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L^p COHOMOLOGY OF CONES AND HORNS

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Abstract

We prove the conjecture of J. Brasselet, M. Goresky, and R. MacPherson on the isomorphism between L^p cohomology and intersection cohomology for a stratified space with a Riemannian metric and conical singularities. We prove the extension of this conjecture to spaces with f-horn singularities, where $f(r)$ is any C^∞ nondecreasing function.

We study the L^p Stokes property which states that the minimal closed extension of d acting on L^p forms coincides with the maximal one. We prove that it implies the Borel-Moorse duality between the complexes of L^p forms and L^q forms. We also prove the converse for spaces with f-horn singularities under the condition that the integral $\int_0^\varepsilon f(r)^{-1}\,dr$ diverges.

1. Introduction

J. Cheeger [4] discovered that the L^2 cohomology of a compact stratified pseudomanifold with respect to a metric which has conical singularities, is isomorphic to its upper middle-perversity itersection cohomology. He showed that this is also the case for the singular metrics which he called *f-horns*; locally they are of the form $dr^2 + f(r)^2 g$ where r is the radial coordinate (i.e., the distance from the singular point), g is the metric on the link of the point, and f is a function of r of the form $f(r) = r^c$ with $c \geq 1$; in case $c = 1$ we get a cone. If L is the link, then the f-horn over it is denoted $C^f L$. (See Definitions 3.1.1 and 3.2.1.)

M. Nagase [7] considered the case $c < 1$ and showed that when L^2 cohomology is isomorphic to the intersection cohomology, although with a different perversity, greater than the middle one and dependent upon the value of c.

J. Brasselet, M. Goresky, and R. MacPherson [2] conjectured that the L^p cohomology of a metric with conical singularities is isomorphic to the intersection cohomology with a perversity $\overline{p}$ which corresponds to L^p cohomology: $\overline{p}(k) = \max\{i \in \mathbb{Z} | i < k/p\}$.

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Here we consider the L^p cohomology of f-horns for any p, $1 < p < \infty$, and for any function f which is nondecreasing and C^∞ for $r > 0$. We introduce the (L^p, f)-perversity $\mathrm{perv}_{L^p, f}$ to generalize $\overline{p}$ (see Definition 3.1.4) and we prove the following generalization of the above conjecture.

Theorem (See Theorems 3.1.2 and 3.3.1.). (a) *Suppose that the maximal closure of the operator d in the Banach space of all L^p forms on the link L has closed range. Then the L^p cohomology of the horn $C^f L$ is given by*

$$H^k_{L^p}(C^f L) = \begin{cases} H^k_{L^p}(L) & \textit{if } k \leq \mathrm{perv}_{L^p, f}(\dim L + 1), \\ 0 & \textit{if } k > \mathrm{perv}_{L^p, f}(\dim L + 1). \end{cases}$$

(b) *If X is a stratified space with a metric and f-horn singularities, then $H^k_{L^p}(X) \cong IH^k_{\mathrm{perv}_{L^p, f}}(X)$.*

The idea behind this theorem is as follows. In the low degrees the radially constant forms (the ones which are pullbacks from the link) on the horn $C^f L$ are L^p integrable, and this is why the L^p cohomology of $C^f L$ are isomorphic to the L^p cohomology of the link. In the higher degrees the radially constant forms are no longer L^p integrable, and the L^p cohomology of $C^f L$ are zero.

The proof uses the explicit integral homotopy operators, introduced by Cheeger [4], which come from the two contractions: the first one contracts the horn (which is topologically a cone) to its vertex and is used to prove the vanishing of L^p cohomology in the higher degrees; the second one contracts the punctured horn (with the vertex removed) to the link and is used to prove the cohomology isomorphism in the lower degrees.

Our second result concerns the L^p *Stokes property*. This property can be formulated as follows: the maximal closed extension of the operator d in the Banach space of all L^p forms coincides with its minimal closed extension, so that no "ideal boundary conditions" at the singularities can be imposed. (See Definition 4.1.1.)

This property was first introduced by Cheeger [4] in case $p = 2$ (he called it L^2 *Stokes theorem*) for the following purpose: he showed that it implies that the natural homomorphism from the space of L^2 harmonic forms to L^2 cohomology is monomorphic.

We reformulate this property in the sheaf-theoretic language (see Definition 4.1.1) and show that it implies the duality between L^p cohomology and L^q cohomology for $1/p + 1/q = 1$, where the duality is understood in the Borel-Moore sense (see Theorem 4.3.1). In case $p = q = 2$ this means that the L^2 Stokes property implies that L^2 cohomology is dual to itself, i.e., satisfies Poincaré duality, in agreement with the results of [4].

In case $p = 2$ [4] it turned out that the converse is true: if the L^2 cohomology is dual to itself, then the L^2 Stokes property holds. More precisely, according to the calculations of [4], the L^2 cohomology is isomorphic to the intersection cohomology with respect to the *upper* middle perversity, hence, it is dual to the intersection cohomology with respect to the *lower* middle perversity; the two are the same on the sheaf theoretic level if and only if the middle-degree L^2 cohomology groups of the links are zero (or the links are odd-dimensional, as in case of complex-analytic spaces). According to [4], this is the condition for the L^2 Stokes property to hold. In other words, although the L^2 Stokes property does not hold always, the only obstruction to it is of cohomological nature: the L^2 Stokes property holds if and only if there is duality in the L^2 cohomology.

We generalize this statement to the case of any p and any f.

Theorem (See Theorem 4.9.1). *Suppose that the integral* $\int_0^\varepsilon f(r)^{-1}\,dr$ *diverges. Then the* L^p *Stokes property holds on a space with* f*-horn singularities if and only if its* L^p *cohomology is Borel-Moore dual to its* L^q *cohomology.*

The condition that the integral $\int_0^\varepsilon f(r)^{-1}\,dr$ diverges, is sharp: if it is not satisfied, then for any space with f-horn singularities and for some p, the L^p Stokes property does not hold (see Example 5.12.1) even if there is no cohomological obstruction to it (i.e., if the duality holds between L^p cohomology and L^q cohomology).

In §2 we develop a general theory of L^p cohomology. We define the relevant complexes of sheaves and apply them to show that the L^p cohomology defined by using only the smooth L^p forms is the same as the L^p cohomology defined by using all L^p forms. We give a condition for our sheaves to be fine ("partitions of unity with bounded differentials").

In §3 we define the horn singularities and formulate our theorems concerning the isomorphism of L^p cohomology and intersection cohomology.

In §4 we define the L^p Stokes property, prove that it implies the cohomological duality in general, and formulate the converse in case of horns.

In §5 we prove our theorems about spaces with horn singularities.

2. Generalities on L^p cohomology

2.1. The two closures of d**.** Let (X, g) be any Riemannian manifold, not necessarily compact, and let $\mathbb{E}$ be a unitary local system on X with the pairing $\mathbb{E}\otimes\overline{\mathbb{E}}\to\mathbb{C}$ fixed. We shall assume that all differential forms take values in $\mathbb{E}$.

Denote by $\Lambda_0^k(X,\mathbb{E})$ the space of C^∞ k-forms on X with compact support. Let p be a real number, $1<p<\infty$, and denote by $\Omega_{L^p}^k(X,g,\mathbb{E})$ the space of such k-forms ω on X with locally summable coefficients for which the integral $\int_X|\omega|^p d\,\mathrm{vol}_g$ converges, where $|\omega|$ is the pointwise norm of ω with respect to g, and $d\,\mathrm{vol}_g$ is the volume form of g. This is a Banach space with the norm $\|\omega\|_{L^p}=(\int_X|\omega|^p d\,\mathrm{vol}_g)^{1/p}$. Denote by $\Lambda_{L^p}^k(X,g,\mathbb{E})$ the space of all C^∞ forms lying in $\Omega_{L^p}^k(X,g,\mathbb{E})$. We shall omit X, g and $\mathbb{E}$ from the notation when it does not cause confusion.

We denote $\Lambda_{L^p}^\bullet(X,g,\mathbb{E})=\bigoplus_k\Lambda_{L^p}^k(X)$, $\Omega_{L^p}^\bullet(X,g,\mathbb{E})=\bigoplus_k\Omega_{L^p}^k(X)$. Let

$$\operatorname{dom}d_{L^p}^\bullet(X,g,\mathbb{E})=\{\omega\in\Lambda_{L^p}^\bullet(X)\,|\,d\omega\in\Lambda_{L^p}^\bullet(X)\},$$
$$\operatorname{dom}d_{L^p}^\bullet(X,g,\mathbb{E})=\bigoplus\operatorname{dom}d_{L^p}^k(X,g,\mathbb{E}).$$

All these topological vector spaces depend only on the quasi-isometry class of g. (Two metrics g and g' are said to be quasi-isometric if there exist global constants $C,C'>0$ such that $Cg<g'<C'g$.)

The operator d in $\Omega_{L^p}^\bullet(X)$ is densely defined on the subspace $\operatorname{dom}d_{L^p}^\bullet(X)$. It has a weak closure: $d\alpha=\beta$ in the weak sense if $\alpha,\beta\in\Omega_{L^p}^\bullet(X)$ and $d\alpha=\beta$ as distributions. The following proposition is well known.

Proposition 2.1.1. *The weak closure of d in $\Omega_{L^p}^\bullet(X)$ is also its strong closure, and the latter is well defined.*

Proof. We need to show that if $d\alpha=\beta$ in the weak sense, then for any $\varepsilon>0$ we can find $\alpha_\varepsilon\in\operatorname{dom}d_{L^p}^\bullet(X)$ such that $\|\alpha-\alpha_\varepsilon\|_{L^p}<\varepsilon$, $\|d\alpha-d\alpha_\varepsilon\|_{L^p}<\varepsilon$. If X is a domain in $\mathbb{R}^n$ with Euclidean metric, then this statement is due to [5]; the general case follows immediately. q.e.d.

Denote the weak (= strong) closure of d in $\Omega_{L^p}^\bullet(X)$ by $\overline{d}$, and its domain by $\operatorname{dom}\overline{d}_{L^p}^\bullet(X,g,\mathbb{E})$.

The Banach spaces $\Omega_{L^p}^\bullet(X,\mathbb{E})$ and $\Omega_{L^q}^\bullet(X,\overline{\mathbb{E}})$ are dual to each other if $1/p+1/q=1$; we define the duality pairing by

$$(\omega,\phi)\mapsto\int_X(-1)^{k(k+1)/2}\omega\wedge\phi,$$

where $\omega\in\Omega_{L^p}^\bullet(X,\mathbb{E})$, $\phi\in\Omega_{L^q}^\bullet(X,\overline{\mathbb{E}})$, $k=\deg\omega$.

Proposition 2.1.2. *The adjoint to the operator $\overline{d}$ in $\Omega_{L^q}^\bullet(X,\overline{\mathbb{E}})$ is the strong closure of $d\colon\Lambda_0^\bullet(X)\to\Lambda_0^\bullet(X)$ in the space $\omega\in\Omega_{L^p}^\bullet(X,\mathbb{E})$. In particular, this strong closure is well defined.*

(The proof goes by a standard argument.)

Denote by $\overline{d}_{\min}$ the strong closure of $d: \Lambda_0^\bullet(X) \to \Lambda_0^\bullet(X)$ in $\Omega_{L^p}^\bullet(X, \mathbb{E})$, and its domain by $\operatorname{dom} \overline{d}_{L^p,\min}^\bullet(X, g, \mathbb{E})$.

We shall consider $\Lambda_0^\bullet(X)$, $\operatorname{dom} d_{L^p}^\bullet(X)$, $\operatorname{dom} \overline{d}_{L^p}^\bullet(X)$, and $\operatorname{dom} \overline{d}_{L^p,\min}^\bullet(X)$ as complexes with differentials d_0, d, $\overline{d}$, and $\overline{d}_{\min}$.

2.2. The cohomological approximation theorem.

Theorem 2.2.1. *The inclusion* $\operatorname{dom} d_{L^p}^\bullet(X) \hookrightarrow \operatorname{dom} \overline{d}_{L^p}^\bullet(X)$ *induces the isomorphism of cohomology.*

The proof of Theorem 2.2.1 is in §2.7.

In case $p = 2$ this theorem was proved by J. Cheeger [4, Appendix]. However, I chose not to follow Cheeger's ideas as I had trouble understanding one point in the proof there, namely, why the operator $R_\varepsilon = \cdots \circ R_{\varepsilon_2,2} \circ R_{\varepsilon_1,1}$ on page 144 of [4] is always smoothing.

My trouble is as follows. Denote the chart in Y on which the operator $R_{\varepsilon_i,i}$ is constructed, by $\phi_i: (-1, 1)^n \xrightarrow{\sim} U_i \subset X$. This operator has the property that for any L^2 form θ on Y, the form $R_{\varepsilon_i,i}\theta$ is smooth on $\phi_i(-1/4, 1/4)^i$ and coincides with θ outside of $\phi_i(-1/2, 1/2)^n$. However, the operator $R_{\varepsilon_i,i}$ may, in principle, destroy the smoothness properties in the region $\phi_i[(-1/2, 1/2)^n \backslash (-1/4, 1/4)^n]$ as it involves the integral operator $\mathscr{H}$. Hence, even if the composite operator $R_{\varepsilon_{i-1},i-1} \circ \cdots \circ R_{\varepsilon_1,1}$ is smoothing on the union of the regions $\phi_j(-1/4, 1/4)^n$ for $j < i$, the application of $R_{\varepsilon_i,i}$ may destroy the smoothing property on the part of this union which intersects with $\phi_i[(-1/2, 1/2)^n \backslash (-1/4, 1/4)^n]$. This is why it is not clear to me why the operator $R_\varepsilon = \cdots \circ R_{\varepsilon_2,2} \circ R_{\varepsilon_1,1}$ is smoothing.

2.3. The sheaves.

Definition 2.3.1. A *singular Riemannian space* is a topological space with an additional structure consisting of an open subset $\mathring{X} \subset X$ ("the top stratum") and the structure of a C^∞ Riemannian manifold on $\mathring{X}$.

By abuse of notation we shall often denote a singular Riemannian space and its underlying topological space by the same symbol.

Let X be a singular Riemannian space, and let $\mathbb{E}$ be a unitary local system on $\mathring{X}$ with the pairing $\mathbb{E} \otimes \overline{\mathbb{E}} \to \mathbb{C}$.

We define the sheaf $\Omega_{L^p,X,\mathbb{E}}^\bullet$ in the following way: for each open set $U \subset X$, the space of sections $\Gamma(U; \Omega_{L^p,X,\mathbb{E}}^\bullet)$ is the space of all forms ω on $U \cap \mathring{X}$ which have the property that for every point $P \in U$ there is some neighborhood V of P in U such that $\omega|_{V\cap \overset{\circ}{X}} \in \Omega^\bullet_{L^p}(V \cap \overset{\circ}{X}, \mathbb{E})$. In other words, $\Omega^\bullet_{L^p,X,\mathbb{E}}$ is the sheaf of forms which are locally L^p in a neighborhood of any point of U, regular or singular. This means that these forms must be L^p at all regular points and satisfy certain growth condition near the singular points; there is no restriction on the growth at the boundary of U.

The sheaves $\Lambda^\bullet_{L^p,X,\mathbb{E}}$, $\operatorname{dom} d^\bullet_{L^p,X,\mathbb{E}}$ and $\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,X,\mathbb{E}}$ are defined in a similar way: a section at P of one of these sheaves is required to lie in the space $\Lambda^\bullet_{L^p}(V \cap \overset{\circ}{X}, \mathbb{E})$, $\operatorname{dom} d^\bullet_{L^p}(V \cap \overset{\circ}{X}, \mathbb{E})$, or $\operatorname{dom} \overrightarrow{d}^\bullet_{L^p}(V \cap \overset{\circ}{X}, \mathbb{E})$ respectively, for some open neighborhood V of P. All these sheaves are the sheaves associated to the presheaves formed by the corresponding vector spaces of L^p forms.

The sheaves $\Lambda^\bullet_{0,X,\mathbb{E}}$ and $\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,\min,X,\mathbb{E}}$ must be defined somewhat more carefully: a section at P must coincide with some element of the space $\Lambda^\bullet_0(V \cap \overset{\circ}{X}, \mathbb{E})$ or $\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,\min}(V \cap \overset{\circ}{X}, \mathbb{E})$ in a smaller neighborhood V' of P. The reason is that the elements of the spaces $\Lambda^\bullet_0(V \cap \overset{\circ}{X})$ and $\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,\min}(V \cap \overset{\circ}{X})$ satisfy certain vanishing conditions at the outside boundary of V which we do not impose on the sections of the sheaves $\Lambda^\bullet_{0,X,\mathbb{E}}$ and $\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,\min,X,\mathbb{E}}$.

All these sheaves are graded by the degree of the forms so that

$$\Omega^\bullet_{L^p,X,\mathbb{E}} = \bigoplus_k \Omega^k_{L^p,X,\mathbb{E}},$$

$$\Lambda^\bullet_{L^p,X,\mathbb{E}} = \bigoplus_k \Lambda^k_{L^p,X,\mathbb{E}},$$

$$\operatorname{dom} d^\bullet_{L^p,X,\mathbb{E}} = \bigoplus_k \operatorname{dom} d^k_{L^p,X,\mathbb{E}},$$

$$\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,X,\mathbb{E}} = \bigoplus_k \operatorname{dom} \overrightarrow{d}^k_{L^p,X,\mathbb{E}},$$

$$\Lambda^\bullet_{0,X,\mathbb{E}} = \bigoplus_k \Lambda^k_{0,X,\mathbb{E}},$$

$$\operatorname{dom} \overrightarrow{d}^\bullet_{L^p,\min,X,\mathbb{E}} = \bigoplus_k \operatorname{dom} \overrightarrow{d}^k_{L^p,\min,X,\mathbb{E}}.$$

As before, we shall drop $\mathbb{E}$, and even X, from the notation when it does not cause confusion.

Note that $\Lambda^\bullet_{0,X} = j_!\Lambda^\bullet_{L^p,\overset{\circ}{X}}$ where $j\colon \overset{\circ}{X} \hookrightarrow X$ is the imbedding map. We also define the sheaf

$$\operatorname{dom}\overline{d}^\bullet_{L^p,0,X,\mathbb{E}} = \bigoplus_k \operatorname{dom}\overline{d}^k_{L^p,0,X,\mathbb{E}}$$

by the formula $\operatorname{dom}\overline{d}^\bullet_{L^p,0,X} = j_!\operatorname{dom}\overline{d}^\bullet_{L^p,\overset{\circ}{X}}$; this is the subsheaf of $\operatorname{dom}\overline{d}^\bullet_{L^p,X}$ consisting of all forms whose support does not intersect $X\backslash\overset{\circ}{X}$.

