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LAGRANGIAN TORI IN R^4

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Introduction

Let (R^4, ω) denote the standard linear symplectic four dimensional space endowed with the (Darboux) coordinates (p_1, p_2, q_1, q_2) in which the symplectic form is represented as

$$\omega = dp \wedge dq = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

An immersed surface is said to be Lagrangian if the symplectic form pulls back under the immersion to the 2-form that is everywhere zero. There are a number of fairly obvious examples: the (p_1, p_2) plane or any domain there of for instance. It is easy to see that the only compact, orientable surface that admits a Lagrangian embedding into (R^4, ω) is the torus: the pairing $v \to \omega(\cdot, v)$ establishes an isomorphism between the normal bundle and cotangent bundle of a Lagrangian submanifold. The only known examples of embedded Lagrangian Tori are all smoothly isotopic to a Clifford torus (i.e., the Cartesian product of the unit circle in the (p_1, q_1) plane with that of the (p_2, q_2) plane). A well known problem in symplectic topology is the question of whether or not any Lagrangian tori are topologically knotted [1]. In this note it is shown that a large number of isotopy classes of embeddings of the torus in 4-space do not contain Lagrangian representatives. In some sense most isotopy classes are in that ensemble; the author knows of a sequence of examples to which the present methods do not apply in any obvious way, but they are all of a rather special nature. Crudely speaking, the results herein will demonstrate that any "sufficiently knotted" torus can not be isotoped to a Lagrangian embedding.

The purpose of the proof is to show that the Lagrangian condition

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puts severe constraints on the topology of an embedding; it is shown that a countable number of "symplectic" Dehn surgeries can be performed on any Lagrangian torus, while on the other hand a theorem of M. Gromov [7] implies that all of the resulting symplectic manifolds are symplectomorphic to linear symplectic 4-space. Thus a Lagrangian embedding can occur in an isotopy class of embeddings only if the aforementioned sequence of surgeries are (differentiably) trivial on knots in that class, i.e., the manifolds resulting from the surgeries are all diffeomorphic to R^4. It is easy to see that the surgeries are trivial on the Clifford torus, and one can see *a priori* that on an arbitrary torus the surgeries must result in punctured homology spheres. As we shall see, the condition that these manifolds are all diffeomorphic to R^4 is an extremely strong restriction on the topology of the embedding in question.

1. Symplectic rigidity

Before elaborating on the surgery construction it is necessary to mention a remarkable symplectic rigidity result of M. Gromov and its extension due to D. McDuff.

Definition. Let (M, w) be a symplectic 4-manifold, where M is noncompact with an end diffeomorphic to $S^3 \times (0, \infty)$. (M, w) is said to be **standard at infinity** if there exists a *symplectomorphism* between an end of M and a neighborhood at infinity in (R^4, w). In other words, there exist compact sets $K \subset M$, $L \subset R^4$ and a diffeomorphism $\psi : (M - K) \to (R^4 - L)$ such that $\psi^\star(\omega) = w$. Gromov has proven the following.

Theorem. [7] *If (M, w) is a symplectic 4-manifold which is standard at infinity, and $H^2(M, R) = 0$, then (M, w) is symplectomorphic to (R^4, w).*

In [10] D. McDuff proves that if the cohomological assumption above is omitted then, (M, w) is either symplectomorphic to (R^4, ω) or (R^4, ω) blown up at finitely many points.

2. Symplectic Dehn surgery

Let T be a 2-torus embedded in R^4. A Dehn surgery on T is, loosely speaking, any process in which a tubular neighborhood of T is removed and then reattached via some diffeomorphism of the boundary of this neighborhood. It is convenient to introduce some notation; let $N(T) = T \times D$ be the tubular neighborhood, $T^3 = \partial N(T)$ and $F : T^3 \to T^3$ be the attaching map. These data, together with a framing (i.e., a parameterization of $N(T)$) define the surgery which gives rise to a smooth manifold

$$M = (R - N(T)) \cup_F (N(T)).$$

The surgery is said to be *trivial* if M is diffeomorphic to R^4.

We will call the surgery *symplectic* if F extends to a symplectomorphism of a tubular neighborhood of T^3. In this case the symplectic structures of $N(T)$ and its complement are collated by F, giving rise to a symplectic structure on M. We will call a symplectic surgery *symplectically trivial* if M is symplectomorphic to R^4. In this paper we are concerned only with symplectic Dehn surgery on a torus in R^4; the notion is obviously well defined in complete generality, i.e., one can easily extend the definition to surgeries on submanifolds of an arbitrary symplectic manifold. Such constructions are extremely local in nature - on R^4, for example, the process (at most) modifies the geometry in a neighborhood of T. Thus, by Gromov's theorem, any symplectic Dehn surgery on R^4 that does not affect second cohomology results in a symplectic manifold symplectomorphic to R^4. This is an observation that will be crucial in what follows, however it is worth mentioning a much stronger version of this statement. For elementary topological reasons, (additivity of signature) it is impossible to obtain a blow up of the ball from a Dehn surgery on an embedded surface. This fact, combined with the theorem of McDuff gives a proof of the following.

