J. DIFFERENTIAL GEOMETRY
39 (1994) 407–431

# CONTRACTION OF CONVEX HYPERSURFACES IN RIEMANNIAN SPACES

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**Abstract**

This paper concerns the deformation of hypersurfaces in Riemannian spaces using fully nonlinear parabolic equations defined in terms of the Weingarten curvature. It is shown that any initial hypersurface satisfying a natural convexity condition produces a solution which converges to a single point in finite time, and becomes spherical as the limit is approached. The result has topological implications including a new proof of the 1/4-pinching sphere theorem of Klingenberg, Berger, and Rauch, and a new "dented sphere theorem" which allows some negative curvature.

## 1. Introduction

An earlier paper by the author [1] considered a general class of fully nonlinear curvature flows of hypersurfaces in Euclidean space. In this paper we adapt these techniques to the more difficult problem of deforming hypersurfaces in Riemannian spaces. We prove that any compact hypersurface satisfying a sharp convexity condition is necessarily the boundary of an immersed disc (Theorem 1-5).

Let $M^n$ be a smooth, connected compact manifold of dimension $n \geq 2$ without boundary, and let $(N^{n+1}, g^N)$ be a complete smooth Riemannian manifold satisfying the following conditions:

$$-K_1 \leq \sigma^N \leq K_2, \qquad |\nabla^N R^N|_{g^N} \leq L \tag{1-1}$$

for some nonnegative constants $K_1$, $K_2$ and $L$. Here $\sigma^N$ is any sectional curvature of $N^{n+1}$, $\nabla^N$ is the Levi-Civita connection corresponding to $g^N$, and $R^N$ is the Riemann tensor on $N^{n+1}$.

Received February 5, 1993. This work was carried out while the author was supported by an Australian Postgraduate Research Award and an ANUTECH scholarship.

Suppose $\varphi_0: M^n \to N^{n+1}$ is a smooth immersion of $M^n$. We seek a solution $\varphi: M^n \times [0, T) \to N^{n+1}$ to an equation of the following form:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(x, t) &= -f(\lambda(\mathscr{W}(x, t)))\nu(x, t), \\ \varphi(x, 0) &= \varphi_0(x), \end{aligned} \tag{1-2}$$

where $\nu(x, t)$ is a unit normal to $\varphi_t(M)$ at $\varphi_t(x)$ in $TN^{n+1}$, $\mathscr{W}(x, t)$ is the Weingarten map on $TM^n$ induced by $\varphi_t$, $\lambda$ is the map from $T^*M^n \otimes TM^n$ to $R^n$ which gives the eigenvalues of a map, and $f$ is a smooth symmetric function. Several further conditions are required of the function $f$; these will be discussed in §3.

Huisken [8] has considered the mean curvature flow in this setting; in this case $f(\lambda) = \sum_{i=1}^n \lambda_i$. The main theorem of [8] may be stated as follows:

**Theorem 1-3.** *Suppose $M^n$, $N^{n+1}$, and $\varphi_0$ are as above, and assume in addition that the injectivity radii $i_y(N)$ of $N^{n+1}$ have a positive lower bound $i(N)$, and that the principal curvatures of $\varphi_0$ satisfy the inequality:*

$$H\lambda - nK_1 > n^2 L/H, \tag{1-4}$$

*where $H = f(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \lambda_i$. Then there exists a unique smooth solution to (1-2) on a maximal time interval $[0, T)$. The immersions $\varphi_t$ converge uniformly to a constant $p \in N^{n+1}$ as $t$ approaches $T$. The rescaled immersions $\tilde{\varphi}_\tau$ obtained by rescaling a neighbourhood of $p$ by a factor $(2n(T-t))^{-1/2}$ converge to the unit sphere $S_1^n(0)$ in Euclidean space, exponentially in $C^\infty$ with respect to the natural time parameter $\tau = -\frac{1}{2}\ln(1 - t/T)$.*

The details of the rescaling process will be explained in §6. This theorem gives optimal results in the case of a locally symmetric background space; the particular case of hypersurfaces of the sphere was developed further in [9]. In more general spaces, the appearance of the derivatives of $R^N$ in Theorem 1-3 is undesirable.

This paper considers a class of fully nonlinear flow equations which does not include the mean curvature flow. The structure of the equations is similar in many respects to the mean curvature flow, and to the class of equations considered in [1]. A typical example is the flow by shifted harmonic mean curvature, for which $f(\lambda) = \left(\frac{1}{n}\sum_{i=1}^n (\lambda_i - \sqrt{K_1})^{-1}\right)^{-1}$. The main result achieved here is the following:

**Theorem 1-5.** *Let $M^n$ and $N^{n+1}$ be as above. Assume that $f$ satisfies the conditions Theorem* 3-1, *and every principal curvature $\lambda$ of $\varphi_0$ satisfies* *the following condition*:

$$\lambda > \sqrt{K_1}. \tag{1-6}$$

*Then there exists a unique smooth solution to* (1–2) *on a maximal time interval* $[0, T)$, *and the immersions* $\varphi_t$ *converge uniformly to a constant* $p$ *in* $N^{n+1}$ *as* $t$ *approaches* $T$. *Expanding a neighbourhood of* $p$ *by a factor* $(2(T-t))^{-1/2}$ *gives rescaled immersions* $\tilde{\varphi}_\tau$ *which converge in* $C^\infty$ *to the unit sphere about the origin in Euclidean space, exponentially with respect to the natural rescaled time parameter* $\tau = -\frac{1}{2}\ln(1 - t/T)$.

The hypotheses of this theorem differ from those in (1–3) in two important respects: No lower bound on the injectivity radius of $N$ is required, and (1–4) is replaced by (1–6). For locally symmetric background spaces ($L = 0$), the new condition is slightly more restrictive than (1–4), but still sharp in the sense that there are counterexamples which satisfy (1–6) with equality. Furthermore, the removal of the dependence on $L$ is a significant improvement in the general case, allowing some useful geometric applications which will be discussed in §7. Note that the condition (1–6) is just enough to ensure that the hypersurface has nonnegative sectional curvatures.

**Corollary 1–7.** *Any compact hypersurface in* $N$ *with principal curvatures greater than* $\sqrt{K_1}$ *is diffeomorphic to a sphere, and bounds an immersed disc.*

The organisation of the paper is as follows: §2 introduces the notation for the paper and gives some useful preliminary results. §3 contains details of the evolution equations—the form of the function $f$, the equivalence of the system (1–2) locally to a scalar equation, short-time existence and uniqueness of solutions, and the induced evolution equations for some geometric quantities. §4 deals with the preservation of convexity and the pinching of principal curvatures; this requires only minor modifications from the proof for the Euclidean case [1]. The application of these estimates, however, is more difficult than in the Euclidean case—the quantities dealt with there can no longer be defined, and one must use more local estimates. These are developed in §5: The local graphical parametrisation of the flow developed in §3 is used to prove local Hölder estimates on the curvature of the immersions, using results from Krylov [10]. This allows us to show in §6 the convergence of appropriately rescaled hypersurfaces on a subsequence of times to a strictly convex pinched hypersurface in Euclidean space. A recent result of Hamilton [6] implies that this limit hypersurface is compact, and the proof of convergence to a point follows directly. The convergence of the rescaled immersions to a sphere follows using techniques similar to those in the analogous section of [1]. §7 concludes with an extension to slightly different flow equations, an application of the main theorem to give a new proof of the $1/4$-pinching sphere theorem, and a generalisation of this proof to give a new "dented sphere" theorem.

The author wishes to thank Gerhard Huisken for his advice and encouragement while this work was undertaken. This work forms part of the author's doctoral thesis for the Australian National University.

## 2. Notation and preliminary results

As far as possible the notation of this paper is consistent with that in [1].

The background space $N^{n+1}$ is supplied with a metric $g^N$, and corresponding connection $\nabla^N$ and Riemann tensor $R^N$. Each immersion $\varphi_t$ of $M^n$ induces a metric $g$, a connection $\nabla$, and a Riemann curvature tensor $R$ on $TM^n$, the tangent bundle of $M$ (the dependence of these on time will not be made explicit):

$$\begin{aligned}
g(u, v) &= g^N(T\varphi(u), T\varphi(v)) \\
\nabla_u v &= T_x\varphi^{-1}\left(\pi_x\left(\nabla^N_{T\varphi(u)}T\varphi(v)\right)\right), \\
R(u, v, w) &= \left(\nabla_v\nabla_u - \nabla_u\nabla_v - \nabla_{[u,v]}\right)w, \\
R(u, v, w, z) &= g^N(R(u, v, w), z)
\end{aligned} \tag{2-1}$$

for all $u$, $v$, $w$ and $z$ in $T_xM^n$. Here $T_x\varphi$ is the derivative of $\varphi$, and $\pi_x$ is the projection of $T_{\varphi(x)}N^{n+1}$ onto the image of $T_x\varphi$. The Riemann tensor is well defined as a symmetric bilinear form on 2-planes: For 2-planes $X = u\wedge v$ and $Y = w\wedge z$, define $R_\Lambda(X, Y) = R(u, v, w, z)$. The sectional curvature $\sigma^N(X)$ of a simple nonzero 2-plane $X = u\wedge v$ is given by $R_\Lambda(X, X)/|X|^2$. It is also convenient to use the Riemann tensor to define a map $\mathscr{R}: T^*M\otimes TM \to T^*M\otimes TM$ generated by the equation $\mathscr{R}(g^*(u\otimes v))(w) = R(u, w, v)$. Note that $\mathscr{R}$ sends symmetric maps to symmetric maps.