The differentials d, $\overline{d}$, d_0, $\overline{d}_{\min}$, and $\overline{d}_0$ make $\operatorname{dom} d^\bullet_{L^p}$, $\operatorname{dom}\overline{d}^\bullet_{L^p}$, $\Lambda^\bullet_0$, $\operatorname{dom}\overline{d}^\bullet_{L^p,\min}$, and $\operatorname{dom}\overline{d}^\bullet_{L^p,0}$ complexes of sheaves.

If X is compact, then

$$\operatorname{dom} d^\bullet_{L^p}(\overset{\circ}{X}) = \Gamma(X; \operatorname{dom} D^\bullet_{L^p,X}),$$
$$\operatorname{dom}\overline{d}^\bullet_{L^p}(\overset{\circ}{X}) = \Gamma(X; \operatorname{dom}\overline{d}^\bullet_{L^p,X}),$$
$$\Lambda^\bullet_0(\overset{\circ}{X}) = \Gamma(X; \Lambda^\bullet_{0,X}),$$
$$\operatorname{dom}\overline{d}^\bullet_{L^p,\min}(\overset{\circ}{X}) = \Gamma(X; \operatorname{dom}\overline{d}^\bullet_{L^p,\min X}).$$

We shall sometimes write $\operatorname{dom} d^\bullet_{L^p}(X)$ and $\operatorname{dom}\overline{d}^\bullet_{L^p}(X)$ instead of $\operatorname{dom} d^\bullet_{L^p}(\overset{\circ}{X})$ and $\operatorname{dom}\overline{d}^\bullet_{L^p}(\overset{\circ}{X})$, respectively.

2.4. Collars and cubes. Consider the open segment $(0, 1)$ with the Euclidean metric dt^2, and the product $(0, 1)\times X$ with the direct product metric and with the local system pulled back from X. By $[0, 1)$, $(0, 1]$, and $[0, 1]$ we shall denote the segment containing one or both of its endpoints.

Theorem 2.4.1. *The homomorphisms of complexes*

$$\operatorname{pr}^*\colon \operatorname{dom} d^\bullet_{L^p}(X) \to \operatorname{dom} d^\bullet_{L^p}((0, 1)\times X)$$

and

$$\operatorname{pr}^*\colon \operatorname{dom}\overline{d}^\bullet_{L^p}(X) \to \operatorname{dom}\overline{d}^\bullet_{L^p}((0, 1)\times X)$$

defined by the projection $\operatorname{pr}\colon (0, 1)\times X \to X$, *induce cohomology isomorphisms.*

For $p = 2$, $\mathbb{E} = \mathbb{C}$, this is Theorem 2.1 of [4]. The proof given there extends to the general case without any changes whatsoever.

Corollary 2.4.2. *For any local system* $\mathbb{E}$ *on* X *as above, the homomorphisms*

$$\operatorname{dom} d^\bullet_{L^p}(X) \to \operatorname{dom} d^\bullet_{L^p}((0, 1)^n\times X)$$

and

$$\operatorname{dom} \overline{d}^{\bullet}_{L^p}(X) \to \operatorname{dom} \overline{d}^{\bullet}_{L^p}((0, 1)^n \times X)$$

induce cohomology isomorphisms.

Consider the case $X = (0, 1)^n$ with the Euclidean metric. Of course, in this case the local system $\mathbb{E}$ has to be trivial, say, $\mathbb{E}$ has to be trivial, say, $\mathbb{E} = \mathbb{C}$.

Corollary 2.4.3. *The complexes* $\operatorname{dom} d^{\bullet}_{L^p}((0, 1)^n, \mathbb{C})$ *and* $\operatorname{dom} \overline{d}^{\bullet}_{L^p}((0, 1)^n, \mathbb{C})$ *are resolutions of the trivial local system* $\mathbb{C}$.

Both corollaries are obvious.

Let us turn to the "sheaf-theoretic" analogues of these results. Let X be a singular Riemannian space. We shall consider $(0, 1] \times X$ as a singular Riemannian space in the following straightforward way: its top stratum is $(0, 1) \times \overset{\circ}{X}$ and the metric is the direct product. As before, we assume the local system is pulled back from $\overset{\circ}{X}$. A homomorphism of complexes of sheaves is called a quasi-isomorphism if it induces isomorphisms on the stalk cohomology.

Corollary 2.4.4. *The homomorphisms of complexes of sheaves* $\operatorname{pr}^* \operatorname{dom} d^{\bullet}_{L^p, X} \hookrightarrow \operatorname{dom} d^{\bullet}_{L^p, (0,1] \times X}$ *and* $\operatorname{pr}^* \operatorname{dom} \overline{d}^{\bullet}_{L^p, X} \hookrightarrow \operatorname{dom} \overline{d}^{\bullet}_{L^p, (0,1] \times X}$ *are quasi-isomorphisms.*

Proof. This follows immediately from Theorem 2.4.1.

Theorem 2.4.5. *The homomorphisms of complexes*

$$\operatorname{pr}^* : \Gamma(X; \operatorname{dom} d^{\bullet}_{L^p, X}) \to \Gamma((0, 1)^n \times X; \operatorname{dom} d^{\bullet}_{L^p, (0,1)^n \times X})$$

and

$$\operatorname{pr}^* : \Gamma(X; \operatorname{dom} \overline{d}^{\bullet}_{L^p, X}) \to \Gamma((0, 1)^n \times X; \operatorname{dom} \overline{d}^{\bullet}_{L^p, (0,1)^n \times X})$$

induce cohomology isomorphisms.

Proof. The statement is similar to Theorem 2.4.1 and Corollary 2.4.2. The same homotopy operators (see §3 of [4] or the proof of Lemma 5.2.3) work equally well in this case.

Theorem 2.4.6. *For any smooth contractible Riemannian manifold* Y, *the homomorphisms of complexes*

$$\operatorname{pr}^* : \Gamma(X; \operatorname{dom} d^{\bullet}_{L^p, X}) \to \Gamma(Y \times X; \operatorname{dom} d^{\bullet}_{L^p, Y \times X})$$

and

$$\operatorname{pr}^* : \Gamma(X; \operatorname{dom} \overline{d}^{\bullet}_{L^p, X}) \to \Gamma(Y \times X; \operatorname{dom} \overline{d}^{\bullet}_{L^p, Y \times X})$$

induce cohomology isomorphisms, where pr *denotes the projection* $Y \times X \to X$.

Proof. Denote by pr_Y the projection $Y \times X \to Y$. It follows from Theorem 2.4.5 that the complex of sheaves $\operatorname{pr}_{Y,*} \operatorname{dom} d^{\bullet}_{L^p, Y \times X}$ on Y

is quasi-isomorphic to the complex of constant sheaves on Y given by $\Gamma(X; \operatorname{dom} d^\bullet_{L^p,X})$.

Since Y is contractible, the hypercohomology of any complex of sheaves $\mathcal{F}^\bullet$ on Y (which is a quasi-isomorphism invariant) is isomorphic to the cohomology of the complex of global sections $\Gamma(Y; \mathcal{F}^\bullet)$, which is, consequently, also a quasi-isomorphism invariant.

Hence, the cohomology of the complex $\Gamma(X; \operatorname{dom} d^\bullet_{L^p,X})$ is isomorphic to the cohomology of $\Gamma(Y; \operatorname{pr}_{Y,*} \operatorname{dom} d^\bullet_{L^p,Y\times X}) = \Gamma(Y\times X; \operatorname{dom} d^\bullet_{L^p,Y\times X})$.

This is the first statement of our theorem, and the second statement is proved similarly. q.e.d.

Consider the imbedding $j\colon (0, 1)\times X \hookrightarrow (0, 1]\times X$.

Corollary 2.4.7. *The imbeddings of complexes of sheaves*

$$\operatorname{dom} d^\bullet_{L^p,(0,1]\times X} \hookrightarrow j_* \operatorname{dom} d^\bullet_{L^p,(0,1)\times X}$$

and

$$\operatorname{dom} \overline{d}^\bullet_{L^p,(0,1]\times X} \hookrightarrow j_* \operatorname{dom} \overline{d}^\bullet_{L^p,(0,1)\times X}$$

are quasi-isomorphisms.

Proof. By Corollary 2.4.4 and Theorem 2.4.5, both complexes of sheaves are quasi-isomorphic to $\operatorname{pr}^* \operatorname{dom} d^\bullet_{L^p,X}$.

2.5. L^p cohomology of compact manifold with boundary. Let X be a compact Riemannian manifold with boundary ∂X. We shall consider it a singular Riemannian space with the top stratum $\mathring{X} = X\backslash\partial X$.

Note that a section of the sheaf $\operatorname{dom} d^\bullet_{L^p,X}$ is a C^∞ form on $\mathring{X}$ which is L^p near ∂X. Because of this, we shall use (both here and in §2.6) the following convention: a C^∞ form on X *means* that it is C^∞ only on $\mathring{X}$; it does *not* mean C^∞ through the boundary ∂X.

Proposition 2.5.1. *The inclusion* $\operatorname{dom} d^\bullet_{L^p,X} \hookrightarrow \operatorname{dom} \overline{d}^\bullet_{L^p,X}$ *induces a quasi-isomorphism of complexes of sheaves.*

Proof. The statement is local on X. At any point of X there is a fundamental system of neighborhoods which have the property that each of them is diffeomorphic to a cube and the restriction of the metric on it is quasi-isometric to the standard metric on the cube $(0, 1)^n$. Consequently, the quasi-isomorphism follows from the results on the cubes (Corollary 2.4.3).

Lemma 2.5.2. *In this case both* $\operatorname{dom} d^\bullet_{L^p,X}$ *and* $\operatorname{dom} \overline{d}^\bullet_{L^p,X}$ *are complexes of fine sheaves.*

(Obvious.)

Corollary 2.5.3. *In this case*

(1) $$H^\bullet(\operatorname{dom} d^\bullet_{L^p}(X)) \cong H^\bullet(\operatorname{dom} \overline{d}^\bullet_{L^p}(X)) \cong H^\bullet(X; \mathbb{E}).$$

(Obvious.)

2.6. Strong approximation on compact manifold with boundary. Let X be a compact Riemannian manifold with boundary ∂X. We shall assume that the boundary is a union of two disjoint (possibly empty) parts, $\partial X = \partial_1 X \cup \partial_2 X$. Let U_1 and U_2 be some neighborhoods of $\partial_1 X$ and $\partial_2 X$, respectively.

Proposition 2.6.1. *Suppose that ω is an L^p form on X, which is C^∞ in U_1 and such that $\overline{d}\omega$ is a C^∞ form everywhere in X. Then for any $\varepsilon > 0$ we can find $\psi_\varepsilon \in \operatorname{dom} \overline{d}^\bullet_{L^p}(X)$ such that $\|\psi_\varepsilon\|_{L^p} < \varepsilon$, $\|\overline{d}\psi_\varepsilon\|_{L^p} < \varepsilon$, $\omega + \overline{d}\psi_\varepsilon$ is C^∞ on $X \backslash U_2$ and* Supp ψ_ε *does not intersect ∂X. (In other words, ψ_ε is supported outside some neighborhood of ∂X, possibly smaller than $U_1 \cup U_2$.)*

Proof. Case 1: ω is an exact form, $\omega = \overline{d}\phi$ with ϕ an L^p form. By the definition of the strong closure of d, for any $\varepsilon' > 0$ we can find $\phi_{\varepsilon'} \in \operatorname{dom} d^\bullet_{L^p}(X)$ such that $\|\phi_{\varepsilon'} - \phi\|_{L^p} < \varepsilon'$ and $\|d\phi_{\varepsilon'} - \overline{d}\phi\|_{L^p} < \varepsilon'$. Choose a C^∞ truncation function f which is equal to 1 on $X \backslash (U_1 \cup U_2)$ and 0 in a neighborhood of ∂X, and take $\psi_\varepsilon = f(\phi_{\varepsilon'} - \phi)$. Then $\omega + \overline{d}\psi_\varepsilon$ is C^∞ in $X \backslash U_2$. If ε' is small enough, then $\|\psi_\varepsilon\|_{L^p} < \varepsilon$, $\|\overline{d}\psi_\varepsilon\|_{L^p} < \varepsilon$.

Case 2: ω is a closed form, i.e., $\overline{d}\omega = 0$. Using the cohomology isomorphism (1), we can find a C^∞ form $\omega' \in \operatorname{dom} d^\bullet_{L^p}(X)$ which is cohomological to ω, i.e., $\omega = \omega' + \overline{d}\phi$ where $\phi \in \operatorname{dom} \overline{d}^\bullet_{L^p}(X)$; in particular, $\overline{d}\phi$ is C^∞ in U_1. Applying the argument of Case 1 above to $\overline{d}\phi$, we get $\psi_\varepsilon \in \operatorname{dom} \overline{d}^\bullet_{L^p}(X)$, such that $\|\psi_\varepsilon\|_{L^p} < \varepsilon$, $\|\overline{d}\psi_\varepsilon\|_{L^p} < \varepsilon$, $\overline{d}\phi + \overline{d}\psi_\varepsilon$ is C^∞ in $X \backslash U_2$, and Supp ψ_ε does not intersect ∂X. As $\overline{d}\phi + \overline{d}\psi_\varepsilon$ is C^∞ in $X \backslash U_2$, $\omega + \overline{d}\psi_\varepsilon = \omega' + \overline{d}\phi + \overline{d}\psi_\varepsilon$ is also C^∞ in $X \backslash U_2$.

Case 3: ω is any form in $\operatorname{dom} \overline{d}^\bullet_{L^p}(X)$ which is C^∞ in U_1 and such that $\overline{d}\omega$ is C^∞ everywhere. As $\overline{d}\omega$ is C^∞, we can use the cohomology isomorphism (1) to find $\phi \in \operatorname{dom} D^\bullet_{L^p}(X)$, such that $d\phi = \overline{d}\omega$. Then $\overline{d}(\omega - \phi) = 0$, $\omega - \phi \in \dim \overline{d}^\bullet_{L^p}(X)$, and $\omega - \phi$ is C^∞ in U_1. The argument of Case 2 above yields $\psi_\varepsilon \in \operatorname{dom} \overline{d}^\bullet_{L^p}(X)$ such that $\|\psi_\varepsilon\|_{L^p} < \varepsilon$, $\|\overline{d}\psi_\varepsilon\|_{L^p} < \varepsilon$, $\omega - \phi + \overline{d}\psi_\varepsilon$ is C^∞ in $X \backslash U_2$, and Supp ψ_ε does not intersect ∂X. As $\omega - \phi + \overline{d}\psi_\varepsilon$ is C^∞ in $X \backslash U_2$ and ϕ is C^∞ everywhere, $\omega + \overline{d}\psi_\varepsilon$ is C^∞ in $X \backslash U_2$.

2.7. Strong approximation on smooth noncompact manifold.

Theorem 2.7.1. *Let X be any Riemannian manifold. Suppose that ω is an L^p form on X whose differential is C^∞; in other words, $\omega \in$*

$\operatorname{dom}\overline{d}^{\bullet}_{L^p}(X)$, $\overline{d}\omega \in \operatorname{dom} d^{\bullet}_{L^p}(X)$. *Then for any* $\varepsilon > 0$ *we can find* $\psi_\varepsilon \in \operatorname{dom}\overline{d}^{\bullet}_{L^p}(X)$ *satisfying* $\|\psi_\varepsilon\|_{L^p} < \varepsilon$, $\|\overline{d}\psi_\varepsilon\|_{L^p} < \varepsilon$, *and* $\omega + \overline{d}\psi_\varepsilon$ *is* C^∞.

Proof. Take a C^∞ exhaustion function $f: X \to \mathbb{R}$ such that $f^{-1}((-\infty, c])$ is compact for any $c \in \mathbb{R}$. The set of critical values of f is closed in $\mathbb{R}$ and is of measure zero; hence, its complement is open and dense in $\mathbb{R}$. We can choose an increasing unbounded sequence $c_1, c_2, \cdots \in \mathbb{R}$ in this complement. Then $f^{-1}(c_i)$ is a smooth compact submanifold in X for any i.

For $i = 1, 2, 3, \cdots$, let $Y_i = f^{-1}([c_{i-2}, c_{i+1}])$; it is a compact smooth manifold with boundary $\partial Y_i = f^{-1}(c_{i-2}) \cup f^{-1}(c_{i+1})$ (we assume $c_0 = c_{-1} = -\infty$ and $f^{-1}(-\infty) = \emptyset$). Let $\overset{\circ}{Y}_i = Y_i \backslash \partial Y_i = f^{-1}((c_{i-2}, c_{i+1}))$.

We shall construct ψ_ε as $\psi_\varepsilon = \chi_1 + \chi_2 + \chi_3 + \cdots$ where $\chi_i \in \operatorname{dom}\overline{d}^{\bullet}_{L^p}(X)$, $\operatorname{Supp}\chi_i \subset \overset{\circ}{Y}_i$, $\|\chi_i\|_{L^p} < \varepsilon_i$, $\|\overline{d}\chi_i\|_{L^p} < \varepsilon_i$, and $\Sigma\varepsilon_i < \varepsilon$. The forms χ_i are constructed inductively in such way that $\phi_i = \omega + \overline{d}(\chi_1 + \chi_2 + \cdots + \chi_i)$ is C^∞ on $f^{-1}((-\infty, c_i])$ (for $i = 0$ this is trivially satisfied as $f^{-1}((-\infty, c_0])$ is empty). In other words, we need to construct χ_{i+1} so that $\phi_{i+1} = \phi_i + \overline{d}\chi_{i+1}$ is C^∞ on $f^{-1}((-\infty, c_{i+1}])$.

Apply Proposition 2.6.1 for the compact manifold Y_{i+1}, whose boundary ∂Y_{i+1} consists of two parts, i.e., $\partial Y_{i+1} = \partial_1 Y_{i+1} \cup \partial_2 Y_{i+1}$, where $\partial_1 Y_{i+1} = f^{-1}(c_{i-1})$ and $\partial_2 Y_{i+1} = f^{-1}(c_{i+2})$. Let $U_1 = f^{-1}([c_{i-1}, c_i))$ and $U_2 = f^{-1}((c_{i+1}, c_{i+2}])$.

