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RESIDUES OF HOLOMORPHIC VECTOR FIELDS RELATIVE TO SINGULAR INVARIANT SUBVARIETIES

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1. Introduction

Let $\mathcal{F}$ be a holomorphic foliation with singularities on a complex manifold W, and V an analytic subvariety (possibly with singularities) of W invariant by $\mathcal{F}$. Here "invariant", or equivalently "saturated" means that if a point of V belongs to the regular part of $\mathcal{F}$, then the whole leaf through this point is included in V. We shall assume furthermore that the normal bundle to the regular part of V in W has a natural extension ν to the whole V, and even a smooth extension $\tilde{\nu}$ to a germ of neighborhood of V in W, making us able to use connections on $\tilde{\nu}$ and to integrate associated differential forms on compact pieces of V. For instance, such a natural extension $\tilde{\nu}$ always exists for complex hypersurfaces, or complete intersections in the projective space, or "strong" local complete intersections (SLCI: see definition below).

Denote the complex dimensions of V, W and the leaves of $\mathcal{F}$ by p, $p+q$ and s respectively. The bundle ν admits a "special" connection away from the singular set $\Sigma = (\text{ Sing }(\mathcal{F}) \cap V) \cup \text{ Sing }(V)$ so that the associated characterictic forms of degree $> 2(p-s)$ vanish. If V is non-singular, we may represent the characteristic classes of ν by characteristic forms on V and see that those classes in dimension $> 2(p-s)$ will "localize" near Σ. In the case of singular V, we work on the characteristic forms of $\tilde{\nu}$ on the ambient space instead, and the characteristic classes of ν in these dimensions will still localize near Σ and give rise to residues for each connected component Σ_α of Σ. In fact, once we know $\tilde{\nu}$ to exist, the definition and the proof of the existence of these residues work similarly as in the case of non-singular

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V (see Théorème 3, p.227, in [8]), and thus we shall omit the theory for $s > 1$. We will concentrate ourselves to the computation of the residues for Chern numbers at an isolated point of Σ in the case $s = 1$. We get then formulas generalizing the ones in [9] and [12] and also, in the spirit of Baum-Bott [1] and [2], the Grothendieck residues already known when V is non-singular ([8]) (see Theorem 1 below, and its third particular case with Theorem 2). Note that the residues of Baum and Bott are localised characteristic classes of the normal sheaf of the foliation $\mathcal{F}$ (or an equivalent virtual bundle), while ours are those of the (extended) normal bundle of V in W.

This residue has first been defined by C. Camacho and P. Sad in [5] when $p = q = s = 1$, V non-singular and Σ_α an isolated point. When the invariant curve V may have singularities, the theory has then been generalized by A. Lins Neto [9] for $W = CP^2$, by M. Soares [11] when the surface W is a complete intersection in CP^n, and in [12] for arbitrary complex surfaces. It has also been studied in higher dimensions when V is non-singular, first in the case $s = p$, $q = 1$ by B. Gmira [6], J.-P. Brasselet (unpublished) and A. Lins Neto [10], and then in [8] for the general case with more precise formulas when $s = 1$.

All these results extend by taking, instead of $\tilde{\nu}$, any C^∞ vector bundle on a germ of neighborhood of V in W, the restriction of which to the regular part of V being holomorphic and equipped with an action of a holomorphic vector field X_0 tangent to this regular part (see Theorem 1' below). In particular, if we take $T(W)$, with the action $[X_0, .]$ on $T(W)|_V$, we get a formula for computing the index defined in Theorem 8 of [8]. (We were wrong when claiming that the index defined there was the same as the index of [9] for $p = q = s = 1$: there was a mistake in the proof of part (iv) of this theorem, the three first parts remaining correct.)

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2. Background on local complete intersections (LCI and SLCI)

Let W be a complex manifold of complex dimension $n = p + q$, and V an analytic irreducible subvariety of pure complex dimension p. We shall call "reduced local defining function" for V every holomorphic map $f : U \to C^q$ defined on an open set U of W, such that:

(i) $V \cap U = f^{-1}(0)$,

(ii) the q components of f generate the ideal $I(V \cap U)$ of holomorphic functions which vanish on $V \cap U$; for instance, if $q = 1$, this condition implies that f may not have factors which are powers.

The subvariety V is said to be a "local complete intersection" (briefly: LCI) if the following condition holds: there exists a family $\{f_h : U_h \to C^q\}_h$ of reduced local defining functions for V, such that $\bigcup_h U_h \supset V$. Such a family will be called a "system of reduced equations" for V. Recall the following proposition, well known to the specialists:

Proposition 1. *(i) Let $f_1 : U \to C^q$ and $f_2 : U \to C^q$ be two reduced local defining functions for V defined on the same open set U. Then, there exists an holomorphic map $\tilde{g} : U \to gl(q, C)$ taking values in the set $gl(q, C)$ of $q \times q$ matrices with complex coefficients, satisfying $f_1 =< \tilde{g}, f_2 >$, such that the restriction g of $\tilde{g}$ to $V \cap U$ is uniquely defined and takes values in the group $GL(q, C)$ of invertible matrices.*

(ii) If V is an LCI, and if $\{f_h : U_h \to C^q\}_h$ denotes a system of reduced equations for V, let $\tilde{g}_{hk} : U_h \cap U_k \to gl(q, C)$ such that $f_h = < \tilde{g}_{hk}, f_k >$ on $U_h \cap U_k$, and denote by g_{hk} the restriction of $\tilde{g}_{hk}$ to $V \cap U_h \cap U_k$. The family $\{g_{hk}\}$ is then a system of transition functions for a holomorphic q vector bundle $\nu \to V$. This vector bundle is well defined (it does not depend on the choice of the given system of reduced equations for V).

(iii) The bundle ν is an extension to V of the (holomorphic) normal bundle to $V -$ Sing (V) in W; more precisely, there exists a natural bundle map $\pi : T_C(W)|_V \to \nu$ which, over the regular part of V, has rank q and the complex tangent bundle to this regular part for kernel (we may therefore identify the restriction of ν to this regular part with the usual normal bundle).

Proof. Let f_1 and f_2 be such as in (i). Since the components $f_{1,\lambda}$ $(1 \leq \lambda \leq q)$ of f_1 and $f_{2,\lambda}$ of f_2 generate the ideal $I(V \cap U)$, there exist $q \times q$ matrices $\tilde{g}$ and $\tilde{h}$ with holomorphic coefficients such that $f_1 =< \tilde{g}, f_2 >$ and $f_2 =< \tilde{h}, f_1 >$. Furthermore, since f_1 and f_2 vanish on $U \cap V$, we get also on $U \cap V$, $df_1 =< g, df_2 >$ and $df_2 =< h, df_1 >$, where g and h denote the restrictions of $\tilde{g}$ and $\tilde{h}$ to $U \cap V$. Since

$df_1 =< g \circ h, df_1 >$ on $V \cap U$, $g \circ h = Id$ on the regular part of $V \cap U$. By continuity, since this regular part is everywhere dense in $V \cap U$, one still has $g \circ h = Id$ on the whole $V \cap U$; g takes values in $GL(q, C)$. The uniqueness of g is obvious since $g = h^{-1}$. This proves part (i) of the proposition.

From the uniqueness of g in part (i), we deduce immediately that the $\{g_{hk}\}$ given in part (ii) satisfy the cocycle condition, and form therefore a system of transition functions for a holomorphic vector bundle $\nu \to V$. Let $\{g'_{hk}\}$ denotes the system of transition functions arising from another system $\{f'_h\}$ of reduced equations for V (with the same open covering $\{U_h\}$ for the moment). From part (i), there exists a family $\{\tilde{g}_h\}$ such that $f_h =< \tilde{g}_h, f'_h >$. Denoting by $\{g_h\}$ the induced family on V, the uniqueness in part (i) implies that the two cocycles $\{g_{hk}\}$ and $\{g'_{hk}\}$ differ by the coboundary of $\{g_h\}$, and therefore define isomorphic bundles. If the coverings are different, we can use a common refinement to both coverings, for coming back to the case of the identical coverings.

Notice that the sections σ of ν may be identified with the families $\{\sigma_h : U_h \to C^q\}_h$ of maps such that $\sigma_h =< g_{hk}, \sigma_k >$ on $V \cap U_h \cap U_k$. On the other hand, there we get also $df_h =< g_{hk}, df_k >$. Therefore the family $\{df_h : T_C(W)|_{V \cap U_h} \to C^q\}$ defines a bundle map $\pi : T_C(W)|_V \to \nu$. Furthermore, the kernel of df_h on the regular part of $V \cap U_h$ is exactly the tangent space to this regular part. This achieves the proof of part (iii).

By continuity and reducing the open sets U_h to smaller ones if necessary, we may assume that the functions $\tilde{g}_{hk}$ themselves take values in $GL(q, C)$. However it is not clear that the cocycle condition remains true off V. This justifies the following definition: an LCI subvariety V of W will be called a "strong" local complete intersection (shortly SLCI), if there exists a C^∞ vector bundle $\tilde{\nu} \to U$, defined over some neighborhood U of V in W, whose restriction to V carries a holomorphic bundle structure compatible with the ambient C^∞ structure and is equal to ν. The last condition implies that in a neighborhood of every point of V, $\tilde{\nu}$ admits a C^∞ trivialization whose restriction to V is holomorphic.

If V is an LCI, the holomorphic bundle ν is trivial on $V \cap U_h$, and there is a trivialization which, on the regular part of $V \cap U_h$, is given by $\pi(\frac{\partial}{\partial f_{h,1}}), \dots, \pi(\frac{\partial}{\partial f_{h,q}})$ taking the components $f_{h,\lambda}$ $(1 \le \lambda \le q)$ of f_h as a part of a local chart on W. We call it the "trivialization associated" to

f_h. If, moreover, V is an SLCI with a C^∞ extension $\tilde{\nu}$ of ν, choosing a smaller U_h if necessary, there is a C^∞ trivialization of $\tilde{\nu}$ on U_h extending the trivialization associated to f_h.

Remarks. 1) Notice that the singular foliations $df_h = 0$ on U_h and $df_k = 0$ on U_k do not coincide in general on $U_h \cap U_k$.