For any point $y$ in $N$, the exponential map $\exp_y: T_yN \to N$ can be defined: For a vector $u$ in $T_yN$, $\exp_y(u)$ is the endpoint of the geodesic from $y$ which has tangent in the direction of $u$ at $y$, and length equal to the length of $u$. This is always a diffeomorphism on a small neighbourhood of the origin in $T_yN$. The injectivity radius $i_y(N)$ is the least upper bound of the set of $r$ for which the exponential map is a diffeomorphism on the ball of radius $r$ about the origin in $T_yN$.

There are several different metrics which will be used in the course of this paper. The norm on tensor bundles associated with a metric $g$ will be denoted by $|.|_g$.

A convenient notation is the following: for a tensor $\mathcal{T}$ in $T^*N$, we write $\mathcal{T}(u)$ in place of $\mathcal{T}(T\varphi(u))$, for any vector field $u$ in $TM$. This generalises in an obvious way to higher tensors.

As in [1], the normal component of the connection on $N^{n+1}$ gives the second fundamental form $I\!I \in T^*M \otimes T^*M$, which is symmetric with respect to the metric $g$:

$$I\!I(u, v) = -g^N\left(\nabla^N_{T\varphi(u)} T\varphi(v), \nu\right) \tag{2–2}$$

for all $u$ and $v$ in $T_xM^n$. The Codazzi and Gauss equations are slightly different from the Euclidean case:

$$\nabla I\!I(u, v, w) = \nabla I\!I(v, u, w) + R^N(\nu, u, v, w), \tag{2–3}$$

$$\begin{aligned} R(u, v, w, z) = I\!I(u, w)I\!I(v, z) - I\!I(v, w)I\!I(u, z) \\ + R^N(u, v, w, z) \end{aligned} \tag{2–4}$$

for all $u$, $v$, $w$ and $z$ in $T_xM^n$.

The Weingarten map $\mathcal{W}: TM^n \to TM^n$ gives the change of the normal with respect to the ambient connection:

$$\mathcal{W}(u) = T\varphi^{-1}\left(\nabla^N_{T\varphi(u)}\nu\right) \tag{2–5}$$

for all $u$ in $T_xM^n$. As in the Euclidean case the Weingarten relation relates the second fundamental form to the Weingarten map:

$$I\!I(u, v) = g(\mathcal{W}(u), v). \tag{2–6}$$

This paper gives particular consideration to convex hypersurfaces. By this we mean local convexity, or positivity of the Weingarten map. We will also refer to a hypersurface as $\alpha$-convex if the Weingarten map has all eigenvalues greater than $\alpha$.

A useful identity involving the second derivatives of the second fundamental form is Simons' identity. This combines the Codazzi equation (2–3), the formula for interchange of derivatives in terms of curvature which derives from (2–1), and the Gauss equation (2–4) for the Riemann tensor:

$$
\begin{aligned}
\mathrm{Hess}_{\nabla} I\!I(u, v, w, z) = {}& \mathrm{Hess}_{\nabla} I\!I(w, z, u, v) + I\!I(u, v) I\!I(\mathscr{W}(w), z) \\
& - I\!I(w, z) I\!I(\mathscr{W}(u), v) \\
& + I\!I(u, z) I\!I(\mathscr{W}(w), v) - I\!I(w, v) I\!I(\mathscr{W}(u), z) \\
& + R^N(u, w, v, \mathscr{W}(z)) - R^N(w, u, z, \mathscr{W}(v)) \\
& + R^N(u, z, v, \mathscr{W}(w)) - R^N(w, v, z, \mathscr{W}(u)) \\
& + I\!I(u, v) R^N(w, \nu, z, \nu) \\
& - I\!I(w, z) R^N(u, \nu, v, \nu) \\
& + \nabla^N R^N(u, v, w, z, \nu) \\
& - \nabla^N R^N(w, z, u, v, \nu)
\end{aligned}
\tag{2–7}
$$

for all vectors $u$, $v$, $w$, and $z$ in $TM$.

In §3 we will make use of special local coordinates on $N^{n+1}$ which are particularly convenient for the local graphical parametrisation of the evolution equations (see Lemma 3–2).

Suppose $\psi_0 \colon \Sigma^n \to N^{n+1}$ is a smooth immersion of a compact manifold $\Sigma$ (possibly with a smooth boundary). We wish to extend $\psi$ to $\Sigma^n \times (-\epsilon, \epsilon)$ by the following equations:

$$
\frac{\partial}{\partial s} \psi(\xi, s) = \hat{\nu}(\xi, s), \qquad \psi(\xi, 0) = \psi_0(\xi) \tag{2–8}
$$

for every $\xi$ in $\Sigma^n$ and every $s$ in $(-\epsilon, \epsilon)$, where $\hat{\nu}(\xi, s)$ is a unit normal to $\psi(\Sigma^n, s)$ at $\psi(\xi, s)$, such that the maps $\psi^{(s)} = \psi(., s)$ are nondegenerate; the corresponding induced metric, connection and second fundamental form on $\Sigma$ are denoted by $g^{(s)}$, $\nabla^{(s)}$, and $I\!I^{(s)}$. The map $\psi$ is called a *graphical coordinate system over* $\psi_0$.

**Lemma 2–9.** *For $\Sigma$ and $\psi_0$ as above, there exists a map $\psi : \Sigma^n \times (-\epsilon, \epsilon)$ satisfying* (2–8) *for some sufficiently small positive $\epsilon$, and also a constant $C$ depending on $\psi_0$ and $N$ such that:*

$$
\begin{gathered}
C^{-1} g^{(0)}(u, u) \le g^{(s)}(u, u) \le C g^{(0)}(u, u), \\
|I\!I^{(s)}(u, u)|_{g^{(0)}} \le C, \qquad |\nabla^{(s)}_u v - \nabla^{(0)}_u v|_{g^{(0)}} \le C
\end{gathered}
\tag{2–10}
$$

*for all $u$ in $T\Sigma^n$.*

*Proof.* This follows from the induced variation equations for geometric quantities, which are given by Theorem 3–15, substituting 1 for $f$. q.e.d.

There is a special case of such graphical coordinates which is very important for proving local estimates: Let $y_0$ be a point in $N$, $P$ an $n$-dimensional subspace of $T_{y_0}N$, and $e_0$ a unit normal to $P$ in $T_{y_0}N$. Define a map $\psi_0 \colon P \to N$ according to the equation:

$$\psi_0(\xi) = \exp_{y_0}(\xi) \tag{2-11}$$

for every $\xi$ in $P$. On a region $\Sigma$ of $P$ where $\psi_0$ is nondegenerate, it can be used as the initial immersion in equation (2–8), where we use the unit normal given by

$$\hat{\nu}(\xi, 0) = \left(T_\xi \exp_{y_0}\right)(e_0). \tag{2-12}$$

The map $\psi$ produced in this way is called the *graphical coordinate system over* $P$.

The metric on $P \subset T_{y_0}N$ will be denoted by $\langle \cdot, \cdot \rangle$, and the corresponding norm by $|\cdot|$. The standard (flat) connection on $P$ is denoted by $d$.

**Lemma 2–13.** *Suppose $N$ satisfies* (1–1) *with $K_1 = K_2 = L = 1$. Then the graphical coordinate system $\psi$ over any $n$-dimensional hyperplane $P$ is nondegenerate on the domain $B_{\rho_0} \times (-\rho_0, \rho_0) \subset P \oplus Re_0$ for some fixed $\rho_0 > 0$ depending only on $n$, where $\Sigma = B_{\rho_0}$ is the ball of radius $\rho_0$ in $P$. Furthermore, the following estimates hold for some fixed constant $C$:*

$$\begin{gathered} C^{-1}|u|^2 \le g^{(s)}(u, u) \le C|u|^2, \qquad |\mathit{II}^{(s)}(u, u)| \le C \\ |\nabla_u^{(s)} v - d_u v| \le C, \qquad |\nabla^{(s)} \mathit{II}^{(s)}| \le C \end{gathered} \tag{2-14}$$

*for all $u$ and $v$ in $P$.*

*Proof.* The assumptions (1–1) give uniform control over the curvature of $N$ and its derivative. This allows control over the Hamilton-Jacobi equations (2–11) and (2–8) which define $\psi$, and the induced variation equations for the metric and curvature. q.e.d.

