beta \circ \mathcal{B}$ is real analytic. So, in order to prove that $\beta \circ \mathcal{B}$ is an analytic diffeomorphism onto its image, it is sufficient to prove that the differential $D(\beta \circ \mathcal{B})$ is bijective at every point of $BQD(\mathbb{D})$ and apply the inverse function theorem. Let Ψ_0, Φ_0 as before and $\|\Psi - \Psi_0\| \leq \delta_1$, where δ_1 is the constant in the above proof. Then by the above, there exists $\Phi \in BQD(\mathbb{D})$ such that $(\beta \circ \mathcal{B})(\Phi) = \Psi$. Let w and h be the harmonic maps described above in the construction of Φ. We would like to show that there exist constants $C_8 > 0$ and $0 < \delta_2 \leq \delta_1$, such that if $\|\Psi - \Psi_0\|_{QD} \leq \delta_2$, then

$$\|\Phi - \Phi_0\|_{QD} \leq C_8 \|\Psi - \Psi_0\|_{QD}. \tag{4.14}$$

Using the above notations and the upper space model for $\mathbb{H}^2$, we write $w_0 = u_0 + iv_0$, $w = u + iv$ and $h \circ w_0 = \tilde{u} + i\tilde{v}$. By comparing the Euclidean distance and the hyperbolic distance, we see that

$$|h \circ w_0 - w| \leq d^* \max\{\tilde{v},\ v\}.$$

where $d^* = \sup_{\mathbb{H}^2} d_{\mathbb{H}^2}(w, h \circ w_0)$. By (4.13), if we choose $0 < \delta_2 \leq \delta_1$ small enough, then for $\|\Psi - \Psi_0\|_{QD} \leq \delta_2$, we have $d^* \leq \frac{1}{2}$. Hence we have $\max\{\tilde{v}, v\} \leq 2\min\{\tilde{v}, v\}$. By (4.13) again, we have

$$|h \circ w_0 - w| \leq C_9 \|\Psi - \Psi_0\|_{QD} \min\{\tilde{v},\ v\}, \tag{4.15}$$

provided $\|\Psi - \Psi_0\|_{QD} \leq \delta_2$, where C_9 is a constant depending only on Φ_0, C_0, C_1 and δ_0. We may assume that $C_9\delta_2 \leq 1$. By Lemma 3.3 and (4.4), there is a constant C_{10} depending only on Φ_0, C_0, C_1 and δ_0, such that

$$\left| \frac{y^2}{\tilde{v}^2} (h \circ w_0)_z \left(\overline{h \circ w_0}\right)_z - \frac{y^2}{v_0^2} (w_0)_z (\overline{w_0})_z \right| \leq C_{10} \|\Psi - \Psi_0\|_{QD}. \tag{4.16}$$

On the other hand, by Lemma 3.4 and (4.15), we have

$$\begin{aligned}
&\left| \frac{y^2}{\tilde{v}^2} (h \circ w_0)_z \left(\overline{h \circ w_0}\right)_z - \frac{y^2}{v^2} (w)_z (\bar{w})_z \right| \\
&\leq \left| \frac{y}{\tilde{v}} \left(\overline{h \circ w}\right)_z \left(\frac{y}{\tilde{v}} (h \circ w)_z - \frac{y}{v} w_z \right) \right| + \left| \frac{y}{v} w_z \left(\frac{y}{\tilde{v}} \left(\overline{h \circ w}\right)_z - \frac{y}{v} \bar{w}_z \right) \right| \\
&\leq C_{11}(a + b),
\end{aligned}$$

where $a = \sup_{\mathbb{H}^2} \|\tau(h \circ w_0)\|$, $b = C_9\|\Psi - \Psi_0\|_{QD}$, and C_{11} depends only on the upper bounds of the energy densities of $h \circ w_0$ and w. The quantity a can be estimated by (4.9), and the energy density of h is bounded by a constant depending only on C_0, C_1 and δ_0 by (4.4) and the results in [32]. The energy density of w_0 is bounded by a constant depending only on Φ_0 by the result of [32] and the energy density of w is bounded by C_7 which is the constant in (4.12). Hence we can find a constant C_{12} depending only on Φ_0, C_0, C_1 and δ_0, such that

$$\left| \frac{y^2}{\tilde{v}^2} (h \circ w_0)_z \left(\overline{h \circ w_0}\right)_z - \frac{y^2}{v^2} (w)_z (\bar{w})_z \right| \leq C_{12} \|\Psi - \Psi_0\|_{QD}, \tag{4.17}$$

provided that $\|\Psi - \Psi_0\|_{QD} \leq \delta_2$. By (4.16), (4.17) and the definition of Hopf differential of a harmonic map, we have

$$\|\Phi - \Phi_0\|_{QD} \leq C\|\Psi - \Psi_0\|_{QD},$$

for some constant C independent of Ψ, provided $\|\Psi - \Psi_0\|_{QD} \leq \delta_2$. This completes the proof of (4.14). In particular, we have proved that $\beta \circ \mathcal{B}$ maps homeomorphically onto its image which is open, by noting that $\beta \circ \mathcal{B}$ is one-to-one.

