J.DIFFERENTIAL GEOMETRY
3 (1995) 515 - 527

INFINITESIMAL RIGIDITY FOR HYPERBOLIC ACTIONS

STEVEN HURDER

1. Statement of results

Let Γ be a finitely-generated group, M a compact manifold and $\varphi: \Gamma \times M \to M$ a C^1-action of Γ on M.

Let $\mathcal{R}(\Gamma, Diff^1(M))$ denote the variety of representations of Γ into $Diff^1(M)$. It is a natural problem to study the local structure of $\mathcal{R}(\Gamma, Diff^1(M))$ in a neighborhood of a given action.

For example, there is a natural "formal tangent space" at the point $[\varphi]$ determined by the action φ, which is given by the 1-cocycles over φ with coefficients in the continuous vector fields on M (cf. Chapter 2, [15]). The 1-coboundaries form a closed subspace of the formal tangent space, and when these two spaces are equal the action is said to be *infinitesimally rigid.*

Every action φ can be perturbed by conjugating it with a diffeomorphism of M, and the set of these conjugates yields a subvariety of $\mathcal{R}(\Gamma, Diff^1(M))$. The action φ is *C^1-locally rigid* if the set of conjugates forms an open neighborhood around $[\varphi]$ – that is, every action φ_1 which is C^1-close to φ on a set of generators of Γ must be C^1-conjugate to φ.

Weil proved that for a representation $\rho: \Gamma \to G$ into a connected Lie group G, infinitesimal rigidity implies rigidity, for the tangent space to the variety of representations is contained in the space of 1-cocycles over the Adjoint representation [20]. The converse is not true: there are rigid representations which are not infinitesimally rigid (cf. proof of Theorem B, [17]; section 2, [15]). For non-isometric group actions on manifolds, only partial results are known connecting infinitesimal rigidity and local rigidity (cf. [1]).

Received July 22, 1993. Supported in part by NSF grant DMS 91-03297.

The purpose of this note is to give a short proof of the infinitesimal rigidity of a hyperbolic group action, whenever the periodic points are dense and the group satisfies a vanishing cohomology condition. Previous results on infinitesimal rigidity for group actions have been obtained by R. Zimmer [21] for cocompact lattices. J. Lewis [10] showed that for $n \geq 7$ and $\Gamma \subset SL(n, Z)$ a subgroup of finite-index, the standard linear action of Γ on T^n is infinitesimally rigid. M. Pollicott proved that the standard action of $SL(3, Z)$ on T^3 is also infinitesimally rigid, using only methods of elementary linear algebra [18]. Both of these latter results follow from our Theorem 1.6 below. The methods used in this paper are based on dynamical systems techniques parallel to those used in [4] to prove deformation rigidity of group actions.

We first establish some basic notation. A set $\Lambda \subseteq M$ is invariant under φ if $\varphi(\gamma)\Lambda = \Lambda$ for all $\gamma \in \Gamma$. We say that φ is hyperbolic on Λ if there exists at least one element, $\gamma_h \in \Gamma$, such that $\varphi(\gamma_h)$ is hyperbolic when restricted to Λ. That is, the restricted tangent bundle admits a continuous direct sum decomposition, $T_\Lambda M \cong E^s \oplus E^-$, which is invariant under $D\varphi(\gamma_h)|\Lambda$ with E^+ uniformly expanding and E^- uniformly contracting. When $\Lambda = M$, the action is said to be *Anosov.*

A C^1-diffeomorphism $f\colon M \to M$ induces a map on continuous vector fields $\sigma \in C^0(T_\Lambda M)$ over a closed invariant set Λ by $f^*\sigma = Df \circ \sigma \circ f^{-1}$.