A hypersurface can be described locally using the graphical coordinates given by Lemma 2–9. For a smooth function $s \colon \Sigma^n \to (-\epsilon, \epsilon)$, define an immersion $\varphi \colon \Sigma^n \to N$ by

$$\varphi(\xi) = \psi(\xi, s(\xi)) \tag{2-15}$$

for all $\xi$ in $\Sigma^n$. For such a graph we can calculate the metric, curvature, and connection of the immersion:

$$\nu = \frac{\hat{\nu}^{(s)} - \nabla^{(s)}s}{\sqrt{1 + |\nabla s|^2_{g^{(s)}}}}, \tag{2–16}$$

$$g = g^{(s)} + \nabla s \otimes \nabla s, \tag{2–17}$$

$$\nabla_u v - \nabla^{(s)}_u v = \frac{\mathrm{Hess}_{\nabla^{(s)}} s(u, v)}{1 + |\nabla s|^2_{g^{(s)}}} \nabla^{(s)} s, \tag{2–18}$$

$$\begin{aligned} I\!I(u, v) = \frac{1}{\sqrt{1 + |\nabla s|^2_{g(s)}}} [ & I\!I^{(s)}(u, v) + I\!I^{(s)}(v, \nabla^{(s)} s)\nabla_u s \\ & + I\!I^{(s)}(u, \nabla^{(s)} s)\nabla_v s - \mathrm{Hess}_{\nabla^{(s)}} s(u, v)] \end{aligned} \tag{2–19}$$

for all vectors $u$ and $v$ in $T_\xi P \cong P$.

To conclude this section we will review some of the properties of symmetric functions which we will use in the paper. Let $f$ be a smooth symmetric function defined on the positive cone $\Gamma_+ = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i > 0\}$ in $\mathbf{R}^n$. Corresponding to this is a function $F = f \circ \lambda$ of positive definite linear maps which applies $f$ to the eigenvalues of a map. This is a smooth function on the space of positive definite maps. If $f$ is monotonic with respect to each of the variables $\lambda_i$, then the derivative $\dot{F}$ of $F$ is positive definite: In local normal coordinates which diagonalise the Weingarten map, $\dot{F} = \mathrm{diag}\{\partial f/\partial\lambda_1, \cdots, \partial f/\partial\lambda_n\}$. If $f$ is convex (concave) with respect to $\lambda$, then $F$ is also convex (concave). In the case where $f$ is concave, the following inequalities hold:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial \lambda_i} \geq 1, \tag{2–20}$$

$$\left(\frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}\right)\left(\lambda_i - \lambda_j\right) \leq 0 \tag{2–21}$$

for every $i$ and $j$, at every point of $\Gamma_+$. For proofs of these results see [1], [2], or [12].

## 3. The evolution equations

The class of allowed speeds $f$ is in many ways similar to that used in [1]:

**Conditions 3–1.**

(1) *$f$ is a symmetric function which is smooth on $\Gamma_\alpha = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i > \alpha\}$, and continuous on $\bar{\Gamma}_\alpha$, where $\alpha = \sqrt{K_1}$.*

(2) $f$ *is strictly increasing in each argument:* $\partial f/\partial\lambda_i > 0$ *for* $i = 1, \cdots, n$ *at every point of* $\Gamma_\alpha$.

(3) $f$ *is homogeneous of degree one in* $(\lambda_1 - \alpha, \cdots, \lambda_n - \alpha)$.

(4) $f$ *is strictly positive on* $\Gamma_\alpha$, *and* $f(1+\alpha, \cdots, 1+\alpha) = 1$.

(5) $f$ *is concave on* $\Gamma_\alpha$.

(6) $f = 0$ *on* $\partial\Gamma_\alpha$.

(7) $\sup_{\lambda\in\Gamma_\alpha} |Df| < \infty$.

For convenience the composition $f \circ \lambda$ will be denoted by $F$, and its derivatives by $\dot{F}$, $\ddot{F}$, etc., as in [1]. Note that the shifted harmonic mean curvature, given by $f = \left(\frac{1}{n}\sum_{i=1}^n(\lambda_i - \alpha)^{-1}\right)^{-1}$ satisfies all of the conditions 3–1.

Condition (2) ensures that equation (1–2) is a degenerate parabolic system of partial differential equations. The second part of condition (4) is only a normalisation condition, and can always be satisfied by rescaling time. Note that condition (6) rules out the mean curvature flow, and condition (7) rules out the other flows previously considered in [3] and [4]. In the case where $N$ has nonnegative sectional curvatures, the allowed flows are a subset of the allowed flows in the Euclidean case [1]. More generally, we require the more complicated homogeneity condition (3) in order to overcome negative curvature of the background space.

The proof of short-time existence and uniqueness of solutions is essentially the same as in [1], but the graphical parametrisation is somewhat more complicated because of the background geometry. Some results concerning the graphical parametrisation of the flow are necessary:

**Lemma 3-2.** *Let* $\psi: \Sigma^n \times (-\epsilon, \epsilon) \to N^{n+1}$ *be a nondegenerate map given by Lemma 2–9, and* $\varphi_0: M^n \to N$ *a smooth* $\alpha$*-convex immersion. Suppose there exist a nondegenerate map* $\chi_0: \Sigma^n \to M^n$, *and a smooth function* $s_0: \Sigma^n \to (-\epsilon, \epsilon)$ *such that*

$$\varphi_0(\chi_0(\xi)) = \psi(\xi, s_0(\xi)), \tag{3–3}$$

$$g^N\left(\nu(\chi_0(\xi)), \hat{\nu}^{(s_0)}(\xi)\right) > 0 \tag{3–4}$$

*for all* $\xi$ *in* $\Sigma^n$. *If* $\varphi: M^n \times [0, T) \to N$ *is a family of* $\alpha$*-convex immersions satisfying* (1–2), *then for sufficiently small* $t_0 > 0$ *there exist a smooth family of nondegenerate maps* $\chi: \Sigma^n \times [0, t_0) \to M^n$ *and a smooth family of functions* $s: \Sigma^n \times [0, t_0) \to (-\epsilon, \epsilon)$ *such that*

$$\varphi_t(\chi_t(\xi)) = \psi(\xi, s_t(\xi)) \tag{3–5}$$

*for all* $(\xi, t)$ *in* $\Sigma^n \times [0, t_0)$. *Furthermore,* $s$ *satisfies the inequality*

$$
\text{(3–6)}\quad \begin{aligned} \frac{1}{\sqrt{1+|\nabla s|^2_{g(s)}}}[\mathit{II}^{(s)}(u, u) + 2\mathit{II}^{(s)}(u, \nabla^{(s)}s)\nabla_u s - \mathrm{Hess}_{\nabla^{(s)}} s(u, u)] \\ \geq \alpha[|u|^2_{g(s)} + (\nabla_u s)^2] \end{aligned}
$$

*for all* $(\xi, t)$ *in* $\Sigma^n \times [0, t_0)$ *and all* $u$ *in* $T_\xi\Sigma^n$. *The following strictly parabolic equation holds on* $\Sigma \times [0, t_0)$:

$$
\frac{\partial}{\partial t}s(\xi, t) = f \circ \lambda(((g^{(s)})^* g)^{-1} \circ \mathscr{A}), \qquad s(\xi, 0) = s_0(\xi),
$$

*where* $g$ *is given in terms of* $s$ *by* (2–17), *and* $\mathscr{A}$ *is the map given by*

$$
\text{(3–8)}\quad \begin{aligned} \mathscr{A} = (g^{(s)})^*(\mathrm{Hess}_{\nabla^{(s)}} s - \mathit{II}^{(s)} - \nabla s \otimes (\mathscr{W}^{(s)})^\dagger(\nabla s) \\ - (\mathscr{W}^{(s)})^\dagger(\nabla s) \otimes \nabla s). \end{aligned}
$$

*Here* $(\mathscr{W}^{(s)})^\dagger$ *is the adjoint of* $\mathscr{W}^{(s)}$.

*Conversely, if* $s: \Sigma^n \times [0, t_0) \to (-\epsilon, \epsilon)$ *is smooth and satisfies* (3–6) *and* (3–7), *then for every point* $(\xi_1, t_1)$ *in* $\Sigma^n \times [0, t_0)$ *there exist a manifold* $\bar{M}$ *and a smooth family of diffeomorphisms* $\bar{\chi}$ *of* $\bar{M} \times [t_1, t_2)$ *onto regions of* $\Sigma^n$ *containing* $\xi_1$, *for some* $t_2 \in (t_1, t_0)$, *such that the map* $\bar{\varphi}: \bar{M} \times [t_1, t_2) \to N$ *given by*

$$
\text{(3–9)}\qquad \bar{\varphi}_t(\bar{x}) = \psi(\bar{\chi}_t(\bar{x}), s_t(\bar{\chi}_t(\bar{x})))
$$

*is a smooth family of* $\alpha$*-convex immersions satisfying* (1–2). *If* $s$ *is produced from* $\varphi$ *as above, then there exists a nondegenerate map* $\phi: \bar{M} \to M$ *such that*

$$
\text{(3–10)}\qquad \varphi_t(\phi(\bar{x})) = \bar{\varphi}_t(\bar{x})
$$

*for all* $(\bar{x}, t)$ *in* $\bar{M} \times [0, t_0)$.