Finally, the fact that the differential $D(\beta \circ \mathcal{B})(\Phi_0)$ is bijective follows from the following lemma, Lemma 4.2, (4.14), and the fact that $\beta \circ \mathcal{B}$ is a real analytic homeomorphism from $BQD(\mathbb{D})$ onto its image which is open in $BQD(\mathbb{D}^*)$.

Lemma 4.2. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Let $U \subset \mathfrak{B}_1$ and $V \subset \mathfrak{B}_2$ be open and $T : U \to V$ be a C^1 map which is also a homeomorphism from U onto V. Let $x \in U$. Suppose there are constants $r > 0$ and $C > 0$ such that $\|x' - x\|_1 \leq r$ implies*

$$\|x' - x\|_1 \leq C\|T(x') - T(x)\|_2. \tag{4.18}$$

Then DT_x is a bijection.

Proof. Since T is differentiable, for any $v \in \mathfrak{B}_1$, such that $\|v\|_1 = 1$, we have

$$\|T(x + tv) - T(x) - DT_x(tv)\|_2 = o(|t|),$$

as $t \to 0$. By (4.18), we conclude that

$$\|DT_x(tv)\|_2 \geq \frac{|t|}{2C},$$

for $|t|$ small enough, which implies, for all $v \in \mathfrak{B}_1$,

$$\|DT_x(v)\|_2 \geq \frac{\|v\|_1}{2C}. \tag{4.19}$$

From this, it is easy to see that DT_x is one-to-one (and has bounded inverse from its image).

To prove that DT_x is onto, let $y = T(x)$, and let $w \in \mathfrak{B}_2$ with $\|w\|_2 = 1$. Since V is open and T^{-1} exists and continuous, we have, if $|t|$ is small, then $T^{-1}(y + tw)$ is well defined, and if

$$v(t) = T^{-1}(y + tw) - T^{-1}(y),$$

then $\|v(t)\|_1 \leq r$, for $|t|$ small. Since $T^{-1}(y) = x$, by (4.18)

$$\begin{aligned}\|v(t)\|_1 &\leq C\|y + tw - y\|_2 \\ &= C|t|.\end{aligned}$$

Therefore, if $|t|$ is small, then

$$\begin{aligned}y + tw &= T(x + v(t)) \\ &= T(x) + DT_x(v(t)) + o(|t|) \\ &= y + DT_x(v(t)) + o(|t|),\end{aligned}$$

as $|t| \to 0$, i.e.,

$$w = DT_x\left(\frac{v(t)}{t}\right) + o(1), \tag{4.20}$$

as $t \to 0$. Hence, together with (4.19)

$$\begin{aligned}\left\|\frac{v(t_1)}{t_1} - \frac{v(t_2)}{t_2}\right\|_1 &\leq 2C\left\|DT_x\left(\frac{v(t_1)}{t_1}\right) - DT_x\left(\frac{v(t_1)}{t_1}\right)\right\|_2 \\ &\longrightarrow 0,\end{aligned}$$

as t_1 and $t_2 \to 0$. That is, $\frac{v(t)}{t}$ is a Cauchy sequence in $\mathfrak{B}_1$. Since $\mathfrak{B}_1$ is complete, there exists $v \in \mathfrak{B}_1$ such that

$$\lim_{t\to 0} \frac{v(t)}{t} = v.$$

Letting $t \to 0$ in (4.20) and using the fact that DT_x is continuous, we have $DT_x(v) = w$. Since w is arbitrary, we see that DT_x is a bijection. This completes the proof of the lemma.

As an application of the theorem, we give another proof of the Teichmüller theorem for compact surfaces with genus greater than 1. Let S be a smooth compact oriented surface without boundary of genus $g > 1$. We pick a particular metric $\mathrm{d}s^2$ of constant curvature -1. Let G be the fundamental group of $(S, \mathrm{d}s^2)$ such that $\mathbb{H}^2/G = (S, \mathrm{d}s^2)$. Then the Teichmüller space of genus g can be identified with $T(G)$ which consists of $[\mu] \in T$ such that there is a representative μ in the class $[\mu]$ satisfying