2) Let $\mathcal{O}_W$ denote the sheaf of germs of holomorphic functions on W, and $\mathcal{I}$ the sheaf of ideals defining the subvariety V in W. Thus $\mathcal{O}_V = \mathcal{O}_W/\mathcal{I}$ is the sheaf of holomorphic functions on V. Denoting by $\Omega_W = \mathcal{O}_W(T_C^*(W))$ the cotangent sheaf of W, we define, as usual, the cotangent sheaf Ω_V of V to be the quotient of $\Omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_V$ by the image of the morphism $\mathcal{I}/\mathcal{I}^2 \to \Omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_V$ given by assigning $df \otimes 1$ to the class of f. Setting $\Theta_W = \mathcal{O}_W(T_C(W))$ and $\Theta_V = \mathcal{H}om_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, we have the exact sequence

$$0 \to \Theta_V \to \Theta_W \otimes_{\mathcal{O}_W} \mathcal{O}_V \to \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_V).$$

If V is an LCI, then the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free, and the sheaf of germs of holomorphic sections of the bundle $\nu \to V$ is identified with $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_V)$. Furthermore, the bundle map $\pi : T_C(W)|_V \to \nu$ corresponds to the third morphism in the above sequence. If f is a reduced local defining function for V, the classes of the components $f_1, \ldots, f_q$ of f in $\mathcal{I}/\mathcal{I}^2$ form a basis (over $\mathcal{O}_V$), and the trivialization of ν associated to f corresponds to its dual basis.

3) We do not know if LCI implies automatically SLCI. In fact, taking a regular neighborhood U of V and using the fact that the classification of continuous vector bundles and that of C^∞ vector bundles coincide on (paracompact) C^∞ manifolds, we see that there exists a C^∞ vector bundle $\tilde{\nu}$ on U such that $\tilde{\nu}|_V$ is isomorphic to ν as a continuous bundle. However, it is not clear if $\tilde{\nu}|_V$ carries a holomorphic bundle structure which is isomorphic to ν and compatible with the ambient C^∞ structure. Note that there are many examples of SLCI.

Example 1. If V is a non-singular subvariety (submanifold) of W, then clearly it is an LCI and moreover an SLCI. In fact let U be a tubular neighborhood of V with C^∞ projection $\rho : U \to V$. Then $\tilde{\nu} = \rho^*\nu$ is an extension of ν with the desired properties.

Example 2. Any hypersurface V of W (subvariety of pure complex codimension 1) is an SLCI. In fact, if we set $\tilde{g}_{hk} = f_h/f_k$, where $\{f_h\}$ denotes a family of reduced local defining functions, then the system

$\{\tilde{g}_{hk}\}$ satisfies the cocycle condition and defines a holomorphic extension $\tilde{\nu}$ of ν on the union of the domains U_h of f_h, which may be assumed to be W. Note that the collection $\{f_h\}$ defines a global section of $\tilde{\nu}$ non-vanishing away from V.

Example 3. Any algebraic set V in $W = CP^n$ which is globally a complete intersection is also an SLCI. In fact, denote by $[X_0, X_1, ..., X_n]$ homogeneous coordinates in CP^n and by $F_1, F_2, ..., F_q$ homogeneous polynomials in the variables $(X_0, X_1, ..., X_n)$ of respective degrees $d_1, d_2, ..., d_q$ such that V has pure complex codimension q, and is defined by the q equations $F_\lambda = 0$ $(1 \leq \lambda \leq q)$. In the affine open subset U_i of CP^n defined by $X_i \neq 0$, $V \cap U_i$ has for equation with respect to the affine coordinates $(\frac{X_j}{X_i})_{j,j\neq i}$: $\frac{1}{(X_i)^{d_\lambda}} F_\lambda = 0, (1 \leq \lambda \leq q)$. Therefore, on $U_i \cap U_j$ the change of equations $\tilde{g}_{ij}$ is equal to the diagonal $q \times q$ matrix $\left(\frac{X_j}{X_i}\right)^{d_1}, ..., \left(\frac{X_j}{X_i}\right)^{d_q}$. (In fact, in this case, it is not necessary to assume that the components $\frac{1}{(X_i)^{d_\lambda}} F_\lambda$ $(1 \leq \lambda \leq q)$ generate the ideal $I(V \cap U_i)$!) Denoting by $\check{L} \to CP^n$ the hyperplane bundle (dual of the tautological bundle), $\tilde{\nu}$ is defined on the whole CP^n by the formula

$$\tilde{\nu} = \oplus_{\lambda=1}^{q} (\check{L})^{\otimes d_\lambda}.$$

Hence: $1 + c_1(\tilde{\nu}) + \cdots + c_q(\tilde{\nu}) = \Pi_{\lambda=1}^{q}(1 + d_\lambda c)$, with $c = c_1(\check{L})$.

Example 4. In general, let $\tilde{\nu}$ be a holomorphic vector bundle of rank q over W, and V the subvariety of W defined by a holomorphic section σ of $\tilde{\nu}$. Suppose σ is a regular section, i.e., a section such that, at each point of V, the germs of its components $(f_1, \ldots, f_q)$ with respect to a local (holomorphic) trivialization of $\tilde{\nu}$ near the point form a regular sequence; in fact, this is the case if and only if the codimension of V is q. Then V is an LCI, locally defined by $f_1 = \cdots = f_q = 0$. Moreover it is an SLCI with $\tilde{\nu}$ itself a holomorphic extension of ν. (We assume that V is reduced and irreducible, to be consistent with the definition in the beginning of this section.)

3. Statement of Theorems 1, 1' and 2

Assume from now on that the subvariety V is invariant by a holomorphic vector field with singularities X_0 on U, a neighborhood of V in W. Note that, by Proposition 1 (iii), any C^∞ section σ of ν over the regular part of V may be written as $\sigma = \pi(Y)$ for some section Y of $T_C(W)|_V$. Let θ_{X_0} be the C-linear operator defined for any section $\pi(Y)$ of ν over the regular part of V by $\theta_{X_0}(\pi(Y)) = \pi([X_0, \tilde{Y}]|_V)$, $\tilde{Y}$ denoting some local extension of Y near V.

In case V is an LCI, let $f_h = 0$ be a local reduced equation of V on U_h. Since V is invariant by X_0, each component $(df_h(X_0))_\lambda$ $(1 \le \lambda \le q)$ of the derivative $df_h(X_0)$ has to vanish on $V \cap U_h$, and must be therefore a linear combination with holomorphic coefficients of the components $(f_h)_\lambda$ of f_h. Thus there exists a $q \times q$ matrix $\tilde{C}_h$ with holomorphic entries such that $df_h(X_0) = <\tilde{C}_h, f_h>$. Denote by $C_h = (C^\mu_{h,\lambda})$ the restriction of $\tilde{C}_h$ to $V \cap U_h$.

Lemma 1.

(i) *$\theta_{X_0}(\pi(Y))$ depends only on $\pi(Y)$, neither on Y nor on $\tilde{Y}$.*

(ii) *$\theta_{X_0}(u\sigma) = u\theta_{X_0}(\sigma) + (X_0.u)\sigma$, for any C^∞ function u on $V -$ Sing (V).*

(iii) *If V is an LCI, and $f_h = 0$ a local reduced equation, denoting by $(\sigma_1, \dots, \sigma_q)$ the trivialization of ν associated to f_h we have:*

$$\theta_{X_0}(\sigma_\lambda) = -\sum_\mu C^\mu_{h,\lambda}\sigma_\mu.$$

In particular, over the regular part of $V_h = V \cap U_h$, C_h depends only on f_h, not on the choice of $\tilde{C}_h$.

Parts (i) and (ii) of the lemma are proved in [8 (Lemma 2-1, p.220)]. For proving part (iii), take a partition $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\}$ of $\{1, \dots, n\}$ such that $\frac{D(f_{h,1}, \dots, f_{h,q})}{D(z_{j_1}, \dots, z_{j_q})} \neq 0$ near some point of the regular part of V_h. Then, near this point, $(z_{i_1}, \dots, z_{i_p}, f_{h,1}, \dots, f_{h,q})$ is a new system of local coordinates denoted by $(x_1, \dots, x_p, y_1, \dots, y_q)$, the local trivialization of ν associated to f_h becoming $\pi(\frac{\partial}{\partial y_\lambda})$, $(1 \le \lambda \le q)$. Hence if locally $X_0 = \sum_{i=1}^p P_i \frac{\partial}{\partial x_i} + \sum_{\mu=1}^q Q_\mu \frac{\partial}{\partial y_\mu}$, then $X_0.f_{h,\mu} = X_0.y_\mu = Q_\mu = \sum_{\lambda=1}^q y_\lambda \tilde{C}^\mu_{h,\lambda}$, and hence, $C^\mu_{h,\lambda} = \frac{\partial Q_\mu}{\partial y_\lambda}|_{y=0}$. On the other hand, $\pi([X_0, \frac{\partial}{\partial y_\lambda}]|_V) = -\sum_{\mu=1}^q \left(\frac{\partial Q_\mu}{\partial y_\lambda}|_{y=0}\right)\pi(\frac{\partial}{\partial y_\mu})$, which proves part (iii) of the lemma.

We denote by Σ the set (Sing $(X_0) \cap V) \cup$ Sing (V) and by $(\Sigma_\alpha)_\alpha$ its connected components. Recall that a singular point of X_0 is either a point where X_0 is not defined, or a point where it vanishes. Now assume Σ_α to be compact, and denote by U_α an open neighborhood of Σ_α in W. We set $V_\alpha = V \cap U_\alpha$. We shall assume furthermore that $U_\alpha \cap U_\beta = \emptyset$, for $\alpha \neq \beta$. Thus, in particular, $V_\alpha - \Sigma_\alpha$ is in the regular part of V. Denote by $\tilde{\mathcal{T}}_\alpha$ a compact real manifold with boundary, of real dimension $2n$, included in U_α, such that Σ_α is in the interior of $\tilde{\mathcal{T}}_\alpha$ and that its boundary $\partial\tilde{\mathcal{T}}_\alpha$ is transverse to $V - \Sigma$. Put $\mathcal{T}_\alpha = \tilde{\mathcal{T}}_\alpha \cap V$, $\partial\mathcal{T}_\alpha = \partial\tilde{\mathcal{T}}_\alpha \cap (V - \Sigma)$.

Assume the following:

(i) U_α is included in the domain of a local holomorphic chart $(z_1, \dots, z_n)$ of W,

(ii) U_α is one of the U_h's above, the index α being one of the indices h. (Write f_α and C_α for the corresponding terms).