Theorem 1. *All symplectic Dehn surgeries on surfaces in R^4 are symplectically and hence differentiably trivial.*

This observation should be thought of as a manifestation of symplectic rigidity - more generally, very few surgery or other topological constructions work in the symplectic category.

To define the surgeries we need to begin with a natural choice of parameterized tubular neighborhood of a Lagrangian torus. For this we will need a corollary of a basic parameterization lemma known as the relative Darboux-Weinstein theorem. A number of good references exist for further details of the symplectic topics discussed here, among them are [2], [3] and [12]. The cotangent bundle of any smooth manifold has a canonical symplectic structure in which the zero section is a Lagrangian submanifold. By the relative Darboux-Weinstein lemma, any Lagrangian torus has tubular neighborhoods symplectomorphic to tubular neighborhoods of the zero section of the cotangent bundle of the torus. Thus one has the following choice for a parameterized tubular neighborhood of a Lagrangian torus: $N(T) = \{(x, y, r, \theta) : r \leq 1\}$ where $(x, y)(mod\, 1)$ are angular coordinates on T, and (r, θ) are polar coordinates on the fibers. Note that one can think of this as the unit disc bundle for a flat Riemannian metric on the torus. For each pair of integers, (m, n) we will define the following diffeomorphism of $\partial N(T) = \{(x, y, 1, \theta)\}$:

$$F_{m,n}(x, y, 1, \theta) = (x + m\theta, y + n\theta, 1, \theta).$$

If $(k, L) \neq (m, n)$ is another integral pair, then $F_{k,L}$ and $F_{m,n}$ induce different maps on homology and are therefore <u>not</u> isotopic. In particular, if $(m, n) \neq (0, 0)$. then $F_{m,n}$ is not isotopic to the identity. In the parameterization we have chosen the symplectic form ω to assume the representation

$$\omega = d(r[cos 2\pi\theta dx + sin 2\pi\theta dy]).$$

One sees by a trivial calculation that $F_{m,n}$ preserves the restriction of ω to the tangent bundle of $\partial N(T)$. Thus $F_{m,n}$ extends to a symplectomorphism of a neighborhood of $\partial N(T)$. This completes the construction of the infinite family of symplectic Dehn surgeries that can be performed on a Lagrangian torus. This *family* of isotopy classes of gluing maps is actually uniquely determined by the meridian element (boundary of a normal disc) which is itself uniquely determined (up to isotopy) by the torus embedding; the meridian determines a product structure $S^1 \times T^2$ of T^3 and therefore a direct sum decomposition $Z + Z^2$ of $H_1(T^3, Z)$. The subgroup of $SL_3(Z)$ stabilizing $0 + Z^2$ coincides with the family of attaching maps. Note that the T^2 factors can be realized as Lagrangian tori which can be thought of as Kolmogorov Tori for the geodesic flow of a flat Riemannian metric on the zero section.

3. Obstructions to Lagrangian embeddings

The topological obstructions to a Lagrangian embedding can be stated in the language of classical knot theory. To do so we need to recall some definitions. Let G be a finitely presented group. An element is said to normally generate G if G is generated by the set of all conjugates of this element. It is a classical result of knot theory that the knot group is normally generated by the meridian curve, and this is true also for the fundamental group of the complement of a closed embedded surface in R^4 (see [11]). There exists a topologically preferred family of framings (i.e., product structure on the 3-torus) for a 2-torus in R^4 consisting of the meridian as S^1 factor and a "longitudinal" 2-torus factor corresponding to the kernel of the surgective homomorphism on one-dimensional homology from Z^3 onto Z induced by the inclusion map of the 3-torus into the knot complement. Note that the map is indeed surjective since the meridian generates the one-dimensional homology of the knot complement. The homomorphism determines a unique isotopy class of 2-torus factor which we can choose as the longitude. We will call an element of the knot group a longitude if it is in the image of the fundamental group of the longitudinal torus. Thus a topologically preferred frame gives us well defined longitudinal elements. It is shown in this paper that a Lagrangian torus has a symplectically preferred family of framings–those corresponding (under a Darboux parameterization) to Kolmogorov foliations for flat Riemannian metrics.