$$\mu(z) = \mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)},$$

in $\mathbb{D}$ for all $\gamma \in G$. Note that the smooth structure of T_g can be identified with the smooth structure of the image of $T(G)$ under the Bers imbedding. Let $BQD_G(\mathbb{D})$ be the space of bounded quadratic differentials $\Phi = \phi \mathrm{d}z^2$ such that $\phi(z) = \phi(\gamma(z))\,(\gamma'(z))^2$ in $\mathbb{D}$ for all $\gamma \in G$. Then, $BQD_G(\mathbb{D})$ can be identified with the space of holomorphic quadratic differentials of $(S, \mathrm{d}s^2)$. By the Riemann-Roch theorem, it is isomorphic to $\mathbb{C}^{3g-3}$. It is proved in [31] that $\mathcal{B}(\Phi) \in T(G)$ if $\Phi \in BQD_G(\mathbb{D})$. On the other hand, every $[\mu] \in T(G)$ has a representative μ and a Riemann surface S' such that μ is the complex dilatation of the lifting of a quasi-conformal map $h : S \to S'$. Equipping S' with a conformal metric of constant curvature -1 and using the result of [27], we can deform h to a quasi-conformal harmonic diffeomorphism. Let w be the lifting to $\mathbb{H}^2 \to \mathbb{H}^2$ of such a harmonic diffeomorphism and let Φ be the Hopf differential of w, then it is easy to see that $\mathcal{B}(\Phi) = [\mu]$. Hence $\mathcal{B}$ maps $BQD_G(\mathbb{D})$ onto $T(G)$. By Theorem 4.1, we have given another proof of

Theorem 4.3. [Teichmüller Theorem] *Let T_g be the Teichmüller space of genus $g > 1$. Then T_g is analytically diffeomorphic to $\mathbb{R}^{6g-6}$. In particular, T_g is finite dimensional.*

Note that $\mathcal{B}$ when restricted to $BQD_G(\mathbb{D})$ is the inverse of the map considered by Wolf [34], where he proved that this restriction is a homeomorphism under the assumption that T_g is of $(6g-6)$-dimensions. Later, Jost [18] was able to prove this without using the fact that dimension of T_g is finite.

We will give more applications in the next section.

5. Decay of Hopf Differentials and Boundary Regularity

In this section, we will discuss the relationship between the decay of the Hopf differential at the ideal boundary of the hyperbolic disc and the boundary regularity of the quasi-conformal harmonic diffeomorphism regarded as a map from $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. We will also give more applications to Theorem 4.1. First, let us introduce some subspaces of $BQD(\mathbb{D})$. For $\alpha \in (0, 2]$, we denote

$$BQD^\alpha(\mathbb{D}) = \left\{ \Phi = \phi(z)\mathrm{d}z^2 \in BQD(\mathbb{D}) \mid \sup_{\mathbb{D}} (\rho(z))^{-2+\alpha} |\phi|(z) < \infty \right\},$$

which is a (not closed) subspace of $BQD(\mathbb{D})$, where $\rho(z) = \frac{2}{1-|z|^2}$.

Note that if $\Phi \in BQD^\alpha(\mathbb{D})$, then $||\Phi||(z) \to 0$ as $z \to \partial\mathbb{D}$. In fact, if we denote

$$BQD_0(\mathbb{D}) = \left\{\Phi = \phi(z)\mathrm{d}z^2 \in BQD(\mathbb{D}) \mid \lim_{z\to\partial\mathbb{D}} ||\Phi||(z) = 0\right\},$$

then we have

Proposition 5.1. $\overline{BQD^\alpha(\mathbb{D})} = BQD_0(\mathbb{D})$ *for all* $\alpha \in (0,2]$, *where the closure is taken with respect to the norm in* $BQD(\mathbb{D})$.

Proof. It is easy to see that $BQD_0(\mathbb{D})$ is closed and contains $BQD^\alpha(\mathbb{D})$ for all $\alpha \in (0,2]$. Let $\Phi = \phi dz^2 \in BQD_0(\mathbb{D})$. For all $0 < R < 1$, let $\Phi_R = \phi_R dz^2$, where $\phi_R(z) = \phi(Rz)$. It is obvious that $\Phi_R \in BQD^\alpha(\mathbb{D})$ for any $0 < \alpha \leq 2$. We want to show that $||\Phi_R - \Phi||_{QD} \to 0$ as $R \to 1$. Since $\Phi \in BQD_0(\mathbb{D})$, for any $\epsilon > 0$, there exists $1 > R_0 > 0$ such that if $|z| \geq R_0$, then $\rho^{-2}(z)|\phi(z)| \leq \epsilon$. Therefore, if $|z| \geq (1+R_0)/2$, and if $1 > R \geq (2R_0)/(1+R_0)$, then $\rho^{-2}(Rz)|\phi(Rz)| \leq \epsilon$, which implies that $\rho^{-2}(z)|\phi(Rz)| \leq \epsilon$. It is also easy to see that $||\Phi_R(z) - \Phi(z)|| \to 0$ uniformly on $|z| \leq (1+R_0)/2$ as $R \to 1$. So, if $1 > R > 0$ is large enough,

