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$SO(3)$-INVARIANTS FOR 4-MANIFOLDS WITH $b_2^+ = 1$. II

D. KOTSCHICK & J. W. MORGAN

1. Introduction

In this paper we extend the definition of Donaldson polynomial invariants to cover the case of manifolds with $b_1 = 0$ and $b_2^+ = 1$. We shall consider $SO(3)$-bundles with w_2 which lifts to an integral class. This paper generalizes [2] where the case of $SU(2)$-bundles with $c_2 = 1$ was treated. It should be viewed as the continuation of [5], where $SO(3)$-bundles with arbitrary p_1 were considered. It extends [5] in two ways. First of all, it completes the proof of the fact that the values of these invariants depend only on the chamber containing the self-dual harmonic 2-form for the metric used to define the anti-self-dual (ASD) equation. Secondly, it establishes more of the general properties of the differences of the values of the invariants as the self-dual 2-form crosses a wall. It follows from the properties that we establish here that, as conjectured in [5], the value of an invariant on every chamber is determined by its value on any one chamber; and in particular, the invariant is defined for all chambers regardless of whether they contain forms which are self-dual harmonic for some metric.

In spite of our progress, there is still more to be done, for we do not give an explicit formula in general, like the one in [2], for the difference term as the self-dual 2-form crosses a wall. We conjecture that there are systematic formulae for these difference terms involving only the classes defining the wall and the self-intersection form of the manifold.

Of course, as has been understood for a long time, the case of $b_2^+ = 1$ is unlike that of $b_2^+ > 1$ in that Donaldson invariants depend on the metric which is used to define the ASD equations. Naively, this gives invariants of Riemannian 4-manifolds. For applications one requires an understanding of the way the invariant depends on the metric. In the case $w_2 = 0$ and $p_1 = -4$ it was shown in [2] that the invariant only depends on the period point in the positive cone in $H^2(M; \mathbf{R})$ of the self-dual

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harmonic 2-form for the metric. Furthermore, the invariant changes only when this period point passes through a wall (a wall here being defined as the space perpendicular to an integral class of square -1), and the change is explicitly known in terms of the class defining the wall. The main result of this paper is to prove a result analogous to the first of these properties for the general $SO(3)$-invariant. This completes the proof of Theorem 3.2 of [5]. In [5] there is a gap in the argument. Contrary to what is asserted on page 432, it is not known whether two metrics whose period points lie in the same chamber can be connected by a path of metrics whose period points lie in that chamber. This is an extremely interesting and unresolved question about the period map, one which we do not address here. We proceed in a different manner, giving an argument which makes no use of detailed properties of the period map. Our argument is based on a generalized gluing construction for gluing concentrated ASD connections over S^4 into not necessarily ASD connections on M. This generalized gluing construction may be of independent interest.

In the last section we give some very partial generalizations of the explicit formulae of [2] and [5] for the difference in the values of the invariant as one crosses a wall, and we list some conjectures about the general properties of the difference terms.

2. Conventions

Throughout this paper X is an arbitrary closed oriented smooth 4-manifold, and M is a closed oriented smooth 4-manifold with $b_2^+ = 1$ and $b_1 = 0$. Fix a principal $SO(3)$-bundle P over X with the property that $w_2(P)$ admits a lift to an integral class. Also, fix a generic Riemannian metric g_0 on X. We denote by $\mathcal{M}(P, g_0)$ the moduli space of equivalence classes of g_0-ASD connections on P, where two connections are defined to be equivalent if they differ by an $SO(3)$-bundle isomorphism which lifts to an isomorphism with trivial determinant of some $U(2)$-bundle covering P. We denote the group of all such $SO(3)$-bundle isomorphisms by $\mathcal{G}(P)$. The moduli space is an orientable smooth manifold, and a lift of $w_2(P)$ to an integral class together with an orientation of $H_+^2(X; \mathbf{R})$ determines an orientation of it; see [5]. We denote by $\overline{\mathcal{M}}(P, g_0)$ the compactification of $\mathcal{M}(P, g_0)$ obtained by adding ideal connections; see [1], [4]. We define $\mu\colon H_2(X; \mathbf{Q}) \to H^2(\mathcal{M}(P, g_0); \mathbf{Q})$ by slanting with $-p_1(\xi)/4$ where ξ is the universal $SO(3)$-bundle over $X \times \mathcal{M}(P, g_0)$. This map has a natural extension to a map $\mu\colon H_2(X) \to$

$H^2(\overline{\mathscr{M}}(P, g_0))$. Following [5] and using perturbations as in the appendix of [2] where necessary to remove nontrivial flat connections, we define the Donaldson multilinear invariant $\Phi(x_1, \cdots, x_d)$ to be the intersection of the classes $2\mu(x_1), \cdots, 2\mu(x_d)$. Our convention for evaluating a monomial $a_1 \dots a_d \in \operatorname{Sym} H^2(X)$ as a multilinear function on $H_2(X)$ is

$$\langle a_1 \dots a_d, (x_1, \cdots, x_d)\rangle = \frac{1}{d!}\sum_{\sigma}\prod_{i=1}^{d}\langle a_i, x_{\sigma(i)}\rangle,$$

where the sum ranges over σ in the permutation group on d letters. In this way any multilinear function on $H_2(X)$ (e.g., the Donaldson invariant) becomes a polynomial in $H^2(X)$. We call the resulting polynomial *the $SO(3)$-Donaldson polynomial of X associated to* (P, g_0).

3. Statement of the main theorem

Let M be a smooth closed oriented 4-manifold with $b_1(M) = 0$ and $b_2^+(M) = 1$. Let $P \to M$ be a principal $SO(3)$-bundle with $w_2(P)$ lifting to an integral class. We shall say that a class $\alpha \in H^2(M; \mathbf{Z})$ *defines a P-wall* if it satisfies

$$\alpha \equiv w_2(P) \pmod 2, \tag{1}$$

$$p_1(P) \le \alpha^2 < 0. \tag{2}$$

For any such α the intersection of $(\alpha)^\perp$ with the positive cone is called a *P-wall* (or the *P-wall defined by* α). According to [3] the set of P-walls is locally finite in the interior of the positive cone of $H^2(M; \mathbf{R})/\mathbf{R}^*$. Let Δ_P be the set of (open) chambers into which this cone is divided by the hyperplanes perpendicular to classes satisfying the above conditions. Fix an orientation of $H^2_+(M; \mathbf{R})$. Let Met be the space of smooth Riemannian metrics on M, and for each $g \in \text{Met}$ let $\omega(g) \in H^2(M; \mathbf{R})/\mathbf{R}^*$ be the ray of self-dual g-harmonic 2-forms contained in the positive component (with respect to the given orientation on $H^2_+(M; \mathbf{R})$) of the positive cone. Let $\mathscr{C}$ be the set of components in Met of the preimage of Δ_P under this mapping. Let $c \in H^2(M; \mathbf{Z})$ be a lift of $w_2(P)$, and let $d = -p_1(P) - 3$.

In the case where $w_2(P)$ is not the pullback of a class in $H^2(K(\pi_1(M), 1); \mathbf{Z}/2\mathbf{Z})$ it is established in [5] that there is a function $\tilde{\Phi}^M_{P,c} : \mathscr{C} \to \operatorname{Sym}^d(H^2(M; \mathbf{Z}))$ (denoted $\Phi^M_{P,c}$ in [5]) which associates to each generic metric g the $SO(3)$-Donaldson polynomial of M associated to (P, g). Without this assumption the moduli space of ASD connections may have ends associated with flat connections with w_2 equal to $w_2(P)$, and therefore its closure may not have a fundamental class if the space of such flat connections is too large. As in the appendix to [2], since $H^1(M; \mathbf{R}) = 0$ one can avoid this problem by perturbing the ASD equations until the only flat solution is the trivial solution. We will always implicitly assume that this has been done if M is not simply connected. In particular, $\widetilde{\Phi}^M_{P,c}$ is defined for all M with $b_1 = 0$ and $b_2^+ = 1$.

Our main theorem is that $\widetilde{\Phi}^M_{P,c}$ factors to induce a map from Δ_P. More precisely, we have

Theorem 3.0.1. *Let M be a closed oriented 4-manifold with $b_2^+(M) = 1$ and $b_1(M) = 0$, and let $P \to M$ be a principal $SO(3)$-bundle with $w_2(P)$ lifting to an integral class. Then there is a function δ_P from the set of classes $\alpha \in H^2(M; \mathbf{Z})$ defining P-walls to $\mathrm{Sym}^d(H^2(M; \mathbf{Q}))$ and a map $\Phi^M_{P,c} : \Delta_P \to \mathrm{Sym}^d(H^2(M; \mathbf{Z}))$ such that the following hold:*

1. *For any generic metric g we have $\widetilde{\Phi}^M_{P,c}(g) = \Phi^M_{P,c}(C_g)$ where C_g is the chamber containing $\omega(g)$.*
2. *Properties 1–4 stated in Theorem 3.2 of [5] hold.*
3. *If C_{-1} and C_1 are chambers in the same component of the positive cone, then*

$$\Phi^M_{P,c}(C_1) - \Phi^M_{P,c}(C_{-1}) = \sum_\alpha \varepsilon(c, \alpha)\delta_P(\alpha), \tag{3}$$

where $\varepsilon(c, \alpha) = (-1)^{((c-\alpha)/2)^2}$, and the sum is taken over all α defining P-walls with $\langle \alpha, C_{-1}\rangle < 0 < \langle \alpha, C_1\rangle$.