Let $\eta_i = \phi_i|_{Y_{i+1}}$; this form is C^∞ on U_1 and $\overline{d}\eta_i$ is C^∞ everywhere on Y_{i+1}. Proposition 2.6.1 yields a form $\psi_{\varepsilon_{i+1}} \in \operatorname{dom}\overline{d}^{\bullet}_{L^p}(Y_{i+1})$ such that $\|\psi_{\varepsilon_{i+1}}\| < \varepsilon_{i+1}$, $\|\overline{d}\psi_{\varepsilon_{i+1}}\| < \varepsilon_{i+1}$, $\eta_i + \overline{d}\psi_{\varepsilon_{i+1}}$ is C^∞ on $Y_{i+1}\backslash U_2 = f^{-1}([c_{i-1}, c_{i+1}])$, and $\operatorname{Supp}\psi_{\varepsilon_{i+1}} \Subset f^{-1}((c_{i-1}, c_{i+2}))$. The latter property shows that $\psi_{\varepsilon_{i+1}}$ can be extended by zero to the entire X; let χ_{i+1} be this extension. Then $\chi_{i+1} \in \operatorname{dom}\overline{d}^{\bullet}_{L^p}(X)$, $\|\chi_{i+1}\| = \|\psi_{\varepsilon_{i+1}}\| < \varepsilon_{i+1}$, $\|\overline{d}\chi_{i+1}\| = \|\overline{d}\psi_{\varepsilon_{i+1}}\| < \varepsilon_{i+1}$. Finally, $\phi_i + \overline{d}\chi_{i+1}$ coincides with ϕ_i on some neighborhood of $f^{-1}((-\infty, c_{i-1}])$, and with $\eta_i + \overline{d}\psi_{\varepsilon_{i+1}}$ on Y_{i+1}; hence, $\phi_i + \overline{d}\chi_{i+1}$ is C^∞ on $f^{-1}((-\infty, c_{i+1}])$. q.e.d.

Proof of Theorem 2.2.1 (The cohomological approximation theorem). Our argument in Cases 2 and 3 in the proof of Proposition 2.6.1 shows that the cohomology isomorphism between the complexes $\operatorname{dom}^{\bullet}_{L^p}(X)$ and $\operatorname{dom}\overline{d}^{\bullet}_{L^p}(X)$ implies the "strong approximation" of the kind asserted in Proposition 2.6.1 and Theorem 2.7.1. It is not hard to see that this argument can be reversed to show that the "strong approximation" of Theorem 2.7.1 implies the cohomology isomorphism of Theorem 2.2.1. q.e.d.

Corollary 2.7.2. *For any singular Riemannian space* X *and any local system* $\mathbb{E}$, *the imbedding homomorphism* $\operatorname{dom} d^{\bullet}_{L^p, X, \mathbb{E}} \hookrightarrow \operatorname{dom} \overline{d}^{\bullet}_{L^p, X, \mathbb{E}}$ *is a quasi-isomorphism of complexes of sheaves.*

(Obvious.)

Definition 2.7.3. For a Riemannian manifold X, its L^p cohomology $H^{\bullet}_{L^p}(X; \mathbb{E})$ is the cohomology of the complex $\operatorname{dom} d^{\bullet}_{L^p}(X; \mathbb{E})$ or, equivalently, $\operatorname{dom} \overline{d}^{\bullet}_{L^p}(X; \mathbb{E})$. For a singular Riemannian space X, its L^p cohomology $H^{\bullet}_{L^p}(X; \mathbb{E})$ is the L^p cohomology of its nonsingular part $\overset{\circ}{X}$.

Clearly, if X is compact, then

(2) $\quad H^{\bullet}_{L^p}(X; \mathbb{E}) \cong H^{\bullet}(\Gamma(X; \operatorname{dom} d^{\bullet}_{L^p, X, \mathbb{E}})) \cong H^{\bullet}(\Gamma(X; \operatorname{dom} \overline{d}^{\bullet}_{L^p, X, \mathbb{E}}))$.

2.8. Partitions of unity with bounded differentials. Let X be a locally compact singular Riemannian space.

Definition 2.8.1. We say that X has *partitions of unity with bounded differentials* if for any point $P \in X$ and any neighborhood $U \subset X$ of P we can find a continuous function $f: X \to \mathbb{R}$ which is equal to 1 in some smaller neighborhood of P, has compact support inside U, is C^{∞} on $\overset{\circ}{X}$ and such that the pointwise norm of its differential $|df|$ is globally bounded.

The reason for the name is that if this condition is satisfied, then for any open covering $X = \bigcup U_i$ we can find a locally finite refinement $X = \bigcup U'_i$ (so that each U'_i lies inside some U_j) and a partition of unity $1 = \sum f_i$ where each f_i is continuous on X, is C^{∞} on $\overset{\circ}{X}$, $\operatorname{Supp} f_i \subset U'_i$ and there exist some bounds $|df_i| < C_i$.

Proposition 2.8.2. *If* X *has partitions of unity with bounded differentials, then the sheaves* $\operatorname{dom} d^{\bullet}_{L^p, X}$, $\operatorname{dom} \overline{d}^{\bullet}_{L^p, X}$ *and* $\operatorname{dom} \overline{d}^{\bullet}_{L^p, \min, X}$ *are fine.*

Proof. Using the partition of unity $1 = \sum f_i$, we can decompose any form ω as $\omega = \sum f_i \omega$ where each summand is bounded in the graph norm as $\|d(f_i\omega)\|_{L^p} \le \|f_i d\omega\|_{L^p} + \|df_i \wedge \omega\|_{L^p}$ and $\|df_i \wedge \omega\|_{L^p} < C_i \|\omega\|_{L^p}$. The rest of the argument is standard. q.e.d.

We shall see (Proposition 3.2.2) that cones and horns have such partitions of unity. Also, any singular Riemannian space X which can be embedded in a smooth Riemannian manifold in such way that the metric on X is locally quasi-isometric to the restriction of the metric on the ambient manifold, has such partitions of unity; they can be obtained by restricting onto X the partitions of unity that exist on the ambient manifold. An example of this kind is given by a complex projective variety with Fubini-Study metric.

Example 2.8.3. Let X be the blowup of the complex plane $\mathbb{C}^2$ centered at the origin, and let the Riemannian metric g_X be the pullback to X of the standard metric on $\mathbb{C}^2$. Then the sheaves $\operatorname{dom} d^{\bullet}_{L^p, X}$ and $\operatorname{dom} \overline{d}^{\bullet}_{L^p, X}$ are not soft. Indeed, take two noninteresting closed subsets of the exceptional divisor, and take a germ of functions on their union which is equal to 0 in a neighborhood of one of the subsets, and to 1 in a neighborhood of the other. This is a germ of sections of each of these sheaves which cannot be extended to a section on X. This shows that both sheaves are not soft, and consequently, not fine; X does not have partitions of unity with bounded differentials.

Moreover, if $p < 4$, this germ of functions can be approximated (in the graph sense) by functions supported away from the exceptional divisor. (To see this, note that a neighborhood of our closed set in X corresponds to a union of two disjoint conical parts of a small ball in $\mathbb{C}^2$.) Hence, for $p < 4$ this germ of functions is a germ of sections of the sheaf $\operatorname{dom} \overline{d}^{\bullet}_{L^p, \min, X}$ which is, consequently, not soft either.

3. Cones and Horns

3.1. Metric horns and their L^p cohomology.

Definition 3.1.1 (Cf. [4]). Let (X, g_X) be a singular Riemannian space. The *metric f-horn* $C^f X$ is the cone $[0, 1) \times X / 0 \times X$ with the structure of a singular Riemannian space given by the top stratum $\overset{\circ}{C}{}^f X = (0, 1) \times \overset{\circ}{X} \subset C^f X$ and the metric $dr^2 + f(r)^2 g_X$ on it, where r is the coordinate on $(0, 1)$ and f is a C^∞ positive nondecreasing function on $(0, 1]$.

In case $f(r) \nrightarrow 0$ as $r \to 0$, the horn metric is quasi-isometric to the product ("collar") metric which has been studied above (see §2.4). All the theorems about the horns formulated below, hold in this trivial case too; nevertheless, we shall leave to the reader to check this and shall always assume that $f(r) \to 0$ as $r \to 0$.

Cheeger [4] required that $f(r) = O(r)$ as $r \to 0$; we do not need this requirement.

Consider the natural projection $\mathrm{pr}\colon (0, 1) \times X \to X$. If $\overset{\circ}{X}$ is equipped with a local system $\mathbb{E}$, then the top stratum of the horn $\overset{\circ}{C}{}^f H = (0, 1) \times \overset{\circ}{X}$ is equipped with its pullback $\mathrm{pr}^* \mathbb{E}$; we shall always assume this implicitly.

Let $m = \dim_{\mathbb{R}} X$.

Theorem 3.1.2. (a) *If the integer k is such that $\int_0^1 f(r)^{m-pk}\,dr$ is convergent, then the operator* $\mathrm{pr}^*: \Omega^k_{L^p}(X) \to \Omega^k_{L^p}(C^f X)$ *is bounded and induces a cohomology isomorphism* $H^k_{L^p}(X) \xrightarrow{\sim} H^k_{L^p}(C^f X)$.

(b) *If the integer k is such that $\int_0^1 f(r)^{m-pk}\,dr$ is divergent, then*

$$H^k_{L^p}(C^f X) = 0,$$

provided that either $k \geq m/p+1$ or $\mathrm{Im}\{\overline{d}: \mathrm{dom}\,\overline{d}^{k-1}_{L^p}(X) \to \mathrm{dom}\,\overline{d}^k_{L^p}(X)\}$ *is closed in* $\Omega^k_{L^p}(X)$.

For example, if $f = r^c$, then part (a) of this theorem covers all $k < m/p + 1/pc$, and part (b)—all $k \geq m/p + 1/pc$.

Generally, let $e = \inf\{\alpha \in \mathbb{R} \mid \int_0^1 f(r)^{-\alpha}\,dr = \infty\}$. Note that the "borderline" integral $\int_0^1 f(r)^{-e}\,dr$ may either converge or diverge.

Part (a) of our theorem covers all $k \leq l$, and part (b)—all $k > l$, where

$$(3)\quad l = \begin{cases} \max\{k \mid k < (m+e)/p\} & \text{if the integral } \int_0^1 f(r)^{-e}\,dr \text{ diverges}, \\ \max\{k \mid k \leq (m+e)/p\} & \text{if } \int_0^1 f(r)^{-e}\,dr \text{ converges}. \end{cases}$$

This integer l depends on p, f and $m = \dim X$.

The proof of Theorem 3.12 is in §5.

Remark 3.1.3. We shall see later that the condition $k \geq m/p+1$ in the part (b) of the theorem can be somewhat weakened. The actual property that we shall use in the proof is too cumbersome to formulate here. It is formulated precisely in Remark 5.9.1.

Definition 3.1.4. We shall write $l = \mathrm{perv}_{L^p,f}(m+1)$. We shall consider $\mathrm{perv}_{L^p,f}$ a function of $m+1$ and call it the *(L^p, f)-perversity.*

Clearly, $e \geq 0$ always and $l = \mathrm{perv}_{L^p,f}(m+1) \geq [m/p]$. Moreover, the (L^p, f)-perversity is *linear*: for some real (or rational) number $e' \geq -1$ we have $\mathrm{perv}_{L^p,f}(s) = [(s+e')/p]$.

In case $f(r) = r$ (the conical metric) we have

$$\mathrm{perv}_{L^p,r}(s) = \max\{k \in \mathbb{Z} \mid k < s/p\}.$$

This perversity was introduced in [2], and Theorem 3.1.2 in case $f(r) = r$ was conjectured there.

3.2. Metric with f-horn singularities. Let $X = X_n \supset X_{n-1} = X_{n-2} \supset \cdots \supset X_1 \supset X_0$ be a stratified pseudomanifold of dimension n. This means the following: X is a topological space, each *closed stratum* X_k is a closed subset of X, each *open stratum* $\overset{\circ}{X}_k = X_k \setminus X_{k-1}$ is a smooth k-manifold, and each point $P \in \overset{\circ}{X}_k$ has a neighborhood $U \subset X$, a compact stratified pseudomanifold L_P of dimension $n-k-1$ ("the link") and a strata-preserving homeomorphism $\phi\colon (U\cap \overset{\circ}{X}_k)\times CL_P \xrightarrow{\sim} U$ which induces a diffeomorphism on each stratum; here $CL_P = [0, 1) \times L_P/0 \times L_P$ is the cone over L_P with the obvious stratification. Note that X is locally compact since each link L_P is compact.

Keeping in line with our previous notation, we denote $\overset{\circ}{X} = \overset{\circ}{X}_n$.

Definition 3.2.1 [4]. Let g be a Riemannian metric on $\overset{\circ}{X}$, and let f be as in Definition 3.1.1. We say that g has *f-horn singularities* if the homeomorphisms ϕ above can be chosen in such way that $\phi^* g$ is quasi-isometric to the product metric on $(U \cap \overset{\circ}{X}_k) \times C^f L_P$. We shall also say that (X, g) is a *singular Riemannian space with f-horn singularities.*

Proposition 3.2.2. *If X is a singular Riemannian space with f-horn singularities, then X has partitions of unity with bounded differentials. In particular, the sheaves* $\operatorname{dom} d^\bullet_{L^p, X}$, $\operatorname{dom} \overline{d}^\bullet_{L^p, X}$, *and* $\operatorname{dom} \overline{d}^\bullet_{L^p, \min, X}$ *are fine.*

Proof. Indeed, any point P has a neighborhood U of the form $U \simeq (U\cap \overset{\circ}{X}_k)\times C^f L_P$ where the metrics on both sides are quasi-isometric. It is fairly obvious that on $(U \cap \overset{\circ}{X}_k) \times C^f L_P$ we can find continuous functions with bounded differentials which are equal to 1 in a neighborhood of P and vanish outside a somewhat larger neighborhood. q.e.d.

3.3. The isomorphism with intersection cohomology. For a perversity $\overline{p}$, we denote by $\mathscr{IC}^\bullet_{\overline{p}, X, \mathbb{E}}$ the complex of intersection chain sheaves with coefficients in $\mathbb{E}$, and by $IH^\bullet_{\overline{p}}(X; \mathbb{E})$ the intersection cohomology of X with coefficients in $\mathbb{E}$ with respect to perversity $\overline{p}$. Indexing notation: we denote by $\mathscr{IC}^k_{\overline{p}, X}$ what is denoted by $\mathbf{IC}^{k-2n}$ in [6].

Theorem 3.3.1. *Let X be a singular Riemannian space with f-horn singularities. Then there is a canonical isomorphism in the derived category of sheaves*

$$\operatorname{dom} d^\bullet_{L^p, X, \mathbb{E}} \cong \mathscr{IC}^\bullet_{\mathrm{perv}_{L^p, f}, X, \mathbb{E}}.$$

If X is compact then, in addition, there is a canonical cohomology isomorphism $H^\bullet_{L^p}(X, \mathbb{E}) \cong IH^\bullet_{\mathrm{perv}_{L^p, f}}(X; \mathbb{E})$.

Proof. Take any $P \in X$, say, $P \in \overset{\circ}{X}_m$, and take $U \subset X$ and L_P as above; we may choose U in such way that $U \cap X_m$ is contractible. Using induction by $n = \dim X$, we may assume that $H^\bullet_{L^p}(L_P) \cong IH^\bullet_{\mathrm{perv}_{L^p, f}}(L_P)$. As L_P is compact, $IH^\bullet_{\mathrm{perv}_{L^p, f}}(L_P)$ is finite dimensional, and so is $H^\bullet_{L^p}(L_P)$. By the Open Mapping Theorem, $\operatorname{Im} \overline{d}$ is closed in $\Omega^\bullet_{L^p}(L_P)$. By Theorem

3.1.2,

$$H^k_{L^p}(C^f L_P) = \begin{cases} H^k_{L^p}(L_P) & \text{if } k \le \operatorname{perv}_{L^p,f}(n-m), \\ 0 & \text{if } k > \operatorname{perv}_{L^p,f}(n-m). \end{cases}$$

Since L_P is compact,

$$H^k_{L^p}(L_P) \cong H^k(\Gamma(L_P; \operatorname{dom} d^\bullet_{L^p, L_P})).$$

By Lemma 3.3.2,

$$H^k_{L^p}(C^f L_P) \cong H^k(\Gamma(C^f L_P; \operatorname{dom} d^\bullet_{L^p, C^f L_P})).$$

As $U \simeq (U \cap X_m) \times C^f L_P$, it follows from Theorem 2.4.6 that

$$H^k(\Gamma(U; \operatorname{dom} d^\bullet_{L^p, X})) \cong H^k(\Gamma(C^f L_P; \operatorname{dom} d^\bullet_{L^p, C^f L_P})).$$

Similarly,

$$H^k(\Gamma(U \backslash X_m; \operatorname{dom} d^\bullet_{L^p, X})) \cong H^k(\Gamma(C^f L_P \backslash P; \operatorname{dom} d^\bullet_{L^p, C^f L_P})).$$

As the metric on $C^f L_P \backslash P$ is *locally*—at every point of $C^f L_P \backslash P$—quasi-isometric to the product metric on $(0, 1) \times L_P$, we have

$$\begin{aligned} H^k(\Gamma(C^f L_P \backslash P; \operatorname{dom} d^\bullet_{L^p, C^f L_P})) &\cong H^k(\Gamma((0, 1) \times L_P; \operatorname{dom} d^\bullet_{L^p})) \\ &\cong H^k_{L^p}(L_P). \end{aligned}$$

Putting all this together, we get

$$\begin{aligned} &H^k(\Gamma(U; \operatorname{dom} d^\bullet_{L^p, X})) \\ &\quad \cong \begin{cases} H^k(\Gamma(U \backslash X_m; \operatorname{dom} d^\bullet_{L^p, X})) & \text{if } k \le \operatorname{perv}_{L^p,f}(n-m), \\ 0 & \text{if } k > \operatorname{perv}_{L^p,f}(n-m). \end{cases} \end{aligned}$$

The first statement of the theorem now immediately follows from the theorem on the uniqueness of the intersection cohomology [6]; the second one—from (2).

Lemma 3.3.2. *Let L be a compact singular Riemannian space. Then*

$$H^k_{L^p}(C^f L) \cong H^k(\Gamma(C^f L; \operatorname{dom} d^\bullet_{L^p, C^f L})).$$

Proof. The difficulty here is that the horn $C^f L$—which is topologically an open cone, $C^f L = [0, 1) \times L/0 \times L$—is not compact.