$$\begin{aligned} ||\Phi_R - \Phi||_{QD} &= \sup_{\mathbb{D}} \rho^{-1}|\phi(Rz) - \phi(z)| \\ &\leq \sup_{|z|\geq R_1} \rho^{-1}\left(|\phi(Rz)| + |\phi(z)|\right) + \sup_{|z|\leq R_1} \rho^{-1}|\phi(Rz) - \phi(z)| \\ &\leq 3\epsilon \end{aligned}$$

where $R_1 = (2R_0)/(1+R_0)$. Hence $||\Phi_R - \Phi||_{QD} \to 0$ as $R \to 1$. This completes the proof of the proposition.

We need the following simple lemma.

Lemma 5.2. *Let f be a smooth function defined on a bounded domain Ω with smooth boundary. If f satisfies the following decay at $\partial\Omega$:*

$$|f| = O(d_x^\alpha), \text{ and}$$

$$|\nabla_0 f| = O(d_x^{-1+\alpha}),$$

for $\alpha \in (0,1]$. Then $f \in C^{0,\alpha}(\overline{\Omega})$.

Proof. Choose a finite open covering of $\partial\Omega$. Then for each open neighborhood in the covering, we can straighten the part of $\partial\Omega$ which lies in the neighborhood by a smooth diffeomorphism. So it is sufficient to show the lemma near a boundary portion of the upper half-plane. Let

d_x be the distance from x to the boundary of $\partial\Omega$ and $d_{x,y} = \min(d_x, d_y)$. We may assume that $d_{x,y} = d_x$. If $|x-y| \leq d_x$, we have, for some θ on the line segment joining x and y and a constant $C > 0$,

$$\begin{aligned}
|f(x) - f(y)| &= |\nabla_0 f(\theta)|\,|x-y| \\
&\leq C d_\theta^{-1+\alpha}|x-y| \\
&\leq C d_x^{-1+\alpha}|x-y| \\
&\leq C|x-y|^{-1+\alpha}|x-y| \\
&= C|x-y|^\alpha.
\end{aligned}$$

If $|x-y| > d_x$, then $d_y \leq d_x + |x-y| < 2|x-y|$. Hence

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x)| + |f(y)| \\
&\leq C\left(d_x^\alpha + d_y^\alpha\right) \\
&\leq 2C d_y^\alpha \\
&\leq 2C2^\alpha|x-y|^\alpha
\end{aligned}$$

Therefore, we have $f \in C^{0,\alpha}(\overline{\Omega})$. This completes the proof of the lemma.

As before, for any $\Phi \in BQD(\mathbb{D})$, we denote by $\mathfrak{w}(\Phi)$ the solution of (1.1). Then we have the following

Lemma 5.3. *Suppose that* $\Phi = \phi dz^2 \in BQD^\alpha(\mathbb{D})$. *Then* $\mu = \rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}$ *satisfies*

$$|\mu|(z) = O(d_z^\alpha),$$

$$|\nabla_0\mu|(z) = O(d_z^{-1+\alpha}),$$

$$|\nabla_0^2\mu|(z) = O(d_z^{-2+\alpha}).$$

Proof. Let $w = \mathfrak{w}(\Phi)$. From Lemma 2.1, we know that $|w|^*_{2,\alpha,\mathbb{D}} < \infty$. In particular, using the fact that $\rho(z) \sim d_z^{-1}$, we have

$$|w| = O(1),$$

$$|\nabla_0 w| = O(\rho), \tag{5.1}$$

$$|\nabla_0^2 w| = O(\rho^2).$$

Similarly, by the gradient estimates for harmonic functions and the fact that ϕ is holomorphic, we have

$$\begin{aligned} \sup_{\mathbb{D}} \rho^{-3+\alpha}|\phi'| < \infty, \\ \sup_{\mathbb{D}} \rho^{-4+\alpha}|\phi''| < \infty, \end{aligned} \tag{5.2}$$

where $\phi' = \phi_z$ and $\phi'' = \phi_{zz}$. By (5.1) and the assumption on ϕ, we have $|\mu| = O(\rho^{-\alpha})$. Direct computations show that

$$\begin{aligned} \mu_z &= -\mu\left(2w_z + \bar{z}\rho\right), \\ \mu_{\bar{z}} &= -\mu\left(2w_{\bar{z}} + z\rho\right) + \rho^{-2}\overline{\phi'}e^{-2w}, \\ \mu_{zz} &= \mu\left(\frac{1}{2}\bar{z}^2\rho^2 + 4\bar{z}\rho w_z + 4w_z^2 - 2w_{zz}\right), \\ \mu_{z\bar{z}} &= -\mu\left[\left(2w_{\bar{z}} + z\rho\right)\left(2w_z + \bar{z}\rho\right) + \rho + |z|^2\rho^2 + 2w_{z\bar{z}}\right] \\ &\quad + \rho^{-2}\overline{\phi'}e^{-2w}\left(2w_z + \bar{z}\rho\right), \end{aligned}$$