Remark 3.0.2. Notice that as a consequence of item 3 the value of $\Phi^M_{P,c}$ on every chamber is determined by the function δ_P and the value of $\Phi^M_{P,c}$ on any one chamber. This verifies Conjecture (3.5) in [5]. It seems to us an interesting problem to compute the difference terms $\delta_P(\alpha)$ in general. Partial results along these lines are contained in the last section of this paper.

4. A generalized gluing construction

We shall consider the question of gluing concentrated ASD connections on S^4 into a not necessarily ASD background connection on X. Of course, the first part of the gluing construction, the patching of connections, does not require that the background connection be ASD. Thus, just as in [6], associated to each open stratum of the k-fold symmetric product of X we can define a space of gluing parameters for connections. The problem is that there is no natural way to amalgamate these pieces as we pass from one stratum to another. However, the embeddings of the various pieces into the space of connections given by performing the gluing differ only by a small error term, and hence by general considerations one can find (noncanonical) deformations of the images of these pieces until they do match. The homotopy type of the result is independent of the choices of deformations. In this section we shall give this construction not only for a single metric and background connection but also for compact families of background connections and metrics.

4.1. A space of generalized connections. Fix a closed oriented Riemannian 4-manifold (X, g_0) and principal $SO(3)$-bundle P over X. Set $k = -p_1(P)/4$ and set $w_2 \in H^2(X; \mathbf{Z}/2\mathbf{Z})$ equal to $w_2(P)$. For any l with $0 \le l \le k$ and with $l \equiv k \pmod{\mathbf{Z}}$ let $\mathscr{B}_l$ be the space of gauge equivalence classes of connections on the principal $SO(3)$-bundle P_l satisfying $p_1(P_l) = -4l$ and $w_2(P_l) = w_2$. Of course, $\mathscr{B}_k = \mathscr{B}(P)$. We form a space of gauge equivalence classes of *ideal connections*

$$\overline{\mathscr{B}}(P) = \mathscr{B}_k \cup (\mathscr{B}_{k-1} \times X) \cup \cdots \cup (\mathscr{B}_{k-[k]} \times \Sigma^{[k]}(X)),$$

where $[k]$ is the greatest integer in k, and $\Sigma^r(X)$ is the rth symmetric product of X. For any ideal connection $(A, [x_1, \cdots, x_r]) \in \mathscr{B}_{k-r} \times \Sigma^r(X)$ we call the x_i its *singular points*, A its *background connection*, and $\|F_A\|^2 + \Sigma_i 8\pi^2 \delta_{x_i}$, its *measure*. The topology on this space is the topology of convergence up to gauge equivalence of background connections on compact subsets in the complement of the singular points together with convergence of measures in the weak topology. Clearly, the topology on this space is independent of the choice of Riemannian metric on X.

We set $\overline{\mathscr{R}}(P) \subset \overline{\mathscr{B}}(P)$ equal to the points whose background connection is reducible to $O(2)$ but not to the trivial subgroup.

The following is a clear consequence of the description in Chapter 3 of [4].

Lemma 4.1.1. *Let $\mathscr{T}$ be a family of Riemannian metrics on X, and let $\overline{\mathscr{M}}(P, \mathscr{T})$ be the compactified parametrized moduli space of ASD connections on P. Then there is natural inclusion $\overline{\mathscr{M}}(P, \mathscr{T}) \subset \overline{\mathscr{B}}(P) \times \mathscr{T}$ which is a homeomorphism onto its image which is a closed subset.*

Lemma 4.1.2. *Let $x \in H_2(X; \mathbf{Z})$. Then there is a well-defined class $\mu(x) \in H^2(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P); \mathbf{Q})$ whose restriction to $\mathscr{B}(P)$ is slant product of x with $-p_1(\xi)/4$ where ξ is the universal $SO(3)$-bundle over $X \times \mathscr{B}(P)$.*

Proof. Fix a surface $C \subset X$ representing x. Let $\mathscr{S}(C) \subset \overline{\mathscr{B}}(P)$ be the (closed) subset of ideal connections with a singularity at some point of C. By choosing a base point $p \in C$ we can form the space of based ideal connections

$$\mathscr{A}^0 = (\overline{\mathscr{B}}(P) - \mathscr{I}(C))^0 .$$

There is a restriction map from this space to the space of gauge equivalence classes of based connections over C. Pulling back the usual $SO(3)$-equivariant two-dimensional class from this space gives us a class β in $H^2_{SO(3)}(\mathscr{A}^0)$. Away from $\overline{\mathscr{R}}(P)$ the $SO(3)$-action has stabilizers with trivial first cohomology, and hence β descends to a class $\mu'(x) \in H^2(\overline{\mathscr{B}}(P) - (\overline{\mathscr{R}}(P) \cup \mathscr{I}(P)))$.

We claim that this class has a unique extension to a class $\mu(x) \in H^2(\mathscr{B}(P) - \overline{\mathscr{R}}(P))$. The argument is similar to the one given in §6 of Chapter 3 of [4] for the case of the moduli space. The point is the following. Consider $\varepsilon > 0$, a collection of open balls of radius less than ε with disjoint closures $B_1, \cdots, B_s$ in X, positive integers $n_1, \cdots, n_s$, and $c_0 > 0$. Associated to these choices there is an open subset of $\overline{\mathscr{B}}(P)$ consisting of all points whose measures ν satisfy

1. $|\nu(B_i) - 8\pi^2 n_i| < \varepsilon$ and
2. on $X - \bigcup_i B_i$ the measure density is less than c_0.

Given c_0, if ε is sufficiently small, then this neighborhood has a small deformation, preserving the singularities of each ideal connection, onto the subset of ideal connections which are ASD in each of the balls. Using this, and following the line of argument in [4], we can deform any two chains in $\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P)$ until they are disjoint from $\mathscr{I}(C)$. Furthermore, as in [4] we also see that any two cycles in $\overline{\mathscr{B}}(P) - (\overline{\mathscr{R}}(P) \cup \mathscr{I}(C))$ which are homologous in $\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P)$ give the same value when paired with $\mu'(x)$. This proves the existence of the unique extension $\mu(x)$ as claimed.

4.2. The space of gluing parameters defined by a single stratum. For this subsection we keep the Riemannian manifold (X, g_0) and principal bundle P from the previous subsection, and we fix a (not necessarily ASD) background connection A on a principal $SO(3)$-bundle $Q \to X$ with $w_2(Q) = w_2(P)$ and $p_1(Q) = p_1(P) + 4r$ for some $r \geq 0$. In this subsection we describe a space of parameters for gluing the concentrated ASD connections of total measure $8\pi^2 r$ on S^4 to A to form connections on P.

Fix a stratum Σ of $\Sigma^r(X)$. Then there is a set $n_1 \geq \cdots \geq n_s$ of positive integers with sum r associated to Σ. These integers describe the multiplicities of the various points in the support of any $\sigma \in \Sigma$.

Fix $\varepsilon > 0$ sufficiently small. As described in §4 of Chapter 3 of [4] associated to Σ there is a bundle E_Σ. It is $\prod_{i=1}^s (Q \times_X TX)$ over

$X \times \cdots \times X$. The bundle E_Σ is a principal $\prod_{i=1}^s (SO(3) \times SO(4))$ bundle on which the permutation group associated to Σ, i.e., the group permuting points of equal multiplicity, acts by bundle automorphisms.

For each $n > 0$ let $\overline{Z}_n^0$ be the space of gauge equivalence classes of based (at the south pole) ideal ASD connections on the round S^4 which are centered at the north pole and concentrated in the ε-ball about the north pole. There are natural commuting actions of $SO(3)$ (on the right, changing the base frame) and $SO(4)$ (on the left, induced by rotating the sphere) on $\overline{Z}_n^0$. We form the bundle over $X \times \cdots \times X$ with fiber $\prod_{i=1}^s \overline{Z}_{n_i}^0$ associated to E_Σ and this action. The symmetry group of Σ acts on this associated bundle covering the natural action on $X \times \cdots \times X$. Let $\widetilde{\Sigma} \subset X \times \cdots \times X$ be the complement of the full diagonal. The space of gluing parameters associated to Σ is the quotient of the preimage of $\widetilde{\Sigma}$ under the action of the symmetry group of Σ. It is denoted by $GP_{(\Sigma,\varepsilon)} = GP_{(\Sigma,\varepsilon)}(A, g_0)$. There is the obvious projection $\pi : GP_{(\Sigma,\varepsilon)} \to \Sigma$. This map is a locally trivial fiber bundle with fiber $\prod_{i=1}^s \overline{Z}_{n_i}^0$.

