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CURVATURE MEASURES AND CHERN CLASSES OF SINGULAR VARIETIES

JOSEPH H. G. FU

The aim of the present article is to show how the approach of [8] to studying the curvature measures of a singular space yields a natural geometric treatment of the theory of Chern homology classes of singular complex analytic varieties. These classes were first considered by M. H. Schwartz [0], but were neglected at the time. Deligne and Grothendieck later introduced axioms for a conjectural theory of Chern homology classes for singular varieties. MacPherson then constructed classes fulfilling these axioms in the seminal paper [16]. Subsequent to MacPherson's work, it was shown by Brylinski, Dubson, and Kashiwara [2] that the MacPherson Chern classes of a singular variety X admit a simple expression involving the characteristic cycle of X from the theory of D-modules. Up to this point, however, a complete treatment of the properties of these classes has rested upon the somewhat indirect approach of [16]. In the meantime, we independently constructed the characteristic cycle of Kashiwara by direct geometric means [8]. The geometric insight from our construction allows us to give a direct and intuitively appealing proof of the Deligne-Grothendieck axioms, which is what we present in these pages.

The advantages of our method over that of [16] are twofold. First, the key covariance axiom of Deligne-Grothendieck for morphisms $f\colon X \to Y$ of varieties was established only indirectly for singular varieties X, using Hironaka's formidable resolution theorem. Our treatment, on the other hand, works with the singular varieties directly, without mention of resolutions. (We have, however, no proof of uniqueness for the Deligne-Grothendieck axioms apart from the original obvious argument using resolution.) Second, certain key coefficients associated to the strata of singular X are in [16] computed somewhat circuitously: viz. by initially defining a certain natural transformation T using a topological "Euler obstruction", and then *inverting* T. The Euler obstruction never enters the treatment of

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the present paper; in fact the crucial role of this invariant in [16] in verifying the key push-forward axiom (C3 below) is taken up here by a far more elementary and intuitive result, namely the Gauss-Bonnet theorem. That our classes agree with those of [16] follows from uniqueness; however, we outline in the final section of this paper some ideas for a direct proof.

Our definition of the MacPherson Chern classes is formally identical with that of [2]. However, the connection with [16] is there made by showing the formal correspondence between MacPherson's Euler obstruction and Kashiwara's definition of the characteristic cycle. In other words, [2] does not offer any new insight into the fundamental theory of [16]. It is interesting also to observe that the basic definition at the bottom of p. 575 of [2] requires some explicit choices of signs that need not be made from our point of view; cf. the lemma of 2.1 below for a possible explanation of this divergence.

Our original motivation for studying these ideas was to understand the curvature of a space with singularities. In [8] this problem is treated in detail. From this point of view, our treatment of Chern classes corresponds to the classical identification of characteristic classes with differential forms arising from polynomials in the curvature tensor. Thus the cycles that we define live in the world of currents, dual to the differential forms. These currents coincide with those identified by Shifrin [19].

Let us now introduce the basic terms of our discussion. Let M be a complex analytic manifold, and $X \subset M$ a closed subanalytic set. The *conormal cycle* $N^*(X)$ is an integral current in the cosphere bundle S^*M of m, coinciding with integration over the conormal sphere bundle to X if X is smooth; its construction is outlined in 1.1 below. The conormal cycle corresponds precisely to Kashiwara's characteristic cycle, by 4.7 of [8]. We will identify "universal" differential forms $\beta, \gamma_0, \gamma_1, \cdots$ on S^*M such that if X is a complex subvariety of M, then the total MacPherson Chern homology class $\hat{c}_*(X)$ is represented by the current $\pi_\#(N^*(X) \llcorner \beta \wedge \gamma_*)$, where π is $S^*M \to M$ is the projection. The Deligne-Grothendieck axioms now take the following form:

C1. If X is smooth, then $\hat{c}_*(X)$ is the Poincaré dual of the total Chern cohomology class of X.

C2. If $Y \subset M$ is a second compact subvariety, then $\hat{c}_*(X \cup Y) = i_*\hat{c}_*(X) + i_*\hat{c}_*(Y) - i_*\hat{c}_*(X \cap Y)$, where i_* denotes the various maps in homology induced by the inclusions into $X \cup Y$.