Let

$$X_0|_{U_\alpha} = \sum_{i=1}^{n} A_i(z_1, \dots, z_n)\frac{\partial}{\partial z_i}.$$

Denote by $\mathcal{V}_i$ $(1 \leq i \leq n)$ the open set of points m in $\partial\mathcal{T}_\alpha$ such that $A_i(m) \neq 0$. These open sets $\mathcal{V}_i$ constitute an open covering $\mathcal{V}$ of $\partial\mathcal{T}_\alpha$. Let $\mathcal{U}$ be any subcovering of $\mathcal{V}$. (Such a $\mathcal{U}$ always exists: take for instance $\mathcal{V}$ itself; see also the particular cases 2 and 3 below). We will denote by (R_i), $(1 \leq i \leq n)$ any system of "honey-cells" adapted to this covering $\mathcal{U}$ (see the definition in [8 (section 1)], under the name of "système d'alvéoles"). For instance, if the real hypersurfaces $|A_i| = |A_j|$ $(i \neq j)$ in U_α are in general position, we may take for R_i the cell defined by $|A_i| \geq |A_j|$ for all $j, j \neq i, \mathcal{V}_j \in \mathcal{U}$.

Denote by $\mathcal{M}$ the set of multiindices $u = (u_1, u_2, \dots, u_p)$ such that $1 \leq u_1 < u_2 < \dots < u_p \leq n$, and by $\mathcal{M}(\mathcal{U})$ the subset of those such that $\mathcal{V}_{u_j} \in \mathcal{U}$ and $\cap_{j=1}^{p}\mathcal{V}_{u_j}$ be not empty (that is the set of p simplices in the "nerve" of $\mathcal{U}$). For any $u \in \mathcal{M}(\mathcal{U})$, define $R_u = R_{u_1 u_2 \dots u_p} = \cap_{j=1}^{p} R_{u_j}$, oriented as in section 1 of [8].

Let $\varphi \in (Z[c_1, \dots, c_q])^{2p}$ be a Chern polynomial having integral coefficients with respect to the Chern classes, and defining a characteristic class of dimension $2p$.

Theorem 1. *Assume V to be LCI. Define*

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = (-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(-C_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

(i) $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ does not depend on the various choices of $(z_1, \dots, z_n)$, $\mathcal{U}, \tilde{\mathcal{T}}_\alpha, f_\alpha, \tilde{C}_\alpha, R_i$, and depends only on the foliation $\mathcal{F}$ defined by

X_0, but not on X_0 itself.

(ii) Assume furthermore V to be compact. $\sum_\alpha I_\alpha(\mathcal{F}, V, \varphi, \nu)$ is then an integer.

(iii) This integer depends only on V and φ , but not on $\mathcal{F}$; it is equal to the evaluation $< \varphi(\nu), V >$ of $\varphi(\nu)$ on the fundamental class $[V]$ of V.

Remark. The index above depends obviously only on $\mathcal{F}$ and not on X_0. If we take uX_0 instead of X_0 u denoting some holomorphic non vanishing function on U, then each A_i is multiplied by $u|_V$, the matrix C_α also, and the term under integration does not change. In fact, we could write the theorem for a foliation $\mathcal{F}$ with singularities, defined only locally by a holomorphic vector field but not necessarily globally.

Particular cases. 1) For $p = q = 1$, $I_\alpha(\mathcal{F}, V, c_1, \nu)$ coincides with the index defined in [9] by A. Lins Neto, if V_α is a locally irreducible curve. For a possibly (locally) reducible V_α, it coincides with the one in [12] (notice that the sum of the indices of Lins Neto over the irreducible components is different from the above index: see [12] (1.3) Remark 1° and (1.4) Proposition). In fact, in this case, the 1-forms $\frac{dz_1}{A_1}$ and $\frac{dz_2}{A_2}$ coincide over $\mathcal{V}_1 \cap \mathcal{V}_2$ and glue therefore together, defining a 1-form η_α on $\partial\mathcal{T}_\alpha$, while $X_0.f_\alpha$ may be written $g_\alpha f_\alpha$ for some holomorphic function g_α. The formula of Theorem 1 becomes now:

$$I_\alpha(\mathcal{F}, V, c_1, \nu) = \frac{-1}{2i\pi}\left[\int_{R_1}(-g_\alpha)\eta_\alpha + \int_{R_2}(-g_\alpha)\eta_\alpha\right] = \frac{1}{2i\pi}\int_{\partial\mathcal{T}_\alpha} g_\alpha\eta_\alpha.$$

On the other hand, when f is irreducible, if $k\omega = \bar{h}.df + f\bar{\alpha}$ according to the notation of [9 (p.198)] (up to the bars for avoiding confusions with our notations), his index is then equal to $\frac{-1}{2i\pi}\int_{\partial\mathcal{T}_\alpha}\frac{\bar{\alpha}}{\bar{h}}$. But $\frac{-\bar{\alpha}}{\bar{h}}$ and $g_\alpha\eta_\alpha$ are equal on $\partial\mathcal{T}_\alpha$, because they both take the same value g_α when applied to the restriction of X_0, Q.E.D. See (1.1) Lemma and (1.2) in [12], when f is possibly reducible. This coincidence is also obvious from Theorem 2 and the remark below. Thus the above Theorem 1 may be seen as a generalization of Theorems A and C of [9] and Theorem (2.1) of [12]. In particular, since the sum of our indices is the self-intersection number of the curve V, the integer $3dg(S) - \chi(S) + \sum_B \mu(B)$, lying in Theorem A of [9], is equal to $dg(S)^2$, if the curve S is locally irreducible at each of its singular points. In general, the integer is different from $dg(S)^2$ (see Theorems (2.1) and (2.5) in

[12], in fact, $dg(S)^2$ is equal to $3dg(S)-\chi(S)+\sum_p c_p(S)$ by the adjunction formula, where, denoting by $B_1,\dots,B_r$ the local branches of S at a singular point p, $c_p(S)=\mu_p(S)+r-1=\sum_{i=1}^r \mu(B_i)+\sum_{i\neq j}(B_i\cdot B_j))$.

More generally, for $p=1$ and any q, there exists a 1-form η_α on $\partial\mathcal{T}_\alpha$, the restriction of which to each $\mathcal{V}_i$ being equal to $\frac{dz_i}{A_i}$. Then, still defining g_α by the same formula $X_0.f_\alpha = g_\alpha f_\alpha$, the formula of Theorem 1 becomes:

$$I_\alpha(\mathcal{F},V,c_1,\nu)=\frac{1}{2i\pi}\int_{\partial\mathcal{T}_\alpha} g_\alpha\eta_\alpha.$$

2) When Σ_α is in the regular part of V, we may take a local chart $(z_1,\dots,z_n)=(x_1,\dots,x_p,y_1,\dots,y_q)$ such that $f_\lambda=y_\lambda$ for any $\lambda=1,\dots,q$. Then $A_{p+\lambda}$ vanishes on V_α, in such a way that all open sets $\mathcal{V}_{p+\lambda}$ are empty, and that we may take $\mathcal{U}=\mathcal{V}_1,\dots,\mathcal{V}_p$: Thus, $u=\{1,\dots,p\}$ is the unique element of $\mathcal{M}(\mathcal{U})$. On the other hand, $C^\mu_{\alpha,\lambda}$ and $\frac{\partial A_{p+\mu}}{\partial y_\lambda}$ are equal on V_α. We recover therefore the formula of Theorem 1 in [8], writing $I_\alpha(\mathcal{F},V,\varphi,\nu)$ as a Grothendieck residue. Note that there are some sign errors in [8]. On the third line of p.237, the factor $(-1)^{[\frac{p}{2}]}$ should be omitted, in Théorème 1 of p.217, the integral giving the residue should be multiplied by $(-1)^{p+[\frac{p}{2}]}=(-1)^{[\frac{p+1}{2}]}$ instead of $(-1)^p$, and in Théorème 1' of p.233, the integral should be multiplied by $(-1)^{[\frac{p}{2}]}$.

3) Assume that Σ_α consists of a point m_α isolated in V, and that X_0 is meromorphic near m_α (thus X_0 has a zero, a pole or both at m_α). Then, we have the following.

Theorem 2. *There exists a local holomorphic chart $(z_1,\dots,z_n)$ near m_α in W, such that $\mathcal{V}_1,\mathcal{V}_2,\dots,\mathcal{V}_p$ cover $\partial\mathcal{T}_\alpha$ ($p=dim_C V$).*

For this covering $\mathcal{U}$, $\mathcal{M}(\mathcal{U})$ has a unique element $u_0=\{1,\dots,p\}$. Writing R instead of R_{u_0}, the formula of Theorem 1 becomes now:

$$I_\alpha(\mathcal{F},V,\varphi,\nu)=(-1)^{[\frac{p}{2}]}\int_R \frac{\varphi(-C_\alpha)dz_1\wedge dz_2\wedge\dots\wedge dz_p}{\prod_{i=1}^p A_i}.$$

Proof. Let us write $X_0=\sum_{i=1}^n A_i\frac{\partial}{\partial z_i}$, $A_i=\frac{P_i}{Q_i}$ with P_i and Q_i holomorphic near m_α. We think of P_i and Q_i as being in the ring $\mathcal{O}_n$ of germs of holomorphic functions at the origin O in C^n, and assume that they are relatively prime for each i. Let Q be the least common multiple of the Q_i's. Then QX_0 is a holomorphic vector field leaving V invariant.

Lemma 2. *The holomorphic vector field QX_0 has an isolated zero at m_α on V.*

In fact suppose QX_0 had a non-isolated zero at m_α on V, and let V' be a positive dimensional irreducible subvariety of V containing m_α and be contained in the zero set of QX_0. For each i, we write $Q = Q_i Q_i'$, where $Q_1', \dots, Q_n'$ have no common factors. Since $QX_0 = \sum_{i=1}^n P_i Q_i' \frac{\partial}{\partial z_i}$, the functions $P_i Q_i'$ are all in the defining ideal $I(V')$ of V'. Hence, since $I(V')$ is prime and X_0 is non-zero away from m_α, there exists i_0 such that $Q_{i_0}' \in I(V')$. Thus there is a prime factor P of Q_{i_0}' such that $P \in I(V')$. Now, since $Q_i Q_i' = Q = Q_{i_0} Q_{i_0}'$, P is a factor of $Q_i Q_i'$ for any i. On the other hand, since the pole of X_0 is the union of the zero sets of the Q_i's, we have $Q_i \notin I(V')$, by the assumption that the pole of X_0 is at most isolated on V. Therefore, P must be a factor of Q_i' for all i. This contradicts the fact that the Q_i''s have no common factors, and the lemma is proved.