A 1-cocycle in $\mathcal{Z}^1(\Gamma; C^0(T_\Lambda M)_{\varphi^*})$ assigns to each $\gamma \in \Gamma$ a continuous vector field $\alpha(\gamma) \in C^0(T_\Lambda M)$ satisfying the cocycle condition

$$(1) \qquad \alpha(\gamma_1 \circ \gamma_2) = \alpha(\gamma_1) + \varphi(\gamma_1)^*\alpha(\gamma_2).$$

A 1-cocycle α is a coboundary if there exists a continuous vector field $\tau \in C^0(\Lambda, T_\Lambda M)$ such that

$$(2) \qquad \alpha(\gamma) = \varphi(\gamma)^*\tau - \tau \qquad \text{for all } \gamma \in \Gamma.$$

$H^1(\Gamma; C^0(T_\Lambda M)_{\varphi^*})$ will denote the quotient group of $\mathcal{Z}^1(\Gamma; C^0(T_\Lambda M)_{\varphi^*})$ by the continuous coboundaries. We say φ is *infinitesimally rigid on* Λ if $H^1(\Gamma; C^0(T_\Lambda M)_{\varphi^*})$ is trivial. When $\Lambda = M$ the reference to Λ is omitted.

When φ is a C^∞-action, it induces an action on the smooth vector fields $C^\infty(TM)$. We say that φ is C^∞-*infinitesimally rigid* if the cohomology group $H^1_\infty(\Gamma; C^\infty(TM)_{\varphi^*})$ of smooth φ_*-cocycles modulo *smooth* coboundaries is trivial.

Definition 1.1. A point $x \in M$ is *periodic* for φ if the set $\Gamma(x) = \{\varphi(\gamma)(x) \mid \gamma \in \Gamma\}$ is finite.

Let $\mathcal{P}(\varphi) \subset M$ denote the set of periodic points for φ.

Definition 1.2. A group Γ satisfies the *strong vanishing cohomology (**SVC**) condition* if $H^1(\Gamma; R^N_\rho)$ is trivial for every finite-dimensional representation $\rho\colon \Gamma \to GL(N, R)$.

Note that if Γ has the **SVC** condition, then every finite-index subgroup $\tilde{\Gamma} \subset \Gamma$ also has the **SVC** condition.

We can now state the main results of this note.

Theorem 1.3. *Let $\varphi\colon \Gamma \times M \to M$ be a C^1-action which is hyperbolic for the invariant set $\Lambda \subseteq M$. If Γ satisfies the **SVC** condition and $\mathcal{P}(\varphi) \cap \Lambda$ is dense in Λ, then φ is infinitesimally rigid on Λ.*

This will be proven in section 2.

A finitely-generated group Γ is a said to be a *higher rank lattice* if Γ is a discrete subgroup of a connected semi-simple algebraic R-group G_R, with the R-split rank of each factor of G_R at least 2, G_R has finite center and G_R^0 has no compact factors, and such that G_R/Γ has finite volume. Theorem 2.1 of Margulis [16] implies that every higher rank lattice satisfies the **SVC** condition.

Corollary 1.4. *Let Γ be a higher rank lattice. Then every Anosov C^1-action $\varphi\colon \Gamma \times M \to M$ with dense periodic orbits is infinitesimally rigid.*

Remark 1.5. The action φ^* also preserves the *bounded* vector fields over Λ. The techniques of section 2 also show that $H^1_b(\Gamma; C^b(T_{\mathcal{P}(\varphi)}M)_{\varphi^*})$ is trivial when the hypotheses of Theorem 1.3 are satisfied.

Theorem 1.3 applied to a C^∞-action φ yields that every C^∞-1-cocycle is the coboundary of a continuous vector field τ, but does not address the question whether τ is C^∞. With some additional dynamical hypotheses, Proposition 3.1 establishes the regularity of τ. A linear representation $\rho\colon \Gamma \to SL(n, Z)$ is *maximal Cartan* if there exists an abelian subgroup $\mathcal{A} \subset \Gamma$ generated (not necessarily freely) by elements $\Delta = \{\gamma_1, \dots, \gamma_n\}$ so that each $\rho(\gamma_i)$ is a semi-simple hyperbolic matrix with exactly one eigenvalue of modulus less than one, and the collection of the corresponding eigenvectors $\{\vec{X}_1, \dots, \vec{X}_n\}$ forms a basis for R^n. For example, a subgroup $\Gamma \subset SL(n, Z)$ of finite-index is maximal Cartan (Lemma 7.5, [4]). Corollary 2 of [5] implies that an affine action of a higher rank lattice has dense periodic orbits, which combined with Theorem 1.3 and Proposition 3.1 yields the following general result on

C^∞-infinitesimal rigidity:

Theorem 1.6. *(C^∞-infinitesimal rigidity) Let $\rho\colon \Gamma \to SL(n,Z)$ be a maximal Cartan representation of a higher rank lattice. Then every affine action $\varphi\colon \Gamma \times T^n \to T^n$ with associated linear representation ρ is C^∞-infinitesimally rigid.*

Corollary 1.7. *Let $\Gamma \subset SL(n,Z)$ be a subgroup of finite index for $n \geq 3$. Then every affine action of Γ on T^n associated to the standard action is C^∞-infinitesimally rigid.*

Examples of affine actions without fixed-points are constructed in [5].