*Proof.* Let $\varphi$ be a solution to (1–2) as above, and suppose $s_0$ and $\chi_0$ give $\varphi_0$ by equation (3–3). Consider the ordinary differential equations

$$
\text{(3–11)}\quad \begin{aligned} \frac{d}{dt}\chi_t(\xi) &= -\frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)), \hat{\nu}^{(s_t)}(\xi))}(T_{\chi_t(\xi)}\varphi_t)^{-1}(\pi_{\chi_t(\xi)}\hat{\nu}^{(s_t)}(\xi)), \\ \frac{d}{dt}s_t(\xi) &= -\frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)), \hat{\nu}^{(s_t)}(\xi))}. \end{aligned}
$$

There exist solutions $\chi$ and $s$ to (3–11) on a short time interval $[0, t_0)$, with $|s_t(\xi)| < \epsilon$. The consistency of the equations is guaranteed by the following calculation:

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_t(\chi_t(\xi))) &= -F(\chi_t(\xi))\nu(\chi_t(\xi)) - \frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)),\, \hat{\nu}^{(s_t)}(\xi))}\pi_{\chi_t(\xi)}\hat{\nu}^{(s_t)}(\xi)\\ &= -\frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)),\, \hat{\nu}^{(s_t)}(\xi))}\hat{\nu}^{(s_t)}(\xi)\\ &= \frac{\partial}{\partial t}\psi(\xi,\, s_t(\xi))\end{aligned}$$

for all $\xi$ and $t$. Hence equation (3–5) holds on the interval $[0, t_0)$. Equations (3–6) and (3–7) follow immediately from the expressions (2–19) and (2–17): The first since $\varphi$ is $\alpha$-convex, and the second from (1–2).

Now consider the converse situation: Suppose $s\colon \Sigma^n \times [0, t_0) \to (-\epsilon, \epsilon)$ is a solution to (3–7), and $(\xi_1, t_1)$ is in $\Sigma^n \times [0, t_0)$. Let $\bar{M}^n$ be a small open neighbourhood of $\xi_1$ in $\Sigma^n$, and define $\bar{\chi}_{t_1}\colon \bar{M}^n \to \Sigma^n \times [0, t_0)$ by $\bar{\chi}_{t_1} = \mathrm{Id} \times \{t_1\}$. Extend $\bar{\chi}$ to a region $\bar{M}^n \times [t_1, t_2)$ (taking $t_2 - t_1$ and $\bar{M}^n$ sufficiently small) by the following differential equation:

$$\frac{\partial}{\partial t}\bar{\chi}_t(\bar{x}) = -\left(\frac{1}{1+|\nabla s|^2_{g^{(s)}}}\frac{\partial}{\partial t}s\right)\nabla^{(s)}s, \tag{3–12}$$

where the right-hand side is evaluated at the point $\bar{\chi}_t(\bar{x})$, for all $(\bar{x}, t)$ in $\bar{M}^n \times [t_1, t_2)$. The definition (3–9) of $\bar{\varphi}$ then gives

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\varphi}(\bar{x}) &= \left(\frac{\partial}{\partial t}s\right)\left(1 - \frac{|\nabla s|^2}{1+|\nabla s|^2}\right)\hat{\nu}^{(s_t)}\left(\bar{\chi}_t(\bar{x})\right) - \left(\frac{\partial}{\partial t}s\right)\frac{T\psi(\nabla^{(s)}s)}{1+|\nabla s|^2}\\ &= \frac{1}{\sqrt{1+|\nabla s|^2}}\left(\frac{\partial}{\partial t}s\right)\nu(\bar{x}) = -F(\bar{\varphi}_t(\bar{x}))\nu(\bar{x})\end{aligned}$$

by equation (2–16), and $\bar{\varphi}$ satisfies (1–2). Finally, if $s$ is produced from a solution $\varphi$, define $\psi\colon \bar{M}^n \to M^n$ by

$$\psi(\bar{x}) = \chi_t \circ \bar{\chi}_t(\bar{x}) \tag{3–13}$$

which is well defined since $\chi_t \circ \bar{\chi}_t$ satisfies the equation $\chi_t\partial/\partial t\chi_t \circ \bar{\chi}_t = 0$. q.e.d.

Now consider the case where $\Sigma^n = M^n$ and $\psi_0 = \varphi_0$. The following result is easily obtained from Lemma 3–2.

**Theorem 3–14.** *There exists a unique smooth solution to equation* (1–2) *on some time interval* $[0, T)$.

*Proof.* There exists a solution for a short time to equation (3–7) with zero initial conditions, since it is strictly parabolic. This gives a solution to (1–2) by the lemma above, satisfying the correct initial conditions.

Suppose there are two solutions $\varphi^1$ and $\varphi^2$ to (1–2) with the same initial condition $\varphi_0$. This gives two solutions to (3–7) with the same initial conditions, which are therefore identical. It follows that $\varphi^1$ and $\varphi^2$ are identical up to a time-independent diffeomorphism, and therefore identical since they have the same initial condition. q.e.d.

The evolution equations satisfied by the metric, normal, and curvature of the immersions $\varphi_t$ of a solution to (1–2) are similar to the Euclidean case:

**Theorem 3–15.**

$$\frac{\partial}{\partial t}g = -2F\mathit{II}, \tag{3–16}$$

$$\frac{\partial}{\partial t}\nu = T\varphi(\nabla F), \tag{3–17}$$

$$\frac{\partial}{\partial t}\mathit{II} = \mathrm{Hess}_\nabla F - F\mathit{II}^2 + FR^N(.,\nu,.,\nu), \tag{3–18}$$

$$\frac{\partial}{\partial t}\mathscr{W} = g^*\mathrm{Hess}_\nabla F + F\mathscr{W}^2 + F\mathscr{R}^N(g^N)^*(\nu\otimes\nu), \tag{3–19}$$

$$\frac{\partial}{\partial t}F = \mathscr{L}F + F\dot{F}(\mathscr{W}^2) + F\dot{F}(\mathscr{R}^N(g^N)^*(\nu\otimes\nu)), \tag{3–20}$$

*where* $\mathscr{L} = \dot{F}g^*\mathrm{Hess}_\nabla$.

*Proof.* The evolution equations for metric and normal follow as in [1]. The evolution of $\mathit{II}$ can be calculated from the definition (2–2):

$$\begin{aligned}
\frac{\partial}{\partial t}\mathit{II}(u,v) &= \nabla^N_{F\nu}g^N\left(\nabla^N_{T\varphi(u)}T\varphi(v),\nu\right)\\
&= g^N\left(\nabla^N_{F\nu}\nabla^N_{T\varphi(u)}T\varphi(v),\nu\right) + g^N\left(\nabla^N_{T\varphi(u)}T\varphi(v),-T\varphi(\nabla F)\right)\\
&= g^N\left(\nabla^N_{T\varphi(u)}\nabla^N_{F\nu}T\varphi(v),\nu\right) + FR^N(u,\nu,v,\nu)\\
&\quad + g(\nabla_u v,-\nabla F)\\
&= g^N\left(\nabla^N_{T\varphi(u)}\nabla^N_{T\varphi(v)}(F\nu),\nu\right) + FR^N(u,\nu,v,\nu) - d_{\nabla_u v}F\\
&= d_u d_v F - Fg(\mathscr{W}(u),\mathscr{W}(v)) + FR^N(u,\nu,v,\nu) - d_{\nabla_u v}F\\
&= \mathrm{Hess}_\nabla F(u,v) - F\mathit{II}^2(u,v) + FR^N(u,\nu,v,\nu).
\end{aligned}$$

The remaining evolution equations follow exactly as in [1].

**Lemma 3–21.**

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathscr{W}(u) = {} & (\mathscr{L}\mathscr{W})(u) + g^*\ddot{F}(\nabla\mathscr{W}, \nabla\mathscr{W})(u) + \dot{F}(\mathscr{W}^2)\mathscr{W}(u) \\
& + \mathscr{R}^N((g^N)^*(\nu\otimes\nu))(\dot{F})\mathscr{W}(u) + 2\mathscr{R}^N(\dot{F}^\dagger\circ\mathscr{W})(u) \\
& - \mathscr{R}^N(\dot{F}^\dagger)(\mathscr{W}(u)) - \mathscr{W}(\mathscr{R}^N(\dot{F}^\dagger)(u)) \\
& + \mathscr{S}(\dot{F}^\dagger)(u) - \alpha\dot{F}(\mathrm{Id})(\mathscr{W}^2(u) + R^N(\nu, u, \nu)),
\end{aligned}
\tag{3–22}
$$

*where* $\mathscr{S}\colon T^*M\otimes TM \to T^*M\otimes TM$ *is defined by the equation*

$$g((\mathscr{S}(u\otimes v))(w), z) = \nabla^N R(w, z, u, v, \nu) - \nabla^N R(u, v, w, z, \nu).$$

*Proof.* Apply Simons' Identity (2–7) to the equation (3–19). q.e.d.

The following result allows us to deduce evolution equations for the graphical parametrisation of Lemma 3–2 from those given above.