and

$$\begin{aligned} \mu_{\bar{z}\bar{z}} = -\mu\left(4w_{\bar{z}}^2 + 2z\rho w_{\bar{z}}z^2\rho^2 - 2w_{\bar{z}\bar{z}}\right) + \rho^{-2}\overline{\phi'}e^{-2w}\left(2w_{\bar{z}} + z\rho\right) \\ + \rho^{-2}\overline{\phi''}e^{-2w}. \end{aligned}$$

Therefore, by (5.1), (5.2), the fact that $|\mu|(z) = O(d_z^\alpha)$, and that $\rho(z) \sim d_z^{-1}$, we have the desired results.

Using Lemma 5.2, we conclude the following

Lemma 5.4. *Let $\mu = \rho^{-2}\bar{\phi}e^{-2\mathfrak{w}(\Phi)}$, where $\Phi = \phi dz^2 \in BQD^\alpha(\mathbb{D})$ with $\alpha \in (0,2]$. Then the function*

$$\tilde{\mu} = \begin{cases} \mu, \; z \in \mathbb{D} \\ 0, \; z \notin \mathbb{D} \end{cases}$$

is in $C^{0,\alpha}(\overline{\mathbb{C}})$ if $\alpha \in (0,1]$ and is in $C^{1,\beta}(\overline{\mathbb{C}})$ if $\alpha = 1+\beta \in (1,2]$.

Proof. This is a direct consequence of Lemmas 5.3 and 5.2 and the observation that for $\alpha = 1+\beta \in (1,2]$, $\tilde{\mu} \in C^1(\overline{\mathbb{D}})$ which is also a easy consequence of Lemma 5.3.

Now we can prove the following boundary regularity of quasi-conformal harmonic diffeomorphisms.

Theorem 5.5. *Let $u : \mathbb{H}^2 \to \mathbb{H}^2$ be a quasi-conformal harmonic diffeomorphism fixing 1, i, -1 in the Poincaré disc model for $\mathbb{H}^2$. Suppose that the Hopf differential Φ of u is in $BQD^\alpha(\mathbb{D})$ for some $\alpha \in (0,2]$, then*

1. *$u \in C^{1,\alpha}(\overline{\mathbb{D}})$ if $\alpha \in (0,1)$,*

2. $u \in C^{1,\gamma}(\overline{\mathbb{D}})$ *for all* $\gamma < 1$ *if* $\alpha = 1$,

3. $u \in C^{2,\beta}(\overline{\mathbb{D}})$ *if* $\alpha = 1 + \beta \in (1,2)$, *and*

4. $u \in C^{2,\gamma}(\overline{\mathbb{D}})$ *for all* $\gamma < 1$ *if* $\alpha = 2$.

Moreover, u has non-vanishing energy density when restricted on $\partial\mathbb{D}$.

Proof. Let $\mu = \rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}$, where ϕ is the coefficient of Φ. Then by Lemma 5.4, the function $\tilde{\mu}$ (defined as in the Lemma 5.4.) is in $C^{0,\alpha}(\overline{\mathbb{C}})$ if $\alpha \in (0,1]$ and is in $C^{1,\beta}(\overline{\mathbb{C}})$ if $\alpha = 1 + \beta \in (1,2]$. Then regularity theory of quasi-conformal maps, see [20], [19], and [28], implies that the quasi-conformal homeomorphism f_μ, constructed as in §1, from $\mathbb{C}$ onto itself is in $C^{1,\alpha}(\overline{\mathbb{C}})$ if $\alpha \in (0,1]$ and is in $C^{2,\beta}(\overline{\mathbb{C}})$ if $\alpha = 1+\beta \in (1,2]$. Moreover, $J(f_\mu) > 0$ for all $z \in \mathbb{C}$, where $J(f_\mu)$ is the Jacobian of f_μ. In particular, $f_\mu(\partial\mathbb{D})$ is a Jordan curve of class $C^{1,\alpha}$ if $\alpha \in (0,1]$ and of class $C^{2,\beta}$ if $\alpha = 1 + \beta \in (1,2]$. Let $\mathcal{R} : f_\mu(\mathbb{D}) \to \mathbb{D}$ be the Riemann map normalized so that $\mathcal{R} \circ f_\mu$ fixes 1, i, and -1. Then u and $\mathcal{R} \circ f_\mu$ are equal, since they have the same complex dilatation μ and fixing 1, i, and -1. The Kellogg-Warschawski theorem (Theorem 3.6 in [24]) implies that $\mathcal{R}^{-1}$ is