As is proved in [4] the fibers of the space of gluing parameters are cone bundles, and the structure group preserves the cone bundle structure. The section given by the cone points is naturally identified with Σ. We represent the point $x \in GP_{(\Sigma,\varepsilon)}$ by an element $(z_1, \cdots, z_s) \in \prod_{i=1}^s \overline{Z}_{n_i}^0$, by identifications of the fibers of the bundles over the south pole of S^4 with the fiber of Q over points p_i and by identifications of the tangent space of S^4 at the north pole with the tangent spaces to X at the p_i. Exponentiating the latter identifications gives us identifications of the ball of radius ε centered at the north pole of S^4 with balls of radius ε in X centered at the p_i. Using these identifications we transfer the singular points of the based connections z_i with their multiplicities to points with multiplicities in X. By definition, these points with multiplicities are the *singularities of* x. The subspace of $GP_{(\Sigma,\varepsilon)}$ consisting of ideal connections whose singularities have total multiplicity r is identified with a neighborhood of Σ in $\Sigma^r(X)$. Clearly, $GP(\Sigma, \varepsilon)$ admits a natural action of the gauge group $\mathscr{G}(Q)$.

4.3. Gluing defined by the stratum Σ. We keep the notation and assumptions of the previous subsection, and fix a compact subset $K \subset \Sigma$. We denote by $GP_{(K,\varepsilon)} = GP_{(K,\varepsilon)}(A, g_0)$ the preimage of K in $GP_{(\Sigma,\varepsilon)}$ under the natural projection mapping π. It contains a neighborhood of $\operatorname{int}(K)$ in $GP_{(\Sigma,\varepsilon)}$. For all ε sufficiently small we shall define a local gluing map $\varphi_{K,\varepsilon} : GP_{(K,\varepsilon)} \to \overline{\mathscr{B}}(P)$, which depends on g_0, A, K, and ε.

Choose ε sufficiently small so that for any $p \in K$ and any points $p_i \neq p_j$ of X in the support of p the distance from p_i to p_j measured by g_0 is at least 4ε, and so that in any ball of radius 2ε the measure of the connection A is at most ε. Fix a C^∞ cutoff function $\beta : [1, 2] \to [0, 1]$ which is identically 1 near 1 and identically 0 near 2. Let $x \in GP_{(K,\varepsilon)}$ be given. Let $p \in K$ be its image under the natural projection, and $\{p_1, \cdots, p_s\}$ its support. As above, we represent the point $x \in GP_{(\Sigma,\varepsilon)}$ by an element $(z_1, \cdots, z_s) \in \prod_{i=1}^s \overline{Z}^0_{n_i}$, by identifications of the fibers of the bundles over the south pole of S^4 with the fibers of Q over points p_i and by identifications of the tangent space of S^4 at the north pole with the tangent spaces to X at the p_i. Let B_i be the ball of radius 2ε centered at p_i. We give a product structure $Q|B_i \cong Q|\{p_i\} \times B_i$ by A-parallel transport along normal geodesics from p_i. In this product structure A is almost a product connection.

As in §4 of Chapter 3 of [4] we use this data (linearly scaling down β to the interval $[\varepsilon, 2\varepsilon]$) to glue together the various connections, forming a family of connections on X which agree with A outside the balls B_i and agree with the ASD connections on the sphere inside concentric balls $B_i' \subset B_i$ of radius ε.

The result is a continuous map $\varphi_{K,\varepsilon} : GP_{(K,\varepsilon)} \to \overline{\mathscr{B}}(P)$ preserving the singularities and their multiplicities.

Clearly, this construction can be done continuously as we vary A in a family of background connections A_T parametrized by a compact space T. We call the resulting map $\varphi_{T,K,\varepsilon}$. If the family A_T is invariant under $\mathrm{Stab}(A) \subset \mathscr{G}(Q)$, then $\varphi_{T,K,\varepsilon}$ is also invariant under $\mathrm{Stab}(A)$.

4.4. Fitting the various spaces of gluing parameters together. In this subsection we shall show how to glue the spaces $GP_{(K,\varepsilon)}$ together as we vary the stratum Σ. Here we cannot directly follow the argument in [6]. The reason is that since A is not assumed to be ASD the glued-up connections are also not necessarily a family of almost ASD connections. Hence, the deformations that we use to match up the various pieces are less natural than those in [6].

Suppose that Σ and Σ' are distinct strata with Σ' contained in the closure of Σ, and also that we have data (K, ε) and (K', ε') related to Σ and Σ' as in the previous subsection.

Definition 4.4.1. Define an open subset $V(\Sigma, \Sigma') \subset \Sigma$ as follows. It consists of all $p \in \Sigma$ for which there is a point $p' \in \operatorname{int} K'$ such that the balls of radius 2ε about the points of support of p are contained in balls of radius ε' centered around the points of support of p'. (See Figure 1.)

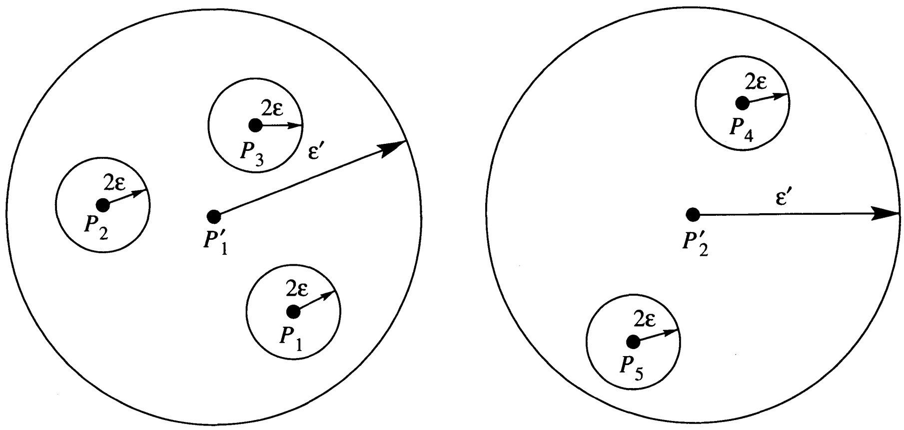

FIGURE 1

For each $p \in V(\Sigma, \Sigma')$ there is a partition of the points of support of p: the subsets of the partition are the subsets contained in a given one of the balls of radius ε'. While the point $p' \in K'$ is not necessarily determined, since distinct points of X in the support of any point in K' are at least distance $4\varepsilon'$ apart, the partition of the points of support of p is unique. We define a map $\rho : V(\Sigma, \Sigma') \to \operatorname{int} K'$ by associating to p the centers of mass of the subsets of the partition of the points of support of p given above (with the obvious induced multiplicities).

The next step is to cover the maps given by the restrictions $\rho|(K \cap V(\Sigma, \Sigma'))$ by maps defined on analogous open subsets

$$U(\Sigma, \Sigma') = \pi^{-1}(V(\Sigma, \Sigma')) \cap GP_{(K,\varepsilon)} \tag{4}$$

of $GP_{(K,\varepsilon)}$. Given a point $\zeta \in U(\Sigma, \Sigma')$ we represent $\varphi_{K,\varepsilon}(\zeta)$ by an ideal connection A' on $P \to X$. Given base frames for the bundle Q at the points of support of $\rho(\pi(\zeta))$, we parallel translate these frames via A to trivializations of Q over the annuli of radii ε' and $2\varepsilon'$ centered at these points.

Over the union C of these annuli the bundles P and Q agree, so that we have a trivialization of $P|C$ as well. Using this trivialization we transfer the restriction of A' to each of the balls of radius 2ε to a connection on the 2ε ball centered at the north pole of the 4-sphere. Using the cutoff function β suitably scaled, we extend the restriction of this connection to the ε ball centered at the north pole to a connection on all of S^4 trivial outside the 2ε ball. The resulting connection on

S^4 is almost ASD with respect to the standard metric. We deform this connection using the deformation described in [6] to an ASD connection. Since these constructions are equivariant with respect to the action of $SO(3)$ by changing the base frames, and the action of $SO(4)$ by rotating the tangent frames, they induce a map

$$\psi'_{\Sigma'} : U(\Sigma, \Sigma') \to \prod_i \overline{\mathscr{M}}^0_{n_i}(S^4),$$

where $\overline{\mathscr{M}}^0_{n_i}(S^4)$ is the moduli space of based ideal ASD connections (ASD with respect to the round metric) on an $SU(2)$-bundle with $c_2 = n_i$. (The south pole is not allowed to be a singular point, and the base frame is taken over the south pole.) The resulting ideal connections will be concentrated near the north pole, but they will not necessarily be centered at the north pole. We translate each of them until they are so centered. This produces a map $\psi_{\Sigma'} : U(\Sigma, \Sigma') \to \prod_i \overline{Z}^0_{n_i}$, which is equivariant with respect to the $SO(3) \times SO(4)$-action changing the frames for Q and TX at the points in the support of $\rho(\pi(\zeta))$. Because of this equivariance, the maps $\rho \circ \pi$ and $\psi_{\Sigma'}$ together induce a map $\tilde{\rho} : U(\Sigma, \Sigma') \to GP_{(K', \varepsilon')}$ covering the restriction of ρ to $V(\Sigma, \Sigma') \cap K$. Clearly, the map $\tilde{\rho}$ is invariant under $\mathrm{Stab}(A)$.