2. Infinitesimal rigidity

The proof of Theorem 1.3 follows from two elementary lemmas.

Lemma 2.1. *Assume that Γ satisfies the **SVC** condition. Let $\alpha\colon \Gamma \to Maps(\mathcal{P}(\varphi), T_{\mathcal{P}(\varphi)}M)$ satisfy the cocycle law (1) for the restriction of φ^* to $\mathcal{P}(\varphi)$. Then α is the coboundary of a set map $\tau\colon \mathcal{P}(\varphi) \to T_{\mathcal{P}(\varphi)}M$.*

Proof. It suffices to prove the claim for each restriction $\alpha_x\colon \Gamma \to Maps(\Gamma(x), T_{\Gamma(x)}M)$ to an orbit $\Gamma(x)$ of $x \in \mathcal{P}(\varphi)$. The action of φ^* on $Maps(\Gamma(x), T_{\Gamma(x)}M)$ defines an N-dimensional linear representation of Γ for $N = n\cdot|\Gamma(x)|$, and α_x is a cocycle over this representation. By the **SVC** condition, α_x is the coboundary of some $\tau_x\colon \Gamma(x) \to T_{\Gamma(x)}M$.

Lemma 2.2. *Assume that φ is hyperbolic on Λ, and $\mathcal{P}(\varphi) \cap \Lambda$ is dense in Λ. Let $\alpha\colon \Lambda \to T_\Lambda M$ be a bounded cocycle over φ^* whose restriction to $\mathcal{P}(\varphi)$ is the coboundary of $\tau\colon \mathcal{P}(\varphi) \to T_{\mathcal{P}(\varphi)}M$. Then τ is a bounded map and uniquely determined by α. Moreover, if α is continuous on Λ, then τ admits a continuous extension $\mathcal{T}\colon \Lambda \to T_\Lambda M$ such that α is the coboundary of $\mathcal{T}$.*

Proof. Let $\varphi(\gamma_h)$ be hyperbolic on Λ. Then $\varphi(\gamma_h)^*$ is a hyperbolic automorphism of the Banach space of bounded sections $C^b(T_\Lambda M)$ (cf. Lemma 6.3, [19]) so that

$$(D_{\varphi_*}(\gamma_h) - Id)^{-1}\colon C^b(T_\Lambda M) \to C^b(T_\Lambda M) \tag{3}$$

is a well-defined bounded linear map.

$D_{\varphi_*}(\gamma_h)$ preserves the closed subspace of bounded sections

$C^b(T_{\mathcal{P}(\varphi)}M) \subset C^b(T_\Lambda M)$ which are non-zero only on $\mathcal{P}(\varphi)$. The restriction of $\alpha(\gamma_h)$ to $\mathcal{P}(\varphi)$ defines such a section, so we define the bounded section

$$\tilde{\tau} = (D\varphi_*(\gamma_h) - Id)^{-1}(\alpha(\gamma_h)|\mathcal{P}(\varphi))$$

which obviously satisfies

$$\alpha(\gamma_h)|\mathcal{P}(\varphi) = (D\varphi_*(\gamma_h) - Id)\tilde{\tau}. \tag{4}$$

The functional equation (4) has a unique solution τ_x when restricted to each each periodic orbit $\Gamma(x)$, by the hyperbolicity of $\varphi(\gamma_h)^*$. Both τ_x and the restriction of $\tilde{\tau}$ to $\Gamma(x)$ satisfy this restricted functional equation, hence $\tau = \tilde{\tau}$ on all of $\mathcal{P}(\varphi)$, and τ is a bounded section.