In the above situation, since the zero set of $P_i Q_i'$ is not smaller than that of P_i, it suffices to prove the proposition for vector fields holomorphic near m_α. Note that the index of X_0 at m_α is equal to that of QX_0, and also that if X_0 has an isolated pole on V, then V is in fact 1-dimensional, since the pole of X_0 has codimension 1 in the ambiant space and in V.

In what follows, for an ideal I in the ring $\mathcal{O}_n$, we denote by ht I its height and by $V(I)$ the (germ of) the analytic set defined by I. Thus ht I = codim $V(I)$. Also, for germs $a_1, \dots, a_r$ in $\mathcal{O}_n$, we denote the ideal generated by them by $(a_1, \dots, a_r)$.

Lemma 3. *Let $A_1, \dots, A_n, f_1, \dots, f_q$ be germs in $\mathcal{O}_n$, $n = p + q$, with* $\operatorname{ht}(f_1, \dots, f_q) = q$ *and* $\operatorname{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$. *Then there exist germs $A_1', \dots, A_p'$ in $\mathcal{O}_n$ such that:*

(i) *$A_1', \dots, A_p'$ are linear combinations of $A_1, \dots, A_n$ with C coefficients,*

(ii) $\operatorname{ht}(A_1', \dots, A_p', f_1, \dots, f_q) = n$.

Since $\operatorname{ht}(f_1, \dots, f_q) = q$, it suffices to show the following for $r = 1, \dots, p$:

(*) if $A_1', \dots, A_{r-1}'$ are linear combinations of $A_1, \dots, A_n$ with C coefficients with $\operatorname{ht}(A_1', \dots, A_{r-1}', f_1, \dots, f_q) = r - 1 + q$, then there exists A_r' which is a linear combination of $A_1, \dots, A_n$ with C coefficients and $\operatorname{ht}(A_1', \dots, A_r', f_1, \dots, f_q) = r + q$.

To show this, let $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q) = V_1 \cup \dots \cup V_s$ be the irreducible decomposition of $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q)$. Since $(A_1, \dots, A_n, f_1, \dots, f_q) = n$, for any point x in $V(A'_1, \dots, A'_{r-1}, f_1, \dots, f_q)$ near O but different from O, $\mathrm{ht}(A_1, \dots, A_n, f_1, \dots, f_q) = n$, we see that there exists A_i with $A_i(x) \neq 0$. Hence there exists A'_r which is a linear combination of $A_1, \dots, A_n$ with $V_k \not\subset V(A'_r)$ for $k = 1, \dots, s$, and thus we have

$$V(A'_1, \dots, A'_r, f_1, \dots, f_q) = (V_1 \cap V(A'_r)) \cup \dots \cup (V_s \cap V(A'_r)).$$

Since each V_k is irreducible and $V_k \not\subset V(A'_r)$, $\dim (V_k \cap V(A'_r)) < \dim V_k$. Therefore, we get $\mathrm{ht}\,(A'_1, \dots, A'_r, f_1, \dots, f_q) = r + q$, hence the lemma.

Note that the condition $\mathrm{ht}\,(f_1, \dots, f_q) = q$ means that the variety V defined by $f_1 = \dots = f_q = 0$ is a complete intersection, and the condition $\mathrm{ht}\,(A_1, \dots, A_n, f_1, \dots, f_q) = n$ means that the singularity of the holomorphic vector field $X = \sum_{i=1}^n A_i \frac{\partial}{\partial z_i}$ is isolated in V.

In the above situation, if we choose a suitable coordinate system $(z_1, \dots, z_n)$ in C^n, then we may suppose that $\mathrm{ht}\,(A_1, \dots, A_p, f_1, \dots, f_q) = n$. Hence Theorem 2 follows.

Remarks. 1) Let V_α be defined by $f_\lambda = 0$, $\lambda = 1, \dots, q$. Suppose that V_α is invariant by a holomorphic vector field X_0 (defined everywhere on U_α) and that Σ_α is an isolated point m_α in V_α. Then as is shown above, there exists a holomorphic chart $(z_1, \dots, z_n)$ near m_α such that when we write $X_0 = \sum_{i=1}^n A_i \frac{\partial}{\partial z_i}$, $\mathrm{ht}\,(A_1, \dots, A_p, f_1, \dots, f_q) = n$, i.e., $A_1, \dots, A_p, f_1, \dots, f_q$ form a regular sequence. We may set

$$\tilde{\mathcal{T}}_\alpha = \{\, z = (z_1, \dots, z_n) \mid |A_i(z)| \le \varepsilon,\ |f_\lambda(z)| \le \varepsilon,\ i = 1, \dots, p,\ \lambda = 1, \dots, q \,\}.$$

Thus we have $\mathcal{T}_\alpha = \{\, z \mid |A_i(z)| \le \varepsilon,\ f_\lambda(z) = 0 \,\}$ and we may also set

$$R_i = \{\, z \in \partial \mathcal{T}_\alpha \mid |A_i(z)| \ge |A_j(z)| \ \text{ for } \ j \neq i \,\}.$$

Then

$$R = R_{12\cdots p} = \{\, z \mid |A_i(z)| = \varepsilon,\ f_\lambda(z) = 0,\ i = 1, \dots, p,\ \lambda = 1, \dots, q \,\},$$

which is a smooth closed submanifold of real dimensiom p in $\partial \mathcal{T}_\alpha$, the link of the singularity V_α. If we set $\theta_i = \arg A_i(z)$, R is oriented so that the form $(-1)^{[\frac{p}{2}]} d\theta_1 \wedge \cdots \wedge d\theta_p$ is positive. Let $R' = (-1)^{[\frac{p}{2}]} R$ so that $d\theta_1 \wedge \cdots \wedge d\theta_p$ is positive on R'. Then

$$I_\alpha(\mathcal{F}, V, \varphi, \nu) = \int_{R'} \frac{\varphi(-C_\alpha) dz_1 \wedge dz_2 \wedge \ldots \wedge dz_p}{\prod_{i=1}^{p} A_i}.$$

2) Theorems 1, 2 and Theorem 1' below could be extended to the case where we take the set $\mathrm{Sing}(X_0) \cap V$ as Σ.

Now Theorem 1 is a special case of the following Theorem 1'. In general, let V be a subvariety of W, and $E \to V$ a continuous complex vector bundle of rank r such that the restriction of E to the regular part of V is holomorphic and that there exists a C^∞ extension $\tilde{E} \to U$ of E to some neighborhood U of V in W. We shall also assume that there exists a holomorphic action of X_0 on $E|_{V-\Sigma}$ in the sense of Bott [4]; a C-linear operator θ_{x_0} from the space of C^∞ sections of $E|_{V-\Sigma}$ into itself is given, such that

$$\begin{cases} \theta_{x_0}(\sigma) \ \text{ is holomorphic whenever } \ \sigma \ \text{ is holomorphic,} \\ \theta_{x_0}(u\sigma) = (X_0.u)\sigma + u\theta_{x_0}(\sigma) \ \text{ for any } \ C^\infty \ \textit{function } u \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and any section } \ \sigma. \end{cases}$$

In order to state Theorem 1', we further assume that U_α is included in the domain of a local chart and that $\tilde{E}|_{U_\alpha}$ is trivial with a trivialization $(\sigma_1, \ldots, \sigma_r)$ whose restriction to $V_\alpha - \Sigma_\alpha$ is holomorphic. We denote by M_α the $r \times r$ matrix with holomorphic entries $M^b_{\alpha,a} : V_\alpha - \Sigma_\alpha \to C$ such that $\theta_{x_0}(\sigma_a) = \sum_b M^b_{\alpha,a} \sigma_b$. Let $\varphi \in (Z[c_1, \ldots, c_r])^{2p}$ as before.

Theorem 1'. *Define*

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = (-1)^{[\frac{p}{2}]} \sum_{u \in \mathcal{M}(\mathcal{U})} \int_{R_u} \frac{\varphi(M_\alpha) dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

Then the following hold:

(i) $I_\alpha(\theta_{X_0}, V, \varphi, E)$ *does not depend on the various choices of* $(z_1, \ldots, z_n), \mathcal{U}, \tilde{\mathcal{T}}_\alpha, (\sigma_1, \ldots, \sigma_r), R_i$.

(ii) *Assume* V *to be compact :* $\sum_\alpha I_\alpha(\theta_{X_0}, V, \varphi, E)$ *is then an integer.*

(iii) *This integer depends only on* V, φ *and* E, *but not on* X_0 *and* θ_{X_0}. *It is in fact equal to the evaluation* $< \varphi(E), V >$ *of* $\varphi(E)$ *on the fundamental class* $[V]$ *of* V.

Remarks. 1) For Theorem 1', V does not need be SLCI even LCI; this assumption was only useful for being sure that ν and $\tilde{\nu}$ exist in the example 1 below. This is still true, even for Theorem 1, if we have some other reason to know that ν and $\tilde{\nu}$ exist.

2) If V is non-singular, we recover Theorem 1' of [8], some particular cases of which being also in Baum-Bott [1] when $E = T_{C}(V)$, and in Bott [4] when X_0 is nondegenerate along Σ_α.

3) Let V_α be defined by $f_\lambda = 0$, $\lambda = 1, \ldots, q$ and invariant by a holomorphic vector field X_0 (defined everywhere on U_α). Suppose that Σ_α is an isolated point m_α in V_α. Then, as in the previous remark 1), there exists a holomorphic chart $(z_1, \ldots, z_n)$ near m_α such that $A_1, \ldots, A_p, f_1, \ldots, f_q$ form a regular sequence. In this case, we have

$$I_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{R'} \frac{\varphi(M_\alpha) dz_1 \wedge dz_2 \wedge \ldots \wedge dz_p}{\prod_{i=1}^{p} A_i},$$

where

$$R' = \{\, z \mid |A_i(z)| = \varepsilon,\ f_\lambda(z) = 0,\ i = 1, \ldots, p,\ \lambda = 1, \ldots, q \,\},$$

which is oriented so that the form $d\theta_1 \wedge \cdots \wedge d\theta_p$ is positive, where $\theta_i = \arg A_i(z)$.

Example 1. Assume V to be SLCI. Take $E = \nu$, and θ_{x_0} defined as in section 3 above, with $M_\alpha = -C_\alpha$. Then we get Theorem 1 from Theorem 1'. We shall write in this case $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ instead of $I_\alpha(\theta_{X_0}, V, \varphi, \nu)$.