Notice that if $\zeta \in U(\Sigma, \Sigma')$, then $\varphi_{K, \varepsilon}(\zeta)$ and $\varphi_{K', \varepsilon'}(\tilde{\rho}(\zeta))$ are close together in $\overline{\mathscr{B}}(P)$, and have the same singularities. In particular, $\tilde{\rho}$ is the natural identification on the subset of $U(\Sigma, \Sigma')$ identified with a subset of $\Sigma^r(X)$.

Here is the theorem which states that we can use maps close to the $\tilde{\rho}$ in order to glue the $GP_{(K, \varepsilon)}$ together to form a space of global gluing parameters. More precisely, the following theorem shows that for appropriate choices of (K, ε) for each stratum of $\Sigma^r(X)$ there is a space covered by open subsets of the $GP_{(K, \varepsilon)}$ such that on the overlap of two pieces the transition function is close to the map $\tilde{\rho}$.

Theorem 4.4.2. *There is a smoothly stratified space $GP(A, g_0)$ with a smooth action of* $\mathrm{Stab}(A)$ *and for each stratum Σ of $\Sigma^r(X)$ a compact subset $K \subset \Sigma$ and $\varepsilon > 0$ so that the local gluing map $\varphi_{K, \varepsilon}$ is defined, satisfying the following properties*:

1. *$GP(A, g_0)$ contains $\Sigma^r(X)$ as a substratified space.*

2. *For each Σ there is a* $\mathrm{Stab}(A)$*-invariant neighborhood W_Σ of $K \subset GP(K, \varepsilon)$ and a smooth* $\mathrm{Stab}(A)$*-equivariant embedding $\iota_\Sigma : W_\Sigma \to GP(A, g_0)$ which is the natural identification on the subset of W_Σ identified with a subset of $\Sigma^r(X)$.*

3. *The union of the* $\iota_\Sigma(W_\Sigma)$ *cover* $GP(A, g_0)$.

4. *For each* Σ *there is a compact subset* $L \subset \operatorname{int} K$ *such that* $Y = W_\Sigma \cap \pi^{-1}(K - L)$ *is covered by open subsets* $\Upsilon_{\Sigma'}$, *indexed by* Σ' *in the closure of* Σ, *with the property that* $\Upsilon_{\Sigma'}$ *is contained in* $U(\Sigma, \Sigma')$ *and* $\iota_\Sigma(\Upsilon_{'})$ *is contained in* $\iota_{\Sigma'}(W_{\Sigma'})$.

5. *For each* Σ' *in the closure of* Σ *the map* $\iota_{\Sigma'}^{-1} \circ \iota_\Sigma : \Upsilon_{\Sigma'} \to W_{\Sigma'}$ *is close in the smooth topology to the restriction of* $\tilde{\rho}$ *to* $\Upsilon_{\Sigma'}$.

Proof. Choose an ordering $\{\Sigma_0, \cdots, \Sigma_t\}$ of the strata of $\Sigma^r(X)$ such that if Σ_j is in the closure of Σ_i, then $j < i$. By induction on i we shall construct

compact subsets $K_i \subset \Sigma_i$ and $L_i \subset \operatorname{int} K_i$,

- $\varepsilon_i > 0$,
- spaces GP_i and W_{Σ_i},
- embeddings $GP_{i-1} \to GP_i$ and $\iota_{\Sigma_i} : W_{\Sigma_i} \to GP_i$

such that the following hold:

(I) The local gluing maps $\varphi_{K_i, \varepsilon_i}$ are defined.

(II) $(\overline{\Sigma}_{i+1} - L_{i+1}) \subset GP_i$.

(III) The union over j such that Σ_j is in the closure of Σ_i of $U(\Sigma_i, \Sigma_j)$ contains the subset $\pi^{-1}(K_i - \operatorname{int} L_i)$ of $GP_{(K_i, \varepsilon_i)}$.

(IV) For the maps $\tilde{\rho} : U(\Sigma_i, \Sigma_j) \to GP_{(K_j, \varepsilon_j)}$, we have $\bigcup_{j<i} (\tilde{\rho})^{-1}(W_{\Sigma_j})$ containing $\pi^{-1}(K_i - \operatorname{int} L_i)$.

It follows by induction that for each $j < i$ we have an embedding $GP_j \subset GP_i$. Hence, for $j < i$ we can view the ι_j as embeddings of W_{Σ_j} into GP_i. We also require inductively that properties 4 and 5 of the theorem hold for $\Sigma = \Sigma_j$ with $j \le i$ and with K, L replaced by the inductively given K_j, L_j.

To begin the induction we take $K_0 = L_0 = \Sigma_0$ and $\varepsilon_0 > 0$ sufficiently small so that $\varphi_{K_0, \varepsilon_0}$ is defined. We take $GP_0 = W_{\Sigma_0} = GP_{(K_0, \varepsilon_0)}$.

Now suppose we have completed the induction through i. It follows by induction from the above properties that the union of the $\iota_{\Sigma_j}(W_{\Sigma_j})$ for $j \le i$ cover GP_i. It is easy to see that we can choose compact subsets $K_{i+1} \supset L_{i+1}$ in Σ_{i+1} and $\varepsilon_{i+1} > 0$ such that items (I)–(IV) hold.

Set $W_{\Sigma_{i+1}} = GP(K_{i+1}, \varepsilon_{i+1})$ and $U'_{i+1,j} = (\tilde{\rho})^{-1}(W_{\Sigma_j}) \subset W_{\Sigma_{i+1}}$. We have the compositions $\iota_{\Sigma_j} \circ \tilde{\rho} : U'_{i+1,j} \to GP_i$. On the overlap $U'_{i+1,j} \cap U'_{i+1,l}$ these compositions do not necessarily agree. But by the construction of the $\tilde{\rho}$ and by the inductive hypothesis on the ι_{Σ_j}, they are closed in the smooth topology. Set $U' = \bigcup_{j\le i} U'_{i+1,j}$. Using a Stab($A$)-invariant partition of unity on U' subordinate to the open cover $\{U'_{i+1,j}\}_j$ we can find a Stab(A)-equivariant smooth map $f: U' \to GP_i$ which for every j is close in the smooth topology to $\iota_j \circ \tilde{p}$ on $U'_{i+1,j}$.

Now we wish to glue $W_{\Sigma_{i+1}}$ to GP_i using the map f. In order for this gluing to produce a Hausdorff space, we may have to shrink GP_i. We find a Stab(A)-invariant open neighborhood GP'_i in GP_i of $(\overline{\Sigma}_{i+1} - L_{i+1}) \cup \bigcup_{j<i+1} \Sigma_j$ with the property that the intersection of its closure with the closure of $f(U')$ is equal to the closure of $f(Y_{i+1})$.

Replace GP_i by GP'_i, and for each $j < i+1$ keep K_j and L_j as before. Replace W_{Σ_j} by the preimage under ι_{Σ_j} of GP'_i. The set of data still satisfies the inductive conditions for i.

Using the embedding $f: Y_{i+1} \to GP'_i$, glue $W_{\Sigma_{i+1}}$ to GP'_i along Y_{i+1} in order to form the space GP_{i+1}. The action of Stab(A) is the one induced from the actions on the two pieces. One checks easily that all inductive conditions hold for $i+1$. This completes the inductive step and hence the proof of the theorem.

Remark 4.4.3. Because the deformations required in this construction do not change the singularities, there is a natural identification of $\Sigma^r(X)$ with the subset of $GP(A, g_0)$ consisting of points with singularities of total multiplicity r. There is a Stab(A)-equivariant deformation retraction of $GP(A, g_0)$ to $\Sigma^r(X)$ well defined up to homotopy.

Definition 4.4.4. Any space $GP(A, g_0)$ satisfying the conclusion of Theorem 4.4.2 is called a *global space of gluing parameters for* (A, g_0).

We can also do this construction for compact families of background connections.

Theorem 4.4.5. *Given a family A_T of connections A_t parametrized by a compact set T, there is a smoothly stratified space $GP(A_T, g_0)$ which fibers over T with the fiber over t being a space of gluing parameters for A_t and g_0. Furthermore, if the family A_T is invariant under a compact subgroup G of $\mathcal{G}(Q)$, then one can perform this construction equivariantly with respect to the action of G so that there will be a resulting action of G on $GP(A_T, g_0)$ covering the given action on A_T.*

Definition 4.4.6. Any space $GP(A_T, g_0)$ satisfying the previous theorem is called a *global space of gluing parameters for* (A_T, g_0).

The subspace of points $x \in GP(A_T, g_0)$ with singularities of total multiplicity r is identified with $T \times \Sigma^r(X)$. As before, there is a Stab(A_T)-equivariant deformation retraction of $GP(A_T, g_0)$ onto $T \times \Sigma^r(X)$.