Observe next that $(\varphi(\gamma_h)^* - Id)^{-1}$ preserves the closed subspace of continuous sections $C^0(T_\Lambda M)$.

For, given a continuous section σ, we can write it as a sum $\sigma = \sigma^+ + \sigma^- \in C^0(T_\Lambda M)$ where $\sigma^\pm$ are continuous sections taking values in the unstable and stable subbundles over Λ for $D\varphi(\gamma_h)$, respectively. Then

$$(\varphi(\gamma_h)^* - Id)^{-1}\sigma^- = -\sum_{n=0}^{\infty} \varphi(\gamma_h^n)^*\sigma^-$$

with uniform convergence, so defines a continuous section. Similarly,

$$(\varphi(\gamma_h)^* - Id)^{-1}\sigma^+ = \sum_{n=1}^{\infty} \varphi(\gamma_h^{-n})^*\sigma^+$$

again with uniform convergence so the sum is continuous on Λ. Thus $(\varphi(\gamma_h)^* - Id)^{-1}\sigma$ is continuous on Λ.

Assume that α is continuous on Λ, and we are given a bounded section $\tau\colon \mathcal{P}(\varphi) \to T_{\mathcal{P}(\varphi)}M$ with $\alpha(\gamma)|\mathcal{P}(\varphi) = (\varphi(\gamma)^* - Id)\tau$ for all $\gamma \in \Gamma$. Apply this coboundary identity to the element γ_h then by the above observation, τ is the restriction of the continuous section $\mathcal{T} = (\varphi(\gamma_h)^* - Id)^{-1}\alpha(\gamma_h) \in C^0(T_\Lambda M)$ to the set $\mathcal{P}(\varphi)$.

The continuous section $\mathcal{T}$ satisfies (4) *for all* $\gamma \in \Gamma$ on the dense subset $\mathcal{P}(\varphi) \subset \Lambda$. By the continuity of the actions, the coboundary identity must hold for $\mathcal{T}$ on all of Λ. This concludes the proof of Theorem 1.3.

The reader familiar with the method of proof of deformation rigidity will see the direct analogy between Proposition 3.7 and Corollary 3.8

of [4] and the above two lemmas. This note uses the Banach space fixed-point principle applied to the linear action $D\varphi_*$ to extend a cohomological solution from the periodic orbits to a continuous solution on the full set Λ, resulting in a very direct proof of infinitesimal rigidity.

3. Cocycle regularity for Cartan actions

In this section we establish the regularity of the coboundary τ constructed in section 2. Proofs of regularity tend to be very delicate, as there are surprising counter-examples when the hypotheses are sufficiently weakened. For example, there exists an $n \times n$-hyperbolic matrix with all eigenvalues distinct so that its standard action on T^n admits C^1-perturbations which are topologically conjugate, but not C^1-conjugate, to the standard action [13]. There are also smooth infinitesimal deformations of these algebraic actions which are coboundaries of continuous vector fields but not of smooth vector fields. We will therefore assume that our action admits a Cartan subaction, which is then sufficient for applying the strongest forms of the regularity techniques (cf. Theorem 2.15, [4], and Theorem 4.1, [9]).

The analysis of regularity for a continuous vector field whose coboundary is a smooth infinitesimal deformation turns out to be more subtle than that of establishing the regularity of a topological conjugacy. An infinitesimal deformation represents the derivative to a putative C^∞-deformation $\{\varphi_t \mid 0 \leq t \leq 1\}$ of smooth actions. Thus, the problem is analogous to establishing that a family of smooth conjugacies $H_t \colon M \to M$ between φ_t and φ_0 depends *differentiably* on the parameter. Theorem 2.12, [4] established the *continuous* dependence in the C^∞-category of maps of H_t on the parameter. For the special case of a *maximal Cartan action*, the conjugacies depend C^1 on the parameter [3]. The main result of this section establishes the counterpart to this latter statement for an infinitesimal conjugacy:

Proposition 3.1. *Let $\rho\colon \Gamma \to SL(n, Z)$ be a maximal Cartan representation of a higher rank lattice, $\varphi\colon \Gamma \times T^n \to T^n$ an affine action with associated linear representation ρ, and $\alpha\colon \Gamma \to C^\infty(TT^n)$ a 1-cocycle over φ^*. Then $\tau \in C^0(TT^n)$ which satisfies the coboundary equation (2) must be C^∞.*