1. in $C^{1,\alpha}(\overline{\mathbb{D}})$ if $\alpha \in (0,1)$,
2. in $C^{1,\gamma}(\overline{\mathbb{D}})$ for all $\gamma < 1$ if $\alpha = 1$
3. in $C^{2,\beta}(\overline{\mathbb{D}})$ if $\alpha = 1 + \beta \in (1,2)$, and
4. in $C^{2,\gamma}(\overline{\mathbb{D}})$ for all $\gamma < 1$ if $\alpha = 2$.

Moreover, $\mathcal{R}^{-1}$ extends to $\overline{\mathbb{D}}$ with non-vanishing derivative up to $\partial\mathbb{D}$. Therefore, we conclude that $\mathcal{R}$, by expressing the derivatives of $\mathcal{R}$ in terms of derivatives of $\mathcal{R}^{-1}$, and hence u is

1. in $C^{1,\alpha}(\overline{\mathbb{D}})$ if $\alpha \in (0,1)$,
2. in $C^{1,\gamma}(\overline{\mathbb{D}})$ for all $\gamma < 1$ if $\alpha = 1$
3. in $C^{2,\beta}(\overline{\mathbb{D}})$ if $\alpha = 1 + \beta \in (1,2)$, and
4. in $C^{2,\gamma}(\overline{\mathbb{D}})$ for all $\gamma < 1$ if $\alpha = 2$;

and has non-vanishing energy density up to $\partial\mathbb{D}$. This completes the proof of the theorem.

Examining the proof of the above theorem, one expects that the Schwarzian derivative of $g = f_\mu|_{\mathbb{D}^*}$ will have the "same" decay as Φ. In order to state it clearly, we denote

$$BQD^{\alpha}(\mathbb{D}^*) = \Big\{\Psi = \psi(z)\mathrm{d}z^2 \in BQD(\mathbb{D}^*) \mid \\ \limsup_{|z|\to 1} (\rho_{\mathbb{D}^*}(z))^{-2+\alpha} |\psi|(z) < \infty\Big\},$$

where $\rho^2_{\mathbb{D}^*} = 4/(|z|^2-1)^2$ is the density of the Poincaré metric on $\mathbb{D}^*$. Then we have the following

Theorem 5.6. *Let $\beta \circ \mathcal{B}$ be the map in Theorem 4.1. Then the image of $BQD^{\alpha}(\mathbb{D})$ under $\beta \circ \mathcal{B}$ is in $BQD^{\alpha}(\mathbb{D}^*)$ if $\alpha \in (0,1) \cup (1,2)$, is in $BQD^{\gamma}(\mathbb{D}^*)$ for all $\gamma < 1$ if $\alpha = 1$, and is in $BQD^{\gamma}(\mathbb{D}^*)$ for all $\gamma < 2$ if $\alpha = 2$.*

Proof. Let $\Phi = \phi\mathrm{d}z^2 \in BQD^{\alpha}(\mathbb{D})$ and $\mu = \rho^{-2}\overline{\phi}e^{-2\mathfrak{w}(\Phi)}$. As in the proof of the Theorem 5.5, we have f_μ is in $C^{1,\alpha}(\overline{\mathbb{C}})$ if $\alpha \in (0,1]$ and is in $C^{2,\beta}(\overline{\mathbb{C}})$ if $\alpha = 1+\beta \in (1,2]$. Therefore, the univalent function $g = f_\mu|_{\mathbb{D}^*}$ is

1. in $C^{1,\alpha}(\overline{\mathbb{D}})$ if $\alpha \in (0,1)$,
2. in $C^{1,\gamma}(\overline{\mathbb{D}})$ for all $\gamma < 1$ if $\alpha = 1$
3. in $C^{2,\beta}(\overline{\mathbb{D}})$ if $\alpha = 1+\beta \in (1,2)$, and
4. in $C^{2,\gamma}(\overline{\mathbb{D}})$ for all $\gamma < 1$ if $\alpha = 2$.

Moreover, $g' \neq 0$ up to boundary. Using the Cauchy integral formula, we conclude, for $\alpha \in (0,1)$, that $g'' = O(d_z^{-1+\alpha})$ and $g''' = O(d_z^{-2+\alpha})$ as $|z| \to 1$. So the Schwarzian derivative $S_g = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = O(d_z^{-2+\alpha})$ since g' continuous and $g' \neq 0$ up to boundary. The other cases are similar. This completes the proof of the theorem.

