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A SIMPLE GEOMETRICAL CONSTRUCTION OF DEFORMATION QUANTIZATION

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Abstract

A construction, providing a canonical star-product associated with any symplectic connection on symplectic manifold, is considered. An action of symplectomorphisms by automorphisms of star-algebra is introduced, as well as a trace construction. Generalizations for regular Poisson manifolds and for coefficients in the bundle $\mathrm{Hom}(E, E)$ are given.

1. Introduction

A manifold M is called a Poisson manifold, if for any two functions $u, v \in C^\infty(M)$, a Poisson bracket is defined by

$$\{u, v\} = t^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j}. \tag{1.1}$$

The bracket is a bilinear skew-symmetric operation, satisfying the Jacobi identity

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0.$$

An important particular case is a symplectic manifold. In this case the matrix t^{ij} has maximal rank $2n$ equal to the manifold dimension. The inverse matrix ω_{ij} defines the exterior 2-form $\omega = \frac{1}{2}\omega_{ij}\,dx^i \wedge dx^j$ which is closed in virtue of Jacobi identity.

In [1] it has been proved that, if the tensor t^{ij} has constant rank $2n > \dim M$, there exists a symplectic foliation of the manifold M, a Poisson manifold with this property being said to be regular. The leaves F of this foliation locally are symplectic manifolds, and a Poisson bracket is defined by the symplectic form ω (closed 2-form of the rank $2n = \dim F$) defined on the leaves.

In the same paper [1] the question of deformation quantization of Poisson and in particular symplectic manifolds is considered. The problem is to define an associative multiplication operation $*$, depending on parameter h (Planck constant), of two functions so that the space $C^\infty(M)$ with

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usual linear operators and $*$-product would be a formal deformation of commutative algebra of functions with a Poisson bracket. More exactly it means the following. Let Z be the linear space, the elements of which are formal series

$$a = a(x, h) = \sum_{k=0}^{\infty} h^k a_k(x), \tag{1.2}$$

where $a_k(x) \in C^\infty(M)$. Further for any $a, b \in Z$ let an associative product operation

$$a * b = c = \sum_{k=0}^{\infty} h^k c_k(x)$$

be defined with the following properties:

(i) c_k are polynomials in a_k, b_k and their derivatives;

(ii) $c_0(x) = a_0(x) b_0(x)$;

(iii) $[a, b] \equiv a * b - b * a = -ih\{a_0, b_0\} + \cdots$, where dots mean the terms of higher orders.

The algebra Z is called the algebra of quantum observables. Property (i) means the locality of $*$-product, property (ii) means that algebra Z is a deformation of the commutative algebra of C^∞ functions, property (iii) is the so-called correspondence principle.

The question of the existence of such a product for symplectic manifolds has been completely solved in [2]. Subsequently, an equivariant generalization of this construction [7] for symplectic manifolds was obtained, as well as a generalization for regular Poisson manifolds [6]. The constructions considered in these works are based on the analysis of Hochschild cohomologies.

In [3] the author, being unaware of the results of [2], proposed another construction of $*$-product for a symplectic manifold. This construction admits straightforward generalizations for both the equivariant case and the case of a regular Poisson manifold. In subsequent papers [4], [5] the author studied the action of symplectic diffeomorphisms, proposed a trace construction in algebra Z, introduced the concept of index, generalizing the index of elliptic operators and obtained an index formula.

Unfortunately, work [3] was published in a local issue of Moscow Institute of Physics and Technology in very few copies, so it remained unknown to most mathematicians. The purpose of the present article, containing the extended exposition of some results of [3], [4], is to introduce the results to broader mathematical circles.

Let us briefly describe the contents of subsequent sections. In §2 we consider the Weyl algebras bundle W, W-valued differential forms, and a connection in the bundle W. These notions give us a basic machinery. In §3 we introduce the notion of Abelian connection and prove the existence of such connections. The sections of the Weyl algebras bundle, which are flat with respect to a fixed Abelian connection, form an associative algebra. We prove that these sections are in one to one correspondence with the functions from Z. This allows us to transfer the associative algebra structure to the set Z and thus to define a $*$-product.

The next sections are concerned with the notion of trace in the algebra of flat sections. First of all we construct isomorphisms of this algebra, corresponding to any symplectic diffeomorphism of symplectic manifolds. This construction, introduced in §4, is used in §§5 and 6 to define a trace by means of localization and reduction to the case of standard symplectic space $\mathbb{R}^{2n}$. In §7 two generalizations are exposed. The first one gives the construction of deformation quantization and the trace for the case where the coefficients are homomorphisms of a vector bundle over M. The second one deals with a generalization of the results obtained in §§2 and 3 for the case of regular Poisson manifolds.

A few years ago there appeared a paper [8], in which quantization is based on the idea of identifying functions on a symplectic manifold with the sections of the Weyl bundle. We use a similar approach. But their means of such an identification is much more complicated than ours.

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2. Weyl algebras bundle

Let (M, ω) be a symplectic manifold of dimension $2n$. The form ω determines a symplectic structure in each tangent space T_xM.

Definition 2.1. The formal Weyl algebra W_x, corresponding to the symplectic space T_xM, is the associative algebra over $\mathbb{C}$ with a unit, its elements being formal series

$$a(y) = \sum_{2k+l\geq 0} h^k a_{k, i_1\cdots i_l} y^{i_1} \cdots y^{i_l}, \tag{2.1}$$

where h is a formal parameter, $y = (y^1, \cdots, y^{2n}) \in T_xM$ is a tangent vector, and $a_{k, i_1\cdots i_l}$ are covariant tensors. The degrees 1 and 2 are prescribed for the variables y^i and h respectively. The product of elements

$a, b \in W_x$ is determined by the Weyl rule

$$(2.2)\qquad \begin{aligned} a \circ b &= \exp\left(-\frac{ih}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(y, h)b(z, h)|_{z=y} \\ &= \sum_{k=0}^{\infty}\left(-\frac{ih}{2}\right)^k \frac{1}{k!}\omega^{i_1 j_1}\cdots\omega^{i_k j_k}\frac{\partial^k a}{\partial y^{i_1}\cdots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\cdots\partial y^{j_k}}. \end{aligned}$$

It is easily seen that the multiplication (2.2) does not depend on the choice of a basis in $T_x M$ and is associative.

Taking the union of the algebras W_x, $x \in M$, we obtain the bundle of formal Weyl algebras whose sections are "functions"

$$(2.3)\qquad a(x, y, h) = \sum_{2k+l\geq 0} h^k a_{k, i_1\cdots i_l}(x) y^{i_1}\cdots y^{i_l},$$

where $a_{k, i_1\cdots i_l}$ are symmetric covariant tensor fields on M. The set of sections also forms an associative algebra with respect to the fiberwise multiplication (2.2). The unit in this algebra is the "function" identically equal to 1. To simplify notation we shall also denote the algebra of the sections by W (instead of the pedantic $C^{\infty}(M, W)$), which, to our mind, should not cause any confusion.

It is easy to see that the center of W consists of the sections not containing y's. Thus the central sections are defined by the series of form (1.2), and consequently the center of W may be identified as a linear space with the space Z mentioned in the introduction. There is a filtration in the algebra W: $W \supset W_1 \supset W_2 \supset \cdots$ with respect to the total degree $2k + l$ of the terms of the series (2.3).

We shall also need differential forms on M with values in W. A differential q-form is defined by the series

$$(2.4)\qquad a = \sum h^k a_{k, i_1\cdots i_p, j_1\cdots j_q}(x) y^{i_1}\cdots y^{i_p}\, dx^{j_1}\wedge\cdots\wedge dx^{j_q},$$

whose coefficients are covariant tensor fields symmetric with respect to indices $i_1, \cdots, i_p$ and antisymmetric with respect to $j_1, \cdots j_q$. The differential forms constitute an algebra $W\otimes\Lambda = \bigoplus_{q=0}^{2n}(W\otimes\Lambda)$, in which the multiplication is defined by means of the exterior product of differentials dx^i and Weyl product (2.2) of polynomials in y^i (dx^i commute with y^i). The product of two forms will be denoted by the same symbol $a\circ b$, such as the product sections of W. A filtration $W\otimes\Lambda \supset W_1\otimes\Lambda \supset W_2\otimes\Lambda \supset \cdots$ is introduced with respect to the total degree $2k + p$ corresponding to the variables h, y^i.

Let us introduce the commutator of two forms $a \in W \otimes \Lambda^{q_1}$, $b \in W \otimes \Lambda^{q_2}$ defined by $[a, b] = a \circ b - (-1)^{q_1 q_2} b \circ a$. A form a is said to be central, if for any $b \in W \otimes \Lambda$ the commutator of a and b vanishes. It is clear that the central forms are just the ones not containing y's, i.e., $Z \otimes \Lambda$ is the center.