Proof. Let $\mathcal{A} \subset \Gamma$ be generated by elements $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ so that each $\rho(\gamma_i)$ is a semi-simple hyperbolic matrix with exactly one eigenvalue of modulus less than one. Let $\vec{X}_i$ be a unit-length contracting eigenvector for $\rho(\gamma_i)$ with eigenvalue $0 < \lambda_i < 1$. Use the basis $\{\vec{X}_1, \dots, \vec{X}_n\}$ of R^n to define a framing of the tangent bundle TT^n, where $\vec{X}_i$ will also denote the parallel vector field which it determines.

Note that for $k \neq \ell$, the maximal Cartan hypotheses implies that

$$\rho(\gamma_k)\vec{X}_\ell = \mu_{k,\ell} \cdot \vec{X}_\ell \quad \text{for} \quad \mu_{k,\ell} > 1.$$

We express α, τ and ρ in terms of the parallel framing $\{\vec{X}_1, \dots, \vec{X}_n\}$ of T^n:

- $A\colon \Gamma \to GL(n, R)$ denotes ρ with respect to the eigenbasis $\{\vec{X}_1, \dots, \vec{X}_n\}$, so that

$$A_i \equiv A(\gamma_i) = \operatorname{diag}(\mu_{i,1}, \dots, \lambda_i, \dots, \mu_{i,n}) \qquad \text{for } 1 \le i \le n.$$

- The differential $D\varphi(\gamma) = A(\gamma)$.
- The coboundary τ can be written as a map $\vec{\tau} = (\tau^1, \dots, \tau^n)\colon T^n \to R^n$ where

$$\tau = \sum_{k=1}^{n} \tau^k \cdot \vec{X}_k \qquad \text{for } C^0\text{-functions} \quad \tau^k\colon T^n \to R.$$

- For each $\gamma \in \Gamma$, the vector field $\alpha(\gamma)$ can be expressed $\vec{\alpha}(\gamma) = (\alpha^1(\gamma), \dots, \alpha^n(\gamma))\colon T^n \to R^n$ where

$$\alpha(\gamma) = \sum_{k=1}^{n} \alpha^k(\gamma) \cdot \vec{X}_k \qquad \text{for } C^\infty\text{-functions} \quad \alpha^k(\gamma)\colon T^n \to R.$$

- The cocycle equation for α becomes

$$\begin{aligned} \sum_{k=1}^{n} \alpha^k(\gamma_1\gamma_2) \cdot \vec{X}_k &= \sum_{k=1}^{n} \alpha^k(\gamma_1) \cdot \vec{X}_k \\ &\quad + \sum_{k,\ell=1}^{n} A(\gamma_1)^k_\ell \cdot \{\alpha^\ell(\gamma_2) \circ \varphi(\gamma_1)^{-1}\} \cdot \vec{X}_k. \end{aligned} \tag{5}$$

- τ satisfies the coboundary equation

$$\sum_{k=1}^{n} \alpha^k(\gamma) \cdot \vec{X}_k = \sum_{k=1}^{n} A^k_\ell \cdot \{\tau^\ell \circ \varphi(\gamma)^{-1}\} \cdot \vec{X}_k - \sum_{k=1}^{n} \tau^k \cdot \vec{X}_k. \tag{6}$$

Set $\varphi_i = \varphi(\gamma_i)$ and $g_i^i = \alpha^i(\gamma_i)$ for $1 \leq i \leq n$. Apply (6) to γ_i and extract the coefficients of $\vec{X}_i$ to obtain the functional equation

$$g_i^i = \lambda_i \cdot \tau^i \circ \varphi_i^{-1} - \tau^i \tag{7}$$

for τ^i, which has the explicit solution

$$\tau^i = -\sum_{k=0}^{\infty} \lambda_i^k \cdot g_i^i \circ \varphi_i^{-k}. \tag{8}$$