**Lemma 3–23.** *Suppose* $Q$ *is a scalar quantity defined on* $M^n\times[0, T)$, *which evolves under* (1–2) *by the evolution equation*

$$\frac{\partial}{\partial t}Q(x, t) = \mathscr{L}Q(x, t) + Z(x, t)$$

*for some* $Z\colon M^n\times[0, T)\to R$, *and let* $\chi\colon \Sigma\times[0, t_0)\to M$ *be the diffeomorphisms given by Lemma* 3–2. *Define* $\bar{Q}\colon \Sigma^n\times[0, t_0)\to R$ *by*

$$\bar{Q}(\xi, t) = Q(\chi_t(\xi), t).$$

*Then the following evolution equation holds*:

$$
\begin{aligned}
\frac{\partial}{\partial t}\bar{Q}(\xi, t) = {} & \bar{\mathscr{L}}\bar{Q}(\xi, t) + \bar{Z}(\xi, t) \\
& + \frac{\dot{F}g^*}{1+|\nabla s|^2_{g^{(s)}}}[\alpha\mathrm{Id} + \mathit{II}^{(s)} + (\mathscr{W}^{(s)})^\dagger\nabla s\otimes\nabla s \\
& \qquad + \nabla s\otimes(\mathscr{W}^{(s)})^\dagger\nabla s)],
\end{aligned}
\tag{3–24}
$$

*where* $\bar{Z}(\xi, t) = Z(\chi_t(\xi), t)$ *and* $\bar{\mathscr{L}} = \dot{F}g^*\mathrm{Hess}_{\nabla^{(s)}}$.

*Proof.* This follows directly from equations (2–18) and (2–19) which give expressions for the difference in the connections $\nabla$ and $\nabla^{(s)}$, and from equation (3–11) which determines the gradient term arising from the diffeomorphism $\chi$ of Lemma 3–2.

## 4. Preserving convexity and pinching

In this section it is proved that a solution to (1–2) remains strictly $\alpha$-convex, where $\alpha = \sqrt{K_1}$, and also that the shifted principal curvatures

$\lambda_i - \alpha$ remain pinched. The proof is very similar to the corresponding estimate in [1], but slightly more complicated.

**Theorem 4–1.** *Let $\varphi$ be a solution of (1–2) on the domain $M^n \times [0, T)$. Then the maximal time of existence $T$ is finite, and there exist constants $C > 0$ and $\beta > \alpha$ depending on $\varphi_0$, $K_1$, and $L$ such that the following estimates hold:*

$$\lambda_i(x, t) - \alpha > C(\lambda_j(x, t) - \alpha), \qquad \lambda_i(x, t) \geq \beta, \tag{4–2}$$

*for all $i$ and $j$, and all $(x, t)$ in $M^n \times [0, T)$.*

*Proof.* Equation (3–20) will enable us to prove both that $\alpha$-convexity is preserved and that the maximal time $T$ is finite: Since $\lambda_i \geq \alpha$, we obtain at a point where $F$ attains its infinum, using a frame $\{e_i\}$ which diagonalises $\mathscr{W}$:

$$\begin{aligned}\frac{\partial}{\partial t}F &\geq F\sum_i \frac{\partial f}{\partial \lambda_i}(\lambda_i^2 + \sigma^N(\nu \otimes e_i)) \\ &\geq F\sum_i \frac{\partial f}{\partial \lambda_i}(\lambda_i - \alpha)^2 \geq \inf_{t=0}(\lambda - \alpha)F^2.\end{aligned}$$

$F$ has an initial strictly positive lower bound. The maximum principle applied to the above equation shows that this is preserved in time. Since $F$ has bounded gradient and is homogeneous, positive, and zero on the boundary of $\Gamma_\alpha$, it is comparable to the smallest shifted eigenvalue $\lambda_{\min} - \alpha$, and strict $\alpha$-convexity is preserved. The maximum principle also proves that the time of existence $T$ is finite, since the above inequality forces $\inf_M F$ to become infinite in finite time.

As in [1], we consider quantities of the form $Q/F$, where $Q = q \circ \lambda$ and $q$ is an appropriate convex, homogeneous degree 1 function of $(\lambda_1 - \alpha, \cdots, \lambda_n - \alpha)$. Note that $Q/F$ approaches infinity on the boundary of the cone $\Gamma_\alpha$, so it is sufficient to find an upper bound. First consider the evolution equation for $Q$, which is calculated by applying the derivative $\dot{Q}$ to equation (3–22):

$$\begin{aligned}\frac{\partial}{\partial t}Q &= \mathscr{L}Q + (\dot{Q}\ddot{F} - \dot{F}\ddot{Q})(\nabla\mathscr{W}, \nabla\mathscr{W}) \\ &\quad + Q\dot{F}(\mathscr{W}^2 + \mathscr{R}^N(g^N)^*(\nu \otimes \nu)) \\ &\quad + \mathscr{S}(\dot{F}, \dot{Q}) + 2\mathscr{R}^N(\dot{F} \circ \mathscr{W})(\dot{Q}) - 2\mathscr{R}^N(\dot{Q} \circ \mathscr{W})(\dot{F}) \\ &\quad + \alpha\big[\dot{Q}(\mathrm{Id})\dot{F}(\mathscr{W}^2 + \mathscr{R}^N(g^N)^*(\nu \otimes \nu)) \\ &\qquad - \dot{F}(\mathrm{Id})\dot{Q}(\mathscr{W}^2 + \mathscr{R}^N(g^N)^*(\nu \otimes \nu))\big],\end{aligned} \tag{4–3}$$

where $\mathscr{S}$ is given in Lemma 3–21. From this it is easy to calculate the derivative of $Q/F$:

(4–4)
$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q}{F} = \mathscr{L}\frac{Q}{F} + \frac{\dot{Q}\ddot{F} - \dot{F}\ddot{Q}}{F}(\nabla\mathscr{W}, \nabla\mathscr{W}) + \frac{2}{F}\dot{F}g^*(\nabla F \otimes \nabla(\frac{Q}{F}))\\ + \frac{1}{F}\mathscr{S}(\dot{F})(\dot{Q}) + \frac{2}{F}\mathscr{R}^N(\dot{F}\otimes\mathscr{W})(\dot{Q}) - \frac{2}{F}\mathscr{R}^N(\dot{Q}\otimes\mathscr{W})(\dot{F})\\ + \frac{\alpha}{F}\left(\dot{Q}(\mathrm{Id})\dot{F} - \dot{F}(\mathrm{Id})\dot{Q}\right)\left(\mathscr{W}^2 + \mathscr{R}^N(g^N)^*(\nu\otimes\nu)\right).\end{aligned}$$

The various terms appearing here are easily estimated: First, concavity of $f$ implies concavity of $F$, and convexity of $q$ implies convexity of $Q$, so we have

$$\frac{\dot{Q}\ddot{F} - \dot{F}\ddot{Q}}{F}(\nabla\mathscr{W}, \nabla\mathscr{W}) \le 0.$$

The next term contains a gradient of $Q/F$, and so can be ignored when applying the maximum principle. The global supremum bound on $\nabla^N R^N$ gives the following estimate:

$$|\mathscr{S}(\dot{F})(\dot{Q})| \le CL\sup_{\Gamma_\alpha}|\dot{F}\otimes\dot{Q}|,$$

where $C$ is a constant depending only on $n$. Note that $\dot{F}$ and $\dot{Q}$ are bounded if we assume both $f$ and $q$ satisfy (7) of Condition 3–1. The next terms can be estimated using the following simple calculation which is valid in a normal coordinate system at a point where $\mathscr{W}$ is diagonal:

$$\begin{aligned}&2\mathscr{R}^N(\dot{F}\circ\mathscr{W})(\dot{Q}) - 2\mathscr{R}^N(\dot{Q}\circ\mathscr{W})(\dot{F})\\ &\quad = \sum_{i,j}(\frac{\partial f}{\partial\lambda_i}\frac{\partial q}{\partial\lambda_j} - \frac{\partial q}{\partial\lambda_i}\frac{\partial f}{\partial\lambda_j})(\lambda_i - \lambda_j)\sigma^N(e_i\wedge e_j),\end{aligned} \tag{4–5}$$

where $e_1, \cdots, e_n$ are unit eigenvectors of $\mathscr{W}$. A similar calculation applies to the last terms:

$$\begin{aligned}&(\dot{Q}(\mathrm{Id})\dot{F} - \dot{F}(\mathrm{Id})\dot{Q})(\mathscr{W}^2 + g^*R^N(.,\nu,.,\nu))\\ &\quad = \frac{1}{2}\sum_{i,j}(\frac{\partial q}{\partial\lambda_i}\frac{\partial f}{\partial\lambda_j} - \frac{\partial f}{\partial\lambda_i}\frac{\partial q}{\partial\lambda_j})(\lambda_i - \lambda_j)(\lambda_i + \lambda_j)\\ &\quad\quad + \sum_{i,j}(\frac{\partial q}{\partial\lambda_i}\frac{\partial f}{\partial\lambda_j} - \frac{\partial f}{\partial\lambda_i}\frac{\partial q}{\partial\lambda_j})\sigma^N(\nu\wedge e_j).\end{aligned}$$

The second term here can be estimated using $K_2$, $K_1$, and (7) of Condition 3–1. The first combines with (4–5) to give

$$\sum_{i,j}\left(\frac{\partial q}{\partial\lambda_i}\frac{\partial f}{\partial\lambda_j} - \frac{\partial f}{\partial\lambda_i}\frac{\partial q}{\partial\lambda_j}\right)(\lambda_i - \lambda_j)\left(\sigma^N(e_i\wedge e_j) + \alpha\frac{\lambda_i + \lambda_j}{2}\right).$$

The last factor here is positive by the assumption of $\alpha$-convexity and the definition of $\alpha$ and $K_1$. The remaining factors are negative since $f$ is concave—compare (2–21). The following estimate is obtained:

$$\frac{\partial}{\partial t}\frac{Q}{F} \le \mathscr{L}\left(\frac{Q}{F}\right) + \frac{2}{F}\dot{F}g^*\left(\nabla F \otimes \nabla\left(\frac{Q}{F}\right)\right) + \frac{C}{\beta - \alpha}. \tag{4–6}$$

The parabolic maximum principle now gives $\sup(Q/F) \le C(1+t)$ which is bounded since we know the interval of existence is finite.