C3. If $f: X \to N$ is an analytic morphism onto a proper analytic subvariety of the complex manifold N, then

$$f_*\hat{c}_*(X) = \sum_i n_i \hat{c}_*(Y_i),$$

where $n_i \in \mathbb{Z}$ and the $Y_i \subset N$ are analytic subvarieties such that

$$\sum_i n_i 1_{Y_i}(q) = \chi(f^{-1}(q)).$$

The plan of the paper is as follows. Section 1 gives a quick survey, without proofs, of the theory of the conormal cycle developed in [8]. The property C2 follows at once from this general theory. Section 2 identifies the differential forms γ_i, and proves C1. Sections 3 and 4 are the body of the paper, devoted to the proof of C3 for $f: M \supset X \to N$. In these sections we work on the graph of $f \subset M \times N$, using its conormal cycle together with pullbacks of the universal forms γ_i associated to M and N. More precisely, we construct in §3 explicit homologies between the images on graph f of the currents $\hat{c}_k(X)$ and $\sum n_i \hat{c}_k(Y_i)$ for certain $Y_i \subset N$ and $n_i \in \mathbb{Z}$. In §4 we use the Gauss-Bonnet theorem to verify that these n_i, Y_i satisfy $\sum n_i 1_{Y_i} = \chi f^{-1}$.

0. Background

0.0. Notation. Given a smooth manifold M, the algebra of all differential forms on M is denoted $\mathscr{E}^*(M)$; the subalgebra consisting of those forms with compact support is $\mathscr{D}^*(M)$. Typically, we will use the same symbol for a form and for its pullback. If M is a complex manifold, then we have the operations $d, d^c: \mathscr{E}^*(M) \to \mathscr{E}^{*+1}(M)$, given in local coordinates by their action on functions:

$$df = \sum \frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i,$$
$$d^c f = \sum \frac{\partial f}{\partial z_i} dz_i - \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i.$$

Given a submanifold V with a canonical orientation, the current defined by integration over it will be denoted by [V]. If $\mathscr{B}$ is a bundle with fiber F and total space E over M, and F is canonically oriented, then we define an injection of the currents on M into the currents on E by

$$T \mapsto T \times_{\mathscr{B}} [F] := \varphi_\#(T \times [F])$$

if T is supported in a trivializing patch of the bundle with chart φ, and extend by linearity using a partition of unity.

This paper is full of commuting canonical maps ("projections") between spaces. Rather than giving a name to each map, it seems less confusing to adopt the following practice: Any such map $M \to N$ will be denoted by π_N. With this notation we have $\pi_N \circ \pi_M = \pi_N$, etc., even though the symbol π_N denotes different maps with different domains on the right and on the left. This convention is superseded by any other explicit labelling scheme; usually these will be confined to small arguments within the text.

The technical foundation of the argument of this paper is the theory of integral and normal currents of Federer and Fleming (cf. Federer [6, Chapter 4]), and its specialization to the semianalytic case due to Hardt [12]. Our notation will be consistent with that of [6], except that we will denote pullback under a map f by the symbol f^* rather than Federer's $f^\#$. The symbol f_* will denote the derivative of f.

Other frequently used notation from the theory of currents includes: (i) $T \llcorner \varphi$ for the restriction of a current T by a form φ. Thus $(T \llcorner \varphi)(\psi) = T(\varphi \wedge \psi)$. If T is representable by integration and U is a set, then we may write also $T \llcorner U$ for the restriction of T to U; cf. [6, 4.1.7]. (ii) $\langle T, f, x \rangle$ for the slice of a current T by a Lipschitz function f at the value x [6, §4.3]. Thus if T is given by integration over an oriented smooth manifold M, f is a smooth map into an oriented manifold, and x is a regular value of f, then $\langle T, f, x \rangle$ is given by integration over $M \cap f^{-1}(x)$, with an orientation induced by f.

0.1. Let us also prove the following basic lemma.

Lemma (Residue theorem). *Let $V \subset \mathbb{C}^m$ be an irreducible complex analytic variety, and let $\varphi\colon \mathbb{C}^m \to \mathbb{C}$ be a holomorphic function such that $V \not\subset \varphi^{-1}(0)$. Then the expression*

$$T := [V] \llcorner (2\pi\sqrt{-1})^{-1} d^c \log|\varphi|^2$$

defines a locally normal current in $\mathbb{G}^m$, with

$$\partial T = \sum n_i [W_i],$$

where W_i are irreducible components of $\varphi^{-1}(0) \cap V$, and $n_i \in \mathbb{Z}$.