We first define the compactification $\overline{C}^f L$ of $C^f L$ as $\overline{C}^f L = [0, 1] \times L/0 \times L$ with the same metric on the top stratum; let j denote the imbedding map $C^f L \hookrightarrow \overline{C}^f L$. Clearly, $H^k_{L^p}(C^f L) = H^k_{L^p}(\overline{C}^f L)$; by compactness,

$$H^k_{L^p}(\overline{C}^f L) \cong H^k(\Gamma(\overline{C}^f L; \operatorname{dom} d^\bullet_{L^p, \overline{C}^f L})).$$

On the other hand, note that the imbedding $\operatorname{dom} d^\bullet_{L^p, \overline{C}^f L} \hookrightarrow j_* \operatorname{dom} d^\bullet_{L^p, C^f L}$ of complexes of sheaves on $\overline{C}^f L$ is a cohomology isomorphism: this is trivial on $C^f L$ and follows from Corollary 2.4.7 on the "outside boundary" $1 \times L$. As both complexes consist of fine sheaves, this implies the cohomology isomorphism on the complexes of global sections:

$$H^k(\Gamma(\overline{C}^f L; \operatorname{dom} d^\bullet_{L^p, \overline{C}^f L})) \cong H^k(\Gamma(\overline{C}^f L; j_* \operatorname{dom} d^\bullet_{L^p, C^f L})).$$

Here

$$H^k(\Gamma(\overline{C}^f L; j_* \operatorname{dom} d^\bullet_{L^p, C^f L})) = H^k(\Gamma(C^f L; \operatorname{dom} d^\bullet_{L^p, C^f L})),$$

and we finally get

$$H^k_{L^p}(C^f L) \cong H^k(\Gamma(C^f L; \operatorname{dom} d^\bullet_{L^p, C^f L})).$$

Remark 3.3.3. Choosing different functions f for different strata, we also get intersection cohomology, but with other perversities, not necessarily linear, and any perversity can be obtained this way; this has already been shown by Nagase [7]. For large enough p, we have the bottom (zero) perversity; choosing either small p or choosing a function f with a large value of e (e.g., $f = r^c$ with small $c > 0$), we get the top perversity $\bar{t}$, $\bar{t}(s) = s - 2$, and even $\bar{t} + 1$ (the latter perversity yields the intersection cohomology which is not a topological invariant of X but is rather isomorphic to the cohomology of $\overset{\circ}{X}$; see [1]).

4. L^p Stokes property

4.1. L^p Stokes property via sheaves. Let X be a singular Riemannian space and $\mathbb{E}$ a unitary local system on $\overset{\circ}{X}$ with a pairing $\mathbb{E} \times \overline{\mathbb{E}} \to \mathbb{C}$ as above.

The sheaf $\operatorname{dom} \overline{d}^\bullet_{L^p, \min, X, \mathbb{E}}$ is always a subsheaf of $\operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}}$.

Definition 4.1.1. We say that X and $\mathbb{E}$ satisfy *L^p Stokes property at a point $P \in X$* if the stalk at P of the sheaf $\operatorname{dom} \overline{d}^\bullet_{L^p, \min, X, \mathbb{E}}$ coincides with the stalk of $\operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}}$. We say that X and $\mathbb{E}$ satisfy *L^p Stokes property everywhere* if they satisfy it for every $P \in X$, i.e., $\operatorname{dom} \overline{d}^\bullet_{L^p, \min, X, \mathbb{E}} = \operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}}$.

Remark 4.1.2. Suppose that X is compact. If L^p Stokes property is satisfied everywhere on X, then $\operatorname{dom} \overline{d}^\bullet_{L^p, \min}(X) = \operatorname{dom} \overline{d}^\bullet_{L^p}(X)$ (both spaces are subspaces of $\Omega^\bullet_{L^p}(X)$). The converse is also true (again, in case X is compact), provided that the sheaves $\operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}}$ and

$\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}}$ are soft, e.g., if X has partitions of unity with bounded differentials.

For $p = 2$, this notion was introduced (in the global form only) by Cheeger [4]; he showed that it implies that the homomorphism from the space of L^2 closed and coclosed forms into the L^2 cohomology has no kernel. Let $1/p + 1/q = 1$; we shall show that this property (for any p) implies the Borel-Moore duality between the complexes of sheaves $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}$ and $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,X,\mathbb{E}}$, and, in case X is compact, the duality between L^p cohomology and L^q cohomology.

4.2. Borel-Moore duality. Recall that for a bounded below complex of c-soft sheaves of vector spaces, say, $\mathscr{F}^\bullet = \{0 \to \mathscr{F}^k \to \mathscr{F}^{k+1} \to \cdots\}$, its Borel-Moore dual [3] is defined as

$$D_X\mathscr{F}^\bullet = \{\cdots \to \mathscr{G}^{-k-1} \to \mathscr{G}^{-k} \to 0\},$$

where $\mathscr{G}^{-l}$ is the sheaf whose sections on an open subset $U \subset X$ are given by

$$\Gamma(U;\mathscr{G}^{-l}) = \operatorname{Hom}(\Gamma_c(U;\mathscr{F}^l), \mathbb{C}).$$

Then by Verdier duality $D_X\mathscr{F}^\bullet$ is isomorphic in derived category to $R\mathscr{H}om^\bullet(\mathscr{F}^\bullet, \mathbb{D}_X)$ where $\mathbb{D}_X$ is the dualizing complex on X, $\mathbb{D}_X = f^!\mathbb{C}$, f being the map from X to the one-point space.

As usual, for a complex $\mathscr{F}^\bullet$ we shall denote by $\mathscr{F}^\bullet[k]$ the same complex with the grading shifted by k degrees, $k \in \mathbb{Z}$, $(\mathscr{F}^\bullet[k])^l = \mathscr{F}^{l+k}$. According to our conventions, on a smooth oriented manifold M of (real) dimension n, $D_M\mathbb{C}_M \cong \mathbb{C}_M[m]$, or $D_M(\mathbb{C}_M[m]) \cong (D_M\mathbb{C}_M)[-m] \cong \mathbb{C}_M$.

4.3. L^p Stokes implies duality. Let X be a stratified pseudomanifold of dimension n which also has the structure of a singular Riemannian space given by a Riemannian metric on $\overset{\circ}{X} = \overset{\circ}{X}_n$.

We say that a complex of sheaves on X has *constructible cohomology* with respect to the given stratification if the restriction of its cohomology sheaves onto the open strata are local systems on these strata.

Theorem 4.3.1. *If the complexes of sheaves* $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}}$ *and* $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}$ *are c-soft, and* $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}$ *has constructible cohomology* (*with respect to the given stratification on* X), *then there is a canonical isomorphism in the derived category*

$$D_X \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}} \cong \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}[n].$$

Corollary 4.3.2. *Suppose that L^p Stokes property is satisfied everywhere on X, X has partitions of unity with bounded differentials, and one of* *the complexes of sheaves* $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,X,\mathbb{E}}$ *and* $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}$ *has constructible cohomology. Then these complexes of sheaves are Borel-Moore dual to each other, i.e.,* $D_X \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,X,\mathbb{E}} \cong \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}[n]$, *and vice versa.*

If, in addition, X *is compact, then there is Poincaré duality between* $H^k_{L^p}(X, \mathbb{E})$ *and* $H^{n-k}_{L^q}(X, \overline{\mathbb{E}})$.

(Obvious.)

4.4. Proof of Theorem 4.3.1. We shall construct a quasi-isomorphism of complexes of sheaves

$$\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}} \to (D_X \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}})[-n].$$

For that purpose, we shall construct the quasi-isomorphisms

$$(4) \qquad \Gamma(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}) \to \Gamma(U; D_X \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}})[-n]$$

for all open neighborhoods U of any point $P \in X$, which have the form $U \simeq (U \cap \mathring{X}_k) \times CL_P$, where $U \cap \mathring{X}_k$ is diffeomorphic to an open ball.

With U chosen this way, the cohomology of the left-hand side of (4) is finite dimensional as the complex of sheaves $\operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}$ has constructible cohomology.

The right-hand side in (4) is, by the definition of the duality functor D_X, the space (or, more precisely, the complex) of all (not only continuous in any sense) linear functionals on $\Gamma_c(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}})[n]$:

$$\Gamma(U; D_X \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}})[-n] = \operatorname{Hom}(\Gamma_c(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}})[n], \mathbb{C}).$$

The homomorphism (4) comes from the pairing

$$(5) \qquad \Gamma_c(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}}) \times \Gamma(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}}) \to \mathbb{C}[-n]$$

given by

$$(6) \qquad (\omega, \phi) \mapsto \int_U (-1)^{k(k+1)/2} \omega \wedge \phi,$$

where $\omega \in \Gamma_c(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^p,\min,X,\mathbb{E}})$, $\phi \in \Gamma(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}})$ and $k = \deg\omega$. The integral is well-defined since $\operatorname{Supp}\omega$ is a compact subset of U, ϕ is L^q on $\operatorname{Supp}\omega$ and we can use a version of Hölder's inequality for forms. The homomorphism (4) thus constructed, commutes with the differential by Proposition 2.1.2. Note that the integral is nonzero only if $\deg\omega + \deg\phi = n$ which accounts for the shift $[-n]$ in (5).

Let us see that (4) is a quasi-isomorphism. Note that $\Gamma(U; \operatorname{dom}\overrightarrow{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}})$ is the domain of the maximal closure $\overline{d}$ of the operator d in the space

$\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ with the topoloogy given by the family of seminorms $\|\bullet\|_{L^q, K}$ where K can be any compact subset in U,

$$\|\omega\|_{L^q, K} = \left(\int_{K \cap \overset{\circ}{X}} |\omega|^q \, d\,\mathrm{vol}\right)^{1/q}.$$

The topological dual to this space (i.e., the space of all continuous linear functionals on it) is $\Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$; we are not concerned with the topology on this vector space.

The adjoint to the (unbounded) operator $\overline{d}$ in $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ is the operator $\overline{d}_{\min}$ in $\Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$. This implies that the orthogonal complement to subspace $\operatorname{Im} \overline{d} \subset \Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ is the subspace $\operatorname{Ker} \overline{d}_{\min} \subset \Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$, and the orthogonal complement to $\operatorname{Ker} \overline{d} \subset \Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ *contains* the subspace $\operatorname{Im} \overline{d}_{\min} \subset \Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$.

Claim. The orthogonal complement to the subspace $\operatorname{Ker} \overline{d}$ in $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}}$ coincides with the subspace $\operatorname{Im} \overline{d}_{\min}$ in $\Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$.

Note that the factorspace $\operatorname{Ker} \overline{d} / \operatorname{Im} \overline{d}$ is the cohomology of the left-hand side of (4), and we have already seen that it is finite dimensional; by the Open Mapping Theorem, the subspace $\operatorname{Im} \overline{d}$ is closed in $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$.

Proof of the Claim. The operator $\overline{d}$ yields a continuous linear operator

$$\frac{\Gamma(U; \operatorname{dom} \overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}})}{\operatorname{Ker} \overline{d}} \to \operatorname{Im} \overline{d}, \tag{7}$$

where $\operatorname{Im} \overline{d} \subset \Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$, and $\Gamma(U; \operatorname{dom} \overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}})$ is understood in the graph topology; the operator (7) is one-to-one. As $\operatorname{Im} \overline{d}$ is closed in $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$, we can apply the Open Mapping Theorem which shows that the operator (7) has a bounded inverse. Hence, there is a bounded operator

$$\overline{d}^{-1} \colon \operatorname{Im} \overline{d} \to \frac{\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})}{\operatorname{Ker} \overline{d}}.$$

Consequently, for any continuous linear functional ϕ on $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ which vanishes on $\operatorname{Ker} \overline{d}$ (i.e., an element of $\Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$ orthogonal to $\operatorname{Ker} \overline{d}$), we have a continuous linear functional $(\overline{d}^{-1})^* \phi$ on $\operatorname{Im} \overline{d} \subset \Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$. Choosing some continuous linear extension ψ of $(\overline{d}^{-1})^* \phi$ on the entire $\Gamma(U; \Omega^\bullet_{L^q, X, \mathbb{E}})$, we get $\psi \in \Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$, $\overline{d}_{\min} \psi = \phi$. q.e.d.

It follows that the subfactor $\operatorname{Ker} \overline{d} / \operatorname{Im} \overline{d}$ in $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ (or in $\Gamma_c(U; \operatorname{dom} \overline{d}^\bullet_{L^p, \min, X, \mathbb{E}})$) is the topological dual to the subfactor

$\operatorname{Ker}\overline{d}/\operatorname{Im}\overline{d}$ in $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ (or in $\Gamma(U; \operatorname{dom}\overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}})$) with respect to the pairing (5). As $\operatorname{Ker}\overline{d}/\operatorname{Im}\overline{d}$ is finite dimensional, the topology is unique and $\operatorname{Ker}\overline{d}_{\min}/\operatorname{Im}\overline{d}_{\min}$ is finite dimensional too.

The right-hand side of (4) is $\operatorname{Hom}(\Gamma_c(U; \operatorname{dom}\overline{d}^\bullet_{L^p, \min, X, \mathbb{E}})[n], \mathbb{C})$. Its cohomology is dual to the cohomology of $\Gamma_c(U; \operatorname{dom}\overline{d}^\bullet_{L^p, \min, X, \mathbb{E}})[n]$, i.e., to $\operatorname{Ker}\overline{d}_{\min}/\operatorname{Im}\overline{d}_{\min}$, and consequently, is isomorphic to $\operatorname{Ker}\overline{d}/\operatorname{Im}\overline{d}$.

This shows that the homomorphism (4), indeed, induces a cohomology isomorphism.

Finally, it is clear that the homomorphisms (4) for different open subsets $U \subset X$ commute with restrictions. This completes the proof of Theorem 4.3.1.

4.5. When L^p Stokes is satisfied on cones and horns. Theorem 4.3.1 implies the following corollary.

Corollary 4.5.1. *Let X be a singular Riemannian space with f-horn singularities. If L^p Stokes property holds on X, then there is an isomorphism in derived category $\mathscr{IC}^\bullet_{\mathrm{perv}_{L^p, f}}(X) \cong \mathscr{IC}^\bullet_{\overline{t}-\mathrm{perv}_{L^q, f}}(X)$ where $\overline{t}$ is the top perversity, $\overline{t}(k) = k - 2$.*

Proof. Indeed, for any perversity $\overline{p}$, $D_X\mathscr{IC}^\bullet_{\overline{p}} \cong \mathscr{IC}^\bullet_{\overline{t}-\overline{p}}[n]$, $n = \dim X$. q.e.d.

Let us see when the isomorphism

(8) $$\mathscr{IC}^\bullet_{\mathrm{perv}_{L^p, f}}(X) \cong \mathscr{IC}^\bullet_{\overline{t}-\mathrm{perv}_{L^q, f}}(X)$$

is possible. The equality $\mathrm{perv}_{L^p, v} = \overline{t} - \mathrm{perv}_{L^q, f}$, i.e., $\mathrm{perv}_{L^p, f}(k) = \overline{t}(k) - \mathrm{perv}_{L^q, f}(k)$ for all integers $k > 0$, would always imply this isomorphism; it can be rewritten as

(9) $$\mathrm{perv}_{L^p, f}(k) + \mathrm{perv}_{L^q, f}(k) = k - 2.$$

However, (8) may hold even if (9) does not.

Remark 4.5.2. Let $\overline{p}$ and $\overline{q}$ be any perversities. If $\overline{p} \neq \overline{q}$, then the isomorphism $\mathscr{IC}^\bullet_{\overline{p}} \cong \mathscr{IC}^\bullet_{\overline{q}}$ is equivalent to the following cohomology vanishing: for every k and every $P \in X_k$, $IH^l_{\overline{p}}(L_P) = 0$ for all integers l satisfying either $\overline{p}(k) < l \le \overline{q}(k)$ or $\overline{q}(k) < l \le \overline{p}(k)$, where L_P is the link of P.

It follows that if (9) is not satisfied, then the isomorphism (8) is equivalent to the cohomology vanishing

$$IH^l_{\mathrm{perv}_{L^p, f}}(L_P) = 0$$

for any k, any $P \in X_k$ and any l satisfying either

(9a) $$\mathrm{perv}_{L^p, f}(k) < l \le k - 2 - \mathrm{perv}_{L^q, f}(k)$$

or

$$k - 2 - \mathrm{perv}_{L^q, f}(k) < l \leq \mathrm{perv}_{L^p, f}(k). \tag{9b}$$

Consider the conical case $f(r) = r$. We have already seen that $\mathrm{perv}_{L^p, r}(k) = \max\{i \in \mathbb{Z} | i < k/p\}$. In this case the equality (9) for all integers $k > 0$ is impossible as clearly

$$\mathrm{perv}_{L^p, r}(k) + \mathrm{perv}_{L^q, r}(k) = \begin{cases} k - 2 & \text{if } k/p \in \mathbb{Z}, \\ k - 1 & \text{otherwise.} \end{cases}$$

Hence, the equality (9) is equivalent to $k/p \in \mathbb{Z}$ which cannot hold for all k as $p > 1$.

In other words, "usually" L^p Stokes is not satisfied for the conical metrics since (8) is not satisfied. The following two phenomena can cause (8) to be satisfied.

First, it may happen that the strata of codimension k such that $k/p \notin \mathbb{Z}$, are simply absent in X; this is the case for complex manifolds and $p = 2$; see [4].

Second, it may happen that the cohomology group $IH^l_{\mathrm{perv}_{L^p, r}}(L_P)$, where L_P is the link at a point P and $l = \mathrm{perv}_{L^p, r}(k)$, vanishes for every point $P \in \overset{\circ}{X}_k$ and every k such that $k/p \notin \mathbb{Z}$. (This value of l is the only one that satisfies (9a) or (9b).)

Any of these two phenomena (or their combination) may cause (8) to be satisfied for a singular Riemannian space with conical singularities. Otherwise, as $\mathrm{perv}_{L^p, r} > \bar{t} - \mathrm{perv}_{L^q, r}$, there is no morphism

$$\mathrm{dom}\, \overrightarrow{d}^{\bullet}_{L^p, X, \mathbb{E}} \to (D_X \,\mathrm{dom}\, \overrightarrow{d}^{\bullet}_{L^q, X, \overline{\mathbb{E}}})[-n],$$

and consequently, there is no pairing in derived category

$$\mathrm{dom}\, \overrightarrow{d}^{\bullet}_{L^p, X, \mathbb{E}} \otimes \mathrm{dom}\, \overrightarrow{d}^{\bullet}_{L^q, X, \overline{\mathbb{E}}} \to \mathbb{D}_X[-n],$$

where $\mathbb{D}_X$ is the dualizing complex of X.

If $f(r) = r^c$ with $c < 1$, then both $\mathrm{perv}_{L^p, f}$ and $\mathrm{perv}_{L^q, f}$ are larger than in the conical case, and for (8) to hold, it may be necessary for the cohomology groups $IH^s_{\mathrm{perv}_{L^p, f}}(L_P)$ to vanish in some range of degrees, as opposed to just one degree $s = l$ in the conical case. However, as we shall see in §5.12, in this case the L^p Stokes property does not have to be satisfied even if (8) holds.