Note that a suitable function $Q$ can always be found—for example $Q = |\mathscr{W} - \alpha \mathrm{Id}|$ satisfies all the required conditions. q.e.d.

An immediate corollary is that the map $\dot{F}$ remains comparable to the identity map throughout the period of existence of the solution.

## 5. Local estimates

In this section Hölder estimates are found for the curvature of the immersions $\varphi$. This is essentially an application of the general results described in [10], but some care is required to apply these in the absence of a lower bound on the injectivity radii of $N$. This is accomplished here by using the graphical coordinates $\psi$ introduced in Lemmas 2–13 and 3–2, which are nondegenerate but may not be diffeomorphic. Once the apparatus of section Lemma 3–2 is in place, the application of [10] presents no difficulties.

The analysis is simplified by considering the scaling properties of equation (1–2), which follow directly from the homogeneity condition (4) of Condition 3–1:

**Lemma 5–1.** *Suppose $\varphi\colon M^n \times [0, T) \to (N^{n+1}, g^N)$ is a solution to equation* (1–2) *with speed $f(\lambda)$. For any constant $A > 0$, define $\varphi^{(A)}\colon M^n \times [0, A^2T) \to (N^{n+1}, A^2g^N)$ by $\varphi_t^{(A)}(x) = \varphi_{A^2t}(x)$. Then $\varphi^{(A)}$ is a solution to* (1–2) *with speed function $f^{(A)}(\lambda) = f(\lambda + (A - 1/A)\alpha)$, which satisfies the homogeneity condition* (4) *of Condition 3-1 with $\alpha^{(A)} = \alpha/A$.*

The first problem is to consider appropriate graphical coordinates, and to estimate the time of existence and other properties of the solution given in Lemma 3–2. The previous lemma assists us by allowing us to consider only solutions which are rescaled to satisfy a curvature bound:

**Lemma 5–2.** *Let $\varphi\colon M^n \times [-1, 1] \to N^{n+1}$ be a solution of the equation* (1–2) *with $\sup_{[-1,0]\times M^n} |\mathscr{W}(x, 0)| = 1$, and suppose $N$ is such that $\max\{K_1, K_2, L\} \le 1$. Choose $x_0$ in $M^n$, and let $P = T_{x_0}\varphi_0(T_{x_0}M^n) \subset T_{\varphi_0(x_0)}N$. Let $\psi$ be the graphical coordinates over $P$. Then on a domain*

$B_\delta(0) \times [-\tau, \tau] \subset P \times \mathbf{R}$ *there exists a smooth function* $s$ *corresponding to* $\varphi$ *by equation* (2–15), *and we have*

$$\sup_{B_\delta \times [-\tau,\tau]} |s| \le \epsilon, \qquad \sup_{B_\delta \times [-\tau,\tau]} |Ds| \le 1, \qquad \sup_{B_\delta \times [-\tau,\tau]} |\mathscr{W}| \le 2. \tag{5–3}$$

*Here* $\delta$ *is a constant depending only on* $n$ *and* $f$.

*Proof.* At the initial time we can construct the required map $\chi$ and function $s$ giving the graphical parametrisation of $\varphi_0$: Set $s(0) = 0$ and $\chi(0) = x_0$, and extend according to the following differential equations:

$$\begin{aligned} \nabla^{(s)} s(\xi) &= (T_\xi \psi^{(s)})^{-1} \left( \hat{\nu}^{(s)}(\xi) - \frac{\nu(\chi(\xi))}{g^N[\nu(\chi(\xi)), \hat{\nu}^{(s)}(\xi)]} \right), \\ T_\xi \chi(u) &= (T_{\chi(\xi)}\varphi_0)^{-1} \left( T_\xi \psi^{(s)}(u) + \nabla_u s(\xi) \hat{\nu}^{(s)}(\xi) \right), \end{aligned} \tag{5–4}$$

where $\nabla_u s$ in the second equation is calculated by the first equation. These expressions can be used to solve for $s$ and $\chi$ along radial curves from the origin in $P$. The solutions $s$ and $\chi$ along such a curve can be extended within the region of definition of $\psi$ as long as $|s| < \epsilon$ and $|\nabla s|_{g^{(s)}}$ remains bounded, since $g^N[\nu(\chi(\xi)), \hat{\nu}^{(s)}(\xi)]^{-1} = \sqrt{1 + |\nabla s|^2_{g^{(s)}}}$. We can estimate these on a small region as follows: The expressions (2–17) and (2–19) can be combined to give an expression for $|\mathscr{W}|$, using the estimates (2–14). Since $|\mathscr{W}| \le 1$, this gives an estimate of the form

$$|D^2 s| \le C(1 + |Ds|^2)^{3/2} \tag{5–5}$$

for some constant $C$. Since $|\nabla s|(0) = 0$, this gives a bound on $|Ds|$ on a ball of radius $r_0$ which does not depend on $x_0$ or $\varphi$. Note that this also implies a bound on $|\nabla s|_{g^{(s)}}$, since $g^{(s)}$ is uniformly equivalent to the metric on $P$ in the region considered. Without loss of generality, let us assume that we have taken $r_0$ sufficiently small to ensure that $|Ds| \le \frac{1}{2}$. By taking $r_0$ smaller if necessary, this also ensures $|s| \le \epsilon/2$.

The next difficulty is to show that this solution $s$ exists for a fixed time interval on a suitable region of $P$, and to estimate $|Ds|$ throughout the time interval. Note that equations (3–11) in the proof of Lemma 3-2 show that the solution can be extended in time as long as $|s| < \epsilon$ and $|Ds|$ is bounded. To control the curvature on a small interval, we can use equation (4–3) with $Q = |\mathscr{W}|$:

$$\frac{\partial}{\partial t}|\mathscr{W}| \le \mathscr{L}|\mathscr{W}| + |\mathscr{W}|\dot{F}(\mathscr{W}^2) + C \le \mathscr{L}|\mathscr{W}| + C|\mathscr{W}|^3 + C,$$

where $C$ depends only on $f$ and the pinching bound of (4–2). Since $\sup_{t=0}|\mathscr{W}| = 1$, we can find a small time interval on which $|\mathscr{W}| \le 2$. On this time interval we also have a bound $F \leq |\mathscr{W}|/\sqrt{n} \leq 2/\sqrt{n}$. On an interval $[0, \tau]$, the solution stays in a neighbourhood of width $2\tau/\sqrt{n}$ of the initial immersion $\varphi_0$. For $\tau$ sufficiently small, and considering only the smaller region of radius $r_0/2$ in $P$, this neighbourhood is contained in a strip about the initial function $s_0$, given by $s_0 - C\tau \leq s \leq s_0 + C\tau$ for some constant $C$, using the bound on $|Ds|$ at the initial time. Clearly we can choose $\tau$ small enough to ensure that $|s| < \epsilon$ on this region. Now we use the bound (5–5) again, in the form

$$|D|Ds|| \leq C(1 + |Ds|^2)^{3/2}.$$

Integrate along a curve $\gamma$ which begins at some point in $B_{r_0/2}(0) \subset P$, and follows the direction of steepest ascent of $s$. First we have an estimate on $|Ds|$ from below for small distances:

$$|Ds|(r) \geq (A - Cr) \Big/ \sqrt{1 - (A - Cr)^2},$$

where $A = |Ds|(0) \Big/ \sqrt{1 + |Ds|^2}$; this holds for $A - Cr \geq 0$. Integrating again we obtain the estimate

$$s(\gamma(r)) - s(\gamma(0)) \geq C^{-1} \left( \sqrt{1 - (A - Cr)^2} - \sqrt{1 - A^2} \right).$$

Suppose $|Ds|(\gamma(0)) > 1$. Then for a distance $r \leq C^{-1}(1/\sqrt{2} - 1/\sqrt{3})$ we have the estimate $s(\gamma(r)) - s(\gamma(0)) \geq r/\sqrt{2}$. However the estimates obtained above ensure that $s(\gamma(r)) - s(\gamma(0)) \leq C\tau + r/2$, using the gradient bound at the initial time. Consider points which are contained in the ball of radius $\frac{1}{3}r_0$, and paths $\gamma$ of fixed length $r$ no greater than the minimum of $C^{-1}\left(1/\sqrt{2} - 1/\sqrt{3}\right)$ and $\frac{1}{6}r_0$. The endpoint of any such curve is still contained in the ball of radius $\frac{1}{2}r_0$, but has $s(\gamma(r)) - s(\gamma(0)) \geq 1/\sqrt{2} > \frac{1}{2}r + C\tau$ provided we restrict to a time interval of length no greater than $C^{-1}r\sqrt{2} - 1/2$.