Theorem 2. *For a Lagrangian torus the topologically preferred framings and symplectically preferred framings coincide.*

Proof. It suffices to show that a symplectic frame is topological– this amounts to showing that the associated 2-torus factor has trivial image in the first homology of the knot complement. If a longitudinal curve δ was homologically non-trivial in the knot complement, then it could be decomposed as a sum: $\delta = n\gamma + l$, where l is a longitudinal element for the topological framing. We can assume , without loss of generality , that δ corresponds to a compact one-dimensional subgroup of the 2-torus. Now by assumption δ is longitudinal for the symplectic framing, so we can define a symplectic Dehn surgery that changes γ to its sum with any multiple of δ we choose. The symplectic manifold arising from this Dehn surgery will not be simply connected , indeed , the first integral homology group will be cyclic since the one-dimensional homology of the knot complement is an infinite cyclic group generated by the meridian, and the surgery kills a positive integral multiple $k\gamma$ of the meridian in homology. Thus the one-dimensional homology of the surgered manifold is the cyclic group of order k. Another proof of the equivalence of framings is found in the paper [6].

We can now state the obstruction to a Lagrangian embedding:

Theorem 3. *Let T be a Lagrangian torus in (R^4, ω) and let G be its knot group. Then G is normally generated by every element of the form $l * \gamma$, where l is any longitude, and γ is the meridian.*

Proof. For each l there exists a symplectic Dehn surgery that takes the meridian γ to its sum with l. This changes the fundamental group by adding the relation $l * \gamma = id$. But this relation must kill the fundamental group, hence the knot group is normally generated by $l * \gamma$.

There are a number of isotopy classes of torus embeddings into R^4, which are very easily seen to be unrealizable by Lagrangian embeddings. An infinite collection of such examples is discussed below.

The examples we will discuss are torus spin knots. Such a surface can be thought of as the orbit of a nontrivial classical (S^1) knot in a 3-dimensional half space under an S^1 action orthogonal to this half space. In this case the torus knot has a rotationally symmetric tubular neighborhood. The boundary of such a neighborhood is the orbit of the boundary of a tubular neighborhood of the classical knot while the meridian of the torus knot can be taken to be a meridian of the classical knot (Note: the meridian determines a rotationally symmetric product structure). An easy application of van Kampen's theorem shows that the fundamental group of the complement of the torus spin knot coincides with that of the classical knot, from which we immediately conclude that there are infinitely many distinct isotopy classes of such embeddings of the torus.

Theorem 4. *No nontrivial torus spin knot admits a Lagrangian realization in (R^4, ω).*

Proof. Assume that a nontrivial Lagrangian torus spin exists.

There are an infinite number of nontrivial symplectic Dehn surgeries that can be performed on a torus spin knot - namely, the rotationally symmetric ones, i.e., those arising from gluing maps induced by translations of the form $(0, n)$ of the 2-torus. (Note: these are the only nontrivial symplectic Dehn surgeries in this case.) By virtually the same application of van Kampen's theorem as above one finds that the 4-manifolds constructed by these surgeries have fundamental groups that are isomorphic to those of the 3-manifolds arising from the restriction of the surgery to the classical knot. The fact that all (but **at most** one!) of these groups are nontrivial follows from the cyclic surgery theorem in classical knot theory (see [4]). By Theorem 1 we therefore conclud that no embedding of this type can possibly be Lagrangian.

There is a sequence of isotopy classes of embeddings of the torus to which Theorem 1 does not apply (in the most obvious way). Define a *localized* knot of the torus to be one obtained by attaching an *unknotted* handle to a knot of the 2-sphere in R^4. The resulting embedding of the torus has the property that the 2-torus fibers in the boundary of a tubular neighborhood carry trivial fundamental group in the knot complement. Consequently, all symplectic Dehn surgeries result in simply connected manifolds, so the methods of this paper do not apply in an obvious fashion. It is also important to realize that great care must be taken in attaching the handle: if the handle is in any sense knotted relative to the 2-sphere knot (there are infinitely many ways to do so), then the resulting torus embedding can easily violate the knot obstructions above and can be ruled out as a Lagrangian embedding. The localized knots are the only examples known to the author at this time that can not be ruled out. At the same time it should be said that we have only applied a small portion of the full strength of these results - the fact that the symplectic manifolds arising from these surgeries are *symplectomorphic* to (R^4, ω) is a strong conclusion; perhaps the full strength of this fact can be applied in some clever fashion to the localized examples. In [5] it is shown that if a Lagrangian knot is in some sense geometrically localized, then it is trivial i.e.,symplectically isotopic to a product torus. The fibered surgery construction has a number of other applications as well. By performing symplectic Dehn surgery on cohomologically nontrivial Lagrangian tori in symplectic 4-manifolds one can create new compact, symplectic (non-Kählerian) manifolds. Indeed, many infinite classes of such examples arise from these constructions [8].

Other applications can be found in [6].

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