4.5. The generalized gluing map. Fix a metric g_0 and a smooth (finite-dimensional) submanifold A_T in the space of connections on Q. We suppose that A_T is invariant under a compact subgroup G of $\mathcal{G}(Q)$, and that the quotient of A_T by the action of G embeds into the gauge equivalence classes of connections on Q. Let us also fix a G-equivariant space of gluing parameters $GP(A_T, g_0)$ as defined in the previous subsection. Our purpose here is to define a G-equivariant gluing map $\lambda_T : GP(A_T, g_0) \to \overline{\mathcal{B}}(P)$.

First let us suppose that the submanifold is a point, represented by a single connection A on Q. We write the space $GP(A, g_0)$ as a union of the images under ι_Σ of the open subsets $W_\Sigma \subset GP_{(K,\varepsilon)}$ as in Theorem 4.4.2. On each W_Σ we have the local gluing map $\lambda_\Sigma = \varphi_{K,\varepsilon}|W_\Sigma$, $\lambda_\Sigma : W_\Sigma \to \overline{\mathcal{B}}(P)$. These maps do not necessarily fit together to form a map on all of $GP(A, g_0)$, but the differences in the overlaps are small in the smooth topology. Thus, using a partition of unity on $GP(A, g_0)$ subordinate to the open cover $\iota_\Sigma(W_\Sigma)$, we can deform the maps $\lambda_\Sigma \circ \iota_\Sigma^{-1}$ until they agree on the overlaps. In this way we construct a map λ : $GP(A_T, g_0) \to \overline{\mathcal{B}}(P)$ which on each W_Σ is close to $\lambda_\Sigma \circ \iota_\Sigma^{-1}$. Any such map is called a *gluing map.* We can perform this construction in such a way that the singularities are preserved. Furthermore, we can perform this construction in a $\mathrm{Stab}(A)$-invariant way, so that it gives an embedding of $GP(A, g_0)/\mathrm{Stab}(A)$ in $\overline{\mathcal{B}}(P)$.

Now suppose that A_T is a compact smooth family of connections invariant under $G \subset \mathcal{G}(P)$. Then there is a gluing map $\lambda_T : GP(A_T, g_0) \to \overline{\mathcal{B}}(P)$ which is invariant under G and restricted to each $\iota_\Sigma(T \times W_\Sigma)$ is close to the local gluing map $\lambda_\Sigma \circ \iota_\Sigma^{-1}$. This map leaves the singularities invariant and hence is the natural identification on $(T/G) \times \Sigma^r(X) \subset GP(A_T, g_0)$. We call any such map a *gluing map for* A_T, g_0.

4.6. Varying the metric. So far we have let many things vary, but we have always fixed the metric. Now it is time to allow the metric to vary in a compact family. Clearly, all the constructions so far can be made to vary smoothly as we vary the metric. Thus, we have

Theorem 4.6.1. *Let g_S be a smooth family of metrics parametrized by a compact smooth manifold S. For each $s \in S$, suppose that A_{T_s} is a smooth submanifold in the space of connections on Q, varying smoothly with s. Suppose that there is a compact subgroup G of $\mathcal{G}(Q)$ such that for each s the submanifold A_{T_s} is G-invariant. Then there is a smooth family $GP(A_{T_s}, g_s)$ of spaces of global gluing parameters and a smooth family of gluing maps $\lambda_{T_s} : GP(A_{T_s}, g_s) \to \overline{\mathcal{B}}(P)$, $s \in S$.*

4.7. The homology class represented by the link of the reducible ideal connections in the space of gluing parameters. For this subsection we fix a smooth closed oriented 4-manifold M with $b_1(M) = 0$ and $b_2^+(M) = 1$ and a principal $SO(3)$-bundle $P \to M$.

Let $\alpha \in H^2(M; \mathbf{Z})$ be a class defining a P-wall. Let Q be the $SO(3)$-bundle obtained by stabilizing the $U(1)$-bundle with $c_1 = \alpha$. Let A be a reducible connection on Q compatible with its given reduction. Since $\operatorname{Stab}(A)$ acts on Q preserving the given reduction, there is a natural identification of $\operatorname{Stab}(A)$ with $U(1)$. (Replacing α by $-\alpha$ changes this identification by complex conjugation.) We suppose that we have a smooth submanifold A_T of the space of connections on Q diffeomorphic to the open unit ball in $\mathbf{C}^N$ with the connection A corresponding to the origin. We suppose that this submanifold is invariant under $\operatorname{Stab}(A)$ and that the action of $\operatorname{Stab}(A)$ is equivalent to the standard complex action on this open ball. We also suppose that the map $T/\operatorname{Stab}(A) \to \mathscr{B}(Q)$ given by $t \mapsto [A_t]$ is an embedding. We give the space A_T its complex orientation. The space $GP(A_T, g_0)$ inherits an orientation from this orientation on T and the orientation of M.

Let $GP(A_T, g_0)$ be a space of global gluing parameters, and let $L \subset GP(A_T, g_0)$ be the boundary of a $\operatorname{Stab}(A)$-invariant regular neighborhood of $T \times \Sigma^r(M)$. It inherits a stratification from that of $GP(A_T, g_0)$. It also inherits the boundary orientation from the given orientation on $GP(A_T, g_0)$. (Replacing α by $-\alpha$ but leaving the family T unchanged affects the orientation of $L/U(1)$ by $(-1)^{N+1}$.)

Remark 4.7.1. In the case where $\alpha^2 = p_1(P)$ the space $GP(A_T, g_0)$ is T and clearly the local orientation on $GP(A_T, g_0)$ that we have chosen is the one induced from the complex structure on T.

The gluing map λ_T of the previous subsection induces an embedding $\overline{\lambda}_T : GP(A_T, g_0)/U(1) \to \overline{\mathscr{B}}(P)$ sending $(T/\operatorname{Stab}(A)) \times \Sigma^r(M)$ to the singular locus of reducible ideal connections $\overline{\mathscr{R}}(P)$. The space L is closed and of dimension $8r+2N-1$. The quotient $L/U(1)$ is then of dimension $8r + 2N - 2$, and the image of its fundamental class $(\overline{\lambda}_T)_*[L/U(1)]$ is an element in $H_{8r+2N-2}(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(\mathscr{P}))$.

We now come to a result which shows that this homology class is independent of all the choices required in forming the gluing map.

Theorem 4.7.2. *Let M, P, A_T, g_0, α be as above. Let $GP(A_T, g_0)$ be any space of gluing parameters invariant under* $\operatorname{Stab}(A)$, *and λ_T : $GP(A_T, g_0) \to \overline{\mathscr{B}}(P)$ be any gluing map. Denote by $L(A_T, g_0) \subset GP(A_T, g_0)$ the boundary of a* $\operatorname{Stab}(A)$*-invariant regular neighborhood* *of* $T \times \Sigma^r(M)$. *The image of the fundamental class* $(\bar{\lambda}_T)_*[L/U(1)]$ *in* $H_*(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P))$ *depends only on the isomorphism class of the* $U(1)$-*reduction of* Q *induced by* A *and the complex dimension* N *of the space* T. *We denote this class by* $D(A, N) \in H_*(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P))$.

Proof. Suppose that we have two sets of data (A_T, g_0) and $(A'_{T'}, g'_0)$ as in the theorem with A and A' inducing isomorphic reductions of Q. By conjugating the second set of data we can assume that A and A' induce the same reduction of Q. Then there is a one-parameter family A_s of connections, all inducing the same reduction of Q, connecting A and A'. It is easy to see that there is in fact a smooth one-parameter family of submanifolds A_{T_s}, each isomorphic to the unit ball in $\mathbf{C}^N$ with the origin corresponding to $A_s \in A_{T_s}$ and each invariant under $\mathrm{Stab}(A)$, connecting A_T to $A'_{T'}$. Lastly, we choose a smooth one-parameter family of metrics connecting g_0 to g'_0. Applying Theorem 4.6.1 to this one-parameter family of data yields the result.

Remark 4.7.3. Even if one assumes that A_T and $A'_{T'}$ are families of g_0- and g'_0-ASD connections, there is no reason to expect that one can choose the one-parameter family of data connecting them to be ASD for each value of s. Because of this, we must allow the A_{T_s} to be families of non-ASD connections. This led us to consider the situation of the general gluing result.

It will be important for our explicit computations to know for $x \in H_2(M)$ the nature of the classes $2\mu(x)$ on the space $L(A_T, g_0)/U(1)$.

The following lemma is the analogue of Proposition (2.13) in [5] and follows from the discussion in [1] concerning the nature of the μ-map near ideal connections.