Define two projections of the form $a = a(x, y, dx, h)$ onto the center: $a_0 = a(x, 0, dx, h)$ and $a_{00} = a(x, 0, 0, h)$. In the particular case where $a = a(x, y, h) \in W$, we shall use the notation $\sigma(a)$ for $a_0 = a(x, 0, h)$ and call $\sigma(a)$ the symbol of the section a.

Consider two important operators on forms:

$$\delta a = dx^k \wedge \frac{\partial a}{\partial y^k}, \qquad \delta^* a = y^k i\left(\frac{\partial}{\partial x^k}\right) a, \tag{2.5}$$

where $i(\partial/\partial x^k)$ means the contraction of the vector field $\partial/\partial x^k$ and the form, multiplication by y^k being the usual commutative product of functions. The operator $\delta\colon W_p \otimes \Lambda^q \to W_{p-1} \otimes \Lambda^{q+1}$ which reduces the filtration by 1 is similar to the exterior derivation. The operator $\delta^*\colon W_p \otimes \Lambda^q \to W_{p+1} \otimes \Lambda^{q-1}$ raises the filtration by 1. In other words the operator δ acts on the monomial

$$y^{i_1} y^{i_2} \cdots y^{i_p}\, dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q} \tag{2.6}$$

by replacing one by one the variables $y^{i_1}, y^{i_2}, \cdots, y^{ip}$ by $dx^{i_1}, dx^{i_2}, \cdots, dx^{i_p}$ respectively; the operator δ^* acts on (2.6) by replacing $dx^{j_1}, dx^{j_2}, \cdots, dx^{j_q}$ by $y^{j_1}, -y^{j_2}, \cdots, (-1)^q y^{j_q}$ respectively.

Lemma 2.2. *The operators δ and δ^* do not depend on the choice of local coordinates and have the following properties*:

(i) $\delta^2 = (\delta^*)^2 = 0$,

(ii) *for monomial* (2.6) *we have* $\delta\delta^* + \delta^*\delta = (p+q)\,\mathrm{id}$.

The lemma is easily proved by a direct check. q.e.d.

Note that δ is an antiderivation, i.e., for $a \in W \otimes \Lambda^{q_1}$ and $b \in W \otimes \Lambda^{q_2}$ we have

$$\delta(a \circ b) = (\delta a) \circ b + (-1)^{q_1} a \circ b, \tag{2.7}$$

(δ^* does not possess this property).

Define the operator δ^{-1} acting on the monomial (2.6) by $\delta^{-1} = \delta^*/(p+q)$ for $p+q > 0$, and $\delta^{-1} = 0$ for $p+q = 0$. By Lemma 2.2 it can be derived that any form $a \in W \otimes \Lambda$ has the representation

$$a = \delta\delta^{-1} a + \delta^{-1}\delta a + a_{00}, \tag{2.8}$$

which is similar to the Hodge–De Rham decomposition.

Definition 2.3. A symplectic connection is a torsion-free connection preserving tensor ω_{ij}, i.e., $\partial_i \omega_{jk} = 0$, ∂_i being a covariant derivative with respect to $\partial/\partial x^i$.

In Darboux local coordinates the coefficients $\Gamma_{ijk} = \omega_{im}\Gamma^m_{jk}$ of the symplectic connection are completely symmetric with respect to indices ijk. The symplectic connection always exists but is not unique, unlike the Riemannian connection [1]. Two symplectic connections differ by a completely symmetric tensor $\Delta\Gamma_{ijk}$.

Let ∂ be a symplectic connection on the manifold M. Using the covariant derivation of tensor fields, which are coefficients in (2.4), define a connection in the bundle W as an operator $\partial\colon W \otimes \Lambda^q \to W \otimes \Lambda^{q+1}$ by

$$\partial a = dx^i \wedge \partial_i a. \tag{2.9}$$

Definition 2.3 implies the following properties of the connection ∂ in the bundle $W \otimes \Lambda$:

(i) $\partial(a \circ b) = \partial a \circ b + (-1)^{q_1} a \circ \partial b$ for $a \in W \otimes \Lambda^{q_1}$.

(ii) For any scalar form $\varphi \in \Lambda^q$, $\partial(\varphi \wedge a) = d\varphi \wedge a + (-1)^q \varphi \wedge \partial a$.

In Darboux local coordinates the connection ∂ can be written in the form

$$\partial a = da + [(i/h)\Gamma, a], \tag{2.10}$$

where $\Gamma = \frac{1}{2}\Gamma_{ijk} y^i y^j dx^k$ is a local 1-form with values in W, $d = dx^i \wedge \partial/\partial x^i$ being the exterior differential with respect to x.

We shall consider more general connections D in the bundle W, namely, connections of the form

$$Da = \partial a + [(i/h)\gamma, a] = da + [(i/h)(\Gamma + \gamma), a], \tag{2.11}$$

where γ is a globally determined 1-form on M with values in W (i.e., section of $W \otimes \Lambda^1$). Note that the operator δ, introduced above, may be written in the form

$$\delta a = -[(i/h)\omega_{ij} y^i dx^j, a]. \tag{2.12}$$

Lemma 2.4. *Let ∂ be a symplectic connection. Then*

$$\partial\delta a + \delta\partial a = 0, \tag{2.13}$$

$$\partial^2 a = \partial(\partial a) = [(i/h)R, a], \tag{2.14}$$

where

$$R = \tfrac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l,$$

$R_{ijkl} = \omega_{im} R^m_{jkl}$ *being the curvature tensor of the symplectic connection.*

Proof. Identities (2.13), (2.14) are obvious consequences of equations (2.10), (2.12). Note that (2.14) is a compact form of the Ricci identity.

Definition 2.5. Let D be a connection in the bundle W of the form (2.11) with

$$\gamma_0 = 0. \tag{2.15}$$

We shall call the 2-form

$$\frac{i}{h}\Omega = \frac{i}{h}\left(R + \partial\gamma + \frac{i}{h}\gamma^2\right) \tag{2.16}$$

the *curvature* of the connection D.

Lemma 2.6. *For any section $a \in W \otimes \Lambda$ we have*

$$D^2 a = [(i/h)\Omega, a]. \tag{2.17}$$

The proof is straightforward.

Remark. Note that the form γ in (2.11) is determined by the connection D not uniquely but up to a central 1-form, because it appears in commutators. For the uniqueness of γ and therefore of the curvature (2.16) some normalizing condition is required. We assume that this condition has the form (2.15) and call it Weyl normalizing condition. The corresponding curvature (2.16) is called Weyl curvature.

3. Abelian connections and quantization

Definition 3.1. A connection D in the bundle W is said to be *Abelian* if for any section $a \in W \otimes \Lambda$,

$$D^2 a = [(i/h)\Omega, a] = 0. \tag{3.1}$$

By (3.1) we can show that the curvature of Abelian connection is a central form and conversely.

In this section we prove the existence of an Abelian connection of the form

$$D = -\delta + \partial + \left[\frac{i}{h}r, \cdot\right] = \partial + \left[\frac{i}{h}(\omega_{ij}y^i dx^j + r), \cdot\right],$$

∂ being a fixed symplectic connection, and $r \in W_3 \otimes \Lambda^1$ being a globally defined 1-form, satisfying Weyl normalizing condition $r_0 = 0$. Calculating the curvature of this connection and using Lemma 2.4, we get

$$\Omega = -\tfrac{1}{2}\omega_{ij}\, dx^i \wedge dx^j + R - \delta r + \partial r + (i/h)r^2. \tag{3.2}$$

The Abelian property will be fulfilled, provided

$$\delta r = R + \partial r + (i/h)r^2. \tag{3.3}$$

Then for the curvature we shall have $\Omega = -\omega$, so that Ω will really be a scalar form.

Theorem 3.2. *Equation* (3.3) *has a unique solution, satisfying the condition*

$$\delta^{-1} r = 0. \tag{3.4}$$

(Note that (3.4) implies $r_0 = 0$).

Proof. Let $r \in W_3 \otimes \Lambda^1$ satisfy (3.3), (3.4). The decomposition (2.8) for the form r becomes $r = \delta^{-1}\delta r$ as $\delta\delta^{-1} r = 0$ by (3.4) and $r_{00} = 0$, since r is a 1-form. Applying the operator δ^{-1} to (3.3) we get

$$r = \delta^{-1} R + \delta^{-1}\left(\partial r + (i/h) r^2\right). \tag{3.5}$$

The operator ∂ preserves the filtration and δ^{-1} raises it by 1, so iteration of equation(3.5) shows that it has a unique solution.