We have seen that the decay of the Hopf differential implies the boundary regularity of the quasi-conformal harmonic diffeomorphism. Conversely, if we have boundary regularity of harmonic map, we also have decay of its Hopf differential. Let $u = f + ig$ be a quasi-conformal diffeomorphism from $\mathbb{H}^2$ onto $\mathbb{H}^2$. Let $\Phi = \phi\mathrm{d}z^2$ be the Hopf differential of u.

Lemma 5.7. *Suppose that u is C^2 up to the boundary and that the boundary data has non-vanishing energy density. If we use the upper half space model for $\mathbb{H}^2$ then $\nabla_0 u_{\bar z} = 0$ at $y = 0$.*

Proof. Let $w = w(z)$ with $z = x+iy$. Then we have

$$g\Delta_0 f - 2\langle\nabla_0 f, \nabla_0 g\rangle = 0, \tag{5.3}$$

and

$$g\Delta_0 g + \langle \nabla_0 f, \nabla_0 f\rangle - \langle \nabla_0 g, \nabla_0 g\rangle = 0.$$

By the result in [22], we have

$$f_x = g_y, \text{ and } f_y = -g_x, \tag{5.4}$$

at $y = 0$. Differentiate (5.3) with respect to y,

$$g_y\Delta_0 g + g\Delta f_y - 2\left(f_{xy}g_x + f_x g_{xy} + f_{yy}g_y + f_y g_{yy}\right) = 0. \tag{5.5}$$

Let p be a point in the boundary. Then by [22], there is a constant $C_1 > 0$, such that in a neighborhood of p,

$$C_1^{-1}y \le g \le C_1 y \tag{5.6}$$

Since $f_y \in C^1$ up to the boundary, by Lemma 1.2 in [22], there is a sequence $q_i = (x_i, y_i) \to p$ such that

$$\lim_{i\to\infty} y_i\Delta_0 f_y(q_i) = 0. \tag{5.7}$$

Evaluating (5.5) at q_i and letting $i \to \infty$, then by using (5.4), (5.6), (5.7), the fact that $g = 0$ at $y = 0$, and the fact that f is C^2 up to the boundary, we have

$$g_y(p)\Delta_0 f(p) = 0.$$

By (5.4) and the assumption that $f_x \neq 0$ at $y = 0$, we have

$$\Delta_0 f = 0, \tag{5.8}$$

at $y = 0$. Using similar argument, one can also show that

$$\Delta_0 g = 0,$$

at $y = 0$. Now $2u_{\bar z} = (f_x - g_y) + i(f_y + g_x)$.

$$\begin{aligned} 2u_{\bar z x} &= (f_{xx} - g_{yx}) + i(f_{yx} + g_{xx}) \\ &= 0 \end{aligned}$$

at $y = 0$, where we have used the fact that w is C^2 up to the boundary and (5.4). Also, by (5.4), (5.7), (5.8), and the fact that w is C^2 up to the boundary,

$$\begin{aligned} 2u_{\bar z y} &= (f_{xy} - g_{yy}) + i(f_{yy} + g_{xy}) \\ &= -\Delta_0 g + i\Delta_0 f \\ &= 0 \end{aligned}$$

at $y = 0$. Hence $\nabla_0 u_{\bar{z}} = 0$ at $y = 0$. This completes the proof of the lemma.

Theorem 5.8. *Let u be a quasi-conformal harmonic diffeomorphism on $\mathbb{H}^2$ such that the boundary map is C^1 with non-vanishing energy density. Let $\Phi = \phi dz^2$ be the Hopf differential of u. If we use the Poincaré disc model for $\mathbb{H}^2$, i.e., $\mathbb{H}^2 = (\mathbb{D}, ds_p^2)$ where $ds_p^2 = \rho^2|dz|^2$ is the Poincaré metric on $\mathbb{D}$, then $\Phi \in BQD_0(\mathbb{D})$. Moreover,*

(i) if u is $C^{1,\alpha}$ up to the boundary for some $0 < \alpha \le 1$, then $\sup_{\mathbb{D}} \rho^{-2+\alpha}|\phi| < \infty$, i.e., $\Phi \in BQD^{\alpha}(\mathbb{D})$;

(ii) if u is $C^{2,\alpha}$ up to the boundary for some $0 < \alpha \le 1$, then $\sup_{\mathbb{D}} \rho^{-1+\alpha}|\phi| < \infty$, i.e., $\Phi \in BQD^{1+\alpha}(\mathbb{D})$. In particular, if u is $C^{2,1}$ up to the boundary, then ϕ is bounded.