The integral curves of each vector field $\vec{X}_\ell$ forms a 1-dimensional linear foliation $\mathcal{F}_\ell$ of T^n, and the collection $\{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ forms a regular C^∞-trellis in the notation of [4]. By Theorem 2.6, [6] the continuous function $\tau^i \colon T^n \to R$ is C^∞ if and only if, for each immersed leaf $\iota \colon R \to T^n$ of $\mathcal{F}_\ell$, the restriction $\tau^i \circ \iota$ is uniformly C^∞ as a function on the line. This is equivalent to proving that for all $1 \leq \ell \leq n$ and all powers $p > 0$, the p-fold iterated partial derivative $\vec{X}_\ell^p(\tau^i)$ exists and is continuous on T^n. For $\ell \neq i$, it is easy to show: apply the chain rule to the absolutely convergent series (8) to calculate

$$\vec{X}_\ell^p(\tau^i) = -\sum_{k=0}^{\infty} \lambda_i^k \, \mu_{i,\ell}^{-pk} \cdot \vec{X}_\ell^p(g_i^i) \circ \varphi_i^{-k},$$

which by the assumption $\mu_{i,\ell} > 1$ is uniformly absolutely convergent on T^n.

The key issue for regularity of τ^i is the existence of the partial derivative $\vec{X}_i(\tau^i)$, as there is no "hyperbolic principle" to guarantee that the formal expression $\vec{X}_i(\tau^i) = -\sum_{\ell=0}^{\infty} \vec{X}_i(g_i^i) \circ \varphi_i^{-1}$ will converge. The approach will be to formulate a functional equation that $\vec{X}_i(\tau^i)$ must satisfy if it exists, show the existence of a C^∞-solution to the functional equation, and then show this solution is the actual derivative.

Suppose that $f_i \equiv \vec{X}_i(\tau^i)$ exists. Then set $G_i = \vec{X}_i(g_i^i)$ and differentiate (7) with respect to $\vec{X}_i$ to obtain

$$G_i = f_i \circ \varphi_i^{-1} - f_i. \tag{9}$$

Cohomology equations of the type (9) over Anosov maps were studied by Livsic [11], [12], [2]. Define the local obstacle at $x \in \mathcal{P}(\varphi_i)$ of period

p to be the sum

$$\mathcal{O}_i(x) = \sum_{k=0}^{p-1} G_i(\varphi_i^k(x)). \tag{10}$$

Livsic proved that a continuous solution f_i for (9) exists if and only if the local obstacles vanish at a *dense set* of periodic points. Transitivity of the Anosov map φ_i implies that a continuous solution to (9) is unique up to a constant. Livsic also proved the much stronger conclusion that if there is a measurable function h_i which satisfies (9) almost everywhere, then all local obstacles vanish and h_i agrees almost everywhere with a continuous solution. It was later shown in [14] (cf. also [6], [7], [8]) that if G_i is a C^∞-function, then a solution f_i to (9) must be a C^∞-function on T^n.

Lemma 3.2. *For each $x_* \in \mathcal{P}(\varphi)$ the local obstacle $\mathcal{O}_i(x_*) = 0$.*

Proof. The coefficients of $\vec{X}_i$ in (5) applied to $\gamma_i^r \circ \gamma_i^s$ yields $\alpha^i(\gamma_i^{r+s}) = \alpha^i(\gamma_i^r) + \lambda_i^r \cdot \alpha^i(\gamma_i^s) \circ \varphi_i^{-r}$. Differentiate with respect to $\vec{X}_i$ to obtain

$$\vec{X}_i\left\{\alpha^i(\gamma_i^{r+s})\right\} = \vec{X}_i\left\{\alpha^i(\gamma_i^r)\right\} + \vec{X}_i\left\{\alpha^i(\gamma_i^s)\right\} \circ \varphi_i^{-r},$$

so that by induction we have

$$\mathcal{O}_i(x_*) = \sum_{k=0}^{p-1} \vec{X}_i\left\{\alpha^i(\gamma_i)\right\}(\varphi_i^{-k}(x_*)) = \vec{X}_i\left\{\alpha^i(\gamma_i^p)\right\}(x_*).$$