Conversely, we will show that the solution of equation (3.5) satisfies (3.3), (3.4). The condition (3.4) is evidently fulfilled because of $(\delta^{-1})^2 = 0$. Let

$$A = \delta r - R - \partial r - (i/h) r^2$$

be the difference between the left-hand and the right-hand sides of (3.3), r being the solution of (3.5). Show that A satisfies the equation

$$\delta A = \partial A + (i/h)[r, A] \tag{3.6}$$

and the "initial" condition

$$\delta^{-1} A = 0. \tag{3.7}$$

From (3.6) and (3.7) it follows that A vanishes. Indeed, applying δ^{-1} to both sides of equation (3.6) and using (3.7) we shall get similar to (3.5)

$$A = \delta^{-1}(\partial A + (i/h)[r, A]),$$

from which by iterations it follows that $A = 0$.

For checking (3.7) we have

$$\delta^{-1} A = \delta^{-1}\delta r - \delta^{-1}\left(R + \partial r + (i/h) r^2\right) = \delta^{-1}\delta r - r = 0.$$

Here we have used (3.5), condition (3.4), and the Hodge-De Rham decomposition.

For checking (3.6) by taking into account that $\delta\delta r = 0$, we obtain

$$\delta A = -\delta R - \delta(\partial r) + [(i/h) r, \delta r],$$

since

$$\delta R = \tfrac{1}{2} R_{ijkl} y^i \, dx^j \wedge dx^k \wedge dx^l,$$

which is equal to 0 because of the relation

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

for the curvature tensor. Further, $\delta(\partial r) = -\partial(\delta r)$ according to Lemma 2.4. Thus,

$$\delta A = \partial(\delta r) + [(i/h)r, R + \partial r + (i/h)r^2]. \tag{3.8}$$

We have $\partial R = 0$ according to the Bianchi identity for the curvature tensor, $\partial\partial r = [(i/h)R, r]$ in virtue of Lemma 2.4 (Ricci identity), $\partial(i/h)r^2 = [\partial r, (i/h)r]$. Taking into account that $[(i/h)r, (i/h)r^2] = 0$, we get that the last two terms in (3.8) would equal to 0, and this proves equality (3.6). q.e.d.

Note that iterating equation (3.5) we can effectively construct the form r and, consequently Abelian connection D. The first two terms are

$$r = \tfrac{1}{8}R_{ijkl}y^i y^j y^k\, dx^l + \tfrac{1}{20}\partial_m R_{ijkl}y^i y^j y^k y^m\, dx^l + \cdots ,$$

∂_m being a covariant derivative with respect to the vector field $\partial/\partial x^m$. Further terms would contain not only y's but also powers of h because of the term $(i/h)r^2$ in (3.5).

Introduce now the main object: the subalgebra $W_D \subset W$, consisting of flat sections, i.e., such that $Da = 0$.

Theorem 3.3. *For any* $a_0 \in Z$ *there exists a unique section* $a \in W_D$ *such that* $\sigma(a) = a_0$.

Recall that, for the section $a(x, y, h) \in W$, $\sigma(a)$ means the projection onto the center, i.e., $\sigma(a) = a(x, 0, h)$.

Proof. The equation $Da = 0$ can be written in the form

$$\delta a = \partial a + [(i/h)r, a]. \tag{3.9}$$

Applying the operator δ^{-1} and using Hodge-De Rham decomposition yield

$$a = a_0 + \delta^{-1}(\partial a + [(i/h)r, a]), \tag{3.10}$$

wherefrom by iterations we should get that equation (3.10) has a unique solution because δ^{-1} increases the filtration.

Conversely, let a be the solution of (3.10). Then evidently we have $\sigma(a) = a_0$ since the result of applying δ^{-1} contains only positive powers of y's. Further, using reasoning similar to the proof of Theorem 3.2, we can show that the difference

$$A = \delta a - \partial a - [(i/h)r, a] \equiv Da$$

between the left-hand and the right-hand sides of (3.9) satisfy the equation

$$\delta A = \partial A + [(i/h)r, A] \tag{3.11}$$

and the trivial "initial" condition

$$\delta^{-1} A = 0. \tag{3.12}$$

Equation (3.11) is fulfilled, since it means that $DA = 0$, so taking into account that $A = Da$ we shall have $DA = D(Da) = 0$ because D is an Abelian connection. Further,

$$\delta^{-1}A = \delta^{-1}\delta a - \delta^{-1}(\partial a + [(i/h)r, a]) = \delta^{-1}\delta a - a + a_0$$

according to equation (3.10). The last expression is equal to 0 by Hodge-De Rham decomposition, since $\delta^{-1}a = 0$. q.e.d.

It is easily seen that for any $a(y, h) \in W_{x_0}$ with fixed $x_0 \in M$ there exists a flat section $a(x, y, h) \in W_D$ (not unique, of course) such that $a(x_0, y, h) = a(y, h)$. This fact implies that the centralizer of W_D in W coincides with the center Z of W. In other words, if a section $b \in W$ commutes with any flat section $a \in W_D$, then $b \in Z$. Similarly, the centralizer of W_D in $W \otimes \Lambda$ is $Z \otimes \Lambda$.

Iterating equation(3.10) we can effectively construct the section $a \in W_D$ by its symbol $a_0 = \sigma(a)$:

$$\begin{aligned} a = a_0 &+ \partial_i a_0 y^i + \tfrac{1}{2}\partial_i\partial_j a_0 y^i y^j + \tfrac{1}{6}\partial_i\partial_j\partial_k a_0 y^i y^j y^k \\ &- \tfrac{1}{24} R_{ijkl}\omega^{lm}\partial_m a_0 y^i y^j y^k + \cdots . \end{aligned}$$

If the curvature tensor is equal to 0, iterations would give the explicit expansion

$$a = \sum_{k=0}^{\infty} \frac{1}{k!}(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} a_0) y^{i_1} y^{i_2} \cdots y^{i_k}.$$

It is clear that, provided Abelian connection D is fixed, flat sections form a subalgebra W_D with respect to fiberwise Weyl multiplication $\circ$ in the algebra W. Theorem 3.3 states that the map $\sigma\colon W_D \to Z$ is bijective. Thus, the inverse map $\sigma^{-1}\colon Z \to W_D$ has been defined.

Now we can explain the construction of $*$-product in the space Z. Namely, by using the bijections σ and σ^{-1} associative product $\circ$ in the algebra W is transferred to the set Z, i.e., we assume for $a, b \in Z$

$$a * b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)). \tag{3.14}$$

Using (3.13) it is easily checked that such defined $*$-product satisfies all the conditions (i)–(iii), formulated in the introduction. However, as will be seen later, it is more convenient to use the subalgebra W_D with $\circ$-product than the $*$-product. Therefore later on we shall not mention $*$-product at all. The subalgebra W_D will be called an algebra of quantum observables.

4. An action of symplectomorphisms

Let M, ω, W, ∂, D, W_D denote, similar to those in the previous sections, a symplectic manifold, a symplectic form, the Weyl algebras bundle (and the algebra of its sections), a symplectic connection on M, the Abelian connection in the bundle W and the subalgebra of flat sections. For a symplectic diffeomorphism $f: M \to M$ the pullbacks f^* of these objects are evidently defined ($f^*\omega = \omega$ since f is a symplectic map). For example, for a section $a(x, y, h) \in W$ we assume

$$(f^*a)(x, y, h) = a(f(x), f'y, h),$$

f' being a differential of the map f. Since f is a diffeomorphism, both pullbacks f^* and pushforwards $f_* = (f^{-1})^*$ are defined for all geometric objects.

Consider in more detail the action of symplectomorphisms $f: M \to M$ on connections. Let ∂ be a symplectic connection considered as the connection in the bundle W according to (2.7), and D be the Abelian connection corresponding to D by Theorem 3.2. Since f^* and f_* are evidently automorphisms of the algebra W, we can define the connections $\tilde{\partial} = f_*\partial$, $\tilde{D} = f_*D$ by the formulas

$$\tilde{\partial}a = f_*(\partial(f^*a)), \qquad \tilde{D}a = f_*(D(f^*a)).$$

It is clear that $\tilde{D}$ is also an Abelian connection. The operators δ, δ^{-1} and Weyl normalizing condition are invariant under diffeomorphisms. (Because of the uniqueness of the solution of (3.3), (3.4), $\tilde{D}$ corresponds to $\tilde{\partial}$, i.e., $\tilde{D}$ is obtained from $\tilde{\partial}$ by Theorem 3.2.