Lemma 4.7.4. *For any* $x \in H_2(M; \mathbf{Q})$ *the class* $2\mu(x) \in H^2(L(A_T, g_0) : \mathbf{Q})$ *is equal to* $2\pi^*\Sigma^r(x) + \langle \alpha, \rangle c_1$ *where* c_1 *is the first Chern class of the* $U(1)$-*bundle* $L(A_T, g_0) \to L(A_T, g_0)/U(1)$, π: $L(A_T, g_0)/U(1) \to \Sigma^r(M)$ *is the map induced by any* $\mathrm{Stab}(A)$-*invariant deformation retraction* $GP(A_T, g_0) \to \Sigma^r(M)$, *and* $\Sigma^r(x) \in H^2(\Sigma^r(M); \mathbf{Q})$ *is the class induced from the Poincaré dual of* x *in* $H^2(M; \mathbf{Q})$ *by symmetrization.*

Remark 4.7.5. In the case where $r = 0$, this formula yields

$$2\mu(x) = \langle \alpha, x \rangle c_1,$$

which is exactly what was proved in Proposition 2.13 of [5] (since the Chern class c_1 is the Chern class of the tautological bundle over $\mathbf{C}P^d$ and is hence equal to minus the Poincaré dual of $\mathbf{C}P^{d-1}$).

5. Proof of the main theorem

For this section we fix a 4-manifold M and a principal $SO(3)$-bundle P over M as in §3. We denote by Δ_P the set of chambers into which $H^2(M;\mathbf{R})/\mathbf{R}^*$ is divided by the P-walls. The proof of the fact that $\widetilde{\Phi}^M_{P,c}$ induces a map on the set of chambers will be deduced from a result about the difference of the values of $\widetilde{\Phi}^M_{P,c}$ on two metrics connected by a path of metrics crossing only one geometric wall.

5.1. Definition of the difference term associated to a class defining a P-wall. Let $\alpha \in H^2(M;\mathbf{Z})$ be a class with $p_1(P) \leq \alpha^2 < 0$ and $\alpha \equiv w_2(P) \pmod 2$. Set $r = (\alpha^2 - p_1(P))/4$ and $N = -\alpha^2 - 2$. We define a polynomial $\delta_P(\alpha)$ of degree $d = -p_1(P) - 3$ in $H^2(M;\mathbf{Q})$ as follows. Let Q_α be the $SO(3)$-bundle obtained by stabilizing the $U(1)$-bundle with first Chern class equal to α, and A_α be any reducible connection on Q_α compatible with the above splitting.

Definition 5.1.1. First, suppose that $\alpha^2 \leq -2$. Then for any classes $x_1, \cdots, x_d \in H_2(M)$ we define

$$\delta_P(\alpha)(x_1, \cdots, x_d) = \langle 2\mu(x_1) \cup \cdots \cup 2\mu(x_d), D(A, N)\rangle,$$

where the class $D(A, N)$ is as in Theorem 4.7.2. Now suppose that $\alpha^2 = -1$, and we set

$$\delta_P(\alpha)(x_1, \cdots, x_d) = \langle c_1 \cup 2\mu(x_1) \cup \cdots \cup 2\mu(x_d), D(A, 0)\rangle,$$

where c_1 is the first Chern class of the $U(1)$-bundle $L(A, g_0) \to L(A, g_0)/U(1)$ as in Lemma 4.7.4.

Since by Theorem 4.7.2 the homology class $D(A, N)$ depends only on the isomorphism class of the $U(1)$-reduction of Q_α determined by α and on the integers N and r, we see that $\delta_P(\alpha)$ is a well-defined function of α, and furthermore we have

$$\delta_P(-\alpha) = -(-1)^{\alpha^2}\delta_P(\alpha). \tag{5}$$

5.2. The local formula for crossing a wall. Consider a one-parameter family γ of metrics on M. We denote by $\mathscr{M}(P, \gamma)$ the parametrized moduli space and by $\overline{\mathscr{M}}(P, \gamma)$ the compactified parametrized moduli space.

In this subsection we shall consider the case where the path of self-dual harmonic 2-forms associated to γ crosses only one geometric P-wall.

There is one case where the result is somewhat different, but already well-understood.

Remark 5.2.1. If $w_2(P) \neq 0$, then $\overline{\mathcal{M}}(P, g)$ does not contain any ideal connection whose background connection is trivial. If $w_2(P) = 0$, and $p_1(P) < -4$, then the stratum of $\overline{\mathcal{M}}(P, g)$ containing ideal connections with trivial background connection is of codimension at least 2 and hence does not affect the arguments that we give below. (If M is not simply connected, then these moduli spaces are the spaces of gauge equivalence classes of solutions of appropriately perturbed ASD equations as in the appendix of [2]. Hence, there are no flat, nontrivial background connections in the moduli spaces that we consider.) In the remaining case where $w_2(P) = 0$ and $p_1(P) = -4$, the moduli space does not carry a fundamental class. However, the issues that we are concerned with here are completely understood in this case; see [2]. So we shall implicitly avoid this case in our discussion.

Here is our main local result.

Theorem 5.2.2. *Suppose that $\gamma = \{g_s, -1 \le s \le 1\}$ is a generic family of metrics on M with the following properties.*

1. *For every $SO(3)$-bundle P' with $w_2(P') = w_2(P)$ and $0 > p_1(P') \geq p_1(P)$, the parametrized moduli space $\mathcal{M}(P', \gamma)$ is a smooth manifold away from the reducible connections.*
2. *For all $s \neq 0$ we have that $\overline{\mathcal{M}}(P, g_s)$ contains no points whose background connections are reducible but not trivial.*
3. *The self-dual form $\omega(g_0)$ lies in a single geometric P-wall; that is to say all the g_0-ASD reducible background connections in $\overline{\mathcal{M}}(P, g_0)$ define complex line bundles whose first Chern classes are rational multiples of each other.*
4. *The path of self-dual two-forms given in the previous item crosses the geometric P-wall transversely at $s = 0$.*

Then
$$\tilde{\Phi}^M_{P,c}(g_1) - \tilde{\Phi}^M_{P,c}(g_{-1}) = \sum_\alpha \varepsilon(c, \alpha)\delta_P(\alpha),$$
where $\varepsilon(c, \alpha) = (-1)^{((c-\alpha)/2)^2}$, and the sum is taken over all α which define the geometric P-wall containing $\omega(g_0)$ and which satisfy $\langle \alpha, w(g_{-1})\rangle < 0 < \langle \alpha, \omega(g_1)\rangle$.

Remark 5.2.3. (i) Notice that because of formula (2.17) of [5] and equation (5) this theorem implies that if one crosses a wall in the opposite direction, the difference term has the opposite sign.

(ii) Even though the path of self-dual 2-forms crosses a single geometric wall, this wall may well be defined by several different classes α (all being rational multiples of each other). Thus, in general the sum given in the theorem is over more than one class.

Proof. Fix a path γ of metrics as in the statement of the theorem. By the proof of Proposition (4.2) in [5], since the path γ is generic, any background connection for a point of $\overline{\mathscr{M}}(P, \gamma)$ which is $O(2)$-reducible is actually $SO(2)$-reducible. Let $\nu \subset \overline{\mathscr{M}}(P, \gamma)$ be a regular neighborhood of the subset of ideal connections whose background connections are reducible but not trivial. From the usual cobordism argument, the perturbation argument from the appendix of [2], and the fact that the $2\mu(x_i)$ are defined away from reducible, nontrivial connections, we have that for any classes $x_1, \cdots, x_d \in H_2(M)$,

$$\begin{aligned}\tilde{\Phi}^M_{P,c}(g_1)(x_1, \cdots, x_d) - \tilde{\Phi}^M_{P,c}(g_{-1})(x_1, \cdots, x_d) \\ = \langle 2\mu(x_1) \cup \cdots \cup 2\mu(x_d), [\partial\nu]\rangle,\end{aligned}$$

where $\partial\nu$ is given the orientation as the boundary of $\overline{\mathscr{M}}(P, \gamma) - \operatorname{int}(\nu)$.

There is one component $\partial\nu_\alpha$ of $\partial\nu$ for each class α defining the P-wall that contains $\omega(g_0)$. Thus,

$$\begin{aligned}\tilde{\Phi}^M_{P,c}(g_1)(x_1, \cdots, x_d) - \tilde{\Phi}^M_{P,c}(g_{-1})(x_1, \cdots, x_d) \\ = \sum_\alpha \langle 2\mu(x_1) \cup \cdots \cup 2\mu(x_d), [\partial\nu_\alpha]\rangle.\end{aligned}$$

We claim that for each α with $\alpha^2 \leq -2$ we have

$$[\partial\nu_\alpha] = \varepsilon(c, \alpha)D(A, N) \in H_*(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P)),$$

where $N = -\alpha^2 - 2$, and $[A]$ is the reducible background connection on an $SO(3)$-bundle associated to the line bundle L_α. We also claim that for $\alpha^2 = -1$ we have

$$[\partial\nu_\alpha] = \varepsilon(c, \alpha)D(\alpha, 0) \cap c_1,$$

where c_1 is the first Chern class of the $U(1)$-bundle $L \to L/U(1)$, with L the boundary of a neighborhood of $\Sigma^r(M)$ in $GP(A_0, g_0)$. The theorem is immediate from these claims and the definition of $\delta_P(\alpha)$.