If $f(r) = r^c$ with $c > 1$, then

$$\mathrm{perv}_{L^p, f} = \max\{l \in \mathbb{Z} | l < (k - 1)/p + 1/pc\}.$$

It is easy to see that then $\mathrm{perv}_{L^p,f} + \mathrm{perv}_{L^q,f}$ is equal to either $k-2$ or $k-1$, and it is trickier to distinguish these two cases explicitly. Similarly to the conical case, the isomorphism (8) is equivalent to the cohomology vanishing

$$IH^l_{\mathrm{perv}_{L^p,f}}(L_P) = 0$$

for any k not satisfying (9), any $P \in X_k$ and $l = \mathrm{perv}_{L^p,f}(k)$. Theorem 4.9.1 shows that in this case the isomorphism (8) is, indeed, equivalent to the L^p Stokes property.

4.6. L^p Stokes for collars.

Proposition 4.6.1. *Let X be a singular Riemannian space satisfying L^p Stokes property. Take $\alpha \in \Gamma((0,1)\times X; \mathrm{dom}\,\overline{d}^\bullet_{L^p,(0,1)\times X})$, $\beta \in \Gamma((0,1)\times X; \mathrm{dom}\,\overline{d}^\bullet_{L^q,(0,1)\times X})$, such that one of the forms α, β is supported inside $(0,1)\times K$ for some compact subset K in X. Then for almost all $a, b \in (0,1)$, $a<b$,*

$$\int_{(a,b)\times X} \overline{d}(\alpha\wedge\beta) = \int_{b\times X - a\times X} \alpha\wedge\beta.$$

This is a straightforward generalization of Lemma 3.1 of [4]. The proof given there, extends to our case without any changes whatsoever.

Proposition 4.6.2. *Let X be a singular Riemannian space which has partitions of unity with bounded differentials. If L^p Stokes property holds on X, it also holds on $(0,1)\times X$ with the metric of direct product. (For the local system on $(0,1)\times X$ we take the pullback of the local system on X.)*

Proof. We know that $\mathrm{dom}\,\overline{d}^\bullet_{L^p,X} = \mathrm{dom}\,\overline{d}^\bullet_{L^p,\min,X}$, and we need to show that $\mathrm{dom}\,\overline{d}^\bullet_{L^p,(0,1)\times X} = \mathrm{dom}\,\overline{d}^\bullet_{L^p,\min,(0,1)\times X}$.

Take any germ of sections of $\mathrm{dom}\,\overline{d}^\bullet_{L^p,(0,1)\times X}$ at some point $P \in (0,1)\times X$, say, ϕ; we may assume it is defined on an open subset of the form $(\alpha,\beta)\times U \subset (0,1)\times X$ where U is an open subset of X. We want to show that ϕ is a section of $\mathrm{dom}\,\overline{d}^\bullet_{L^p,\min,(0,1)\times X}$, possibly, in a smaller neighborhood of P.

Using partitions of unity, we may assume tht ϕ has compact support inside $(\alpha,\beta)\times U$. Proposition 4.6.1 implies that for any $\psi \in \Gamma((\alpha,\beta)\times U; \mathrm{dom}\,\overline{d}^\bullet_{L^q,(0,1)\times X})$ we have

$$\int_{(\alpha,\beta)\times U} \overline{d}(\phi\wedge\psi) = 0.$$

In terms of the duality pairing (5), it means that for any $\psi \in \Gamma((\alpha,\beta)\times U; \mathrm{dom}\,\overline{d}^\bullet_{L^q,(0,1)\times X})$ we have $(\overline{d}\phi, \psi) = (\phi, \overline{d}\psi)$.

On the other hand, we know that the adjoint to the operator $\overline{d}$ in $\Gamma((\alpha, \beta) \times U; \Omega^{\bullet}_{L^q,(0,1)\times X})$ is the operator $\overline{d}_{\min}$ in

$$\Gamma_c((\alpha, \beta) \times U; \Omega^{\bullet}_{L^p,(0,1)\times X}).$$

Hence, $\phi \in \operatorname{dom} \overline{d}_{\min}$, i.e., $\phi \in \Gamma_c((\alpha, \beta) \times U; \operatorname{dom} \overline{d}^{\bullet}_{L^q,\min,(0,1)\times X})$. q.e.d.

4.7. When duality implies L^p Stokes. Let X be any singular Riemannian space.

Proposition 4.7.1. *Suppose that the imbedding morphism of complexes of sheaves* $\operatorname{dom} \overline{d}^{\bullet}_{L^p,\min,X} \hookrightarrow \operatorname{dom} \overline{d}^{\bullet}_{L^p,X}$ *is a quasi-isomorphism, and, in addition,*

$$\operatorname{Im} \overline{d} \subset \operatorname{dom} \overline{d}^{\bullet}_{L^p,\min,X}, \tag{10}$$

where $\operatorname{Im} \overline{d} = \operatorname{Im}\{\operatorname{dom} \overline{d}^{\bullet}_{L^p,X} \to \operatorname{dom} \overline{d}^{\bullet}_{L^p,X}\}$. *Then L^p Stokes property holds on* X.

Proof. Indeed, consider the complex of sheaves

$$\operatorname{dom} \overline{d}^{\bullet}_{L^p,X} / \operatorname{dom} \overline{d}^{\bullet}_{L^p,\min,X}.$$

As the imbedding $\operatorname{dom} \overline{d}^{\bullet}_{L^p,\min,X} \hookrightarrow \operatorname{dom} \overline{d}^{\bullet}_{L^p,X}$ is a quasi-isomorphism, the quotient complex is acyclic. On the other hand, the property (10) shows that the differential in the quotient complex has zero image, i.e., is equal to zero. Hence, the quotient complex is zero. q.e.d.

Clearly, the L^p Stokes property implies the inclusion (10).

4.8. The noncohomological obstruction to L^p Stokes. Now we wish to analyze property (10). Let X be a singular Riemannian space which has partitions of unity with bounded differentials.

Proposition 4.8.1. *Property* (10) *is equivalent to each of the following properties*:

(a) *For any open subset* $U \subset X$, *if* $\omega \in \Gamma_c(U; \operatorname{dom} \overline{d}^{\bullet}_{L^p,X,\mathbb{E}})$, $\phi \in \Gamma(U; \operatorname{dom} \overline{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}})$, *then* $\int_U \overline{d}\omega \wedge \overline{d}\phi = 0$.

(b) *Same for* $\omega \in \Gamma_c(U; \operatorname{dom} \overline{d}^{\bullet}_{L^p,X,\mathbb{E}})$, $\phi \in \Gamma_c(U; \operatorname{dom} \overline{d}^{\bullet}_{L^q,X,\overline{\mathbb{E}}})$.

(c) *Same for* $\omega \in \Gamma_c(U; \operatorname{dom} d^{\bullet}_{L^p,X,\mathbb{E}})$, $\phi \in \Gamma_c(U; \operatorname{dom} d^{\bullet}_{L^q,X,\overline{\mathbb{E}}})$.

Proof. Obviously, property (10) is equivalent to $\operatorname{Im} \overline{d} \subset \operatorname{Ker} \overline{d}_{\min}$ where $\operatorname{Ker} \overline{d}_{\min} = \operatorname{Ker}\{\operatorname{dom} \overline{d}^{\bullet}_{L^p,\min,X,\mathbb{E}} \to \operatorname{dom} \overline{d}^{\bullet}_{L^p,\min,X,\mathbb{E}}\}$. Using partitions of unity, we see that the inclusion of sheaves $\operatorname{Im} \overline{d} \subset \operatorname{Ker} \overline{d}_{\min}$ is equivalent to the inclusion of vector spaces

$$\operatorname{Im} \overline{d} = \operatorname{Im}\{\overline{d} : \Gamma_c(U; \Omega^{\bullet}_{L^p,X,\mathbb{E}}) \to \Gamma_c(U; \Omega^{\bullet}_{L^p,X,\mathbb{E}})\}$$
$$\subset \operatorname{Ker} \overline{d}_{\min} = \operatorname{Ker}\{\overline{d}_{\min} : \Gamma_c(U; \Omega^{\bullet}_{L^p,X,\mathbb{E}}) \to \Gamma_c(U; \Omega^{\bullet}_{L^p,X,\mathbb{E}})\}$$

for each open $U \subset X$.

We know that $\Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$ is the topological dual space to the space $\Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$, and $\operatorname{Ker} \overline{d}_{\min}$ in the first space is the orthogonal complement to $\operatorname{Im} \overline{d}$ in the second. Hence, the inclusion $\operatorname{Im} \overline{d} \subset \operatorname{Ker} \overline{d}_{\min}$ as subspaces of $\Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$ is equivalent to the orthogonality between $\operatorname{Im} \overline{d} \subset \Gamma_c(U; \Omega^\bullet_{L^p, X, \mathbb{E}})$ and $\operatorname{Im} \overline{d} \subset \Gamma(U; \Omega^\bullet_{L^q, X, \overline{\mathbb{E}}})$ which is precisely the statement (a).

Obviously, (a) implies (b). To see the opposite, take ω and ϕ as in (a), and find a truncation function f which is equal to 1 on $\operatorname{Supp} \omega$ and has compact support inside U. Then $\operatorname{Supp} f\phi$ is compact and

$$\int_U \overline{d}\omega \wedge \overline{d}\phi = \int_U \overline{d}\omega \wedge \overline{d}(f\phi) = 0.$$

Finally, (b) and (c) are equivalent because $\operatorname{dom} d^\bullet_{L^p, X, \mathbb{E}}$ and $\operatorname{dom} d^\bullet_{L^q, X, \overline{\mathbb{E}}}$ are dense in $\operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}}$ and $\operatorname{dom} \overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}}$ with respect to the graph norms.

Example 4.8.2. Take $X = [0, 1]$ with Euclidean metric, so that $\overset{\circ}{X} = (0, 1)$. Properties 4.8.1(a)-(c) are satisfied on X for any p simply because $\dim X = 1$ and $\deg(d\omega \wedge d\phi) \geq 2$.

On the other hand, these properties are not satisfied on $(0, 1) \times X = (0, 1) \times [0, 1]$ for any p. A counterexample to 4.8.1(a) is given by $\omega = yh(x)$, $\phi = x$ where x is the coordinate on the first factor $(0, 1)$, y is the coordinate on the second factor $[0, 1]$, and $h(x)$ is a C^∞ function on $(0, 1)$ with compact support.

Remark 4.8.3. This shows that although L^p Stokes property on X implies L^p Stokes property on $(0, 1) \times X$, the similar assertion is not true for the properties 4.8.1(a)-(c).

These properties are known to hold in the following cases: cones and horns, $p = 2$, see the precise conditions in [4]; cones and horns, any p, see the precise conditions in Theorem 4.9.1; complex algebraic varieties with Fubini-Study metric, $p = 2$, see [8, §4, assertion P_2]. In all cases these properties follow from estimates similar to the ones used to prove cohomology vanishing, but somewhat more delicate.

In Example 5.12.1 these properties do not hold but the imbedding $\operatorname{dom} \overline{d}^\bullet_{L^p, \min, X} \hookrightarrow \operatorname{dom} \overline{d}^\bullet_{L^p, X}$ is a quasi-isomorphism.

4.9. The L^p Stokes property on the horns.

Theorem 4.9.1. *Let X be a singular Riemannian space of dimension n with f-horn singularities, and suppose that f is such that the integral $\int_0^1 f(r)^{-1}\, dr$ diverges. Then the isomorphism in the derived category $D_X \operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}} \cong \operatorname{dom} \overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}}[n]$ implies the L^p Stokes property on X.*

The proof of this theorem is in §5.10.

If $f(r) = O(r)$ (this was the assumption of Cheeger [4]) then, of course, the integral $\int_0^1 f(r)^{-1}\,dr$ diverges. In particular, the conical case $f(r) = r$ falls into this category. We shall see later in Example 5.12.1 that the condition that the integral $\int_0^1 f(r)^{-1}\,dr$ diverges, is sharp as otherwise the L^p Stokes property does not hold on X for at least some p, whether there is an isomorphism $D_X \operatorname{dom} \overline{d}^\bullet_{L^p, X, \mathbb{E}} \cong \operatorname{dom} \overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}}[n]$ or not.

Nagase [7] considered the case $f(r) = r^c$ with $c < 1$; in this case the integral $\int_0^1 f(r)^{-1}\,dr$ does converge.

5. Calculations on cones and horns

Here we prove Theorems 3.1.2 and 4.9.1. Our main instrument is the two homotopy operators $\mathscr{H}^0$ and $\mathscr{H}^1$. The operator $\mathscr{H}^0$ corresponds to the contraction of the horn to its vertex; the explicit contraction T_ε appears in §5.6. The operator $\mathscr{H}^1$ corresponds to the contraction of the horn to the link. We prove the homotopy formulas for these operators which, first, yield the cohomology isomorphisms, and second, give certain decompositions which we use to prove the L^p Stokes property.

5.1. Notation. We shall write $a \lesssim b$ if $a \le Cb$ for some constant C, and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

Let X be a singular Riemannian space of dimension m, and let g_X be its Riemannian metric. Denote by l the maximal integer k such that $\int_0^1 f(r)^{m-pk}\,dr$ converges; then $l = \operatorname{perv}_{L^p, f}(m+1)$.

The top stratum ('the regular part") of $C^f X$ is diffeomorphic to $(0, 1) \times \mathring{X}$; we shall denote the coordinate on the first factor $(0, 1)$ by r. If ω is a k-form on $(0, 1) \times \mathring{X}$, we denote its L^p norm by $\|\omega\|_{L^p, C^f X}$. For $r \in (0, 1)$, we denote by $\|\omega\|_{L^p, r}$ the L^p norm of $\omega|_{r \times X}$ with respect to the metric g_X on $r \times X$.

Let $\omega = \omega(r, x) = \omega_1(r, x) + dr \wedge \omega_2(r, x)$ where ω_1 and ω_2 do not involve dr. Then for the pointwise norms with respect to the metric $dr^2 + f(r)^2 g_X$ on $C^f X$ we have $|\omega_1| \le |\omega|$, $|\omega_2| \le |\omega|$, $|\omega|^2 = |\omega_1|^2 + |\omega_2|^2$, $|\omega|^p \sim |\omega_1|^p + |\omega_2|^p$.

Let $d\operatorname{vol}_X$ and $d\operatorname{vol}$ denote the volume forms on X and $C^f X$ respectively. Then

$$d\operatorname{vol} = f(r)^m\,dr \wedge d\operatorname{vol}_X$$

and

$$(11)\qquad \begin{aligned} \|\omega\|^p_{L^p, C^f X} &= \int_{C^f X} |\omega|^p d\,\text{vol} \\ &\sim \int_{C^f X} (|\omega_1|^p + |\omega_2|^p) f(r)^m \, dr \wedge d\,\text{vol}_X \\ &= \int_0^1 (f(r)^{m-pk} \|\omega_1\|^p_{L^p, r} + f(r)^{m-p(k-1)} \|\omega_2\|^p_{L^p, r}) \, dr. \end{aligned}$$

Note that (11) becomes an exact equality if either $\omega_1 = 0$ or $\omega_2 = 0$.

5.2. Radially constant forms.

Lemma 5.2.1. *The operator* $\text{pr}^* : \Omega^k_{L^p}(X) \to \Omega^k_{L^p}(C^f X)$ *is bounded in* L^p *norm if and only if the integral* $\int_0^1 f(r)^{m-pk} \, dr$ *is convergent.*

Proof. Take any $\omega \in \Omega^k_{L^p}(X)$. Then from (11),

$$\begin{aligned} \|\text{pr}^* \omega\|^p_{L^p, C^f X} &= \int_0^1 f(r)^{m-pk} \|\text{pr}^* \omega\|^p_{L^p, r} \, dr \\ &= \|\omega\|^p_{L^p, X} \int_0^1 f(r)^{m-pk} \, dr. \qquad \text{q.e.d.} \end{aligned}$$

We shall say that the form $\text{pr}^* \omega$ is *radially constant.*

Remark 5.2.2. We see that the *existence* of a nonzero radially constant form of degree k which is L^p integrable, is equivalent to the convergence of the integral $\int_0^1 f(r)^{m-pk} \, dr$.

Denote by $\text{dom}\, \tilde{d}^\bullet_{L^p}(C^f X)$ the subcomplex of $\text{dom}\, d^\bullet_{L^p}(C^f X)$ consisting of all those forms which are radially constant for $2/3 < r < 1$, i.e., their restrictions onto $(2/3, 1) \times X \subset C^f X$ are pullbacks of some forms on X. Similarly, we shall denote by $\tilde{\Lambda}^\bullet_{L^p}(C^f X)$ and $\tilde{\Omega}^\bullet_{L^p}(C^f X)$ the subspaces of $\Lambda^\bullet_{L^p}(C^f X)$ and $\Omega^\bullet_{L^p}(C^f X)$ respectively, consisting of all those forms which are radially constant for $2/3 < r < 1$.

Lemma 5.2.3. *The inclusion of complexes of vector spaces* $\text{dom}\, \tilde{d}^\bullet_{L^p}(C^f X) \hookrightarrow \text{dom}\, d^\bullet_{L^p}(C^f X)$ *induces a cohomology isomorphism.*

Proof. By Theorem 2.4.1, there is a cohomology isomorphism $\text{pr}^* : \text{dom}\, d^\bullet_{L^p}(X) \xrightarrow{\sim} \text{dom}\, d^\bullet_{L^p}((1/2, 1) \times X)$. Moreover, there are explicit homotopy operators $\mathscr{H}_a$, $a \in (1/2, 1)$ which act on a form

$$\omega \in \Omega^\bullet_{L^p}((1/2, 1) \times X)$$

by the formula

$$(\mathscr{H}_a \omega)(r, x) = \int_a^r \omega_2(t, x) \, dt,$$

where $\omega = \omega(r, x) = \omega_1(r, x) + dr \wedge \omega_2(r, x)$. For any

$$\omega \in \operatorname{dom} d^{\bullet}_{L^p}((1/2, 1) \times X),$$

for almost any $a \in (1/2, 1)$, they satisfy the equality $(d\mathscr{H}_a + \mathscr{H}_a d)\omega = (\mathrm{Id} - \mathrm{pr}^* P_a)\omega$ where the operator P_a: $\operatorname{dom} d^{\bullet}_{L^p}((1/2, 1) \times X) \to \operatorname{dom} d^{\bullet}_{L^p}(X)$ acts by the formula $P_a\omega = \omega|_{a\times X}$. (See [4, §3].)

Let $u(r)$ be a C^∞ truncation function which is equal to 0 for $r \le 1/2$ and to 1 for $r \ge 2/3$.