Theorem 4.1. *The automorphism $f_*: W \to W$ isomorphically maps the subalgebra W_D onto the subalgebra $W_{\tilde{D}}$. Besides, if the symplectic connection ∂ is invariant under f, i.e., if $\tilde{\partial} \equiv f_*\partial = \partial$, then f_* defines the automorphism of the algebra W_D.*

Proof. The proof directly follows from the definitions. If $a \in W_D$, then $f_*a \in W_{\tilde{D}}$, since $\tilde{D}f_*a = f_*(Da) = 0$. Further on, if $\tilde{\partial} = \partial$, the property of the uniqueness of Theorem 3.2 implies that $\tilde{D} = D$, $W_{\tilde{D}} = W_D$, i.e., f_* is an automorphism of the algebra of quantum observables W_D. q.e.d.

In particular, if G is a group of symplectomorphisms of M, and ∂ is a G-invariant symplectic connection, then the corresponding Abelian connection is also G-invariant and the group G acts by automorphisms on the algebra of quantum observables W_D.

In a general case, when ∂ is not invariant with respect to f, it is nevertheless possible to define automorphism A_f, corresponding to f, by using the fiberwise conjugation automorphisms. To do so introduce an extension W^+ of the algebra W as follows:

(i) Elements $U \in W^+$ are given by the series (2.1), but the powers of h can be both positive and negative.

(ii) The total degree $2k + l$ of any term of the series is nonnegative.

(iii) There exists a finite number of terms with a given nonnegative total degree.

It is clear that W^+ is also an algebra with respect to Weyl fiberwise multiplication, and the connections ∂ and D act on sections $a \in W^+$.

Lemma 4.2. *Let $a \in W^+$ and $Da = 0$. Then a does not contain negative powers of h, i.e., $a \in W_D \subset W \subset W^+$.*

Proof. Let $\sigma(a) = a(x, 0, h)$. Nonnegativeness of the total degree of series terms implies $\sigma(a) \in Z$, i.e., it does not contain negative powers of h. According to Theorem 3.3 a flat section is uniquely defined by its symbol $\sigma(a) \in Z$ and thus belongs to W_D. q.e.d.

Like W the algebra W^+ has the filtration with respect to the total degree $2k + l$ of series terms (2.3).

Introduce a group, consisting of invertible elements of the algebra W^+ with the leading term 1 having the form

$$U = \exp\left(\frac{i}{h}H\right) = \sum_{k=0}^{\infty}\left(\frac{i}{h}\right)^k \frac{1}{k!}\underbrace{H \circ H \circ \cdots \circ H}_{k}, \tag{4.1}$$

where $H \in W_3$. It follows from the Campbell-Hausdorff formula that such elements form a group. It is clear that the map

$$a \mapsto U \circ a \circ U^{-1} = \sum_{k=0}^{\infty}\left(\frac{i}{h}\right)^k \frac{1}{k!}[H, [H, \cdots, [H, a], \cdots]] \tag{4.2}$$

(the commutator is taken k times) is an automorphism of W^+, which maps the algebra W onto itself. It is also clear that this map preserves the filtration but not the degrees of the series terms.

Let D be the Abelian connection in the bundle W of the form (2.11), γ satisfying Weyl normalizing condition $\gamma_0 = 0$. Automorphism (4.2) defines a new Abelian connection $\widetilde{D}$ by the rule

(4.3) $$\widetilde{D}a = U \circ D(U^{-1} \circ a \circ U) \circ U^{-1} = Da - [DU \circ U^{-1}, a].$$

From (4.3) it follows that the form $\tilde{\gamma}$, corresponding to the connection $\widetilde{D}$ and satisfying normalizing condition, has the form

(4.4) $$\tilde{\gamma} = \gamma + \Delta\gamma = \gamma - DU \circ U^{-1} + (DU \circ U^{-1})_0.$$

Hence for the curvature we shall have

$$\frac{i}{h}\widetilde{\Omega} = \frac{i}{h}\Omega + D\left(\frac{i}{h}\Delta\gamma\right) + \left(\frac{i}{\Delta}\gamma\right)^2 = -\frac{i}{h}\omega + d(DU \circ U^{-1})_0.$$

The last equality is obtained by using the relation

$$-D(DU \circ U^{-1}) + (DU \circ U^{-1})^2 = 0.$$

The scalar form $(DU \circ U^{-1})_0$ belongs to $W_2 \otimes \Lambda^1 \cap Z$, i.e., begins with the first power of h. Thus we obtain that the curvatures of these two connections D and $\widetilde{D}$ differ by an exact 2-form belonging to $(W_2 \otimes \Lambda^2) \cap Z$.

Theorem 4.3. *Let ∂, $\tilde{\partial}$ be two symplectic connections, and D, $\widetilde{D}$ be the Abelian connections corresponding to ∂, $\tilde{\partial}$ by Theorem* 3.2. *Then there exists a section $U \in W^+$ of the form* (4.1) *such that*

$$\widetilde{D} = D - [DU \circ U^{-1}, \cdot].$$

Proof. The connection $\widetilde{D}$ can be written in the form

$$\widetilde{D} = D + [(i/h)\Delta\gamma, \cdot],$$

where

$$\Delta\gamma = \widetilde{\Gamma} - \Gamma + \tilde{r} - r \in W_2 \otimes \Lambda^1$$

satisfies Weyl normalizing condition $(\Delta\gamma)_0 = 0$. Hence for the curvature $(i/h)\widetilde{\Omega}$ of the connection $\widetilde{D}$,

$$\frac{i}{h}\widetilde{\Omega} = \frac{i}{h}\Omega + \frac{i}{h}D(\Delta\gamma) + \left(\frac{i}{h}\Delta\gamma\right)^2.$$

Since this expression is to be equal to $(i/h)\Omega$, we have

(4.5) $$D(\Delta\gamma) + (i/h)(\Delta\gamma)^2 = 0.$$

Find the section $U \in W^+$ as a solution of the equation

$$DU \circ U^{-1} = -(i/h)\Delta\gamma,$$

which is equivalent to

$$(4.6) \qquad DU = -(i/h)\Delta\gamma \circ U\,.$$

Condition (4.5) is necessary for the solvability of equation (4.6) in W^+. Indeed, applying operator D, we get 0 on the left-hand side, since $D^2U = 0$. Thus

$$0 = -(i/h)D(\Delta\gamma) \circ U + (i/h)\Delta\gamma \circ DU\,.$$

Substituting (4.6) for DU in the above equation, we obtain

$$0 = -\{(i/h)D(\Delta\gamma) + ((i/h)\Delta\gamma)^2\} \circ U\,,$$

which is fulfilled according to (4.5).

Let us show that condition (4.5) is also sufficient for the solvability of equation (4.6). Rewrite (4.6) in the form

$$\delta U = (D+\delta)U + (i/h)\Delta\gamma \circ U$$

and apply the operator δ^{-1} to both sides of the equation. Taking $U_0 = 1$ and using the Hodge-De Rham decomposition, we get

$$(4.7) \qquad U = 1 + \delta^{-1}\{(D+\delta)U + (i/h)\Delta\gamma \circ U\}\,.$$

Since the operator $D + \delta = \partial + [(i/h)r, \cdot]$ does not change the filtration, multiplication by $(i/h)\Delta\gamma$ in $W_0^+ \otimes \Lambda^1$ does not change the filtration either, and δ^{-1} raises the filtration by 1, the iterations of equation (4.7) give a unique solution. The resulting solution is an invertible element of the algebra W^+, since its leading term is equal to 1.

Conversely, let us show that the solution of equation (4.7) satisfies (4.6). Let

$$A = DU + (i/h)\Delta\gamma \circ U\,.$$

Then

$$(4.8) \qquad \delta^{-1}A = 0$$

according to (4.7). Further we have

$$\begin{aligned} DA &= \frac{i}{h}(D\Delta\gamma) \circ U - \frac{i}{h}\Delta\gamma \circ DU \\ &= \frac{i}{h}\left\{(D\Delta\gamma) + \frac{i}{h}(\Delta\gamma)^2\right\} \circ U - \frac{i}{h}\Delta\gamma \circ \left\{DU + \frac{i}{h}\Delta\gamma \circ U\right\}, \end{aligned}$$

Hence, in consequence of (4.5), A satisfies the equation

$$(4.9) \qquad DA + (i/h)\Delta\gamma \circ A = 0\,.$$

So, using reasoning similar to Theorem 3.2, we get $A = 0$. Indeed, equation (4.9) together with condition (4.8) gives

$$A = \delta^{-1}((D+\delta)A + (i/h)\Delta\gamma \circ A).$$

Thus the iterations yield a trivial solution.