Let us consider more generally the result of differentiating (5) by the vector field $\vec{X}_\ell$,

$$\begin{aligned} &\vec{X}_\ell\left\{\vec{\alpha}(\gamma_1\gamma_2)\right\} = \\ &= \sum_{k=1}^{n} \vec{X}_\ell\left\{\alpha^k(\gamma_1\gamma_2)\right\} \cdot \vec{X}_k = \sum_{k=1}^{n} \vec{X}_\ell\left\{\alpha^k(\gamma_1)\right\} \cdot \vec{X}_k \\ &\quad + \sum_{k,r=1}^{n} A(\gamma_1)_r^k \cdot \vec{X}_\ell \cdot \left\{\alpha^r(\gamma_2) \circ \varphi(\gamma_1)^{-1}\right\} \cdot \vec{X}_k \\ &= \sum_{k=1}^{n} \vec{X}_\ell\left\{\alpha^k(\gamma_1)\right\} \cdot \vec{X}_k \\ &\quad + \sum_{k,r,s=1}^{n} A(\gamma_1)_r^k \cdot \vec{X}_s\left\{\alpha^r(\gamma_2)\right\} \circ \varphi(\gamma_1)^{-1} \cdot A(\gamma_1^{-1})_\ell^s \cdot \vec{X}_k. \end{aligned} \tag{11}$$

Let $\Gamma_* \subset \Gamma$ be the isotropy subgroup of x_*. Define a map

$$\text{(12)} \qquad \mathcal{O}_{x_*} : \Gamma_* \longrightarrow gl(n,R)$$

$$\text{(13)} \qquad \mathcal{O}_{x_*}(\gamma)^k_\ell = \{\vec{X}_\ell \{\alpha^k(\gamma)\}(x_*)$$

which by (11) is a 1-cocycle over the *Adjoint representation* $Ad \circ A\colon \Gamma_* \to GL(gl(n,R))$.

Γ_* has finite index in Γ as the $\varphi(\Gamma)$-orbit of x_* is finite. Thus, Γ_* has the **SVC** condition so the 1-cocycle $\mathcal{O}_{x_*}$ must be trivial. That is, there exists a matrix $b \in gl(n,R)$ so that for all $\gamma \in \Gamma_*$,

$$\mathcal{O}_{x_*}(\gamma) = A(\gamma) \cdot b \cdot A(\gamma)^{-1} - b.$$

In particular, $\gamma_i^p \in \Gamma_*$ so we have

$$\begin{aligned} \mathcal{O}_i(x_*) &= \mathcal{O}_{x_*}(\gamma_i^p)_i^i \\ &= \left\{A_i^p \cdot b \cdot A_i^{-p} - b\right\}_i^i \\ &= b_i^i - b_i^i = 0 \end{aligned}$$

as conjugation by a diagonal matrix acts as the identity along the diagonal. This concludes the proof of Lemma 3.2.

Remark 3.3. The above proof has an intuitive geometric interpretation when α is the tangent cocycle associated to a path of actions $\varphi_t\colon \Gamma \times T^n \to T^n$. Take the total derivative of $\varphi_t(\gamma)$ with respect to the framing of T^n to get the derivative cocycle $D\varphi_t\colon \Gamma \times T^n \to GL(n,R)$. Suppose that the group Γ has the **SVC** condition. Then an isolated periodic point $x_* \in \mathcal{P}(\varphi_0)$ of the action is stable under perturbation, so there is a smooth path of periodic points $\{x_t \mid 0 \le t \le \epsilon\}$ with $x_0 = x_*$. The derivative representation $D_{x_t}\varphi_t\colon \Gamma_* \to GL(n,R)$ at x_t of the isotropy subgroup Γ_* is stable by Weil's theorem [20] and the **SVC** condition. That is, there is a path of inner automorphisms which conjugates $D_{x_t}\varphi_t(\Gamma_*)$ to $D_{x_*}\varphi_0(\Gamma_*)$. In particular, the exponents at x_t of the hyperbolic element $\gamma_i^p \in \Gamma_{x_*}$ are constant under the path of actions. Hence the diagonal entries of the derivative $d\{D_{x_t}\varphi(\gamma_i^p)\}/dt$ vanish; the local obstacle $\mathcal{O}_i(x)$ is the derivative of the i^{th}-diagonal entry.