Let $\mathscr{H}'_a = u(r)\mathscr{H}_a$; this operator preserves the subcomplex $\operatorname{dom} \tilde{d}^{\bullet}_{L^p}(C^f X)$. It satisfies $(d\mathscr{H}'_a + \mathscr{H}'_a d)\omega = (\mathrm{Id} - P'_a)\omega$ where P'_a is some operator $\operatorname{dom} d^{\bullet}_{L^p}(C^f X) \to \operatorname{dom} \tilde{d}^{\bullet}_{L^p}(C^f X)$. Although P'_a is not defined on all forms in $\operatorname{dom} d^{\bullet}_{L^p}(C^f X)$, for any $\omega \in \operatorname{dom} d^{\bullet}_{L^p}(C^f X)$, $P'_a\omega$ is defined for almost all a. It is clear from this that the inclusion $\operatorname{dom} \tilde{d}^{\bullet}_{L^p}(C^f X) \hookrightarrow \operatorname{dom} d^{\bullet}_{L^p}(C^f X)$ is a cohomology isomorphism. q.e.d.

Here is the reason we need the complex $\operatorname{dom} \tilde{d}^{\bullet}_{L^p}(C^f X)$ and the spaces $\tilde{\Lambda}^{\bullet}_{L^p}(C^f X)$ and $\tilde{\Omega}^{\bullet}_{L^p}(C^f X)$: there is an operator P: $\tilde{\Omega}^{\bullet}_{L^p}(C^f X) \to \Omega^{\bullet}_{L^p}(X)$ which acts by

$$P\omega = \omega|_{1\times X}.$$

The meaning of this formula is as follows: $\omega|_{(2/3, 1)\times X}$ is a pullback of some form on X, and we take $P\omega$ to be equal to this form.

Lemma 5.2.4. *The operator* P: $\tilde{\Omega}^{\bullet}_{L^p}(C^f X) \to \Omega^{\bullet}_{L^p}(X)$ *is bounded in* L^p *norm.*

(Obvious.)

5.3. The homotopy operators. In order to prove Theorems 3.1.2 and 4.9.1, we introduce the two homotopy operators, $\mathscr{H}^1$: $\tilde{\Lambda}^k_{L^p}(C^f X) \to \tilde{\Lambda}^{k-1}_{L^p}(C^f X)$ and $\mathscr{H}^0$: $\tilde{\Omega}^k_{L^p}(C^f X) \to \tilde{\Omega}^{k-1}_{L^p}(C^f X)$. They act on a form $\omega = \omega(r, x) = \omega_1(r, x) + dr \wedge \omega_2(r, x)$ by the formula

$$(\mathscr{H}^a\omega)(r, x) = \int_a^r \omega_2(t, x)\, dt \tag{12}$$

for $a = 0, 1$; they transform any form which is radially constant for $2/3 < r < 1$, into a form of the same kind.

Proposition 5.3.1. *The operator* $\mathscr{H}^a$: $\tilde{\Omega}^k_{L^p}(C^f X) \to \tilde{\Omega}^{k-1}_{L^p}(C^f X)$ *where* $a = 0$ *or* $a = 1$, *is well-defined and continuous with respect to* L^p *norm, if and only if one of the following conditions is satisfied:*

(a) *If* $a = 0$, *then the integral*

$$\int_0^1 f(r)^{-(m-p(k-1))/(p-1)}\, dr \tag{13}$$

converges.

(b) *If* $a = 1$, *then the integral*

$$\int_0^1 f(r)^{m-p(k-1)}\,dt \tag{14}$$

converges.

Proof. Let $h(r) = f(r)^{(m-p(k-1))/p}\|\omega_2\|_{L^p,r}$. Then by (11),

$$\|\omega\|^p_{L^p, C^f X} \geq \|\omega_2\|^p_{L^p, C^f X} = \int_0^1 h(r)^p\,dr. \tag{15}$$

In particular, ω_2 is L^p if and only if h is L^p.

As $\mathscr{H}^a\omega$ does not involve dr, by (11),

$$\|\mathscr{H}^a\omega\|^p_{L^p, C^f X} = \int_0^1 \|\mathscr{H}^a\omega\|^p_{L^p,r} f(r)^{m-p(k-1)}\,dr.$$

Further,

$$\begin{aligned} \|\mathscr{H}^a\omega\|_{L^p,r} &\leq \left|\int_a^r \|\omega_2\|_{L^p,t}\,dt\right| \\ &= \left|\int_a^r h(t)f(t)^{-(m-p(k-1))/p}\,dt\right|, \end{aligned} \tag{16}$$

where the equality takes place if ω_2 is of the form $\tilde{h}(r)\phi(x)$; in particular, for any nonnegative L^p function h the equality here does take place for some L^p form ω satisfying $h(r) = f(r)^{(m-p(k-1))/p}\|\omega_2\|_{L^p,r}$. Hence,

$$\begin{aligned} &\|\mathscr{H}^a\omega\|^p_{L^p, C^f X} \\ &\quad\leq \int_0^1 \left|\int_a^r h(t)f(t)^{-(m-p(k-1)/p}\,dt\right|^p f(r)^{m-p(k-1)}\,dr, \end{aligned} \tag{17}$$

where, as before, the equality can take place for any nonnegative L^p function h.

Comparing (15) with (17), we see that the operator $\mathscr{H}^a$ is bounded if and only if for some constant C, for any L^p function h, we have the inequality

$$\begin{aligned} &\int_0^1 \left|\int_a^r h(t)f(t)^{-(m-p(k-1))/p}\,dt\right|^p f(r)^{m-p(k-1)}\,dr \\ &\qquad\qquad \leq C\int_0^1 |h(r)|^p\,dr. \end{aligned} \tag{18}$$

It will be helpful for us to rewrite the left-hand side of (18) as

$$\int_0^1 \left|\int_a^r h(t)f(t)^{-(m-p(k-1))/p}\,dt\right|^p f(r)^{m-p(k-1)}\,dr = \int_0^1 \left|\int_a^r h(t)\left(\frac{f(t)}{f(r)}\right)^{-(m-p(k-1))/p} dt\right|^p dr. \tag{19}$$

Case $a = 0$: we need to show that the inequality (18) for all L^p functions h is equivalent to the convergence of the integral (13).

If $k \geq m/p+1$, then we claim that the inequality (18) is always satisfied and the integral (13) always converges. Indeed, in this case the exponent $-(m-p(k-1))/p \geq 0$, so the integral (13) clearly converges. Let us show that the inequality (18) is satisfied. Note that in (19) $t \leq r$ as $a = 0$, and consequently, $f(t) \leq f(r)$ since f is nondecreasing, and so

$$(f(t)/f(r))^{-(m-p(k-1))/p} \leq 1,$$

$$\int_0^1 \left|\int_0^r h(t)f(t)^{-(m-p(k-1))/p}\,dt\right|^p f(r)^{m-p(k-1)}\,dr \leq \int_0^1 \left|\int_0^r h(t)\,dt\right|^p dr.$$

It is not hard to see that

$$\int_0^1 \left|\int_0^r h(t)\,dt\right|^p dr \lesssim \int_0^1 |h(r)|^p\,dr;$$

the inequality (18) follows immediately.

If $k < m/p+1$, then $m-p(k-1) > 0$. If the integral (13) converges, then the inside integral in (18) is a bounded function of r:

$$\begin{aligned}\left|\int_a^r h(t)f(t)^{-(m-p(k-1))/p}\,dt\right| &\leq \int_0^1 |h(t)||f(t)|^{-(m-p(k-1))/p}\,dt \\ &\leq \left(\int_0^1 |h(t)|^p\,dt\right)^{1/p}\left(\int_0^1 f(t)^{-(m-p(k-1))/(p-1)}\,dt\right)^{1-1/p}.\end{aligned}$$

As the term $f(r)^{m-p(k-1)}$ is also bounded, the inequality (18) follows.

Conversely, suppose that inequality (18) holds for any L^p function h. In particular, this means that the inside integral $\int_0^1 h(t)f(t)^{-(m-p(k-1))/p}\,dt$ in (18) converges for any L^p function h. This means that the function $f^{-(m-p(k-1))/p}$ is L^q; the latter property is the same as the convergence of the integral (13).

Case $a = 1$: we need to show that the inequality (18) for all L^p functions h is equivalent to the convergence of the integral (14).

If $k \leq m/p+1$, then we claim that the inequality (18) is always satisfied and the integral (14) always converges. Indeed, in this case the exponent $m-p(k-1) \geq 0$, so the integral (14) clearly converges. Let us show that the inequality (18) is satisfied. In (19) $t \geq r$ as $a = 1$, and consequently, $f(t) \geq f(r)$ since f is nondecreasing, $-(m-p(k-1))/p \leq 0$, and so again

$$(f(t)/f(r))^{-(m-p(k-1))/p} \leq 1.$$

The same argument as above yields the inequality (18).

If $k > m/p+1$, then $-(m-p(k-1))/p > 0$. If the integral (14) converges, then the inequality (18) holds because, clearly, the inside integral in (18) is a bounded function of r.

Conversely, suppose that the inequality (18) holds for any L^p function h. In particular, we can take $h(r) = -1$; then for all $r \leq 1/2$ we have $\int_1^r h(t)f(t)^{-(m-p(k-1))/p}\,dt > \varepsilon$ for some $\varepsilon > 0$. Hence, the convergence of the left-hand side of (18) implies the convergence of the integral $\int_0^{1/2} f(r)^{m-p(k-1)}\,dr$, and the latter is equivalent to the convergence of (14).

Remark 5.3.2. Let l_0 be the minimal integer k for which the operator $\mathscr{H}^0: \widetilde{\Omega}^k_{L^p}(C^fX) \to \widetilde{\Omega}^{k-1}_{L^p}(C^fX)$ is bounded. Then it is bounded for all $k \geq l_0$; this is due to the fact that if the integral (13) converges for some value of k, then it converges for all larger values.

We have already seen that this integral converges for all $k \geq m/p+1$; hence, $l_0 < m/p+2$.

Let l_1 be the maximal integer k for which the operator $\mathscr{H}^1: \widetilde{\Lambda}^k_{L^p}(C^fX) \to \widetilde{\Lambda}^{k-1}_{L^p}(C^fX)$ is bounded. It is bounded for all $k \leq l_1$, since the integral (14) converges for all values smaller than some value of k for which it converges.

We have already known that this integral converges for all $k \leq m/p+1$; hence, $l_1 > m/p$.

As $\mathscr{H}^0$ is bounded for all $k \geq m/p+1$, and $\mathscr{H}^1$ for all $k \leq m/p+1$, for any value of k at least one of these operators is bounded. Hence, $l_1 \geq l_0 - 1$. If the inequality here is strict, i.e., $l_1 \geq l_0$, then there are "overlap degrees" k (satisfying $l_0 \leq k \leq l_1$) in which both operators $\mathscr{H}^0$ and $\mathscr{H}^1$ are bounded.

We have denoted by l the maximal k for which the operator $\mathrm{pr}^*: \Omega^k_{L^p}(X) \to \Omega^k_{L^p}(C^fX)$ is bounded, or equivalently, there exist nonzero radially constant L^p forms of degree k. By Lemma 5.2.1, this l is the maximal k for which the integral $\int_0^1 f(r)^{m-pk}\,dr$ converges. Clearly, $l_1 = l+1$.

Remark 5.3.3. The "overlap case": if $l_0 \le k \le l_1$ and $\deg\omega = k$, then both $\mathscr{H}^0\omega$ and $\mathscr{H}^1\omega$ are defined and

$$(\mathscr{H}^0 - \mathscr{H}^1)\omega = \mathrm{pr}^* \int_0^1 \omega_2(t, x)\,dt = \mathrm{pr}^* \int_0^r \omega_2(t, x)\,dt|_{r=1}$$
$$= \mathrm{pr}^* P\mathscr{H}^0\omega.$$

5.4. The homotopy formula in degrees $\le l$. We have seen above that $l = l_1 - 1$ and so $k \le l$ is equivalent to $k < l_1$.

Lemma 5.4.1. *Let $k \le l$. For $\omega \in \mathrm{dom}\,\tilde{d}^k_{L^p}(C^f X)$ we have*

$$(d\mathscr{H}^1 + \mathscr{H}^1 d)\omega = \omega - \mathrm{pr}^* P\omega. \tag{20}$$

In particular, $d\mathscr{H}^1\omega$ is L^p integrable.

Proof. Denote by $\tilde{d}$ the operator which acts on forms on $C^f X$ by exterior differentiation along X, i.e., by the x variables only, as opposed to the r variable; then

$$\begin{aligned} d\omega &= d(\omega_1(r, x) + dr \wedge \omega_2(r, x)) \\ &= \tilde{d}\omega_1(r, x) + dr \wedge \left(\frac{\partial \omega_1(r, x)}{\partial r} - \tilde{d}\omega_2(r, x)\right). \end{aligned}$$

As ω is a C^∞ form on $(0, 1) \times \mathring{X}$, we have

$$\begin{aligned} (d\mathscr{H}^1\omega)(r, x) &= d\int_1^r \omega_2(t, x)\,dt \\ &= dr \wedge \omega_2(r, x) + \int_1^r \tilde{d}\omega_2(t, x)\,dt, \\ (\mathscr{H}^1 d\omega)(r, x) &= \int_1^r \left(\frac{\partial\omega_1(t, x)}{\partial t} - \tilde{d}\omega_2(t, x)\right) dt \\ &= \omega_1(r, x) - \omega_1(1, x) - \int_1^r \tilde{d}\omega_2(t, x)\,dt, \\ ((d\mathscr{H}^1 + \mathscr{H}^1 d)\omega)(r, x) &= \omega(r, x) - \omega_1(1, x) = \omega - \mathrm{pr}^* P\omega. \end{aligned} \tag{21}$$

Proof of Theorem 3.1.2(a). This follows immediately from the homotopy formula (20).

5.5. The operators $\mathscr{H}_\varepsilon$ and the maps T_ε. Suppose we are given a family of homeomorphisms $T_\varepsilon\colon [0, 1] \to [0, T_\varepsilon(1)]$ depending on a parameter $\varepsilon > 0$ and satisfying $0 < T_\varepsilon(r) \le r$ for any $r \in [0, 1]$; in particular, $0 < T_\varepsilon(1) \le 1$. We shall assume that these homeomorphisms are actually diffeomorphisms between $(0, 1]$ and $(0, T_\varepsilon(1)]$, and that $T_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. By abuse of notation, we shall also denote by T_ε the map $C^f X \to C^f X$ given by $(r, x) \mapsto (T_\varepsilon(r), x)$; then the family T_ε becomes a contraction of the cone $C^f X$ to its vertex.

We define the operators $\mathcal{H}_\varepsilon\colon \Lambda^k_{L^p}(C^f X) \to \Lambda^{k-1}_{L^p}(C^f X)$ by

$$(\mathcal{H}_\varepsilon \omega)(r, x) = \int_{T_\varepsilon(r)}^r \omega_2(t, x)\, dt.$$

Lemma 5.5.1. *If $k \geq l_0$, then $\mathcal{H}_\varepsilon$ is bounded in L^p norm, and for any $\omega \in \Omega^k_{L^p}(C^f X)$ we have $\|\mathcal{H}_\varepsilon \omega - \mathcal{H}^0 \omega\|_{L^p, C^f X} \to 0$ as $\varepsilon \to 0$.*

Proof. The argument is similar to the one in the proof of Proposition 5.3.1.

Lemma 5.5.2. *For $\omega \in \operatorname{dom} d^\bullet_{L^p}(C^f X)$ we have $(\mathcal{H}_\varepsilon d + d\mathcal{H}_\varepsilon)\omega = \omega - T^*_\varepsilon \omega$; in particular, if, in addition, $T^*_\varepsilon \omega$ is L^p, then $d\mathcal{H}_\varepsilon \omega$ is L^p too.*

Proof. This is similar to the proof of Lemma 5.4.1:

$$\begin{aligned}(d\mathcal{H}_\varepsilon \omega)(r, x) = {} & dr \wedge \omega_2(r, x) - dT_\varepsilon(r) \wedge \omega_2(T_\varepsilon(r), x) \\ & + \int_{T_\varepsilon(r)}^r \tilde{d}\omega_2(t, x)\, dt,\end{aligned}$$

$$\begin{aligned}(\mathcal{H}_\varepsilon d\omega)(r, x) &= \int_{T_\varepsilon(r)}^r \left(\frac{\partial \omega_1(t, x)}{\partial t} - \tilde{d}\omega_2(t, x)\right) dt \\ &= \omega_1(r, x) - \omega_1(T_\varepsilon(r), x) - \int_{T_\varepsilon(r)}^r \tilde{d}\omega_2(t, x)\, dt.\end{aligned} \tag{22}$$

Adding up, we get

$$(d\mathcal{H}_\varepsilon + \mathcal{H}_\varepsilon d)\omega = \omega - T^*_\varepsilon \omega.$$

5.6. The definition of T_ε. Take $k \geq l + 1$ so that the integral $\int_0^1 f(r)^{m-pk}\, dr$ diverges.

We make a coordinate change on $C^f X$ from (r, x) to (u, x) where

$$u(r) = \int_1^r f(t)^{m-pk}\, dt.$$

As r varies from 0 to 1, u varies from $-\infty$ to 0. Denote by $r(u)$ the function inverse to $u(r)$.

We define the maps $T_\varepsilon\colon (-\infty, 0] \to (-\infty, -1/\varepsilon]$ by $T_\varepsilon(u) = u - 1/\varepsilon$.

As $u(r)$ identifies $[-\infty, 0]$ and $[0, 1]$, we get the maps $T_\varepsilon\colon [0, 1] \to [0, T_\varepsilon(1)]$ and $T_\varepsilon\colon C^f X \to C^f X$.

Take $\omega = \omega_1 + dr \wedge \omega_2 \in \operatorname{dom} d^\bullet_{L^p}(C^f X)$ and write $\omega = \omega_1 + du \wedge \tilde{\omega}_2$ where $\tilde{\omega} = \omega_2 / f(r)^{m-pk}$. Then by (11),

$$\begin{aligned}\|\omega\|^p_{L^p, C^f X} &\sim \int_0^1 (f(r)^{m-pk} \|\omega_1\|^p_{L^p, r} + f(r)^{m-p(k-1)} \|\omega_2\|^p_{L^p, r})\, dr \\ &= \int_{-\infty}^0 (\|\omega_1\|^p_{L^p, r(u)} + f(r(u))^\alpha \|\tilde{\omega}_2\|^p_{L^p, r(u)})\, du,\end{aligned} \tag{23}$$

where $\alpha = m - p(k-1) + p(m-pk) - (m-pk) = p(m+1-pk)$.