Example 2. Take $E = T_C(W)|_V$, and define $\theta_{x_0}(Y) = [X_0, \tilde{Y}]|_V$, depending only on the vector field Y tangent to W along V, and not on its extension $\tilde{Y}$ to some neighbourhood of V. Then we have $M_\alpha = -\frac{D(A_1, \ldots, A_n)}{D(z_1, \ldots, z_n)}$. The index now is the one defined in section 8 of [8], Theorem 1' (and the above remark 3)) giving a formula for computing it. In this case, we shall write $I_\alpha(X_0, V, \varphi, T_C(W))$ instead of $I_\alpha(\theta_{X_0}, V, \varphi, T_C(W)|_V)$. (Notice that if we replace here X_0 by uX_0 as in Theorem 1, the index is now changing!)

4. Proof of Theorem 1'

We use the notation Δ_ω for the Chern-Weil homomorphism defined by a connection ω, and $\Delta_{\omega_0 \omega_1 \cdots \omega_k}$ for the Bott's operator for iterated differences [3] so that $d \circ \Delta_{\omega_0\omega_1\cdots\omega_k} = \sum_{j=0}^{k}(-1)^j \Delta_{\omega_0\cdots\hat{\omega}_j\cdots\omega_k}$. In particular, $d \circ \Delta_{\omega\omega'} = \Delta_{\omega'} - \Delta_{\omega}$. Thus for $\varphi \in (Z[c_1, \dots, c_r])^{2p}$, $\Delta_{\omega_0\omega_1\cdots\omega_k}(\varphi)$ is a differential form of degree $2p-k$ on the common domain of definition of the connections $\omega_0, \omega_1, \dots, \omega_k$.

We shall say that a connection ω on $E|_{V-\Sigma}$ is *special* relative to θ_{X_0} if it is defined by a derivation law ∇ satisfying:

$$\begin{cases} \nabla_{X_0}\sigma = \theta_{X_0}\sigma & \text{for every section } \sigma \text{ of } E|_{V-\Sigma}, \\ \nabla_{Z}\,\sigma = 0 & \text{for every section } Z \text{ of the anti-holomorphic tangent} \\ & \text{bundle } \bar{T}(V-\Sigma) \text{ of } V-\Sigma \text{ and every holomorphic} \\ & \text{section } \sigma \text{ of } E|_{V-\Sigma}. \end{cases}$$

For special connections, we have the "vanishing theorem" (see Lemma 4 below for more general statement): If ω is special relative to θ_{X_0}, then $\Delta_\omega(\varphi) = 0$.

Let U_0 be a sufficiently small tubular neighborhood of $V - \Sigma$ in W with (C^∞) projection $\rho : U_0 \to V - \Sigma$. Then the C^∞ vector bundles $\tilde{E}|_{U_0}$ and $\rho^*(E|_{V-\Sigma})$ are isomorphic, since their restrictions to $V - \Sigma$ are both equal to $E|_{V-\Sigma}$. We denote by ω the connection on $\tilde{E}|_{U_0}$, which is equivalent to the pull-back of a special connection on $E|_{V-\Sigma}$ by ρ. We give also an arbitrary connection ω_α on $\tilde{E}|_{U_\alpha}$.

Proposition 2. *Let*

$$J_\alpha(\theta_{X_0}, V, \varphi, E) = \int_{\mathcal{T}_\alpha} \Delta_{\omega_\alpha}(\varphi) + \int_{\partial\mathcal{T}_\alpha} \Delta_{\omega_\alpha\omega}(\varphi).$$

Then the following hold:

(i) $J_\alpha(\mathcal{F}, V, \varphi, E)$ *does not depend on the choices of* $\tilde{\mathcal{T}}_\alpha$, ω , ω_α.

(ii) *Assume* V *to be compact* $\sum_\alpha J_\alpha(\theta_{X_0}, V, \varphi, E)$ *is then an integer.*

(iii) *This integer depends only on* V *and* φ , *but not on* $\mathcal{F}$. *It is in fact nothing else but the evaluation* $< \varphi(E), V >$ *of* $\varphi(E)$ *on the fundamental class* $[V]$ *of* V.

Notice that, in Proposition 2, we do not have to assume either that U_α is included in the domain of a local chart, or that $\tilde{E}|_{U_\alpha}$ is trivial.

The proof is similar to that for the first three parts in Theorem 8 of [8], if we replace $\nabla_{X_0} Y = [X_0, Y]$ by $\nabla_{X_0}\sigma = \theta_{X_0}\sigma$.

Theorem 1' (hence Theorem 1) will follow immediately from Proposition 2 above and

Proposition 3. *Suppose that U_α is included in the domain of a local chart and that $\tilde{E}|_{U_\alpha}$ is trivial with a trivialization whose restriction to $V_\alpha - \Sigma_\alpha$ is holomorphic. Then we have*

$$I_\alpha(\theta_{x_0}, V, \varphi, E) = J_\alpha(\theta_{x_0}, V, \varphi, E).$$

In what follows, we fix a trivialization $(\sigma_1, \dots, \sigma_r)$ of $\tilde{E}|_{U_\alpha}$ as in Proposition 3 and compute the matrix M_α in terms of this trivialization. We also choose ω_α equal to a trivial connection ω_0 whose connection form with respect to this trivialization is the matrix 0. Hence, in the formula of Proposition 2, we have

$$J_\alpha(\theta_{x_0}, V, \varphi, E) \;=\; \int_{\partial\mathcal{T}_\alpha} \Delta_{\omega_0\omega}(\varphi).$$

Remarks 1) Notice that the integration of the same expression over only one of the connected components of $\partial\mathcal{T}_\alpha \cap V$ would give the partial index corresponding to the corresponding "sheet" or "branch" through Σ_α.

2) If V is not LCI, we still can define $I_\alpha(\mathcal{F}, V, \varphi, \nu)$ and $J_\alpha(\mathcal{F}, V, \varphi, \nu)$ under the condition that the bundle $\nu|_{V_\alpha - \Sigma_\alpha}$ is trivializable, and conclusion of Proposition 3 will still remain true. But this index will now depend on the choice of the homotopy class of the trivialization. Furthermore, even if this is possible at any point of Σ, the sum of these indices has now no reason to be either an integer or independent of $\mathcal{F}$.

There are three steps in the proof of Proposition 3: 1) We first study the properties of the holomorphic connections ω_i on $E|_{V_i}$, the connection form of which with respect to the given trivialization being $\frac{dz_i}{A_i} M_\alpha$. 2) Then we prove that $\Delta_{\omega_0\omega}(\varphi)$, which is a cocycle on $\partial\mathcal{T}_\alpha$, is cohomologous, when imbedded in the total Čech-de Rham complex $CDR^*(\mathcal{U})$, to the element μ in $CDR^{2p-1}(\mathcal{U})$ defined by

$$\begin{cases} \mu_u = & \Delta_{\omega_0\, \omega_{u_1}\, \omega_{u_2} \dots \omega_{u_p}}(\varphi) \quad \text{for } u \in \mathcal{M}(\mathcal{U}), \\ \mu_I = & 0 \ \text{ for any simplex } I \text{ of dimension } \neq p-1 \text{ in the nerve of } \mathcal{U}. \end{cases}$$

3) Finally, we prove that

$$\mu_u = \frac{\varphi(M_\alpha)\; dz_{u_1} \wedge dz_{u_2} \wedge \dots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

Using integration on $CDR^*(\mathcal{U})$ as recalled in Lemma 6 below, this will achieve the proof of Proposition 3.

First step. Let Ω be an open set in $V_\alpha - \Sigma_\alpha$, Y a holomorphic non-vanishing vector field tangent to Ω, and Γ a holomorphic map from Ω into the space of $r \times r$ matrices with complex entries. A connection $\bar{\omega}$ on $E|_\Omega$ will be said to be "adapted" to (Y, Γ) if its connection form relative to the trivialization $(\sigma_1, \ldots, \sigma_r)$ of $E|_\Omega$, still denoted by $\bar{\omega}$, satisfies:

$$\begin{cases} \bar{\omega}(Y) = \Gamma, \\ \bar{\omega}(Z) = 0 \quad \text{for every section } Z \text{ of } \bar{T}(V_\alpha - \Sigma_\alpha). \end{cases}$$

Hence the restriction to Ω of a "special" connection, such as defined for Proposition 2, is adapted to (X_0, M_α), while the restriction to Ω of the trivial connection ω_0 is adapted to any $(Y, \textit{matrix } 0)$ for Y holomorphic tangent to Ω. From the usual vanishing theorem (Bott [3], Kamber-Tondeur [7]), we deduce the

Lemma 4. *Let* $\dim \varphi = 2p$. *Then the following hold:*

$$\begin{cases} \textit{If } \bar{\omega} \textit{ is adapted to some } (Y, \Gamma),\ \Delta_{\bar{\omega}}(\varphi) = 0. \\ \textit{If } \bar{\omega}_1, \ldots, \bar{\omega}_k \textit{ are adapted to the same } (Y, \Gamma),\ \Delta_{\bar{\omega}_1 \ldots \bar{\omega}_k}(\varphi) = 0. \end{cases}$$

For any q multiindex $I = (1 \leq i_1, i_2, \ldots, i_q \leq n)$, the i_j's being all distinct, define

$$D_I = \det \frac{D(f_1, \ldots, f_q)}{D(z_{i_1}, \ldots, z_{i_q})}.$$

For any $u \in \mathcal{M}$, define the q multiindex $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_q)$ so that $1 \leq \bar{u}_1 < \bar{u}_2 < \ldots < \bar{u}_q \leq n$, and $\{1, 2, \ldots, n\} = \{u_1, u_2, \ldots, u_p\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_q\}$, and by $\Omega_{\bar{u}}$ the open set of points in V_α where $D_{\bar{u}} \neq 0$: $\Omega_{\bar{u}}$ is a union of open sets where the restrictions of the functions $z_{u_1}, \ldots, z_{u_p}$ constitute a system of local coordinates. For any $q+1$ multiindex $I = (1 \leq i_0, i_1, \ldots, i_q \leq n)$, Y_I will denote the holomorphic vector field:

$$Y_I = \sum_{k=0}^{q} (-1)^k D_{I - i_k} \frac{\partial}{\partial z_{i_k}}.$$

Lemma 5.

(i) Y_I *is tangent to* V.