The proof of Proposition 3.1 is completed by:

Lemma 3.4. *For each $1 \leq i \leq n$, the p-fold partial derivative $\vec{X}_i^p(\tau^i)$ exists and is uniformly continuous on T^n.*

Proof. τ^i is the uniform limit of the partial sums

$$\tau_N^i = -\sum_{k=0}^{N} \lambda_i^k \cdot g_i^i \circ \varphi_i^{-k}.$$

Let f_i be a solution to (9) which exists by the Livsic theorem and Lemma 3.2. Then estimate the derivative of each τ_N^i:

$$\begin{aligned}\vec{X}_i(\tau_N^i) &= -\sum_{k=0}^{N} \vec{X}_i(g_i^i) \circ \varphi_i^{-k} \\ &= -\sum_{k=0}^{N} \{f_i \circ \varphi_i^{-1} - f_i\} \circ \varphi_i^{-k} \\ &= f_i - f_i \circ \varphi_i^{-N-1}.\end{aligned}$$

This is uniformly bounded for all N, so the limit function τ^i is uniformly Lipshitz when restricted to the integral curves of the vector field $\vec{X}_i$ and therefore has a derivative with respect to $\vec{X}_i$ almost everywhere on T^n. Moreover, $\vec{X}_i(\tau^i)$ gives a measurable solution to (9) so by the measurable uniqueness of solutions to (9) there must exist a constant C_i so that $\vec{X}_i(\tau^i) = C_i \cdot f_i$ almost everywhere. Now note that f_i is C^∞ and Lemma 3.4 is proven.

References

[1] F.Chen & S.Hurder, *Local rigidity for hyperbolic actions*, in preparation, 1994.

[2] V.Guillemin & D.Kazhdan, *On the cohomology of certain dynamical systems*, Topology **19** (1980) 291–299.

[3] S.Hurder, *Deformation rigidity for subgroups of $SL(n, Z)$ acting on the n-torus*, Bull. Amer. Math. Soc. **23** (1990) 107–113.

[4] _____, *Rigidity for Anosov actions of higher rank lattices*, Ann. of Math. **135** (1992) 361–410.

[5] _____, *Affine Anosov actions*, Michigan Math. J. **40** (1993) 561-575.

[6] S.Hurder & A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows.* Inst. Hautes Études Sci. Publ. Math. **72** (1990) 5–64.

[7] J.-L. Journé, *On a regularity problem occurring in connection with anosov diffeomorphisms*, Comm. Math. Phys. **106** (1986) 345–352.

[8] _____, *A regularity lemma for functions of several variables*, Reoam Mat. Ibericana. **4(2)** (1988) 187–193.

[9] A.Katok & J. Lewis, *Local rigidity for certain groups of toral automorphisms*, Israel J. Math. **75** (1991) 203–241.

[10] J.Lewis, *Infinitesimal rigidity for the action of $SL(n, Z)$ on T^n*, Trans. Amer. Math. Soc. **234** (1991) 421–445.

[11] A.Livsic, *Homology properties of U-systems*, Math. Notes of U.S.S.R. **10** (1971) 758–763.

[12] _____, *Cohomology of dynamical systems*, Math. USSR Izv. **6** (1972) 1278–1301.

[13] R. de la Llavé, *Smooth conjugacies and SRB measures*, Comm. Math. Phys. **150** (1994) 289-320.

[14] R. de la Llavé, J. Marco & R. Moriyon, *Canonical perturbation theory of Anosov systems and regularity results for Livsic cohomology equation*, Ann. of Math. **123** (1986) 537–612.

[15] A.Lubotzsky & A.Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. No. 336, 1985.

[16] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer, Berlin, 1991.

[17] J.J. Millson, *Deformations of representations of finitely generated groups*, (Geometry of Group Representations, W. Goldman & A. Magid, editors), Contemp. Math. **74** (1988) 237–253.

[18] M.Pollicott, *Infinitesimal rigidity of group actions with hyperbolic generators*, Preprint, 1993.

[19] M.Shub, *Global Stability of Dynamical Systems*, Springer, Berlin, 1987.

[20] A.Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964) 149-157.

[21] R.J. Zimmer, *Infinitesimal rigidity for smooth actions of discrete subgroups of Lie groups.* J. Differential Geom. **31** (1990) 301-322.

UNIVERSITY OF ILLINOIS AT CHICAGO