5.7. The homotopy formula in degrees $\geq \max((m+1)/p, l+1, l_0)$.

Proposition 5.7.1. *If $k \geq \max((m+1)/p, l+1)$, then for any $\omega \in \Omega^k_{L^p}(C^f X)$ the form $T^*_\varepsilon \omega$ is L^p integrable and its norm $\|T^*_\varepsilon \omega\|_{L^p, C^f X} \to 0$ as $\varepsilon \to 0$.*

Proof. Note that the exponent $\alpha = p(m+1-pk) \leq 0$ in (23) since $k \geq (m+1)/p$.

Clearly, $T^*_\varepsilon \omega = T^*_\varepsilon \omega_1 + du \wedge T^*_\varepsilon \tilde{\omega}_2$, so

$$\begin{aligned}\|T^*_\varepsilon \omega\|^p_{L^p, C^f X} &\sim \int_{-\infty}^0 (\|T^*_\varepsilon \omega_1\|^p_{L^p, r(u)} + f(r)^\alpha \|T^*_\varepsilon \tilde{\omega}_2\|^p_{L^p, r(u)})\, du \\ &\leq \int_{-\infty}^0 (\|T^*_\varepsilon \omega_1\|^p_{L^p, r(u)} + T^*_\varepsilon f(r(u))^\alpha \|T^*_\varepsilon \tilde{\omega}_2\|^p_{L^p, r(u)})\, du,\end{aligned} \tag{24}$$

because $T^*_\varepsilon f(r(u))^\alpha = f(r(u - 1/\varepsilon))^\alpha \geq f(r(u))^\alpha$ since $\alpha \leq 0$ and f is nondecreasing. Performing a coordinate change in (24) yields

$$\|T^*_\varepsilon \omega\|^p_{L^p, C^f X} \lesssim \int_{-\infty}^{-1/\varepsilon} (\|\omega_1\|^p_{L^p, r(u)} + f(r(u))^\alpha \|\tilde{\omega}_2\|^p_{L^p, r(u)})\, du. \tag{25}$$

As the integral (23) converges, the right-hand side in (25) is finite and approaches zero as $\varepsilon \to 0$.

Corollary 5.7.2. *If $k \geq \max((m+1)/p, l+1, l_0)$, then for any $\omega \in \operatorname{dom} \overline{d}^k_{L^p}(C^f X)$ we have*

$$(\mathscr{H}^0 \overline{d} + \overline{d} \mathscr{H}^0)\omega = \omega. \tag{26}$$

In particular, $\overline{d}\mathscr{H}^0 \omega$ is L^p integrable.

Proof. This immediately follows from Proposition 5.7.1 and Lemmas 5.5.2 and 5.5.1.

5.8. The homotopy formula in the borderline degree. Here we consider the only case which has not been covered yet, namely, when the degree k satisfies

$$l < k < \max((m+1)/p, l+1, l_0). \tag{27}$$

Lemma 5.8.1. $l+1 \le \max((m+1)/p, l+1, l_0) \le l+2$.

Proof. Obviously, $\max((m+1)/p, l+1, l_0) \ge l+1$.

As we have already noted in Remark 5.3.2, $l_1 = l+1$, $l_1 > m/p$, and $l_0 \le l_1 + 1$. This shows that, first,

$$\frac{m+1}{p} - (l+1) = \frac{m+1}{p} - l_1 < \frac{m+1}{p} - \frac{m}{p} = \frac{1}{p} < 1,$$

and second,

$$l_0 - (l+1) = l_0 - l_1 \le 1.$$

Consequently, in any case $\max((m+1)/p, l+1, l_0) \le l+2$. q.e.d.

Suppose that $\max((m+1)/p, l+1, l_0) > l+1$. This is possible only if either

$$(m+1)/p > l+1 \tag{28}$$

or

$$l_0 > l+1. \tag{29}$$

(Both possibilities may hold together.)

The inequality (27) is possible only if $\max((m+1)/p, l+1, l_0) > l+1$ and

$$k = l+1 = l_1. \tag{30}$$

Since $\max((m+1)/p, l+1, l_0) > l+1$, one of the possibilities (28) or (29) takes place.

It follows that $k < m/p + 1$. Indeed, we have either $k < (m+1)/p$ (28) or $k < l_0$ (29), and $l_0 < m/p + 2$ by Remark 5.3.2. Clearly, in both cases $k < m/p + 1$.

In the assumptions of Theorem 3.1.2, we are in case (b) since $k > l$ and the integral $\int_0^1 f(r)^{m-pk}\,dr$ is divergent; since $k < m/p + 1$, the subspace $\operatorname{Im}\{\overline{d}\colon \operatorname{dom}\overline{d}_{L^p}^{k-1}(X) \to \operatorname{dom}\overline{d}_{L^p}^{k}(X)\}$ must be closed in $\Omega_{L^p}^k(X)$.

The equality (30) means that if $\omega \in \operatorname{dom}\tilde{d}_{L^p}^k(C^f X)$, then $\mathscr{H}^1\omega$ is defined, $\mathscr{H}^1 d\omega$ is not necessarily defined as $\mathscr{H}^1$ is not bounded in degree $k+1 > l_1$, and consequently, $\mathscr{H}^0 d\omega$ is defined. If $k < (m+1)/p$ (28), then possibly $\|T_\varepsilon^*\omega\|_{L^p, C^f X} \nrightarrow 0$; if $k < l_0$ (29), then $\mathscr{H}^0\omega$ may be undefined. Note that the operator pr^* is defined and bounded in degree $k-1 = l$ but not in degree $k = l+1$.

Proposition 5.8.2. *In any of these cases,*

$$\overline{d}(\mathscr{H}^1\omega + \mathrm{pr}^*\psi) + \mathscr{H}^0 d\omega = \omega \tag{31}$$

for some $\psi \in \operatorname{dom} \overline{d}_{L^p}^{k-1}(X)$, *provided that* $\operatorname{Im} \overline{d}$ *is closed in* $\Omega_{L^p}^k(X)$. *In particular,* $\mathscr{H}^1 \omega + \operatorname{pr}^* \psi \in \operatorname{dom} \overline{d}_{L^p}^{k-1}(C^f X)$.

Proof. (Cf. [4, Lemma 3.3].) Let $\omega = \omega_1 + dr \wedge \omega_2$. From (21) we get

$$(d\mathscr{H}^1 \omega)(r, x) = dr \wedge \omega_2(r, x) + \int_1^r \tilde{d}\omega_2(t, x)\, dt.$$

As $\mathscr{H}^0 d\omega$ is defined, $\mathscr{H}_\varepsilon d\omega \to \mathscr{H}^0 d\omega$ as $\varepsilon \to 0$. From (22) it follows that

$$(\mathscr{H}_\varepsilon d\omega)(r, x) = \omega_1 - T_\varepsilon^* \omega_1 - \int_{T_\varepsilon(r)}^r \tilde{d}\omega_2(t, x)\, dt. \tag{32}$$

By (23),

$$\|\omega_1\|_{L^p, C^f X}^p = \int_{-\infty}^0 \|\omega_1\|_{L^p, r(u)}^p\, du,$$

and, similarly to (24) and (25), we have

$$\|T_\varepsilon^* \omega_1\|_{L^p, C^f X}^p = \int_{-\infty}^0 \|T_\varepsilon^* \omega_1\|_{L^p, r(u)}^p\, du = \int_{-\infty}^{-1/\varepsilon} \|\omega_1\|_{L^p, r(u)}^p\, du \to 0$$

as $\varepsilon \to 0$.

Hence, we can take the limit in (32) and get

$$(\mathscr{H}^0 d\omega)(r, x) = \omega_1 - \lim_{\varepsilon \to 0} \int_{T_\varepsilon(r)}^r \tilde{d}\omega_2(t, x)\, dt.$$

In particular, the limit exists here in the strong sense.

Adding up yields

$$\begin{aligned}(d\mathscr{H}^1 \omega + \mathscr{H}^0 d\omega)(r, x) &= \omega + \int_1^r \tilde{d}\omega_2(t, x)\, dt - \lim_{\varepsilon \to 0} \int_{T_\varepsilon(r)}^r \tilde{d}\omega_2(t, x)\, dt \\ &= \omega - \lim_{\varepsilon \to 0} \int_{T_\varepsilon(r)}^1 \tilde{d}\omega_2(t, x)\, dt.\end{aligned}$$

Let

$$\phi(r, x) = \lim_{\varepsilon \to 0} \int_{T_\varepsilon(r)}^1 \tilde{d}\omega_2(t, x)\, dt.$$

Clearly, ϕ is independent of r and we can write

$$\phi = \phi(x) = \lim_{\delta \to 0} \int_\delta^1 \tilde{d}\omega_2(t, x)\, dt = \lim_{\delta \to 0} \tilde{d} \int_\delta^1 \omega_2(t, x)\, dt. \tag{33}$$

Hence, ϕ lies in the closure of $\operatorname{Im}\{\overline{d}\colon \Omega_{L^p}^{k-1}(X) \to \Omega_{L^p}^k(X)\}$ in $\Omega_{L^p}^k(X)$. By our assumptions, $\operatorname{Im} \overline{d}$ is closed in $\Omega_{L^p}^k(X)$; hence, $\phi = \overline{d}\psi$ for some

$\psi \in \Omega^{k-1}_{L^p}(X)$, and

$$d\mathscr{H}^1\omega + \mathscr{H}^0 d\omega = \omega - \mathrm{pr}^* \overline{d}\psi$$

or

$$\overline{d}(\mathscr{H}^1\omega + \mathrm{pr}^*\psi) + \mathscr{H}^0 d\omega = \omega.$$

Remark 5.8.3. In case (28) does hold but (29) does not, the operator $\mathscr{H}^0$ is bounded in degree k, so $\mathscr{H}^0\omega$ is defined and the integral $\int_\delta^1 \omega_2(t, x)\,dt$ converges to $P\mathscr{H}^0\omega = \int_0^1 \omega_2(t, x)\,dt$ as $\delta \to 0$. Together with (33) this shows that $P\mathscr{H}^0\omega = \lim_{\delta\to 0}\int_\delta^1 \omega_2(t, x)\,dt$ lies in $\mathrm{dom}\,\overline{d}^{k-1}_{L^p}(X)$ and we can take $\psi = P\mathscr{H}^0\omega$ in (31). By Remark 5.3.3, $\mathscr{H}^1\omega + \mathrm{pr}^* P\mathscr{H}^0\omega = \mathscr{H}^0\omega$, so we can rewrite (31) as

$$(\mathscr{H}^0 d + \overline{d}\mathscr{H}^0)\omega = \omega.$$

In other words, in this case the homotopy formula (26) holds too. Altogether, (26) holds in the degrees k satisfying $k \geq \max(l+1, l_0)$.

5.9. Proof of Theorem 3.1.2(b). Indeed, this follows immediately from the homotopy formulas (26) and (31).

Remark 5.9.1. In the statement of Theorem 3.1.2 we required that if k is such that the integral $\int_0^1 f(r)^{m-pk}\,dr$ is divergent (i.e., $k \geq l+1$), then either $k \geq m/p + 1$ or $\mathrm{Im}\{\overline{d}\colon \mathrm{dom}\,\overline{d}^{k-1}_{L^p}(X) \to \mathrm{dom}\,\overline{d}^{k}_{L^p}(X)\}$ is closed in $\Omega^k_{L^p}(X)$. As we mentioned in Remark 3.1.3, we actually need a weaker condition which we can formulate now: if

$$k = l + 1 = l_1 < \max((m+1)/p, l_0)$$

(see (30), (27)), then $\mathrm{Im}\{\overline{d}\colon \mathrm{dom}\,\overline{d}^{k-1}_{L^p}(X) \to \mathrm{dom}\,\overline{d}^{k}_{L^p}(X)\}$ is closed in $\Omega^k_{L^p}(X)$.

5.10. Proof of Theorem 4.9.1. We assume that $\dim X = n$, X has f-horn singularities, $\int_0^1 f(r)^{-1}\,dr = \infty$, and there is a duality isomorphism in the derived category $D_X\,\mathrm{dom}\,\overline{d}^\bullet_{L^p, X, \mathbb{E}} \cong \mathrm{dom}\,\overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}}[n]$. We need to show that the L^p Stokes property holds at every point $P \in X$. It follows from Proposition 4.6.2 that it is enough to show that the L^p Stokes property holds on $C^f L_P$ where L_P is the link of P.

Since both complexes $\mathrm{dom}\,\overline{d}^\bullet_{L^p, X, \mathbb{E}}$ and $\mathrm{dom}\,\overline{d}^\bullet_{L^q, X, \overline{\mathbb{E}}}$ have constructible cohomology, the duality isomorphism implies the local duality isomorphisms: for every point $P \in X$ we have $D_{C^f L_P}\,\mathrm{dom}\,\overline{d}^\bullet_{L^p, C^f L_P, \mathbb{E}} \cong \mathrm{dom}\,\overline{d}^\bullet_{L^q, C^f L_P, \overline{\mathbb{E}}}[m+1]$ where $m = \dim L_P$.

By Proposition 4.7.1, it is enough to show that

$$\operatorname{Im}\overline{d} \subset \operatorname{dom}\overline{d}^{\bullet}_{L^p,\min,C^fL_P,\mathbb{E}}$$

where

$$\operatorname{Im}\overline{d} = \operatorname{Im}\{\operatorname{dom}\overline{d}^{\bullet}_{L^p,C^fL_P,\mathbb{E}} \to \operatorname{dom}\overline{d}^{\bullet}_{L^p,C^fL^P,\mathbb{E}}\}.$$

By Proposition 4.8.1, this is equivalent to $\int_{C^fL_P} d\omega \wedge d\phi = 0$ for $\omega \in \operatorname{dom} d^{\bullet}_{L^p}(C^fL_P)$, $\phi \in \operatorname{dom} d^{\bullet}_{L^q}(C^fL_P)$ such that $\operatorname{Supp}\omega$ and $\operatorname{Supp}\phi$ are compact subsets of $C^fL_P = [0, 1) \times L_P/0 \times L_P$.

Using induction, we may assume that the L^p Stokes property holds on the singular Riemannian space L_P; in particular, we shall use the fact that

$$\int_{L_P} \overline{d}\omega \wedge \overline{d}\phi = 0$$

for $\omega \in \operatorname{dom}\overline{d}^{\bullet}_{L^p}(L_P)$, $\phi \in \operatorname{dom}\overline{d}^{\bullet}_{L^q}(L_P)$. Consequently, if $\omega \in \operatorname{dom}\overline{d}^{\bullet}_{L^p}(C^fL_P)$, $\phi \in \operatorname{dom}\overline{d}^{\bullet}_{L^q}(C^fL_P)$, then for almost all ε we have

$$\int_{\varepsilon\times L_P} \overline{d}\omega \wedge \overline{d}\phi = 0.$$

Applying Proposition 4.6.1 to $(\varepsilon, 1) \times L_P$, we get

$$\int_{C^fL_P} \overline{d}\omega \wedge \overline{d}\phi = \lim_{\varepsilon\to 0}\int_{(\varepsilon,1)\times L_P} \overline{d}\omega \wedge \overline{d}\phi = -\lim_{\varepsilon-\to 0}\int_{\varepsilon\times L_P} \omega \wedge \overline{d}\phi$$
$$= \mp \lim_{\varepsilon-\to 0}\int_{\varepsilon\times L_P} \overline{d}\omega \wedge \phi,$$

where $\varepsilon - \to 0$ means that the limit is taken as ε goes to zero, possibly avoiding a subset of measure zero in $[0, 1)$. In particular, all these limits exist.

Hence, we need to prove that for $\omega \in \operatorname{dom} d^{\bullet}_{L^p}(C^fL_P)$, $\phi \in \operatorname{dom} d^{\bullet}_{L^q}(C^fL_P)$ we have

$$\int_{\varepsilon\times L_P} \omega \wedge d\phi \to 0,$$

as $\varepsilon - \to 0$.

We can assume that $\operatorname{Supp}\omega$ and $\operatorname{Supp}\phi$ lie in $[0, 2/3) \times L_P/0 \times L_P \subset C^fL_P$. In particular, this means that ω and ϕ are radially constant for $2/3 \le r \le 1$.

Lemma 5.10.1. (i) *In the assumptions of Theorem* 4.9.1, *we have* $l_1 \le l_0$, *i.e., the operators* $\mathscr{H}^0$ *and* $\mathscr{H}^1$ *can be both defined in no more than one degree. In particular, this means that* $l = l_1 - 1 \le l_0 - 1$.

(ii) *If* $l \le k < l_0 - 1$, *then* $k = l$ *and this is possible only if* $l_1 = l_0 - 1$ (*"no overlap between* $\mathscr{H}^0$ *and* $\mathscr{H}^1$*"*).

(iii) *Let* $\omega \in \operatorname{dom} \tilde{d}^k_{L^p}(C^f L_p)$. *Then we can decompose it into the sum*

$$\omega = d\zeta_1 + \zeta_2 + \zeta_3,$$

where $\zeta_1, \zeta_2, \zeta_3 \in \operatorname{dom} \overline{d}^{\bullet}_{L^p}(C^f L_p)$, ζ_2 *is radially constant, and* ζ_3 *is as follows*:

(a) *If* $k \ge l_0 - 1$ (*so that* $\mathscr{H}^0$ *is defined in degree* $k+1$), *then* $\zeta_3 = \mathscr{H}^0 d\omega$.

(b) *If* $k < l$, *then* $\zeta_3 = \mathscr{H}^1 d\omega$.

(c) *If* $l \le k < l_0 - 1$ (*in this case* $k = l$ *by part* (ii)), *then* $\zeta_3 = \mathscr{H}^1 d\omega + \mathrm{pr}^* \psi$ *for some* $\psi \in \operatorname{dom} \overline{d}^k_{L^p}(X)$.

Note that cases (a) and (b) do not overlap due to the part (i) of the Lemma.

Proof. Part (i): Let e be the same as in §3.1. By our assumptions, the integral $\int_0^1 f(r)^{-1}\,dr$ diverges. Hence, $e \le 1$, and if $e = 1$, then the integral $\int_0^1 f(r)^{-e}\,dr$ diverges.

As $l_1 = l + 1$ (see Remark 5.3.2), we get from (3) the following formula for l_1:

$$l_1 = \begin{cases} \max\{k \mid k < m/p + 1 + e/p\} \\ \qquad\qquad \text{if the integral } \int_0^1 f(r)^{-e}\,dr \text{ diverges}, \\ \max\{k \mid k \le m/p + 1 + e/p\} \\ \qquad\qquad \text{if } \int_0^1 f(r)^{-e}\,dr \text{ converges.} \end{cases}$$

Similarly,

$$l_0 = \begin{cases} \min\{k \mid -(m - p(k-1))/(p-1) > -e\} \\ \qquad\qquad \text{if the integral } \int_0^1 f(r)^{-e}\,dr \text{ diverges}, \\ \min\{k \mid -(m - p(k-1))/(p-1) \ge -e\} \\ \qquad\qquad \text{if } \int_0^1 f(r)^{-e}\,dr \text{ converges}, \end{cases}$$

$$= \begin{cases} \min\{k \mid k > m/p + 1 + e/p - e\} \\ \qquad\qquad \text{if the integral } \int_0^1 f(r)^{-e}\,dr \text{ diverges}, \\ \min\{k \mid k \ge m/p + 1 + e/p - e\} \\ \qquad\qquad \text{if } \int_0^1 f(r)^{-e}\,dr \text{ converges.} \end{cases}$$

By comparing, we see that $l_1 - l_0 < 1$, i.e., $l_1 \le l_0$.