(ii) *For $m \in \mathcal{V}_i$ $(1 \leq i \leq n)$, there exists $u \in \mathcal{M}$ containing i such that $D_{\bar{u}} \neq 0$ at the point m.*

(iii) *For any i $(1 \leq i \leq n)$, the connection $\omega_i = \frac{dz_i}{A_i} M_\alpha$ on $E|_{\mathcal{V}_i}$ satisfies the following condition: for any $u \in \mathcal{M}$ containing i, the restriction of ω_i to $\Omega_{\bar{u}}$ is simultaneously adapted to (X_0, M_α) and any $(Y_{u_j+\bar{u}}, matrix\ 0)$ such that $u_j \neq i$.*

In fact let I be some $q+1$ multi index such that $D_{I-i_k} \neq 0$ at some point m in V for some $i_k \in I$, so that the restrictions $\tilde{z}_i$ to V of the functions z_i constitute, for i belonging to $\{1, 2, \dots, n\} - \{I - i_k\}$ (in particular for $i = i_k$), a system of local coordinates on V near m. But then, the restriction of Y_I to the domain of such a local chart is equal to $(-1)^k D_{I-i_k} \frac{\partial}{\partial \tilde{z}_{i_k}}$ and is therefore tangent to V, hence part (i) of the lemma.

The condition for X_0 to be tangent to V may be written:

$$\sum_{j=1}^{n} A_j (f_\lambda)'_{z_j} = 0 \quad \text{on} \quad V_\alpha \quad \text{for all} \quad \lambda = 1, \dots, q.$$

Hence, if $m \in \mathcal{V}_i$, the q dimensional vector $\left((f_\lambda)'_{z_i}\right)_{\lambda=1,\dots,q}$ is, on V_α, a linear combination of the others $\left((f_\lambda)'_{z_j}\right)_{\lambda=1,\dots,q}$, $(j \neq i)$; D_J must be zero at m for any q multiindex J containing i. But, since $\mathcal{V}_i$ is in the regular part of V, one at least of the D_J must be $\neq 0$; the only possibility is therefore that $i \notin J$ for such an J, hence part (ii) of the lemma.

On $\Omega_{\bar{u}}$, $X_0 = \sum_{j=1}^{p} A_{u_j} \frac{\partial}{\partial \tilde{z}_{u_j}} = \frac{1}{D_{\bar{u}}} \sum_{j=1}^{p} A_{u_j} Y_{u_j+\bar{u}}$ and, on $\mathcal{V}_i \cap \Omega_{\bar{u}}$, the p holomorphic vector fields X_0 and $\left(Y_{u_j+\bar{u}}\right)_{u_j \neq i}$ are linearly independant. The part (iii) of the lemma becomes now obvious to check, since $\mathcal{V}_i$ is covered by the $\Omega_{\bar{u}}$ such that $i \in u$.

Second step. For any k simplex $I = (i_0 \cdots i_k)$ in the nerve of $\mathcal{U}$, write $\Delta_{\omega_0\ \omega\ \omega_I}(\varphi) = \Delta_{\omega_0\ \omega\ \omega_{i_0} \cdots \omega_{i_k}}(\varphi)$, $\Delta_{\omega\ \omega_I}(\varphi) = \Delta_{\omega\ \omega_{i_0} \cdots \omega_{i_k}}(\varphi)$, and $\Delta_{\omega_0\ \omega_I}(\varphi) = \Delta_{\omega_0\ \omega_{i_0} \cdots \omega_{i_k}}(\varphi)$.

Define $\gamma \in CDR^{2p-1}(\mathcal{U})$ as the family $(\gamma_I)_I$ given by $\gamma_I = (-1)^{[\frac{k+1}{2}]} \Delta_{\omega_0\ \omega\ \omega_I}(\varphi)$, where k denotes the dimension $|I|$ of I.

Then, the total differential $D\gamma$ of γ in $CDR^*(\mathcal{U})$ is given by:

$$\begin{aligned}\Big(D\gamma\Big)_I &= (-1)^{[\frac{k+1}{2}]+k}\Big(\Delta_{\omega\ \omega_I}(\varphi) - \Delta_{\omega_0\ \omega_I}(\varphi)\\ &\quad + \textstyle\sum_{\alpha=0}^{k}(-1)^{\alpha}\Delta_{\omega_0\ \omega\ \omega_{I-i_\alpha}}(\varphi)\Big)\\ &\quad + \textstyle\sum_{\alpha=0}^{k}(-1)^{[\frac{k}{2}]+\alpha+1}\Delta_{\omega_0\ \omega\ \omega_{I-i_\alpha}}(\varphi)\\ &= (-1)^{[\frac{k+1}{2}]+k}\Big(\Delta_{\omega\ \omega_I}(\varphi) - \Delta_{\omega_0\ \omega_I}(\varphi)\Big), \text{ for } |I|>0,\\ \Big(D\gamma\Big)_i &= \Delta_{\omega\ \omega_i}(\varphi) - \Delta_{\omega_0\ \omega_i}(\varphi) + \Delta_{\omega_0\ \omega}(\varphi) \text{ for } |I|=0.\end{aligned}$$

But all terms $\Delta_{\omega\ \omega_I}(\varphi)$ vanish because the connections $\omega, \omega_{i_0}, \cdots, \omega_{i_k}$ are all adapted to the same (X_0, M_α), all terms $\Delta_{\omega_0\ \omega_I}(\varphi)$ vanish for $|I| < p-1$ because the connections $\omega_0, \omega_{i_0}, \cdots, \omega_{i_k}$ are all adapted to a same $(Y, matrix\ 0)$, and all terms of $\Big(D\gamma\Big)_I$ vanish for $|I| \geq p$ because $\Delta_{\tilde{\omega}_0\cdots\tilde{\omega}_r}(\varphi)$ is always 0 for any family of $r+1$ connections when $r > p$. Therefore, it remains only: $\Big(D\gamma\Big)_i = \Delta_{\omega_0\ \omega}(\varphi)$ for $I = \{i\}$ of dimension 0, $\Big(D\gamma\Big)_u = -\mu_u$ for $u \in \mathcal{M}(\mathcal{U})$ of dimension $p-1$, all others $\Big(D\gamma\Big)_I$'s being 0. This proves: $D\gamma = \iota\Big(\Delta_{\omega_0\ \omega}(\varphi)\Big) - \mu$, where ι denotes the natural imbedding of the de Rham complex $\Omega^*_{DR}(\partial\mathcal{T}_\alpha)$ into $CDR^*(\mathcal{U})$.

Third step. The set $\mathcal{V}_u$ equal to $\cap_{j=1}^{p}\mathcal{V}_{u_j}$ is included into $\Omega_{\bar{u}}$. In fact, as already seen in Lemma 5, if m belongs to $\mathcal{V}_i$, D_I must be zero when $i \in I$. So if $m \in \mathcal{V}_u$, u is the only possible element v in $\mathcal{M}(\mathcal{U})$ such that $D_v \neq 0$.

For computing $\Delta_{\omega_0\ \omega_{u_1}\cdots\omega_{u_p}}$, we introduce (Bott [3]) the connection $\tilde{\omega}$ on $(E|_{\mathcal{V}_u}) \times \Delta^p \to \mathcal{V}_u \times \Delta^p$, ($\Delta^p$ denoting the p-simplex $0 \leq \sum_{i=1}^{p} t_i \leq 1$, $0 \leq t_i \leq 1$, in R^p), defined by

$$[\tilde{\omega} = \sum_{i=1}^{p} t_i\omega_i + \Big[1 - (\sum_{i=1}^{p} t_i)\Big]\omega_0 = \Big(\sum_{j=1}^{p} \frac{t_j}{A_{u_j}} dz_{u_j}\Big)M_\alpha].$$

The curvature $\tilde{\Omega}$ of this connection is then equal to

$$\tilde{\Omega} = \Big(\sum_{j=1}^{p} dt_j \wedge \frac{1}{A_{u_j}} dz_{u_j}\Big)M_\alpha + \quad (\text{terms without any} \quad dt_k)\ .$$

Therefore, for every polynomial φ in $Chern^{2p}[c_1 \ldots c_n]$,

$$\begin{aligned}\Delta_{\tilde{\omega}}(\varphi) &= p!(-1)^{[\frac{p}{2}]} dt_1 \wedge dt_2 \wedge \cdots \wedge dt_p \wedge \tfrac{\varphi(M_\alpha) dz_{u_1} \wedge \cdots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}\\ &\quad + (\text{terms of degree} < p \quad \text{in} \quad dt_j).\end{aligned}$$

By integration over Δ^p, and using the equality $\int_{\Delta^p} dt_1 \wedge \cdots \wedge dt_p = \frac{1}{p!}$, we get [3 (p.64)]:

$$\Delta_{\omega_0\omega_1\cdots\omega_p}(\varphi) = \frac{\varphi(M_\alpha)\ dz_{u_1} \wedge dz_{u_2} \wedge \ldots \wedge dz_{u_p}}{\prod_{j=1}^{p} A_{u_j}}.$$

This achieves the proof of proposition 3, hence of Theorems 1' and 1, once using:

Lemma 6. *There exists a linear map $L : CDR^{2p-1}(\mathcal{U}) \to C$ with the following properties:*

(i) L vanishes on the total coboundaries $D\Big(CDR^{2p-2}(\mathcal{U})\Big)$,

(ii) L extends simultaneously the integration $\int_{\partial T_\alpha} : \Omega_{DR}^{2p-1}(\partial T_\alpha) \to C$, and the map: $(-1)^{\left[\frac{p}{2}\right]} \sum_{u\in\mathcal{M}(\mathcal{U})} \int_{R_u} : C^{p-1}(\mathcal{U}, \Omega_{DR}^p) \to C$.

Proof. See section 6 of [8].