Let us consider the case where $\alpha^2 \leq -2$ first. Let A_T be the intersection of the space of γ-ASD connections with the slice centered at A_α for the action of the gauge group $\mathscr{G}(Q_\alpha)$ on the space of connections on Q_α. Since γ is generic, A_T is a smooth submanifold of the slice of real dimension $2N$. It is invariant under the action of $\operatorname{Stab}(A_\alpha)$, with A_α being the only point fixed by any nontrivial element of $\operatorname{Stab}(A_\alpha)$. Thus, this family is equivariantly diffeomorphic to an open ball in $\mathbf{C}^N$ with its standard $U(1)$-action. In this case by Taubes's gluing theory for ASD connections [6], the space $\partial\nu_\alpha$ is the quotient of a space of gluing parameters for (A_T, g_0) by the action of $\mathrm{Stab}(A_\alpha)$, and its inclusion in $\overline{\mathscr{B}}(P)$ is a gluing map. If we orient the moduli space $\mathscr{M}(P, \gamma)$ using α (this is an admissible choice since $\alpha \equiv w_2(P) \pmod 2$), then the orientation on ν_α as a boundary component of the parametrized moduli space is the one opposite to its orientation as the boundary of the space of gluing parameters modulo the action of $U(1)$. Thus, using the given class c to orient the moduli space, the difference in orientations is $-\varepsilon(c, \alpha)$ (see Lemma (2.17) in [5].) Consequently,

$$[\partial\nu]_\alpha = \varepsilon(c, \alpha) D(A_\alpha, N) \in H_{2d}(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P))$$

(see Theorem 4.7.2).

In the case where $\alpha^2 = -1$ the connection A_α is an isolated point of the space of γ-ASD connections in the slice. Furthermore, the obstruction space $H^2_+(M; A_\alpha)$ is three-dimensional, and the action of $\mathrm{Stab}(A_\alpha)$ on this space is isomorphic to the standard action of $U(1)$ on $\mathbf{R} \oplus \mathbf{C}$. Since the path of metrics γ is generic, the obstruction space at A_α for the parametrized moduli space for the bundle Q_α is two-dimensional, and the action of $\mathrm{Stab}(A_\alpha)$ on it is equivalent to the standard action of $U(1)$ on $\mathbf{C}$. We consider the space R of gauge equivalence classes of ideal connections in $\overline{\mathscr{B}}(P)$ satisfying the following conditions:

- The self-dual part of the curvature lies in the eigenspace for the smallest positive eigenvalue of the Laplacian on self-dual 2-forms on L_α, and
- On a large subset of M the connection is close to A_α.

According to Taubes's gluing theorem [6] R is the image under a gluing map of the quotient of a space $GP(A_\alpha, g_0)$ of gluing parameters by the action of $\mathrm{Stab}(A_\alpha)$. Restricting to the preimage of ∂R, this determines a principal $U(1)$-bundle $L \to \partial R$ and identifies R with $L/U(1)$. In particular, the fundamental class of ∂R is equal to $D(A_\alpha, 0)$ in $H_*(\overline{\mathscr{B}}(P) - \overline{\mathscr{R}}(P))$. According to [6] $\partial\nu_\alpha$ is the zero locus of a section of a vector bundle over ∂R. This vector bundle is the bundle associated to $L \to \partial R$ and the action of $U(1)$ on the obstruction space for the parametrized moduli space on Q_α at $[A_\alpha]$. Thus, the fundamental class $[\partial\nu_\alpha]$ is equal to $\varepsilon(c, \alpha) D(A_\alpha, 0) \cap c_1$ where c_1 is the first Chern class of the natural $U(1)$-bundle $L \to \partial R$. (The comparison of orientations is as in the previous case.) This completes the proof of the result in this case as well.

5.3. The general wall-crossing formula. Now we can pass from this result about paths of metrics whose self-dual 2-forms cross a single geometric wall to a similar result about an arbitrary path of metrics.

Theorem 5.3.1. *Suppose that* $\gamma = \{g_s, -1 \le s \le 1\}$ *is a generic path of metrics on* M *with the following properties.*

1. *For every* $SO(3)$*-bundle* P' *with* $w_2(P') = w_2(P)$ *and* $0 > p_1(P') \ge p_1(P)$, *the parametrized moduli space* $\mathcal{M}(P', \gamma)$ *is a smooth manifold away from the reducible connections.*
2. *The intersection of the path of self-dual 2-forms* $\omega(g_s)$ *with any* P*-wall is transverse.*

Then

$$\widetilde{\Phi}^M_{P,c}(g_1) - \widetilde{\Phi}^M_{P,c}(g_{-1}) = \sum_\alpha \varepsilon(c, \alpha)\delta_P(\alpha),$$

where the sum is taken over all α, *which define* P*-walls and which satisfy* $\langle \alpha, \omega(g_{-1})\rangle < 0 < \langle \alpha, \omega(g_1)\rangle$.

Proof. Since the set of P-walls is locally finite, this follows immediately by additivity from the previous result together with the usual cobordism argument for smooth parameterized moduli spaces (for the perturbed equations). Notice that if the path crosses a wall twice in opposite directions then the changes in the value of $\widetilde{\Phi}^M_{P,c}$ associated to these two crossings cancel out since the difference term $\delta_P(\alpha)$ depends only on the wall the path is crossing, not where it crosses that wall and changes sign when the direction of crossing is reversed (see Remark 5.2.3).

5.4. Completion of the proof of Theorem 3.0.1. It follows immediately from the above result that if g_0 and g_1 are metrics whose period points $\omega(g_i)$ lie in the same element of Δ_P, then $\widetilde{\Phi}^M_{P,c}(g_0) = \widetilde{\Phi}^M_{P,c}(g_1)$. Thus, $\widetilde{\Phi}^M_{P,c}$ defines a function $\Phi^M_{P,c}$ on the subset of Δ_P of chambers containing period points of metrics. Since the space of metrics is connected, it also follows from the above theorem that for any two chambers in this subset equation (3) holds. Since $\delta_P(\alpha)$ is defined for every α defining a P-wall, this equation allows us to uniquely extend the definition of $\Phi^M_{P,c}$ to the entire set Δ_P so that equation (3) holds for all pairs of chambers.

Given this, properties 1–4 of Theorem (3.2) of [5] were established in [5].

6. Properties of the difference term $\delta_P(\alpha)$

While we are far from a general understanding of the nature of the difference terms $\delta_P(\alpha)$, in this section we give some conjectures and partial results towards such an understanding.

6.1. A partial computation of the difference term $\delta_P(\alpha)$. In this subsection we give a partial computation of the difference term. In particular, we shall compute the lowest order term in the expansion of $\delta_P(\alpha)$ in powers of the class α. Our computation generalizes (3) of Theorem (3.2) in [5].

According to our conventions on the symmetric algebra, if $\alpha \in V^*$ and $q \in \mathrm{Sym}^2(V^*)$, then the monomial $\alpha^r q^s$ evaluated on $(x_1, \cdots, x_{r+2s}) \in V^{r+2s}$ gives the result

$$\frac{1}{(r+2s)!}\sum_{\sigma}\prod_{i=1}^{r}\langle\alpha, x_{\sigma(i)}\rangle\prod_{j=1}^{s}q(x_{\sigma(r+2j-1)}, x_{\sigma(r+2j)}),$$

where σ ranges over the symmetric group on $r+2s$ elements.

Theorem 6.1.1. *Let $P \to M$ be a principal $SO(3)$-bundle, and suppose that $\alpha \in H^2(M;\mathbf{Z})$ defines a P-wall. Let $d = -p_1(P) - 3$ and $r = (\alpha^2 - p_1(P))/4$. If $\alpha^2 < -1$, then in the polynomial ring on $H^2(M;\mathbf{Q})$ we have*

$$\delta_P(\alpha) \equiv (-1)^d 2^{2r}\frac{d!}{r!\,(d-2r)!}\alpha^{d-2r}q^r \pmod{\alpha^{d-2r+2}}. \tag{6}$$

If $\alpha^2 = -1$, then in the polynomial ring on $H^2(M;\mathbf{Q})$ we have

$$\delta_P(\alpha) \equiv -2^{2r}\frac{d!}{r!\,(d-2r)!}\alpha^{d-2r}q^r \pmod{\alpha^{d-2r+2}}. \tag{7}$$

Proof. To prove this we shall use the following elementary lemma in multilinear algebra.