Part (ii): By Remark 5.3.2, $l + 1 = l_1 \ge l_0 - 1$. Hence, if $l \le k < l_0 - 1$, then $l \le k < l + 1$, i.e., $k = l$.

Part (iii): Case $k < l$: we have the homotopy formula $(d\mathscr{H}^1 + \mathscr{H}^1 d)\omega = \omega - \mathrm{pr}^* P\omega$ (20), where $d\mathscr{H}^1\omega$ is L^p integrable by Lemma 5.4.1; applying the same statement to $d\omega$ (this is possible since $\deg d\omega = k+1 \leq l$), we find that $d\mathscr{H}^1 d\omega$ is L^p integrable. Take $\zeta_1 = \mathscr{H}^1\omega$, $\zeta_2 = \mathrm{pr}^* P\omega$ and $\zeta_3 = \mathscr{H}^1 d\omega$; clearly, $\zeta_1, \zeta_2, \zeta_3 \in \mathrm{dom}\,\overline{d}^{\bullet}_{L^p}(C^f L_P)$.

Case $k \geq \max(l+1, l_0)$: we have the homotopy formula $(\mathscr{H}^0 d + \overline{d}\mathscr{H}^0)\omega = \omega$ (26), where $\overline{d}\mathscr{H}^0\omega$ is L^p integrable by Corollary 5.7.2 (and by Proposition 5.8.2 and Remark 5.8.3 in case $k < (m+1)/p$). Applying Corollary 5.7.2 to $d\omega$ yields $\overline{d}\mathscr{H}^0 d\omega$ is L^p integrable. Take $\zeta_1 = \mathscr{H}^0\omega$, $\zeta_2 = 0$, and $\zeta_3 = \mathscr{H}^0 d\omega$; clearly, $\zeta_1, \zeta_2, \zeta_3 \in \mathrm{dom}\,\overline{d}^{\bullet}_{L^p}(C^f L_P)$.

What is left, is the cases where $l \leq k < \max(l+1, l_0)$. We have already seen in Lemma 5.8.1 that $\max(l+1, l_0) \leq l+2$; hence, we are left with two cases: first, $k = l$, and second, $k = l+1 < \max(l+1, l_0)$, i.e., $k = l+1 < l_0$.

Case $k = l$: again, we have the homotopy formula $(d\mathscr{H}^1 + \mathscr{H}^1 d)\omega = \omega - \mathrm{pr}^* P\omega$ (20), where $d\mathscr{H}^1\omega$ is L^p integrable. If $k+1 < \max((m+1)/p, l+1, l_0)$, then by Proposition 5.8.2, $\mathscr{H}^1 d\omega + \mathrm{pr}^*\psi \in \mathrm{dom}\,\overline{d}^{k}_{L^p}(C^f X)$; otherwise, $\mathscr{H}^0 d\omega \in \mathrm{dom}\,\overline{d}^{k}_{L^p}(C^f X)$. In the latter case both $\mathscr{H}^0 d\omega$ and $\mathscr{H}^1 d\omega$ are defined, so by Remark 5.3.3 $\mathscr{H}^0 d\omega = \mathscr{H}^1 d\omega + \mathrm{pr}^*\psi$ where $\psi = P\mathscr{H}^0 d\omega$. Take $\zeta_1 = \mathscr{H}^1\omega$, $\zeta_2 = \mathrm{pr}^*(P\omega - \psi)$, and $\zeta_3 = \mathscr{H}^1 d\omega + \mathrm{pr}^*\psi$. Again, it follows that $\zeta_1, \zeta_3 \in \mathrm{dom}\,\overline{d}^{\bullet}_{L^p}(C^f L_P)$; consequently, $\zeta_2 \in \mathrm{dom}\,\overline{d}^{\bullet}_{L^p}(C^f L_P)$ since $\omega = d\zeta_1 + \zeta_2 + \zeta_3$; in particular, ζ_2 is closed as otherwise its differential would be a nonzero radially constant form of degree $l+1$ (it is also easy to see directly that ζ_2 is closed).

Case $k = l+1 < l_0$: we have the homotopy formula $\overline{d}(\mathscr{H}^1\omega + \mathrm{pr}^*\psi) + \mathscr{H}^0 d\omega = \omega$ (31), where $\mathscr{H}^1\omega + \mathrm{pr}^*\psi \in \mathrm{dom}\,\overline{d}^{k-1}_{L^p}(C^f X)$. By Corollary 5.7.2, $\mathscr{H}^0 d\omega \in \mathrm{dom}\,\overline{d}^{k}_{L^p}(C^f X)$. Take $\zeta_1 = \mathscr{H}^1\omega + \mathrm{pr}^*\psi$, $\zeta_2 = 0$, and $\zeta_3 = \mathscr{H}^0 d\omega$; clearly, $\zeta_1, \zeta_2, \zeta_3 \in \mathrm{dom}\,\overline{d}^{\bullet}_{L^p}(C^f L_P)$. q.e.d.

The following is the key estimate in our proof.

Lemma 5.10.2. *For any* $\rho \in \mathrm{dom}\,\overline{d}^{k'}_{L^p}(C^f L_P)$, *let*

$$\zeta = \begin{cases} \mathscr{H}^0\rho & \text{if } k' \geq l_0, \\ \mathscr{H}^1\rho & \text{if } k' \leq l, \\ \mathscr{H}^1\rho + \mathrm{pr}^*\psi & \text{if } k' = l+1 < l_0, \end{cases}$$

where ψ *is a form in* $\mathrm{dom}\,\overline{d}^{k'-1}_{L^p}(L_P)$ *such that* ζ *lies in* $\mathrm{dom}\,\overline{d}^{k'-1}_{L^p}(C^f L_P)$.

Then for any $\eta \in \operatorname{dom} d_{L^q}^{m+1-k'}(C^f L_P)$ *we have*

$$\int_{\varepsilon \times L_P} \zeta \wedge \eta \to 0,$$

as $\varepsilon \dashrightarrow 0$.

For the proof, see below.

Applying this lemma to the case $\rho = d\omega$ and $\eta = d\phi$, we find that if ζ_3 is as above, then

$$\lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} \zeta_3 \wedge d\phi = 0.$$

Hence,

$$\begin{aligned} \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} \omega \wedge d\phi &= \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} (d\zeta_1 + \zeta_2 + \zeta_3) \wedge d\phi \\ &= \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} \zeta_2 \wedge d\phi = \pm \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} d\zeta_2 \wedge \phi, \end{aligned}$$

where ζ_2 is radially constant, $\zeta_2 \in \operatorname{dom} \overline{d}_{L^q}^k(C^f L_P)$.

In a similar way we can decompose $\phi = d\eta_1 + \eta_2 + \eta_3$ where $\eta_1, \eta_2, \eta_3 \in \operatorname{dom} \overline{d}_{L^q}^{\bullet}(C^f L_P)$, η_2 is radially constant, and η_3 is defined in a way similar to ζ_3. We then have

$$\begin{aligned} \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} \omega \wedge d\phi &= \pm \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} d\zeta_2 \wedge \phi = \pm \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} d\zeta_2 \wedge \eta_2 \\ &= \lim_{\varepsilon \dashrightarrow 0} \int_{\varepsilon \times L_P} \zeta_2 \wedge d\eta_2. \end{aligned}$$

Here ζ_2 and η_2 are radially constant forms, $\zeta_2 \in \operatorname{dom} \overline{d}_{L^p}^k(C^f L_P)$, $\eta_2 \in \operatorname{dom} \overline{d}_{L^q}^{m-k-1}(C^f L_P)$. If either $d\zeta_2 = 0$ or $d\eta_2 = 0$, then, clearly, all our limits are zero. Otherwise, $d\zeta_2 \neq 0$ and $d\eta_2 \neq 0$ are radially constant forms which are L^p and L^q integrable, respectively. By Lemma 5.2.1, their degrees $k+1$ and $m-k$ are such that the integrals $\int_0^1 f(r)^{m-p(k+1)}\,dr$ and $\int_0^1 f(r)^{m-q(m-k)}\,dr$ are convergent. By our assumptions, $\int_0^1 f(r)^{-1}\,dr$ is divergent, hence, $m - p(k+1) > -1$ and $m - q(m-k) > -1$. Then $k+1 < (m+1)/p$ and $m-k < (m+1)/q$; adding these inequalities, we get $m+1 < (m+1)/p + (m+1)/q = m+1$, which is a contradiction. Consequently, either $d\zeta_2 = 0$ or $d\eta_2 = 0$ and our limits are zero. (Cf. [4, §2].)

5.11. Proof of Lemma 5.10.2. Let us first estimate $\mathscr{H}^a \rho$ where $\deg \rho = k'$ and $a = 0$ if $k' \geq l_0$, $a = 1$ if $k' < l_0$.

Let $\rho = \rho_1 + dr \wedge \rho_2$ where ρ_1 and ρ_2 do not involve dr. As in §5.3, let $h(r) = f(r)^{(m-p(k'-1))/p} \|\rho_2\|_{L^p, r}$. Then, similarly to (16),

$$\|\mathscr{H}^a \rho\|_{L^p, r} \leq \left| \int_a^r h(t) f(t)^{-(m-p(k'-1))/p} \, dt \right|$$

and hence,

$$\|\mathscr{H}^a \rho\|_{L^p, r} \leq \left| \int_a^r h(t)^p \, dt \right|^{1/p} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p}$$
$$\leq \left(\int_0^1 h(t)^p \, dt \right)^{1/p} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p}.$$

Similarly to (15), we have $\int_0^1 h(r)^p \, dr = \|\rho_2\|^p_{L^p, C^f X} \leq \|\rho\|^p_{L^p, C^f X}$, so

$$\|\mathscr{H}^a \rho\|_{L^p, r} \leq \|\rho\|_{L^p, C^f X} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p}. \tag{34}$$

Now let us estimate ζ. We claim that in any case

$$\|\zeta\|_{L^p, r} \lesssim \|\rho\|_{L^p, C^f X} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p} \tag{35}$$

for r close enough to zero. Indeed, $\zeta = \mathscr{H}^a \rho$ unless $k' = l + 1 < l_0$, and $\zeta = \mathscr{H}^a \rho + \mathrm{pr}^* \psi$ if $k' = l + 1 < l_0$. In the first case the inequality (35) follows immediately from (34). In the second $a = 1$ and $k' < l_0$; the latter means that the integral $\int_0^1 f(t)^{-(m-p(k'-1))/(p-1)} \, dt$ diverges. Consequently, in (35) we have

$$\left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p} \to \infty$$

as $r \to 0$, and the inequality (35) follows from

$$\|\zeta\|_{L^p, r} = \|\mathscr{H}^a \rho + \mathrm{pr}^* \psi\|_{L^p, r} \leq \|\mathscr{H}^a \rho\|_{L^p, r} + \|\mathrm{pr}^* \psi\|_{L^p, r}$$
$$\leq \|\rho\|_{L^p, C^f X} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p} + \mathrm{const}$$
$$\lesssim \|\rho\|_{L^p, C^f X} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)} \, dt \right|^{1-1/p}$$

for r close enough to zero.

For any form $\chi = \chi_1 + dr \wedge \chi_2$ on $C^f L_P$, denote by $\|\chi\|_{L^p, r \times L_P}$ the L^p norm of the restriction of χ onto the slice $r \times L_P$, with respect to the metric on the slice which is induced by the metric on $C^f L_P$. This induced metric is clearly equal to $f(r)^2 g_{L_P}$ where g_{L_P} is the metric on L_P. Consequently, if $\deg \chi = s$ then

$$\|\chi\|_{L^p, r\times L_P} = \|\chi_1\|_{L^p, r} f(r)^{(m-ps)/p}. \tag{36}$$

Combining this with (11), we get

$$\|\chi\|^p_{L^p, C^f L_P} \geq \int_0^1 \|\chi\|^p_{L^p, r\times L_P}\, dr. \tag{37}$$

As $\deg \zeta = k' - 1$, (36) and (35) yield

$$\begin{aligned}(38)\quad & \|\zeta\|_{L^p, r\times L_P} \\ & \lesssim \|\rho\|_{L^p, C^f L_P} \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)}\, dt \right|^{1-1/p} f(r)^{(m-p(k'-1))/p}\end{aligned}$$

for r close enough to zero.

Suppose that $\lim_{\varepsilon \longrightarrow 0} \int_{\varepsilon \times L_P} \zeta \wedge \eta \neq 0$. Since

$$\int_{r\times L_P} \zeta \wedge \eta \leq \|\zeta\|_{L^p, r\times L_P} \|\eta\|_{L^q, r\times L_P},$$

the inequality (38) implies that

$$\|\eta\|_{L^q, r\times L_P} \gtrsim \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)}\, dt \right|^{-(1-1/p)} f(r)^{-(m-p(k'-1))/p}$$

for some $\varepsilon > 0$ and almost all r satisfying $0 < r < \varepsilon$. Noting that $1 - 1/p = 1/q$, we get

$$\|\eta\|^q_{L^q, r\times L_P} \gtrsim \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)}\, dt \right|^{-1} f(r)^{-(m-p(k'-1))/(p-1)}$$

for almost all r satisfying $0 < r < \varepsilon$. Similarly using (37), we obtain

$$\begin{aligned}\|\eta\|^q_{L^q, C^f L_P} &\geq \int_0^\varepsilon \|\eta\|^q_{L^q, r\times L_P}\, dr \\ &\gtrsim \int_0^\varepsilon \left| \int_a^r f(t)^{-(m-p(k'-1))/(p-1)}\, dt \right|^{-1} f(r)^{-(m-p(k'-1))/(p-1)}\, dr.\end{aligned}$$

Denote $g(r) = \int_a^r f(t)^{-(m-p(k'-1))/(p-1)}\, dt$; then

$$\|\eta\|^q_{L^q, C^f L_P} \gtrsim \int_0^\varepsilon |g(r)|^{-1} g'(r)\, dr = |\ln(|g(r)|)|_{r=0}^{r=\varepsilon}|.$$

In case $a = 0$ we have $k' \geq l_0$, so the integral $\int_0^1 f(t)^{-(m-p(k'-1))/(p-1)}\, dt$ converges and $g(r) \to 0$ as $r \to 0$, which means that $\ln g(r) \to -\infty$ as $r \to 0$, and $\|\eta\|_{L^q, C^f L_P} = \infty$.

In case $a=1$ we have $k'<l_0$, so the integral $\int_0^1 f(t)^{-(m-p(k'-1))/(p-1)}\,dt$ diverges and $|g(r)|\to\infty$ as $r\to 0$, which also means that $\|\eta\|_{L^q, C^f L_P} = \infty$.

We see that in both cases η is not L^q integrable on $C^f L_P$, which is a contradiction.

5.12. The case where $\int_0^1 f(r)^{-1}\,dr$ converges.

Example 5.12.1. Consider a singular Riemannian space X with f-horn singularities which also has the following properties.

First, we assume that the duality isomorphism $D_X \operatorname{dom} d^{\bullet}_{L^p, X, \mathbb{E}} \cong \operatorname{dom} \overline{d}^{\bullet}_{L^q, X, \overline{\mathbb{E}}}[n]$ holds on X; this may be achieved by making sure that the appropriate cohomology groups of the links vanish; see §4.5.

Second, we assume that for some point $P\in X$ the L^p Stokes is satisfied on the link L_P; for example, we may assume that L_P is smooth. Let $m=\dim L_P$.

Third, we assume that the function f is such that the integral $\int_0^1 f(r)^{-1}\,dr$ converges.

Fourth, suppose that $(m+1)/p\in\mathbb{Z}$.

We claim that under these assumptions, the L^p Stokes property, as well as the property (10) ("the noncohomological obstruction to L^p Stokes"), is not satisfied at P.

Indeed, let $k=(m+1)/p-1$ and $s=(m+1)/q-1=m-k-1$. Take a k-form ψ on L_P, which is C^∞ and has compact support inside $\mathring{L}_P$; we shall assume $d\psi\neq 0$. Let $\omega=\operatorname{pr}^*\psi$. Formula (3) shows that $\int_0^1 f(r)^{m-pk}\,dr$ and $\int_0^1 f(r)^{m-p(k+1)}\,dr$ converge (cf. the calculations at the end of §5.10). Hence, both ω and $d\omega$ are L^p integrable on $C^f L_P$.

Let $\phi=\operatorname{pr}^*(*d\psi)$ where $*$ is taken with respect to the metric g_{L_P}. Then $\deg\phi=m-(k+1)=s$, and the similar argument shows that ϕ and $d\phi$ are L^q integrable on $C^f L_P$. Thus

$$\int_{\varepsilon\times L_P} d\omega\wedge\phi=\int_{L_P} d\psi\wedge *d\psi>0,$$

which is a nonzero constant, independent of ε. As we have seen at the beginning of §5.10,

$$\int_{C^f L_P} d\omega\wedge d\phi=\mp\lim_{\varepsilon\longrightarrow 0}\int_{\varepsilon\times L_P} d\omega\wedge\phi;$$

consequently,

$$\int_{C^f L_P} d\omega\wedge d\phi\neq 0.$$

Hence, the properties 4.8.1(a)-(c) and (10), as well as L^p Stokes, do not hold at P. In this case, the failure of L^p Stokes has noncohomological nature.

This example also shows that the condition $\int_0^1 f(r)^{-1}\,dr = \infty$ in Theorem 4.9.1 is sharp.

Notes added in proof.

1. After this paper was written, I found out that a different proof of Theorem 2.2.1 appeared in the paper: V. M. Gol′dshtein, V. I. Kuz′minov & I. A. Shvedov, *A property of de Rham regularization operators*, Sibirsk. Mat. Ž. **25** (1984) 104–111 (in Russian; English translation in Siberian Math. J. **27** (1986) 35–44), Corollary 2. A different proof of Theorem 3.1.2 appeared in the paper: V. M. Gol′dshtein, V. I. Kuz′minov & I. A. Shvedov, *On Cheeger's theorem: extensions to L_p-cohomology of warped cylinders*, Siberian Advances in Math. **2** (1992) 114–122.

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3. Professor Oshawa advised me recently that the arguments in [8] contained a gap.

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