5. Examples. Let W be the 3-dimensional complex projective space CP^3, with homogeneous coordinates $[X, Y, Z, T]$. Take for V the cone V_l of equation

$$X^l + Y^l + Z^l = 0 \quad (l \text{ being any integer } \geq 1),$$

which has a single isolated singular point $O = [0,0,0,1]$. Denote by U_T, U_Z and U_Y the affine spaces $T \neq 0, Z \neq 0$ and $Y \neq 0$ with respective coordinates $(x = \frac{X}{T}, y = \frac{Y}{T}, z = \frac{Z}{T}), (x' = \frac{X}{Z}, y' = \frac{Y}{Z}, t' = \frac{T}{Z})$ and $(x" = \frac{X"}{Y}, z" = \frac{Z}{Y}, t" = \frac{T}{Y})$. The three open sets U_T, U_Z, U_Y cover V_l since the point $[1,0,0,0]$ does not belong to V_l. The corresponding equations of V_l may be written respectively: $f_T = 0$, $f_Z = 0$, $f_Y = 0$, with: $f_T(x,y,z) = x^l + y^l + z^l$, $f_Z(x',y',t') = x'^l + y'^l + 1$, and $f_Y(x",z",t") = x"^l + z"^l + 1$. The bundle $\tilde{\nu}$ is defined by the cocycle

$$(g_{TZ} = z^l = \frac{1}{t'^l},\ \ g_{TY} = y^l = \frac{1}{t"^l},\ \ g_{ZY} = y'^l = \frac{1}{z"^l}).$$

In general, for a hypersurface V_l of degree l in CP^n ($\dim_C V_l = p = n-1$), we have (see Example 3 in section 2)

$$< (c_1)^p(\nu), V_l >= l^{n-1} \int_{V_l} c^{n-1} = l^n.$$

Also, from $T_C(CP^n) \oplus 1 = (n+1)\check{L}$, we obtain

$$1 + c_1(T_C) + c_2(T_C) + \cdots = (1+c)^{n+1},$$

hence

$$c_1(T_C(CP^n)) = (n+1)c, \quad c_2(T_C(CP^n)) = \frac{(n+1)n}{2}c^2, \ldots.$$

In particular, for $p = 2$, $q = 1$,

$$< (c_1)^2(T_C(CP^3)), V_l >= (3+1)^2 \int_{V_l} c^2 = 16l,$$
$$< c_2(T_C(CP^3)), V_l >= \tfrac{4\cdot 3}{2} \int_{V_l} c^2 = 6l.$$

Example 1. Take for X_0 the extension H to the whole CP^3 of the vector field of infinitesimal homotheties $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ in U_T. (In U_Z and U_Y, H is equal respectively to $-t'\frac{\partial}{\partial t'}$ and $-t"\frac{\partial}{\partial t"}$). This vector field has for singular set the union of $\{O\}$ and of the hyperplane $T = 0$, and Σ has 2 connected components: Σ_1 is the isolated point $\{O\}$, and Σ_2 the curve $(X^l + Y^l + Z^l = 0,\ T = 0)$. Notice however that Σ_2 does not contain any singularity for the foliation $\mathcal{F}$ generated by H, so that we can already assert

$$I_2(\mathcal{F}, V_l, (c_1)^2, \nu) = 0.$$

1) Computation of $I_1(\mathcal{F}, V_l, (c_1)^2, \nu)$ and $I_1(H, V_l, \varphi, T_C(W))$ ($\varphi = (c_1)^2$ or c_2):

For $E = \nu$, $H.f_T = lf_T$ and $M_0 = -C_0$ is the 1×1 constant matrix $(-l)$. For $E = T_C(W)|_V$, $M_0 = -\frac{D(x,y,z)}{D(x,y,z)}$ is equal to the opposite of the 3×3 identity matrix, in such a way that for $E = \nu$, $(c_1)^2(M_0)$ is a constant equal to $\frac{-l^2}{4\pi^2}$, while for $E = T_C(W)|_V$, $\varphi(M_0)$ is also a constant equal to $\frac{-9}{4\pi^2}$ if $\varphi = (c_1)^2$, and $\frac{-3}{4\pi^2}$ if $\varphi = c_2$. (Recall that, c_k applied to some matrix is equal to $(\frac{i}{2\pi})^k$ times the kth elementary symmetric function of the eigenvalues).

We compute the indices in two ways; first directly by the definition in Theorem 1 or 1' and then applying Theorem 2.

(i) Take for $\tilde{\mathcal{T}}$ the ball Sup $(|x|, |y|, |z|) \leq \varepsilon$ for some positive constant ε. Let R_z be the region in the boundary $\partial\mathcal{T}$ defined by $|z| \geq |x|, |z| \geq |y|$, and define R_x and R_y similarly. The index $I_1(\theta_H, V_l, \varphi, E)$ at the origin O is equal in both cases to

$$-\varphi(M_0)\left(\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} + \int_{R_{yz}} \frac{dy}{y} \wedge \frac{dz}{z} + \int_{R_{xz}} \frac{dx}{x} \wedge \frac{dz}{z}\right).$$

On R_{xy}, we may write $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$, and $\frac{dx}{x} \wedge \frac{dy}{y} = -d\theta \wedge d\sigma$, which is positive on R_{xy}. In fact, remember ([8]) the convention about the orientation of R_{xy} by the normal from R_x to R_y. Let us write $x = re^{i\theta}$ and $y = se^{i\sigma}$ on $\mathcal{T}$; then $dr \wedge d\theta \wedge ds \wedge d\sigma$ is positive on $\mathcal{T}$ with r increasing when approaching $\partial\mathcal{T} \cap R_x$, $r = \varepsilon$ and $d\theta \wedge ds \wedge d\sigma$ is positive on R_x with s increasing when approaching the boundary near R_{xy}, in such a way that $-d\theta \wedge d\sigma$ is positive on R_{xy}. But there, we have $z^l = -(x^l + y^l) = -2\varepsilon^l cos\ \frac{l(\sigma-\theta)}{2} e^{i\frac{l(\sigma+\theta)}{2}}$, so that R_{xy} is an l-fold covering of the set of (θ, σ) such that $2\varepsilon^l |cos(\sigma - \theta)| \leq \varepsilon^l$ (because $|z| \leq \varepsilon$ on R_{xy}). It is easy to check that the set of (θ, σ) in the square $[0, 2\pi]^2$ where the previous condition holds is made of l strips, the area of each one being $\frac{2\pi}{3} \times 2\pi = \frac{4\pi^2}{3}$. Then, because of the l sheets of the covering, we get: $\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} = \frac{4l\pi^2}{3}$. The computation is the same for the two others integrals, so that

$$\int_{R_{xy}} \frac{dx}{x} \wedge \frac{dy}{y} + \int_{R_{yz}} \frac{dy}{y} \wedge \frac{dz}{z} + \int_{R_{xz}} \frac{dx}{x} \wedge \frac{dz}{z} = 4l\pi^2.$$

(ii) We observe that, in this case, x, y and f_T form a regular sequence (see Remark 1) after Theorem 2 and Remark 3) after Theorem 1'), and we may take for $\tilde{\mathcal{T}}$ the ball Sup $(|x|, |y|, |f_T|) \leq \varepsilon$. The index $I_1(\theta_H, V_l, \varphi, E)$ at the origin O is equal to

$$\varphi(M_0) \int_{R'} \frac{dx}{x} \wedge \frac{dy}{y},$$

where R' is the 2-submanifold in the boundary $\partial\mathcal{T}$ given by

$$R' = \{\, (x, y, z) \mid |x| = |y| = \varepsilon,\ x^l + y^l + z^l = 0 \,\}.$$

On R', we may write: $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$, and $\frac{dx}{x} \wedge \frac{dy}{y} = -d\theta \wedge d\sigma$, which is negative on R'. But there, we have $z^l = -(x^l + y^l)$, so that R' is an l-fold covering of the set of (θ, σ) in the square $[0, 2\pi]^2$. Thus

$$\int_{R'} \frac{dx}{x} \wedge \frac{dy}{y} = -4l\pi^2.$$

In either way we get:

$$I_1(\mathcal{F}, V_l, (c_1)^2, \nu) = l^3, \quad \text{and}$$

$$I_1(H, V, \varphi, T_C(W)) = \begin{cases} 9l & \text{if } \varphi = (c_1)^2, \\ 3l & \text{if } \varphi = c_2, \end{cases}$$

2) Computation of $I_2(H, V_l, \varphi, T_C(W))$.

Since Σ_2 is a smooth compact holomorphic manifold in the regular part of V_l, we may use the Bott's theorem ([4 (p.314)]) for computing the index, under the condition that the infinitesimal action of H on the bundle N normal to Σ_2 in V_l be non degenerate. Since V_l is compact, this action will be of constant type along Σ_2, and the same thing is true for the action $\theta_H|_{\Sigma_2}$ of H. So, it is enough to calculate them for instance along $\Sigma_2 \cap U_Z$. Since $\frac{\partial f_Z}{\partial x'} = lx'^{l-1}$, and $\frac{\partial f_Z}{\partial y'} = ly'^{l-1}$, and because both coordinates x' and y' may not vanish simultaneously over $\Sigma_2 \cap U_Z$, we may assume for instance $x' \neq 0$. Near such a point in $\Sigma_2 \cap U_Z$, we may replace the coordinates (x', y', t') by $(u = f_Z(x', y', t'), v = y', w = t')$, so that V_l has now $u = 0$ for local equation, while Σ_2 is now locally defined by $u = 0, w = 0$. The bundle N is generated by $\frac{\partial}{\partial w}$, $H = -w\frac{\partial}{\partial w}$, and $[H, \frac{\partial}{\partial w}] = \frac{\partial}{\partial w}$. Therefore this action, represented by the constant 1×1 matrix $(+1)$, is effectively nondegenerate. On the other hand, ν is generated by $\frac{\partial}{\partial u}$, so that $[H, \frac{\partial}{\partial u}] = 0$, while the third bracket $[H, \frac{\partial}{\partial v}]$ being also 0, the action $\theta_H|_{\Sigma_2}$ on $T_C(W)$ will be represented by the constant matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote a, b, c the formal classes such that the kth Chern class of W is equal to the kth elementary symmetric function of a, b, c. After Bott, we have

$$I_2(H, V_l, \varphi, T_C(W)) = < \frac{\hat{\varphi}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c+1 \end{pmatrix}}{1 + c_1(N)}, \Sigma_2 >,$$

where $\hat{\varphi}$ denotes $(a + b + c + 1)^2$ for $\varphi = (c_1)^2$, and $ab + (a + b)(c + 1)$ for $\varphi = c_2$. Hence

$$I_2(H, V, \varphi, T_C(W)) = \begin{cases} < 2c_1(T_C(W)) - c_1(N), \Sigma_2 >, & \text{for } (c_1)^2, \\ \text{and } < a + b, \Sigma_2 > & \text{for } c_2. \end{cases}$$

Notice that N coincides with the restriction to Σ_2 of the hyperplane bundle $\check{L} \to CP^2$ after identification of CP^2 with the hyperplane $T = 0$ in CP^3, while $T_C(W)$ is stably equivalent to $4\check{L}$, and $(a+b)|_{CP^2} = c_1(CP^2) = 3c_1(\check{L})$. We get therefore $7 < c_1(\check{L}), \Sigma_2 >= 7l$ for $(c_1)^2$, and $3 < c_1(\check{L}), \Sigma_2 >= 3l$ for c_2.