The same techniques show that the solution can be extended backward in time to $-\tau$, since we have assumed a curvature bound on $[-\tau, 0]$. q.e.d.

Now we are in a position to begin applying estimates from [10]. Note that we have existence of $s$ on a region which is independent of any bound on the injectivity radii. The first estimate we obtain is a bound on the oscillation of the curvature:

**Lemma 5–6.** *Under the conditions as in Lemma* 5–2, *there exists a positive function* $\sigma\colon (0, 1] \to \mathbf{R}$ *such that*

$$\inf_{(\xi, t)\in B_{\delta/2}\times[\gamma\tau, \tau]} F(\xi, t) \geq \sigma(\gamma)F(0, 0) \tag{5–7}$$

*for all* $\gamma \in (0, 1]$.

*Proof.* The previous lemma allows us to apply directly the following Harnack inequality due to Krylov and Safonov ([11] (see also [10, §3.1]):

**Lemma 5–8.** *Let* $u$ *be a positive solution in* $W^{1,2}(B_1(0) \times [-1, 1])$ *to the equation*

$$\frac{\partial}{\partial t}u = a^{ij}(x, t)D_iD_ju + b^i(x, t)D_iu + c(x, t)u \tag{5–9}$$

*on the domain* $B_1(0) \times [-1, 1] \subset \mathbf{R}^n \times \mathbf{R}$, *where the coefficients are measureable, bounded, and uniformly elliptic*:

$$\underline{C}|v|^2 \leq a^{ij}v_iv_j \leq \overline{C}|v|^2, \qquad |b| \leq C, \quad |c| \leq C, \tag{5–10}$$

*for all* $(x, t)$ *in* $B_1(0)\times[-1, 1]$ *and* $v \in \mathbf{R}^n$. *Then there exists a constant* $K$ *depending only on* $n$, $\underline{C}$, $\overline{C}$, *and* $C$ *such that*

$$\inf_{B_{1/2}(0)} u(x, 1) \geq \frac{u(0, 0)}{K}. \tag{5–11}$$

*If* $\underline{C}$, $\overline{C}$, *and* $C$ *change within a bounded range, then so does* $K$.

An application of this lemma followed by rescaling of either space or time variables gives a more general result. In view of the estimates (2–14) which control the map $\psi$, and the bounds on height, gradient and curvature (5–3), this lemma can be used immediately to obtain the desired result.

**Lemma 5–12.** *Under the conditions of Lemma* 5–2 *the following estimate holds if* $x_0$ *is chosen so that* $\sup_{M^n} |\mathscr{W}(x, 0)| = |\mathscr{W}(x_0, 0)| = 1$:

$$\inf_{(y, t)\in B_{\delta/2}(x)\times[\tau/2, \tau]} F(y, t) \geq C, \tag{5–13}$$

*where* $C$ *is a function of* $n$, $f$, *and* $d_0(x, x_0)$, $d_0$ *being the distance in* $M^n$ *with respect to* $g$ *at time* $0$.

*Proof.* This result follows by repeated application of the previous lemma—for points near $x_0$, a single application suffices. For points further away, several applications on shorter time intervals give the result.

**Lemma 5–14.** *Under the conditions of Lemma* 5–12,

$$\|\mathscr{W}\|_{C^{0,\beta(x)}(B_\delta(x)\times[3\tau/4, \tau])} \leq C(x), \tag{5–15}$$

*where* $\beta(x)$ *and* $C(x)$ *are functions of* $n$, $f$, *and* $d_0(x, x_0)$.

*Proof.* For this result we can apply a more sophisticated result from [10, §5.5] which gives Hölder estimates for solutions to uniformly parabolic equations

$$\frac{\partial}{\partial t}u = \mathscr{F}(D^2u, Du, u, x, t),$$

where $F$ is convex (or concave) in the second derivatives, provided some other conditions are satisfied involving boundedness of the derivatives of $\mathscr{F}$ with respect to other arguments. Our previous lemma ensures that the curvature is bounded above and below on each region we consider. This guarantees that all the required conditions are satisfied, and the result follows.

## 6. Convergence

In this section we apply the estimates from the previous chapter to complete the proof of Theorem 1–5. This involves showing convergence to a sphere on a subsequence of times under an appropriate rescaling (which uses a recent result of Hamilton [6]), and then deducing the convergence for other times (which is in most respects analogous to the proof in the Euclidean case [1]). Before we can carry out this program, we require the following result which guarantees the existence of the solution as long as the curvature remains bounded:

**Theorem 6–1.** *Suppose* $\varphi\colon M^n \times [0, t_0) \to N^{n+1}$ *is a smooth* $\alpha$*-convex solution to* (1–2)*, and* $\sup_{M^n\times[0,t_0)} F < \infty$. *Then* $\varphi$ *extends uniquely to* $M^n \times [0, t_1)$ *for some* $t_1 > t_0$.

*Proof.* Lemma 5–12 ensures that we have $C^\alpha$ estimates for the curvature of $\varphi$ on the domain $M^n \times [0, t_0)$. Note that the distance moved by any point is bounded by $t_0 \sup_{M^n\times[0,t_0)} F < \infty$, so the image of $\varphi$ is contained in a compact set of $N$ on this time interval. Consequently we have bounds on all the higher derivatives of the Riemann tensor of $N$. Standard Schauder estimates therefore provide bounds on all the derivatives of the curvature of $\varphi$. This ensures $C^\infty$ convergence to an immersion $\varphi_{t_0}$ (see, for example, [7, §8]). The short time existence result Theorem 3–14 now applies to extend the solution to a longer time interval. q.e.d.

The estimates of §5 are enough to prove the convergence in a restricted sense: We consider a subsequence of times $\{t_k\}$ approaching the maximal time of existence $T$ of $\varphi$, chosen such that the following holds for a corresponding sequence of points $x_k$ in $N$:

$$\sup_{M^n\times[0,t_k]} |\mathscr{W}|(x, t) = |\mathscr{W}|(x_k, t_k). \tag{6–2}$$

The existence of such a sequence is guaranteed by Theorem 6–1.

For each $k$ we rescale the metric $g^N$ on a time interval about $t_k$ to make $\varphi$ satisfy the curvature bound required for the application of Lemma 5–1. Then we use Lemma 5–1 with $A = A_k = \sup_{[0,t_k]} |\mathscr{W}|$, and proceed with the estimates of §5, obtaining Hölder estimates on the curvature on a time interval of rescaled duration $\tau$, depending only on the rescaled distance from the point $x_k$.

For each $k$, we choose an isometry from $\mathbf{R}^{n+1}$ to $T_{\varphi(x_k)}N$. In this way we identify the tangent spaces to $N$ at each of these points. Note that the exponential map at $\varphi(x_k)$ is nondegenerate on a ball of (rescaled) radius $r_0 A_k$ for some fixed $r_0 > 0$ depending on $K_1$ and $K_2$. Since $A_k$ is unbounded as $k$ becomes large, the exponential map is eventually nondegenerate on arbitrarily large regions of $\mathbf{R}^{n+1}$ under this identification. Furthermore, the curvature bounds (1–1) show that the metric induced on $\mathbf{R}^{n+1}$ by the exponential maps converges in $C^3$ to the flat metric as $k$ tends to infinity. Although the exponential map may not be diffeomorphic on these regions, we can use the nondegeneracy to obtain a family of hypersurfaces in $B_{r_0 A_k}(0) \subset \mathbf{R}^{n+1}$ which corresponds to the family $\varphi(M)$ under the exponential map. This is given by the solution to the following differential equation for immersions $\tilde{\varphi}$ into $\mathbf{R}^{n+1}$:

$$(6\text{–}3) \qquad T\tilde{\varphi} = \left(T_{\tilde{\varphi}} \exp_{x_k}\right)^{-1} \circ T\varphi, \qquad \tilde{\varphi}(x_k, t_k) = 0.$$

The estimates from Lemma 5–14 give $C^{2+\beta}$ estimates on each ball $B_r$ in $\mathbf{R}^{n+1}$, independently of $k$. Hence for each positive integer $R$ we can find a subsequence $\{t_{k_R}\}$ of $\{t_k\}$ for which the families of hypersurfaces converge to a $C^{2+\beta}$ family of hypersurfaces of $R^{n+1}$. Furthermore we can arrange that $\{t_{k_{R+1}}\}$ is a subsequence of $\{t_{k_R}\}$ for each $R$. Taking a diagonal subsequence $\{t_{k_k}\}$, we obtain convergence to a limiting family of complete hypersurfaces in $\mathbf{R}^{n+1}$. Each hypersurface in this family satisfies the estimates (5–15), depending only on the distance from the origin. Furthermore, the limiting family consists of strictly convex hypersurfaces with curvature bounded below by the estimate (5–13), depending only on distance from the origin. The curvature of the hypersurfaces is also bounded ($|\mathscr{W}| \le 2$), and the family is a solution to equation (1–2) with $\alpha = 0$. It follows (again using the estimates of §5 and Schauder theory) that the limit hypersurfaces are smooth.