Lemma 6.1.2. *Let V be a finite-dimensional $\mathbf{Q}$-vector space, and let $p \in \mathrm{Sym}^d(V^*)$ be a homogeneous polynomial function. Suppose that $\alpha \in V^*$ is nonzero, and set $K \subset V$ equal to the kernel of α. Then the following hold:*

1. *$p \equiv 0 \pmod{\alpha^k}$ if and only if for elements $(x_1, \cdots, x_d)$ of V we have $p(x_1, \cdots, x_d) = 0$ whenever at least $d-k+1$ of the x_i are in K.*
2. *If $p \equiv 0 \pmod{\alpha^k}$ define $p' \in \mathrm{Sym}^{d-k}(K^*)$ by the formula*

$$p'(x_1, \cdots, x_{d-k}) = \frac{d!}{k!\,(d-k)!}p(x_1, \cdots, x_{d-k}, a, \cdots, a),$$

where $a \in V$ is any class such that $\langle\alpha, a\rangle = 1$. Then

$$p \equiv p'\alpha^k \pmod{\alpha^{k+1}}.$$

Let us first show that $\delta_P(\alpha) \equiv 0 \pmod{\alpha^{d-2r}}$. According to the lemma it suffices to show that $\delta_P(\alpha)(x_1, \cdots, x_d) = 0$ whenever $x_1, \cdots, x_{2r+1}$

are in the kernel of α. Let $L = L(A_T, g_0)$ be as before. By Lemma 4.7.4 for each i we have

$$2\mu(x_i)|(L/U(1)) = 2\pi^*\Sigma^r(x_i) + \langle\alpha, x_i\rangle c_1.$$

Under our hypothesis it follows that

$$2\mu(x_i) = 2\pi^*\Sigma^r(x_i)$$

for $i \leq 2r+1$. Since the dimension of $\Sigma^r(M)$ is $4r$, it is clear that

$$\Sigma^r(x_1) \cup \cdots \cup \Sigma^r(x_{2r+1}) = 0.$$

Thus,

$$(2\mu(x_1) \cup \cdots \cup 2\mu(x_{2r+1}))|(L/U(1)) = 0,$$

and hence

$$\delta_P(\alpha)(x_1, \cdots, x_d) = 0.$$

Now we turn to the first nontrivial power of α. Let us first consider the unobstructed case, this is the case where $\alpha^2 < -1$. In this case we show that

$$\delta_P(\alpha) - (-1)^d 2^{2r} \frac{d!}{r!\,(d-2r)!} q^r \alpha^{d-2r}$$

is equal to zero modulo α^{d-2r+1}. To do this we consider classes $(x_1, \cdots, x_d)$ in $H_2(M; \mathbf{Z})$ such that $x_1, \cdots, x_{2r}$ are in the kernel of α. We have

$$\begin{aligned}(2\mu(x_1) &\cup \cdots \cup 2\mu(x_d))|(L/U(1)) \\ &= \sum_I 2^{|I|}\pi^* \left(\prod_{i\in I}(\Sigma^r(x_i))\right) \prod_{j\notin I}\langle\alpha, x_j\rangle c_1^{d-2|I|}\end{aligned}$$

where the sum runs over all subsets I of $\{1, \cdots, d\}$, $|I|$ is the cardinality of I, and c_1 is the Chern class of the principal $U(1)$-bundle $L \to L/U(1)$. As above, if $|I| > 2r$, then $\prod_{i\in I}\Sigma^r(x_i) = 0$. Hence, it suffices to consider only the terms where $|I| \leq 2r$. On the other hand, if $|I| < 2r$, then for at least one $j \notin I$ we have $j < 2r$, and hence $\langle\alpha, x_j\rangle = 0$. Thus, in fact we need only consider the terms where $|I| = 2r$ and every $j \notin |I|$ is greater than $2r$. There is only one such term, which is

$$2^{2r}\pi^*(\Sigma^r(x_1) \cup \cdots \cup \Sigma^r(x_{2r})) \prod_{j=2r+1}^{d} \langle\alpha, x_j\rangle c_1^{d-2r}.$$

A direct computation shows that $\Sigma^r(x_1) \cup \cdots \cup \Sigma^r(x_{2r}) \in H^{4r}(\Sigma^r(M); \mathbf{Z})$ is $((2r)!/r!\,2^r)q^r(x_1, \cdots, x_{2r})$ times the fundamental cohomology class of

$\Sigma^r(M)$. Thus,

$$\delta_P(\alpha)(x_1, \cdots, x_d) = 2^r \frac{(2r)!}{r!} q^r(x_1, \cdots, x_{2r}) \prod_{j=2r+1}^{d} \langle \alpha, x_j \rangle \langle c_1^{d-2r}, [F] \rangle,$$

where $F \subset L/U(1)$ is a generic fiber of the map $L/U(1) \to \Sigma^r(M)$. The space F is the quotient of the link of the central point of $\mathbf{C}^{d-4r+1} \times \prod_r c(SO(3))$ by the natural $U(1)$ action. An easy computation shows that

$$\langle c_1^{d-2r}, [F] \rangle = (-1)^d 2^r.$$

Thus, we have shown that

$$\delta_P(\alpha)(x_1, \cdots, x_d) = (-1)^d 2^{2r} \frac{(2r)!}{r!} q^r(x_1, \cdots, x_{2r}) \prod_{i=2r+1}^{d} \langle \alpha, x_i \rangle.$$

Invoking part 2 of Lemma 6.1.2 gives the result in this case.

The case where $\alpha^2 = -1$ follows by a similar argument. The sign is evaluated to be $-(-1)^d$. Since d must be even in this case we can rewrite the sign as in the statement.

Remark 6.1.3. The leading coefficient in the case $d = 2$ and $\alpha^2 = -1$ was computed in Theorem 3.11 of [5] by a different argument. The result there agrees with what we establish here, though because of a difference in conventions the coefficient there is given as -4, whereas here plugging in $d = 2$ and $r = 1$ gives a coefficient of -8.

Now we have established the congruences claimed in equations (6) and (7) modulo α^{d-2r+1}. It remains to establish them modulo α^{d-2r+2}. The two cases are similar, so we restrict ourselves to the case where $\alpha^2 < -1$. We wish to show that

$$\delta_P(\alpha)(x_1, \cdots, x_d) - (-1)^d 2^{2r} \frac{(2r)!}{r!} q^r(x_1, \cdots, x_{2r}) \prod_{i=2r+1}^{d} \langle \alpha, x_i \rangle$$

vanishes if $x_1, \cdots, x_{2r-1}$ are orthogonal to α. Consider $\Sigma^r(x_1) \cup \cdots \cup \Sigma^r(x_{2r-1})$. This is a cohomology class dual to a two-dimensional homology class in $\Sigma^r(M)$ represented by a surface S embedded in the top stratum of $\Sigma^r(M)$. Up to a multiplicative factor, to evaluate the above difference on the classes we must evaluate the remaining $d - 2r + 1$ classes on the top cycle of the preimage B of S in $L/U(1)$. The map $B \to S$ is a locally trivial fiber bundle with fiber the link of the central point in $\mathbf{C}^{d-4r+1} \times \prod_{i=1}^r c(SO(3))/U(1)$. Because of what we have already established, the result will follow if we can show that c_1^{d-2r+1} vanishes on B. The fiber bundle over S is associated with the symmetric product of the product of the principal $SO(3)$-bundle $P \to M$ and the frame bundle of the tangent bundle of M. Thus, at the expense of replacing the x_i by positive even multiples, we can assume that $B \to S$ is a trivial fiber bundle. It is then clear that the restriction of the circle bundle to B is isomorphic to the product of a circle bundle with S

$$\mathbf{C}^{d-4r+1} \times \prod_{i=1}^{r} c(SO(3)) \times S \to \left(\mathbf{C}^{d-4r+1} \times \prod_{i=1}^{r} c(SO(3))\right) / U(1) \times S .$$

As a consequence, c_1^{d-2r+1} vanishes on B.

6.2. Conjectural properties of the difference term. We end with two general conjectures about the difference terms $\delta_P(\alpha)$.

Conjecture 6.2.1. The difference term $\delta_P(\alpha)$ is a homotopy invariant of the triple (P, M, α).

We make the following apparently stronger conjecture:

Conjecture 6.2.2. The difference term $\delta_P(\alpha)$ is a polynomial in α and the quadratic form q_M of the manifold with coefficients depending only on α^2, $p_1(P)$, and the homotopy type of M.

Both these conjectures are true in the case where $\alpha^2 = p_1(P)$ (see Theorem 3.6 of [5]) and in the case where $\alpha^2 = -4$, $p_1(P) = -8$, and α is divisible by two (see [7]). Notice that, unlike the leading coefficient which depends only on α^2, the general coefficients will depend on $b_2(M)$.

References

[1] S. K. Donaldson, *Connections, cohomology, and the intersection forms of 4-manifolds*, J. Differential Geometry **24** (1986) 275–341.

[2] ——, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. **26** (1987) 141–168.

[3] R. Friedman & J. W. Morgan, *On the diffeomorphism types of certain algebraic surfaces.* I, J. Differential Geometry **27** (1988) 297–369.

[4] ——, *Smooth 4-manifolds and complex surfaces*, Springer, Berlin, to appear.

[5] D. Kotschick, *$SO(3)$-invariants for 4-manifolds with $b_2^+ = 1$*, Proc. London Math. Soc. (3) **63** (1991) 426–428.

[6] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geometry **17** (1982) 139–170.

[7] H-J. Yang, *Transition functions for Donaldson polynomials of manifolds with $b_2^+ = 1$*, Columbia Univ. thesis, 1992.

Universität Basel

Columbia University