Finally, we recover

$$< (c_1)^2(\nu), V_l >= l^3 + 0 = l^3,$$

$$< (c_1)^2(T_C(W)), V_l >= 9l + 7l = 16l,$$

$$< c_2(T_C(W)), V_l >= 3l + 3l = 6l.$$

In particular, for $l = 2$,

$$< (c_1)^2(\nu), V_2 >= 8, \quad \text{and}$$

$$< (c_1)^2(T_C(W)), V_2 >= 32, \ < c_2(T_C(W)), V_2 >= 12.$$

Example 2. Take $l = 2$, with now for X_0 the extension $\mathcal{R}$ to the whole CP^3 of the vector field of infinetisimal "complex rotations" $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ in U_T.

In U_Z (resp. in U_Y), $\mathcal{R}$ may be written as $y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}$ (resp. $(x"^2 + 1)\frac{\partial}{\partial x"} + x"z"\frac{\partial}{\partial z"} + x"t"\frac{\partial}{\partial t"}$). Now Σ is made of 3 isolated points: $m_1 = [0,0,0,1]$,
$m_2 = [i,1,0,0]$ and $m_3 = [-i,1,0,0]$. Notice that V_2 is regular at m_2 and m_3. We have $\mathcal{R}.f_T = 0$, $\mathcal{R}.f_Z = 0$, and $\mathcal{R}.f_Y = 2x"f_Y$, which prove that $\mathcal{R}$ still preserves V, and that $I_1(\mathcal{R}, V, (c_1)^2, \nu) = 0$ since $m_1 \in U_T$.

1) Computation of $I_1(\mathcal{R}, V_2, \varphi, T_C(W))$

In this case, y, $-x$ and f_T form a regular sequence and we may take for $\tilde{\mathcal{T}}$ the ball Sup $(|x|, |y|, |f_T|) \leq \varepsilon$ for some positive constant ε. The index $I_1(\theta_{X_0}, V, \varphi, E)$ at the origin O is then equal to

$$\int_{R'} \varphi(M_1)\frac{dx \wedge dy}{-xy},$$

where R' is the 2-submanifold in the boundary $\partial\mathcal{T}$ given by

$$R' = \{ (x,y,z) \mid |y| = |-x| = \varepsilon, \ x^2 + y^2 + z^2 = 0 \}.$$

If we write $x = \varepsilon e^{i\theta}$, $y = \varepsilon e^{i\sigma}$ on R', then $d\sigma \wedge d\theta$ is positive on R'. Hence we have $\int_{R'} \frac{dx\wedge dy}{-xy} = -8\pi^2$. When $E = T_C(W)|_V$, M_1 is now the matrix $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\varphi(M_1)$ is still a constant, now equal to 0 for $\varphi = (c_1)^2$, and to $\frac{-1}{4\pi^2}$ for $\varphi = c_2$. Then we have, $I_1(\mathcal{F}, V_2, (c_1)^2, \nu) = I_1(X_0, V_2, (c_1)^2, T_C(W)) = 0$, and $I_0(X_0, V_2, c_2, T_C(W)) = 2$.

2) Computation of indices at points m_2 and m_3.

Observe that $\frac{\partial f_Y}{\partial x''} = 2x'' \neq 0$ near these points. Then we may use $(u = f_Y, v = z'', w = t'')$ instead of (x'', z'', t'') as local coordinates, with $\mathcal{R} = x(2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w})$. The tangent space to V is generated by $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$. Since the restriction $x(v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w})$ is nondegenerate at m_2 and m_3, with eigenvalues $(\varepsilon i, \varepsilon i)$ with $\varepsilon = 1$ (resp. -1) at m_2 (resp. m_3), we may use the Bott's formula. The normal bundle ν is generated by $\frac{\partial}{\partial u}$, and the action of R on ν at points m_2 and m_3 is given by the 1×1 matrix $(-2\varepsilon i)$, and

$$I_2(\mathcal{F}, V, (c_1)^2, \nu) = I_3(\mathcal{F}, V, (c_1)^2, \nu) = 4.$$

The action of $\mathcal{R}$ on $T_C(W)$ is given by the matrix $-\varepsilon i \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and

$$I_2(\mathcal{R}, V_2, (c_1)^2, T_C(W)) = I_3(\mathcal{R}, V_2, (c_1)^2, T_C(W)) = 16,$$

$$I_2(\mathcal{R}, V_2, c_2, T_C(W)) = I_3(\mathcal{R}, V_2, c_2, T_C(W)) = 5.$$

We may notice that we still have, as in example 1:

$$\begin{aligned} &< (c_1)^2(\nu), V_2 >= 0 + 4 + 4 = 8, \\ &< (c_1)^2(T_C(W)), V_2 >= 0 + 16 + 16 = 32, \\ &< c_2(T_C(W)), V_2 >= 2 + 5 + 5 = 12. \end{aligned}$$

Example 3. Take still $l = 2$, with now for X_0 the linear combination $X_\omega = aH + b\mathcal{R}$ of Examples 1 and 2, where $\omega \in [0, \frac{\pi}{2}[$, $a = \cos\omega$, $b = \sin\omega$, $(a \neq 0)$. In U_T, $X_\omega = a\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right] + b\left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right]$ has only for singular point the origin m_1. In U_Z, $X_\omega = b(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}) - at'\frac{\partial}{\partial t'}$, has no singular point on V_2. In U_Y, $X_\omega = b(x''^2 + 1)\frac{\partial}{\partial x''} +$

$bx"z"\frac{\partial}{\partial z"}) + t"(bx" - a)\frac{\partial}{\partial t"}$ has the same singular points m_2 and m_3 as in Example 2.

1) Computation of indexes at point m_1.

Since $X_\omega.f_T = 2af_T$, the 1×1 matrix C_1 is constant equal to $((-2a))$, so that $(c_1)^2(C_1) = \frac{-a^2}{\pi^2}$.

Write: $A = ax + by$, $B = -bx + ay$ and $C = az$. We have $\frac{D(A,B,C)}{D(x,y,z)} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & a \end{pmatrix}$, and $\varphi(-\frac{D(A,B,C)}{D(x,y,z)})$ is still a constant equal to $\frac{-9a^2}{4\pi^2}$ if $\varphi = (c_1)^2$, and $\frac{-(3a^2+b^2)}{4\pi^2}$ if $\varphi = c_2$.

In this case, A, B and f_T form a regular sequence, and we may take for $\tilde{\mathcal{T}}$ the ball Sup $(|A|, |B|, |f_T|) \leq \varepsilon$ for some positive constant ε. Then the index $I_1(\mathcal{F}, V_2, \varphi, E)$ at the origin O is equal to

$$\varphi(M_1)\int_{R'} \frac{dx \wedge dy}{AB},$$

where R' is the 2-submanifold in the boundary $\partial\mathcal{T}$ given by

$$R' = \{\,(x,y,z) \mid |A| = |B| = \varepsilon,\ x^2 + y^2 + z^2 = 0\,\}.$$

Since $dx \wedge dy = dA \wedge dB$, the integral is computed as in Example 1 to get: $\int_{R'} \frac{dx\wedge dy}{AB} = -8\pi^2$. Thus we have

$$I_1(\mathcal{F}, V_2, \varphi, E) = \begin{cases} 8a^2 \text{ for } E = \nu \text{ and } \varphi = (c_1)^2, \\ 18a^2 \text{ for } E = T_C W \text{ and } \varphi = (c_1)^2, \\ 2(3a^2 + b^2) \text{ for } E = T_C W \text{ and } \varphi = c_2. \end{cases}$$

2) Computation of indices at points m_2 and m_3.

We already observed that $\frac{\partial f_Y}{\partial x"} = 2x" \neq 0$ near these points, so that we may use $(u = f_Y, v = z", w = t")$ instead of $(x", z", t")$ as local coordinates, with $X_\omega = bx"(2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}) + (bx" - a)w\frac{\partial}{\partial w}$. The tangent space to V_2 is generated by $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial w}$. The restriction

$$bx"v\frac{\partial}{\partial v} + (bx" - a)w\frac{\partial}{\partial w}$$

of X_ω to V_2 has for eigenvalues $(b\varepsilon i, b\varepsilon i - a)$ with $\varepsilon = 1$ (resp. -1) at m_2 (resp. m_3). It is therefore nondegenerate at these points, and we may use the Bott's formula.

The normal bundle ν is generated by $\frac{\partial}{\partial u}$, the action of X_ω on ν at points m_2 and m_3 is given by the 1×1 matrix $((-2b\varepsilon i))$, and $I_2(\mathcal{F}, V, (c_1)^2, \nu) = -\frac{4b^2}{ib(ib-a)} = 4b(b-ai)$, while $I_3(\mathcal{F}, V, (c_1)^2, \nu) = 4b(b+ai)$. We recover:

$$< (c_1)^2(\nu), V_2 >= 8a^2 + 4b(b-ai) + 4b(b+ai) = 8.$$

The action of X_ω on $T_C(W)$ has $(-2b\varepsilon i, -b\varepsilon i, -(b\varepsilon i - a))$ for eigenvalues.
$I_2(X_\omega, V_2, (c_1)^2, T_C(W)) = \frac{(4ib-a)^2}{ib(ib-a)} = (16b^2 + 7a^2) - i\frac{a(8b^2-a^2)}{b}$, while $I_3(X_\omega, V_2, (c_1)^2, T_C(W)) = (16b^2 + 7a^2) + i\frac{a(8b^2-a^2)}{b}$. We recover:

$$< (c_1)^2(T_C(W)), V_2 >= 18a^2 + 2(16b^2 + 7a^2) = 32.$$

$I_2(X_\omega, V_2, c_2, T_C(W)) = \frac{2(bi)^2+2bi(bi-a)+bi(bi-a)}{ib(ib-a)} = 5b^2 + 3a^2 - 2iab$, while $I_3(X_\omega, V_2, c_2, T_C(W)) = 5b^2 + 3a^2 + 2iab$. We recover:

$$< c_2(T_C(W)), V_2 >= 2(3a^2 + b^2) + 2(5b^2 + 3a^2) = 12.$$

We may notice, in accordance with the theory, that the indices themselves are not necessarily integers and depend on a, b, contrary to their sum, and also that we recover the values of Example 1 ($l = 2$) for $\omega = 0$, and that of Example 2 for $\omega = \frac{\pi}{2}$. However the calculation for this last case had to be done separately, because we assumed explicitely $C \neq 0$ near m_0 in the calculation of Example 3.

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