The solution of equation (4.6) is not unique. Let V be another solution. Then for $U^{-1} \circ V$ we get

$$\begin{aligned} D(U^{-1} \circ V) &= -U^{-1} \circ DU \circ U^{-1} \circ V + U^{-1} \circ DV \\ &= U^{-1} \circ (i/h)\Delta\gamma \circ V - U^{-1} \circ (i/h)\Delta\gamma \circ V = 0. \end{aligned}$$

Consequently, $U^{-1} \circ V = C \in W_D$, and the two solutions of equation (4.6) differ by the factor C, which is an invertible flat section.

Let us finally show that the solution can always be chosen in the form (4.1) with $H \in W_3$. Introduce the section

$$H = -ih \ln U = -ih \ln(1 + (U-1)) = -ih \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(U-1)^k, \tag{4.10}$$

where powers are understood with respect to the multiplication $\circ$. Since $U - 1 \in W_1$, the series converges with respect to filtration in W^+ and defines the section $H \in W_3^+$. Multiplying U by the proper factor $C = e^a$ to the right ($a \in W_D$ and the exponent is calculated in the algebra W_D) we can always achieve $H_0 = 0$. Indeed, if $H_0 \in W_4 \cap Z$ is not equal to 0, then by Theorem 3.3, we can construct $a \in W_D \cap W_4$, such that $\sigma(a) = H_0$. Taking in (4.10) $U \circ e^{-a}$ instead of U, we get a new section H, for which $H_0 \in W_6 \cap Z$. Repeating this procedure, we shall get sections H with H_0 having higher and higher degree so that in the limit we obtain H with $H_0 = 0$. Let us show that this section belongs not only to W_3^+ but also to W_3 as well. Indeed, derivating the exponent $U = \exp((i/h)H)$ and substituting into equation (4.6) we obtain

$$DU \circ U^{-1} = \frac{\exp(\mathrm{ad}((i/h)H)) - 1}{\mathrm{ad}((i/h)H)} \frac{i}{h} DH = -\frac{i}{h}\Delta\gamma,$$

where $\mathrm{ad}((i/h)H) = [(i/h)H, \cdot]$. This gives an equation for H, which can be written in the form

$$\delta H = \frac{\mathrm{ad}((i/h)H)}{\exp(\mathrm{ad}((i/h)H)) - 1}\Delta\gamma + (D+\delta)H.$$

Applying δ^{-1} and using $H_0 = 0$, we shall get, according to Hodge-DeRham decomposition,

$$H = \delta^{-1}\left(\frac{\mathrm{ad}((i/h)H)}{\exp(\mathrm{ad}((i/h)H)) - 1}\Delta\gamma + (D+\delta)H\right).$$

Since $\Delta\gamma \in W_2 \otimes \Lambda^1$, all the iterations will give elements of W_3, so that the section H belongs to W_3. Hence the theorem has been proved. q.e.d.

Now if we are given a symplectic diffeomorphism $f: M \to M$, then an automorphism $A_f: W \to W$, mapping the subalgebra W_D onto itself, can be associated with f in the following way.

Let ∂ be a symplectic connection on M, and D be the Abelian connection in the bundle W, corresponding to ∂ by Theorem 3.2. Let, further, $\tilde{\partial} = f_*\partial$ and $\tilde{D} = f_*D$ be pushforwards of ∂ and D under diffeomorphism f. Since Weyl curvatures of D and $\tilde{D}$ both are equal to $-(i/h)\omega$ according to Theorem 3.2, by Theorem 4.3 there exists the section $U \in W^+$ of the form (4.1) such that the connection D goes to the connection D under conjugation automorphism (4.2). So automorphism A_f defined by the relation

$$A_f: a \mapsto U \circ (f_* a) \circ U^{-1} \tag{4.11}$$

maps the sections of W_D to the sections belonging to W_D.

Generally speaking, these automorphisms do not satisfy natural cocycle condition

$$A_{f_1} A_{f_2} A_{f_3} = \mathrm{id}, \tag{4.12}$$

if $f_1 f_2 f_3 = \mathrm{id}$. However, we can state that the left-hand side of (4.12) is the conjugation automorphism by the section $U = U_1 \circ f_{1*}(U_2 \circ f_{2*} U_3) \in W^+$ of the form (4.1).

Lemma 4.4. *Let conjugation automorphism* (4.2) *map the subalgebra* W_D *onto itself. Then locally there exists a function* φ *such that* $Ue^{\varphi} \in W_D$.

Proof. For any $U \in W_D$ we have $Aa = U \circ a \circ U^{-1} \in W_D$. Then

$$0 = D(U \circ a \circ U^{-1}) = [DU \circ U^{-1}, U \circ a \circ U^{-1}] + U \circ Da \circ U^{-1},$$

wherefrom it follows that

$$[DU \circ U^{-1}, U \circ a \circ U^{-1}] = 0.$$

Thus, the form $\psi = DU \circ U^{-1}$ commutes with any section $U \circ a \circ U^{-1} \in W_D$, i.e., it is the central form. This form is closed since

$$d\psi = D(DU \circ U^{-1}) = (DU \circ U^{-1})^2 = \psi \wedge \psi = 0.$$

So, locally $\psi = d\varphi$, and

$$D(Ue^{-\varphi})e^{\varphi} \circ U^{-1} = DU \circ U^{-1} - d\varphi = 0,$$

which means that the section $Ue^{-\varphi}$ is flat in W^+ and then, according to Lemma 4.2, it automatically belongs to W.

Corollary 4.5. *Automorphism* $A_{f_1}A_{f_2}A_{f_3}$, *where* $f_1 f_2 f_3 = \mathrm{id}$, *locally is an inner automorphism of the algebra* W.

5. A trace in the algebra of quantum observables on $\mathbf{R}^{2n}$

In this section we shall consider the case $M = \mathbf{R}^{2n}$ with the standard symplectic structure. The standard symplectic form has constant coefficients, therefore we can take the operator of exterior derivation as a symplectic connection. The Abelian connection D in the bundle $W = W(\mathbf{R}^{2n})$, corresponding to d by Theorem 3.2, has the form $D = -\delta + D$. In this case the isomorphism σ^{-1} and the multiplication $*$ have a very simple explicit form

$$\sigma^{-1}: a(x, h) \mapsto a(x+y, h) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\partial^{(\alpha)}a(x, h)y^{\alpha},$$

$$a * b = \exp\left(-\frac{ih}{2}\omega^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right)a(x, h)b(y, h)|_{y=x} \tag{5.1}$$

$$= \sum_{k=0}^{\infty}\left(-\frac{ih}{2}\right)^k \frac{1}{k!}\omega^{i_1 j_1}\cdots\omega^{i_k j_k}\frac{\partial^k a}{\partial x^{i_1}\cdots\partial x^{j_1}}\frac{\partial^k b}{\partial x^{j_1}\cdots\partial x^{j_k}}.$$

Definition 5.1. The trace in the algebra $W_D(\mathbf{R}^{2n})$ is the linear functional defined on the ideal $W_D^{\mathrm{comp}}(\mathbf{R}^{2n})$, consisting of the flat sections with compact support by formula

$$\operatorname{tr} a = \frac{1}{(2\pi h)^n}\int_{\mathbf{R}^{2n}} \sigma(a)\frac{\omega^n}{n!}. \tag{5.2}$$

Thus the trace has values in Laurent formal series in h with negative powers of h not greater than n, i.e.,

$$\operatorname{tr} a = \sum_{k=0}^{\infty} h^{k-n}c_k.$$

Lemma 5.2. *The trace has the property*

$$\operatorname{tr} a \circ b = \operatorname{tr} b \circ a, \tag{5.3}$$

where $a \in W_D^{\mathrm{comp}}(\mathbf{R}^{2n})$, $b \in W(\mathbf{R}^{2n})$.

Proof. Since $\sigma(a \circ b) = \sigma(a) * \sigma(b)$, according to (5.1) it is sufficient to check the equality

$$\int_{\mathbf{R}^{2n}} \frac{\partial^k a}{\partial x^{i_1}\cdots\partial x^{i_k}}\frac{\partial^k b}{\partial x^{j_1}\cdots\partial^{j_k}}\omega^n = \int_{\mathbf{R}^{2n}} \frac{\partial^k a}{\partial x^{j_1}\cdots\partial x^{j_k}}\frac{\partial^k b}{\partial x^{i_1}\cdots\partial x^{i_k}}\omega^n,$$

which is easily verified by integrating by parts. q.e.d.