Proof. Using the upper half space model as in Lemma 5.7, then $\phi = \frac{1}{g^2} w_z \bar{w}_z$. By (5.6) and the fact that u is at least C^1 up to the boundary, we have

$$|\phi| = O\left(\left|\frac{w_{\bar{z}}}{y^2}\right|\right), \tag{5.9}$$

near $y = 0$. If u is C^1 up to the boundary, then $u_{\bar{z}} = 0$ at $y = 0$ by [22], and near a point $(x_0, 0)$ at $y = 0$, $C^{-1} \le y/g \le C$ for some positive constant C. These imply that $\Phi \in BQD_0(\mathbb{D})$. If u is $C^{1,\alpha}$ up to the boundary, then $|u_{\bar{z}}| = O(y^{\alpha})$ near $y = 0$. Hence, by using (5.9), (i) is true. If $u \in C^{2,\alpha}$ up to the boundary, then we also have $\nabla_0 w = 0$ at $y = 0$ by Lemma 5.7. Hence, using (5.9) again, it is easy to see that (ii) is also true.

Remark 5.1. It is easy to see that the results in Theorem 5.8 is purely local.

Using the Theorem 5.5, 5.8 and the results in §4, we can say a little more on the image of the map $\mathcal{B}$ defined in §1. By Theorem 4.1, we know that if $[\mu]$ is in the image of $\mathcal{B}$, then there is a $\delta > 0$ such that the ball in the universal Teichmüller space of radius δ with center $[\mu]$ is also in the image. Here the radius δ may depend on $[\mu]$. On the other hand, it was proved in [21, 22, 23], if the normalized quasi-symmetric function on $\partial\mathbb{D}$ corresponding to $[\mu]$ is C^1 with non-vanishing energy density, then $[\mu]$ is in the image. Now, let N be the subset of the universal Teichmüller space T consisting of those $[\mu]$ such that the corresponding normalized quasi-symmetric function on $\partial\mathbb{D}$ is C^1 with non-vanishing energy density. By Theorem 5.8, if $[\mu] \in N$, then $\Phi \in BQD_0(\mathbb{D})$, where $\mathcal{B}(\Phi) = [\mu]$. Therefore, if $[\nu]$ is also in the image of $\mathcal{B}$, then by Lemma 3.2 and the proof of Theorem 4.1, we see that $[\nu] \circ [\mu]$ is also in the image. Hence, using Theorem 2.6, we have

Proposition 5.9. *There is a constant $\delta_0 > 0$ such that if $[\mu] \in N$, then $[\nu]$ lies in the image of $\mathcal{B}$ for all $[\nu]$ such that $\lambda([\nu], [\mu]) < \delta_0$, where λ is the Teichmüller metric. In particular, $\overline{N}$ is also in the image of $\mathcal{B}$.*

We should emphasis that in this case, the number δ_0 does not depend on $[\mu]$ as long as $[\mu] \in N$. By the proposition, $\mathcal{B}$ would be onto if N were dense in T. Unfortunately, this is not the case. In fact, let N^α, $\alpha \in [0, 2]$, be the subset of the universal Teichmüller space corresponding to the set of quasi-symmetric functions on $\partial\mathbb{D}$ with non-vanishing energy density which is $C^{1,\alpha}$, if $\alpha \in [0, 1]$ and is $C^{2,\beta}$, if $\alpha = 1 + \beta \in (1, 2]$. Note that $N^0 = N$. Then, we have the following corollary of Theorem 5.5 and Theorem 5.8.

Corollary 5.10. *We have the following subset relations in T.*

$$\begin{aligned}\mathcal{B}\left(BQD_0(\mathbb{D})\right) &= \overline{N},\\ \mathcal{B}\left(BQD^\alpha(\mathbb{D})\right) &= N^\alpha \textit{ for } \alpha \in (0,1) \cup (1,2)\end{aligned}$$

In particular N is not dense in the image of $\mathcal{B}$, hence, not dense in the universal Teichmüller space.

Proof. The subset relations in T follow from the fact that $BQD_0(\mathbb{D})$ is closed, Theorem 5.5, Theorem 5.8, and the results in [22, 23]. To prove that N is not dense in T, we observe that there is a $\Phi \in BQD(\mathbb{D})$ such that there is a sequence of point $z_i \to \partial\mathbb{D}$ and a positive number $\epsilon > 0$ such that

$$\|\Phi\|(z_i) \geq \epsilon. \tag{5.10}$$

For example, one can choose $\Phi \neq 0$ which comes from a holomorphic quadratic differential on a compact Riemann surface of genus $g \geq 2$. Therefore, $BQD_0(\mathbb{D})$ is not dense in $BQD(\mathbb{D})$. Since $\mathcal{B}$ maps $BQD(\mathbb{D})$ homeomorphically onto its image, the rest of the corollary follows.

Remark 5.2. Using the proof of Theorem 4.1 and Remark 5.1, it might be possible to construct harmonic maps (in two or higher dimensions) with more general boundary data than those considered in [21, 22, 23, 33].

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