We can now employ the following recent result due to Hamilton [6]:

**Theorem 6–4.** *A complete, smooth, strictly convex hypersurface with pinched principal curvatures in Euclidean space is compact.*

It follows immediately that the solutions $\varphi$ are boundaries of small immersed balls in $N$ for sufficiently large times. In particular, the solution remains in a compact subset of $N$ for the length of its existence. This implies that the solution converges for a subsequence of times to some point of $N$, since the hypersurfaces approach a compact hypersurface after arbitrarily large rescaling, and so have diameter tending to zero. It follows that we have convergence to a point of the whole solution, since later hypersurfaces are contained by earlier hypersurfaces.

Note that this result immediately gives us uniform estimates in $C^\infty$ for the rescaled hypersurfaces, since the solution remains in a compact subset of $N$, and we have uniform estimates on all the derivatives of the Riemann tensor of $N$ on this region. This also implies that we have the convergence to the limiting hypersurfaces in $C^\infty$ on the subsequence of times.

The limit hypersurfaces must in fact be spheres. This follows from the evolution equation (4–4) for the pinching quotient $Q/F$: In the limit, the maximum of this quotient is nonincreasing. By the strong maximum principle, the maximum is strictly decreasing unless $Q/F$ is constant. But if the maximum decreases on the family of limiting hypersurfaces, we have a contradiction to the convergence (note that the quantity $Q/F$ is unaffected by the rescaling process). Hence $Q/F$ is constant in the limit, for any $Q$ satisfying the conditions of §4. But then in equation (4–4), the negative second term must also vanish, which implies that the limiting hypersurfaces have constant curvature and are therefore spheres.

The Harnack estimate (5–13) gives bounds below on the rescaled curvature at each of the times $t_k$, since the diameter of the hypersurface is finite. Since the (unrescaled) minumum of the curvature is nondecreasing by the maximum principle applied to equation (3–20), this ensures that after some sufficiently large time, the hypersurfaces are strictly convex and pinched with respect to the flat metric on $\mathbf{R}^{n+1}$. The proof now proceeds exactly as in [1], §7.

## 7. Extensions and applications

In this section we conclude with some extensions to slightly different flows, and some applications to geometry.

**Theorem 7–1.** *For any strictly $\alpha$-convex initial immersion $\varphi_0$, there exists a unique smooth solution $\varphi$ on a finite time interval $[0, T)$ to equa-*

*tion* (1–2) *with speed* $f$ *satisfying Conditions* 3–1 *with* (4) *replaced by homogeneity of degree* 1 *in* $\lambda$. *The immersions* $\varphi_t$ *converge to a point of* $N$ *and become spherical as in Theorem* 1–5.

*Proof.* Equation (3–20) still ensures that the convexity is preserved (although $\alpha$-convexity need not be preserved), with a bound below on the principal curvatures decaying exponentially in time. Theorem 6–1 still holds, showing that a solution which has bounded curvature on a finite time interval can be extended further. On any finite time interval equation (4–4) still yields a pinching estimate. Therefore it is sufficient to show that the interval of existence of the solution is finite—the proof then proceeds exactly as before.

First note that $\varphi_0$ encloses an immersed disc, by Theorem 1–5; we can consider the evolution as taking place on the disc itself, in which $\varphi_0$ is embedded. The solution $\varphi$ to this equation immediately becomes enclosed by the solution $\varphi^{(\alpha)}$ of the nonhomogeneous equation. The solutions also remain disjoint: Suppose the two solutions touched again. At the point where this occurs the curvature of the outer hypersurface $\varphi^{(\alpha)}$ is no greater than the curvature of the inner hypersurface $\varphi$. Hence the rate of change of the distance between the hypersurfaces at such a point can be estimated as follows:

$$\begin{aligned}\frac{\partial}{\partial t}d(\varphi, \varphi^{(\alpha)}) &\geq f(\mathscr{W}) - f^{(\alpha)}(\mathscr{W}^{(\alpha)})\\ &\geq f(\mathscr{W}^{(\alpha)}) - f(\mathscr{W}^{(\alpha)} - \alpha\mathrm{Id}) \geq \alpha \inf_{\Gamma_+} \dot{F}(\mathrm{Id}) \geq \alpha,\end{aligned}$$

where we have used the inequality (2–20) in the last step. This is a contradiction since $\partial d/\partial t \leq 0$ at a newly attained minimum of $d$. Since $\varphi^{(\alpha)}$ contracts to a point in finite time, $\varphi$ can only exist for a finite time. q.e.d.

Note that this proof depends very strongly upon the result for the nonhomogeneous equation proved in this paper. I know of no way to prove this result directly.

The first application I wish to discuss is a simple proof of the 1/4-pinching sphere theorem of Klingenberg, Berger, and Rauch. This proof uses a method devised by Gromov and employed by Eschenburg [5].

**Theorem 7–2.** *Let* $N$ *be a compact simply connected smooth Riemannian manifold with sectional curvatures in the range* $\frac{1}{4} < \sigma^N \leq 1$. *Then* $N$ *is diffeomorphic to a twisted sphere.*

*Proof.* Choose a point $x_0$ in $N$, and consider exponential spheres about $x_0$. We consider these as immersed spheres given by immersions $\varphi_s$ where $s$ is the distance parameter. These immersions are related by the equations

$$\frac{\partial}{\partial s}\varphi = \nu.$$

The change in the curvature and the metric on these spheres in given by the following equations, the proof of which is identical to that of Theorem 3–15:

$$\frac{\partial}{\partial s}g(u, v) = 2\mathit{II}(u, v), \tag{7–3}$$

$$\frac{\partial}{\partial s}\mathscr{W}(u) = -\mathscr{W}^2(u) - R^N(\nu, u, \nu). \tag{7–4}$$

Using the assumptions on the curvature of $N$, we obtain the following estimates for the maximum and minumum principal curvatures of the exponential spheres:

$$\lambda_{\max} < \tfrac{1}{2}\cot(\tfrac{1}{2}s), \tag{7–5}$$

$$\lambda_{\min} \geq \cot(s). \tag{7–6}$$

It follows that the exponential spheres are nondegenerate for any $s < \pi$. Equation (7–3) for the metric gives a bound on the metric as long as $|\mathscr{W}|$ remains finite for expanding exponential spheres. The strict inequality in (7–5) implies that there is some distance $s < \pi$ for which $0 > \lambda_{\max} \geq \lambda_{\min} > -\infty$, and hence the exponential sphere at this distance is strictly convex in the outward direction. It follows from Theorem 1–5 that this sphere bounds a disc in $N$. This gives an expression for $N$ as a union of two discs by a diffeomorphism from one boundary to the other. q.e.d.

The result from Theorem 1–5 in the general case allows negative curvature in $N$. We can use this to prove the following "dented sphere theorem" which generalises the $\frac{1}{4}$-pinching theorem above:

**Theorem 7–7.** *Let $N$ be a compact smooth simply connected Riemannian manifold with sectional curvatures bounded below by some constant $-\alpha^2$. Let $\epsilon \in (\frac{1}{2}, 1)$ be such that $\epsilon \cot(\epsilon\pi) < -\alpha$, and let $\rho \in [\pi/2, \pi)$ be such that $\epsilon \cot(\epsilon\rho) = -\alpha$. If there is a point $x_0$ in $N$ such that $\epsilon < \sigma^N \leq 1$ on the ball $B_\rho(x_0)$, then $N$ is diffeomorphic to a twisted sphere.*

Note that for any bound below for the sectional curvatures of $N$, one can find a pinching ratio $\epsilon$ and a radius $\rho$ which satisfy the conditions here. If $\alpha$ becomes very large, then $\epsilon$ and $\rho$ must be taken very close to 1 and $\pi$ respectively.

*Proof.* This is exactly analogous to the previous theorem. If we take expanding exponential spheres about the point $x_0$, the evolution of minimum and maximum principal curvatures can be estimated by the following equations:

$$\frac{\partial}{\partial s}\lambda_{\max} < -\lambda_{\max}^2 - \epsilon^2, \tag{7–8}$$

$$\frac{\partial}{\partial s}\lambda_{\min} \geq -\lambda_{\min}^2 - 1. \tag{7–9}$$

This gives the following estimates for balls of radius less than or equal to $\rho$:

$$\epsilon\cot(\epsilon s) > \lambda_{\max} \geq \lambda_{\min} \geq \cot(s). \tag{7–10}$$

At distance $\rho$ the hypersurface is still nondegenerate, and is $\alpha$-convex in the outward direction, but possibly not strictly $\alpha$-convex. However, since $N$ is smooth, there is some short distance beyond $\rho$ on which the sectional curvatures are positive. Hence by taking a distance $s$ slightly larger than $\rho$, we obtain a nondegenerate, strictly outward $\alpha$-convex hypersurface. By Theorem 1–5, this is the boundary of a disc, and the result follows.

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