We shall prove that property (5.3) implies the invariance of the trace under isomorphisms A_f, considered in §4. For an open contractible set $O \subset \mathbf{R}^{2n}$ we shall denote the algebra of flat sections with support in O by $W_D^{\mathrm{comp}}(O)$. Let f be a symplectic diffeomorphism, defined on O and mapping it onto the open set $f(O)$, and let $A_f\colon W_D^{\mathrm{comp}}(O) \to W_D^{\mathrm{comp}}(f(O))$ be an isomorphism corresponding to f by formula (4.11).

Theorem 5.3. *For any* $a \in W_D^{\mathrm{comp}}(O)$,

$$\operatorname{tr} a = \operatorname{tr}(A_f a). \tag{5.4}$$

Proof. For the proof we construct a family $a(t)$, $t \in [0, 1]$, of flat sections with compact support such that $a(0) = a$, $a(1) = A_f a$, satisfying the Heisenberg equation

$$\dot{a}(t) = (i/h)[H(t), a(t)], \tag{5.5}$$

with the Hamiltonian $H(t) \in W_D(\mathbf{R}^{2n})$. Then according to (5.3) we have

$$\frac{d}{dt} \operatorname{tr} a(t) = \frac{i}{h} \operatorname{tr}[H(t), a(t)] = 0,$$

wherefrom it follows that $\operatorname{tr} a(t) = \mathrm{const}$.

Lemma 5.4. *Let f_t be a family of symplectic diffeomorphisms of the open set O and let*

$$A_t\colon W_D^{\mathrm{comp}}(O) \to W_D^{\mathrm{comp}}(f_t(O)) \subset W_D^{\mathrm{comp}}(\mathbf{R}^{2n})$$

be the corresponding family of isomorphisms

$$A_t a = U_t \circ (f_{t*} a) \circ U^{-1}. \tag{5.6}$$

Then $a(t) = A_t a$ satisfies equation (5.5) *with the Hamiltonian $H(t) \in W_D(\mathbf{R}^{2n})$.*

Proof of the Lemma. We have

$$\dot{a}(t) = \dot{U}_t \circ U_t^{-1} \circ a(t) - a(t) \circ \dot{U}_t \circ U_t^{-1} + U_t \circ (f_{t*} a) \circ U_t^{-1}. \tag{5.7}$$

Denoting the map, inverse to f_t by g_t, we shall have

$$(f_{t*} a) = \frac{d}{dt} a\left(g_t(x), \frac{\partial g_t(x)}{\partial x^i} y^i, h\right) = \left(f_{t*} \frac{\partial a}{\partial x^j}\right) \dot{g}_t^j \left(f_{t*} \frac{\partial a}{\partial y^j}\right) \frac{\partial \dot{g}_t^j}{\partial x^i} y^i .$$

Since the section a is flat, $\partial a/\partial x^j = \partial a/\partial y^j$, so that the last expression can be written in the form

$$f_{t*}\left(\frac{\partial a}{\partial y^j}\right)\left(\dot{g}_t^j+\frac{\partial \dot{g}_t^j}{\partial x^i}y^i\right)=\frac{\partial}{\partial y^k}(f_{t*}a)\frac{\partial f^k}{\partial x^j}\left(\dot{g}^j+\frac{\partial \dot{g}_t^j}{\partial x^i}y^l\right)$$
$$=\frac{i}{h}\left[f_{t*}a\,,\,\omega_{ik}y^i\frac{\partial f^k}{\partial x^j}\left(\dot{g}^j+\frac{1}{2}\frac{\partial \dot{g}_t^j}{\partial x^l}y^l\right)\right].$$

Here we have used the fact that for a symplectic map f the expression $\omega_{ij}u^i(\partial f^k/\partial x^j)(\partial \dot{g}_t^j/\partial x^l)v^l$ is symmetric with respect to $u, v \in \mathbf{R}^{2n}$. Substituting into (5.7), we get equation (5.5), where $H(t)$ belongs to W_1^+ but is, generally speaking, not a flat section. We will show that it is possible to pick $H(t) \in W_D(\mathbf{R}^{2n})$. Applying the operator D to the both sides of (5.5) and taking into account that $Da(t) = 0$ we get $(i/h)[DH(t), A_t a] = 0$. Being fulfilled for any $a \in W_D^{\text{comp}}(O)$, this equation means that $DH(t)$ is a central (i.e., scalar) 1-form ψ, which is closed because

$$d\psi = D\psi = D(DH) = 0.$$

Hence, $\psi = d\varphi$, where $\varphi_t = \varphi_t(x, h)$ is a scalar function, which is uniquely defined if subjected to the normalizing condition $\varphi_t(f_t(x_0)) = 0$, where $x_0 \in O$ is any fixed point. Replacing $H(t)$ by $H(t) - \varphi_t$ we do not change equation (5.5), as φ_t belongs to the center, and, on the other hand, $D(H(t) - \varphi_t) = 0$, i.e., $H(t) = \varphi_t$ is a flat section. q.e.d.

Let us proceed to prove the theorem. According to the lemma, it is sufficient to construct a family of symplectic diffeomorphisms f_t, so that $f_0 = \text{id}$, and $f_1 = f$. Besides, we may confine ourselves to a sufficiently small neighborhood O_{x_0} of an arbitrary fixed point $x_0 \in O$. A general case would be obtained by using a partition of unit subordinated to a sufficiently fine covering of a compact set $\operatorname{supp} a$.

The desired deformation f_t is constructed in two steps. At the first step, consider the linear part L_f of the map f at the point x_0 given by

$$L_f x = f(x_0) + \frac{\partial f(x_0)}{\partial x^i}(x^i - x_0^i).$$

Since the group of linear symplectic transformations is connected, there exists a deformation, connecting the identity map with L_f. At the second step, consider a nonlinear map $L_f^{-1}f(x)$. In a sufficiently small neighborhood of x_0 it is close to the identity map, so it may be given by a generating function $S(z)$ according to the formulas

$$x^i = z^i + \omega^{ij}\frac{\partial S(z)}{\partial z^j}, \qquad f^i(x) = z^i - \omega^{ij}\frac{\partial S(z)}{\partial z^j}. \tag{5.8}$$

From (5.8) we get

$$z = \frac{x + f(x)}{2}, \qquad dS(z) = \frac{1}{2}\omega_{ij}(f^j(x) - x^j)\,dz^j. \tag{5.9}$$

It is easy to see that the 1-form on the right-hand side of the second equation of (5.9) is exact. Indeed, its exterior differential is equal to

$$\begin{aligned}\omega_{ij}(df^i - dx^i) \wedge (df^j + dx^j) = \omega_{ij} df^i \wedge df^j - \omega_{ij}\, dx^i \wedge dx^j \\ + \omega_{ij} df^i \wedge dx^j - \omega_{ij}\, dx^i \wedge df^j.\end{aligned}$$

The first two summands give 0, since f is a symplectic diffeomorphism. The second two summands also give 0, because

$$\omega_{ij}\, dx^i \wedge df^j = -\omega_{ij}\, df^j \wedge dx^i = \omega_{ji}\, df^j \wedge dx^i.$$

Thus, (5.9) determine the generating function $S(z)$, provided the first equation of (5.9) determines a diffeomorphism $x \mapsto z$, which is just so in a sufficiently small neighborhood O_{x_0}. Besides, we have $S(z) = 0(|z|^3)$. Replacing the function $S(z)$ in (5.8) by the functions $tS(z)$, $t \in [0, 1]$, we shall have the desired deformation $f_t(x)$ in the sufficiently small neighborhood O_{x_0} by formulas (5.8).

Remark. If f is a linear transformation $f^j(x) = A^i_j x^j$ with a symplectic matrix A^i_j, formulas (5.8), (5.9) give the Cayley transformation.

6. A localization and a trace

In this section we construct a trace in the algebra $W_D(M)$ on an arbitrary symplectic manifold M. The basic tool is a localization, i.e., a representation of the algebra $W_D(M)$ by a compatible family of the algebras of quantum observables in $\mathbf{R}^{2n}$. We shall denote the standard symplectic form on $\mathbf{R}^{2n}$ by ω_0 and the Abelian connection $-\delta + d$ in $W(\mathbf{R}^{2n})$ by D_0.

Let $\{O_i\}$ be a locally finite covering of the manifold M by local Darboux charts, $\{\rho_i(x)\}$ be a partition of unity subordinated to this covering, and $\chi_i\colon O_i \to \mathbf{R}^{2n}$ be coordinate maps. For a given symplectic connection ∂ and the corresponding Abelian connection D in the bundle $W = W(M)$ consider the algebra $W_D(M)$ of flat sections determined on M and its subalgebra $W_D^{\mathrm{comp}}(O_i)$, consisting of flat sections with supports in O_i. Using the constructions of §4 we may define isomorphisms

$$A_i\colon W_D^{\mathrm{comp}}(O_i) \to W_{D_0}^{\mathrm{comp}}(\chi_i(O_i)) \tag{6.1}$$

of the form (4.11). More precisely, for $a \in W_D^{\mathrm{comp}}(O_i)$ we take its push-forward $\chi_{i*}a$, which is a flat section in $W^{\mathrm{comp}}(\mathbf{R}^{2n})$ with respect to the connection $\widetilde{D}_i = \chi_{i*}D$, i.e., $\chi_{i*}a \in W_{\widetilde{D}_i}^{\mathrm{comp}}(\chi_i(O_i))$. After that we pass from the connection $\widetilde{D}_i$ to the connection D_0 using the conjugation automorphism. Finally we get

$$A_i\colon W_D^{\mathrm{comp}}(O_i) \ni a \mapsto U_i \circ (\chi_{i*}a) \circ U_i^{-1} \in W_{D_0}^{\mathrm{comp}}(\chi_i(O_i)).$$

We shall call A_i coordinate isomorphisms.

For the algebra $W_D^{\mathrm{comp}}(O_i \cap O_j)$ we have two coordinate isomorphisms A_i and A_j, and thus transition isomorphisms are defined as follows:

$$(6.2) \qquad A_{ij} = A_i A_j^{-1}\colon W_{D_0}^{\mathrm{comp}}(\chi_j(O_i \cap O_j)) \to W_{D_0}^{\mathrm{comp}}(\chi_i(O_i \cap O_j)).$$

From (6.2) it immediately follows that A_{ij} satisfies a cocycle condition

$$(6.3) \qquad A_{ij}A_{jk}A_{ki} = \mathrm{id}$$

in the algebra $W_{D_0}^{\mathrm{comp}}(O_i \cap O_j \cap O_k)$.

Using Theorem 3.3 we can construct the flat sections $\hat{\rho}_i = \sigma^{-1}(\rho_i) \in W_D^{\mathrm{comp}}(O_i)$, which form a partition of unity in the algebra $W_D(M)$. Indeed

$$\sum_i \hat{\rho}_i = \sum_i \sigma^{-1}(\rho_i) = \sigma^{-1}\left(\sum_i \rho_u\right) = \sigma^{-1}(1) = 1.$$

So, we obtain a set of flat sections

$$(6.4) \qquad a_i = A_i(\hat{\rho}_i \circ a) \in W_{D_0}^{\mathrm{comp}}(\chi_i(O_i)) \subset W_{D_0}(\mathbf{R}^{2n})$$

corresponding to the flat section $a \in W_D(O_i)$. We shall call this set a local representation of the section a, or shorter a localization. It is clear that $a = \sum_i A_i^{-1} a^i$.

Definition 6.1. A trace in the algebra $W_D(M)$ is a linear functional defined on an ideal $W_D^{\mathrm{comp}}(M)$ with values in Laurent formal series, containing negative powers of h not greater than $n = \frac{1}{2}\dim M$. For any $a \in W_D^{\mathrm{comp}}(M)$ and $b \in W_D(M)$ the equality

$$(6.5) \qquad \mathrm{tr}\, a \circ b = \mathrm{tr}\, b \circ a$$

must be fulfilled.

Theorem 6.2. *A trace in the algebra $W_D(M)$ does exist.*

Proof. For a given coordinate covering $\{O_i\}$ and a partition of unity $\{\rho_i(x)\}$ take

$$(6.6) \qquad \mathrm{tr}\, a = \sum_i \mathrm{tr}\, a_i = \sum_i \mathrm{tr}\, A_i(\hat{\rho}_i \circ a),$$

where $a \in W_D(M)$ and the traces $\operatorname{tr} a_i$ are given by formula (5.2) in $W_{D_0}^{\text{comp}}(\mathbf{R}^{2n})$. We should check the correctness of the definition, i.e., independence of the choice of a covering, a partition of unity, and coordinate isomorphisms A_i, and then prove property (6.5).

Let us prove the independence of the choice of coordinate isomorphisms. Let $a \in W_D^{\text{comp}}(O)$ be a flat section with a support in the coordinate neighborhood $O \subset M$, χ and χ' be two coordinate diffeomorphisms $O \to \mathbf{R}^{2n}$, and A and A' be the corresponding coordinate isomorphisms, mapping $W_D^{\text{comp}}(O)$ onto $W_{D_O}^{\text{comp}}(\chi(O))$, $W_{D_0}^{\text{comp}}(\chi'(O))$ respectively. Then the symplectic map $f = \chi'\chi^{-1}\colon \chi(O) \to \chi'(O)$ and the corresponding isomorphism $A_f\colon W_{D_0}^{\text{comp}}(\chi(O)) \to W_{D_0}^{\text{comp}}(\chi'(O))$ are defined. According to Theorem 5.3 we have

$$\operatorname{tr} A_f A a = \operatorname{tr} A a. \tag{6.7}$$

Generally speaking, the automorphisms $A_f A$ and $A'\colon W_D^{\text{comp}}(O) \to W_{D_0}^{\text{comp}}(\chi'(O))$ do not coincide. However they differ by an inner automorphism of the algebra $W_{D_0}^{\text{comp}}(\chi'(O))$ (see Corollary 4.5), i.e., there exists a section $S \in W_{D_0}^{\text{comp}}(\chi'(O))$, such that $A_f A a = S \circ (A'a) \circ S^{-1}$. Hence, according to property (5.3) of the trace in the algebra $W_{D_0}(\mathbf{R}^{2n})$ we get

$$\operatorname{tr} A_f A a = \operatorname{tr} S \circ (A'a) \circ S^{-1} = \operatorname{tr} A'a,$$

as desired. The independence of a covering and a partition of unity is now proved in a standard way, i.e., by passing to a refined covering $\{O_i \cap O_j\}$ and the corresponding partition of unity $\{\rho_i \rho_j\}$.

Let us prove the equality (6.5). We have

$$a \circ b = \sum_{i,j} (\hat{\rho}_i \circ a) \circ (\hat{\rho}_j \circ b),$$

so

$$\begin{aligned}
\operatorname{tr} a \circ b &= \sum_{i,j} \operatorname{tr} A_i((\hat{\rho}_i \circ a) \circ (\hat{\rho}_j \circ b)) \\
&= \sum_{i,j} \operatorname{tr} A_j((\hat{\rho}_i \circ a) \circ (\hat{\rho}_j \circ b)) = \sum_{i,j} \operatorname{tr}(A_j(\hat{\rho}_i \circ a)) \circ (A_j(\hat{\rho}_j \circ b)) \\
&= \sum_{i,j} \operatorname{tr}(A_j(\hat{\rho}_j \circ b)) \circ (A_j(\hat{\rho}_i \circ a)) = \sum_{i,j} \operatorname{tr} A_j((\hat{\rho}_j \circ b) \circ (\hat{\rho}_i \circ a)) \\
&= \operatorname{tr} b \circ a,
\end{aligned}$$

which proves the theorem.

7. Generalizations

The above constructions of the deformation quantization and the trace in the algebra of quantum observables allow different generalizations.

Quantization with coefficients in $\mathrm{Hom}(E, E)$. An evident generalization consists in considering matrix coefficients. No change is necessary, except that in the definition of the trace (5.2) a matrix trace under integral sign must be taken. A less evident generalization is obtained if we admit that the coefficients $a_{k, i_1 \cdots i_p, j_1 \cdots j_q}(x)$ of series (2.3), (3.4) take values in a bundle $\mathrm{Hom}(E, E)$, where E is a vector bundle over M. Let us consider this case in more detail.

Let ∂_s be a symplectic connection on M, and ∂_E be a connection in the vector bundle E. Then $\partial = \partial_s \otimes 1 + 1 \otimes \partial_E$ defines the connection in Weyl algebras bundle $W \otimes \mathrm{Hom}(E, E)$; we will denote this bundle by W as before for short. We shall look for Abelian connection D in the bundle W in the same form as in (3.1). The same equation (3.3) is obtained for r, R being now equal to

$$(h/2)R^E_{ij}\, dx^i \wedge dx^j + \tfrac{1}{4}R^s_{ijkl} y^i y^j\, dx^k \wedge dx^l,$$

where the first term is the curvature of ∂_E, and the second one is the same as in (2.11); the superscript s means "symplectic". Theorems 3.2 and 3.3 are completely valid in this case.

As to the action of symplectic diffeomorphisms, the results of §4 are also valid with some modifications. Let $f_s\colon M \to M$ be a symplectic diffeomorphism of M, and $f_{s*}E = (f_s^{-1})^*E$ be a pushforward of the bundle E under f_s, i.e., $(f_s^{-1})^*$ is an induced bundle. Let a fiberwise homomorphism $\varphi\colon (f_{s*}E) \to E$ be given as well. Then the formula

$$(f_8 a)(x) = \varphi a(f^{-1}(x))\varphi^{-1} \tag{7.1}$$

defines a lifting of the map f^s onto a bundle space $\mathrm{Hom}(E, E)$. We define the pushforward of a section $a(x, y, h) \in W \otimes \mathrm{Hom}(E, E)$, by assuming

$$(f_* a)(x, y, h) = \varphi a(f_s^{-1}(x), (f_s')^{-1}y, h)\varphi^{-1}. \tag{7.2}$$

So, if lifting (7.1) is given, the pushforwards and pullbacks of the sections and the connections ∂ and D are defined as in §4.

In the case of coefficients in $\mathrm{Hom}(E, E)$ localization, considered in §6, is constructed as before with some modifications. More exactly, not only a coordinate mapping $\chi_s\colon O \to \mathbf{R}^{2n}$ is to be given, but the trivialization of the bundle E as well. The trivialization defines a lifting χ of χ_s, so that sections of $\operatorname{Hom}(E, E)$ over O go to matrix-valued functions on $\chi_s(O)$ and allow us to define coordinate isomorphisms

$$A\colon W_D^{\text{comp}} \to W_{D_0}^{\text{comp}}(\chi_s(O)),$$

and then a trace can be defined as before by formulas (6.6), (5.2) with the matrix trace under integral sign in (5.2).

Deformation quantization of regular Poisson manifolds. As mentioned in the introduction, a regular Poisson manifold has a symplectic foliation. It means that it is possible to introduce local coordinates (Darboux coordinates)

$$x^1, x^2, \cdots, x^{2n}, x^{2n+1}, \cdots, x^m; \qquad 2n = \operatorname{rank}(t^{ij}), \quad m = \dim M,$$

in which the components of Poisson tensor t^{ij} have the form

$$t^{i,i+n} = 1, \quad t^{i+n,i} = -1; \qquad i = 1, 2, \cdots, n,$$

and the rest of its components are equal to 0. The leaves F of the foliation are locally defined by equations $x^k = \text{const}$, $k = 2n+1, \cdots, m$, and the form $\omega = \sum_{i=1}^n dx^i \wedge dx^{n+i}$ defines a symplectic structure on the leaves. Thus the regular Poisson manifold can be locally considered as a family of symplectic manifolds depending on parameters $x^{2n+1}, \cdots, x^m$. The quantization construction, given in §§2, 3, smoothly depends on parameters and is local, so it is evidently valid for the case of regular Poisson manifolds.

More precisely, the construction looks as follows. We consider a tangent bundle TF along the leaves and the exterior algebra $\Lambda_F = \Lambda(T^*F)$. A homomorphism $i^*\colon T^*M \to T^*F$ is defined, induced by a local embedding of the leaf $i\colon F \to M$. In Darboux local coordinates we introduce the natural basis of vector fields $e_k = \partial/\partial x^k$ $(k = 1, \cdots, 2n)$ tangent to the leaves and the dual basis $\theta = (\theta^1, \cdots, \theta^{2n})$ in T^*F. Instead of series (2.4) we will now consider the series

$$\begin{aligned} a &= \sum_{2k+p\geq 0} h^k a_{k,p,q} \\ &= \sum_{2k+p\geq 0} h^k a_{k,i_1\cdots i_p,j_1\cdots j_q}(x) y^{i_1}\cdots y^{i_p} dx^{j_1}\wedge\cdots\wedge dx^{j_q}, \end{aligned} \tag{7.3}$$

where the terms $a_{k,p,q}$ are the sections of $S^p(TF)\otimes\Lambda^q(T^*F)$, $x\in M$, the range of the indices i_k, j_k is from 1 to $2n$. Such series form an algebra with respect to the fiberwise multiplication $\circ$ (the exterior product of θ^j and Weyl product (2.2) for monomials in y's). The algebra of such series will be denoted by $W \otimes \Lambda_F$. The operators δ, δ^*, δ^{-1} are introduced similar to (2.5) as follows:

$$\delta a = \theta^k \wedge \frac{\partial a}{\partial y^k}; \qquad \delta^* a = y^k i(e_k) a,$$
$$(7.4) \qquad \delta^{-1} a_{k,p,q} = \frac{1}{p+q}\delta^* a_{k,p,q}, \quad (p+q>0); \qquad \delta^{-1} a_{k,0,0} = 0,$$

and have the same properties, including Hodge–De Rham decomposition.

We shall also need a Poisson connection along the leaves $\partial : C^\infty(TF) \to C^\infty(TF \otimes \Lambda_F^1)$. For such a connection, its local restriction on each leaf gives a symplectic connection on the leaf. For the sake of completeness let us give the construction of such a connection.

Let ∇ be an arbitrary connection in the bundle TF over M. Let us denote the indices ranging from 1 to $2n$ by Roman letters and those ranging from 1 to m by Greek letters. In Darboux local coordinates we have

$$\nabla e^i = e^j \Gamma^i_{j\alpha} dx^\alpha .$$

Restricting it to the vectors tangent to the leaves, we get a connection ∇_F along the foliation

$$(7.5) \qquad \nabla_F e^i = e^j \Gamma^i_{jk} \theta^k ,$$

the Jacobian matrix of the transition diffeomorphism between two Darboux local charts

$$f^\alpha_\beta = \partial x'^\alpha / \partial x^\beta$$

has a triangular form, because

$$\partial x'^\alpha / \partial x^i = 0, \qquad \alpha = 2n+1, \cdots, m,$$

and its upper left block (f^i_j) gives a transition function of the bundle TF. Hence, a skew-symmetric part of the connection coefficients defines a tensor $\Gamma^i_{\{jk\}}$ in the bundle TF (a torsion tensor), since $\partial f^i_j / \partial x^k = \partial^2 f^i / \partial x^j \partial x^k$ are symmetric with respect to j, k. Thus, symmetrizing the coefficients Γ^i_{jk} in (7.5), we get a new torsion-free connection $\widetilde{\nabla}_F$ along the foliation.

Finally we find a tensor $\Delta\Gamma^i_{jk}$, which is symmetric in lower indices and such that the connection $\partial = \widetilde{\nabla}_F + \Gamma^i_{jk}\theta^k$ preserves the tensor ω_{ij} inverse to t^{ij}. We have

$$\partial_k \omega_{ij} = (\widetilde{\nabla}_F)_k \omega_{ij} - \Delta\Gamma^p_{ik}\omega_{pj} - \Delta\Gamma^p_{jk}\omega_{ip} = 0,$$

wherefrom we obtain the equations

$$\Delta\Gamma_{ijk} - \Delta\Gamma_{jik} = (\widetilde{\nabla}_F)_k \omega_{ij} \tag{7.6}$$

for $\Delta\Gamma_{ijk} = \omega_{ip}\Delta\Gamma^p_{jk}$. In Darboux local coordinates

$$(\widetilde{\nabla}_F)_k \omega_{ij} = \widetilde{\Gamma}^p_{ik}\omega_{pj} + \widetilde{\Gamma}^p_{jk}\omega_{ip} = \widetilde{\Gamma}_{ijk} - \widetilde{\Gamma}_{jik},$$

$\widetilde{\Gamma}^i_{jk}$ being the coefficients of $\widetilde{\nabla}_F$. A partial solution of system (7.6) is given by

$$\Delta\Gamma_{ijk} = \tfrac{1}{3}(2\widetilde{\Gamma}_{ijk} - \widetilde{\Gamma}_{jki} - \widetilde{\Gamma}_{kij}), \tag{7.7}$$

(the general solution is obtained by adding to (7.7) any completely symmetric 3-tensor).

Thus, we obtain the connection ∂ along the leaves in the bundle TF such that its restriction to any leaf gives a symplectic connection on the leaf. According to (7.7) it smoothly depends on the coordinates $x^{2n+1}, \cdots, x^m$, which are parameters, defining the leaf. Theorems 3.2 and 3.3 give a smooth dependence on these parameters and thus define quantization for regular Poisson manifolds.

As for the results of §§4, 5, 6 it is not quite clear whether a reasonable generalization of these results for regular Poisson manifolds could be made.

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