If $\varphi^{-1}(0)$ does not contain any irreducible component of the variety of singular points of V, and $d(\varphi|S)(x) \neq 0$ for any regular point $x \in V \cap \varphi^{-1}(0)$, then

$$\partial T = [V \cap \varphi^{-1}(0)].$$

Proof. Consider the map $\gamma\colon V - \varphi^{-1}(0) \to \mathbb{C}^m \times S^1$,

$$\gamma(p) := (p, \varphi(p)/|\varphi(p)|^{-1}).$$

Then the closure of the image of γ is the real analytic subvariety

$$\begin{aligned}\widetilde{V} &:= \{(p, e^{i\theta}) : p \in V,\ \operatorname{Im}(\varphi(p)e^{-i\theta}) = 0\} \\ &\subset \mathbb{C}^m \times S^1.\end{aligned}$$

The variety $\widetilde{V}$ inherits an orientation from V so that the corresponding current $[\widetilde{V}]$ satisfies

$$\begin{aligned}\pi_{\mathbb{C}^m\#}[\widetilde{V}] &= [V], \\ [\widetilde{V}] \llcorner (\varphi^{-1}(0) \times S^1) &= 0, \\ \operatorname{spt} \partial[\widetilde{V}] &\subset (V \cap \varphi^{-1}(0)) \times S^1.\end{aligned}$$

In fact, [6, 4.2.28] implies that $[\widetilde{V}]$ is a locally integral current in $\mathbb{C}^m \times S^1$, and invoking the support theorem (4.1.20, op. cite.) we may characterize its boundary as

$$\partial[\widetilde{V}] = \sum n_i [W_i] \times [S^1]$$

for some integers n_i, where the W_i are the irreducible components of $V \cap \varphi^{-1}(0)$.

Now we observe that

$$\begin{aligned}&(2\pi\sqrt{-1})^{-1} d^c \log|\varphi|^2 | V \\ &= (2\pi)^{-1} \gamma^* \pi_{S^1}^* \, d\theta\end{aligned}$$

and

$$\begin{aligned}&\pi_{\mathbb{C}^m}^*((2\pi\sqrt{-1})^{-1} d^c \log|\varphi|^2) | (\widetilde{V} - \varphi^{-1}(0) \times S^1)) \\ &= (2\pi)^{-1} \pi_{S^1}^* \, d\theta | (\widetilde{V} - (\varphi^{-1}(0) \times S^1).\end{aligned}$$

Thus

$$T := \pi_{\mathbb{C}^m\#}([\widetilde{V}] \llcorner (2\pi)^{-1} \, d\theta) = [V] \llcorner (2\pi\sqrt{-1})^{-1} d^c \log|\varphi|^2$$

is well-defined and normal, with boundary

$$\begin{aligned}\partial T &= \pi_{\mathbb{C}^m\#}(\partial[\widetilde{V}] \llcorner (2\pi)^{-1} \, d\theta) \\ &= \pi_{\mathbb{C}^m\#}\left(\sum n_i([W_i] \times [S^1]) \llcorner (2\pi)^{-1} \, d\theta\right) \\ &= \sum n_i [W_i].\end{aligned}$$

To prove the second assertion, we observe that under the additional hypothesis we have

$$\dim_{\mathbb{R}}(\varphi^{-1}(0) \cap \operatorname{sing} V) \le \dim \partial T - 2,$$

so the support theorem implies that

$$\partial T = \partial T \llcorner (V - \operatorname{sing} V).$$

Let $p \in (V - \operatorname{sing} V) \cap \varphi^{-1}(0)$, and let $\mathscr{U}$ be a neighborhood of p in V-sing V such that $\varphi = \varphi_1, \cdots, \varphi_k$ form a holomorphic coordinate system $\Phi: \mathscr{U} \to \mathbb{C}^k$, $k := \dim V$. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}^m \times S^1 \supset \widetilde{\mathscr{U}} & \longrightarrow & \mathscr{U} & & \\ & & & \searrow^{\varphi} & \\ \Big\downarrow \tilde{\Phi} & & \Big\downarrow \Phi & & \mathbb{C} \\ & & & \nearrow_{\pi_{\mathbb{C}}} & \\ \mathbb{C} \times S^1 \times \mathbb{C}^{k-1} \supset \widetilde{\mathbb{C}} \times \mathbb{C}^{k-1} & \longrightarrow & \mathbb{C} \times \mathbb{C}^{k-1} & & \end{array}$$

where

$$\begin{aligned} \widetilde{\mathbb{C}} &= \{(z, e^{i\theta}) \in \mathbb{C} \times S^1 : \operatorname{Im} z e^{-i\theta} = 0\} \\ &= \{(z, z/|z|) : z \in \mathbb{C},\ z \neq 0\} \cup \{0\} \times S^1, \\ \partial[\widetilde{\mathbb{C}} \times \mathbb{C}^{k-1}] &= [0] \times [S^1] \times [\mathbb{C}^{k-1}]. \end{aligned}$$

Since $\Phi_\#[\mathscr{U}] = [\mathbb{C} \times \mathbb{C}^{k-1}] \llcorner \Phi(\mathscr{U})$, we obtain

$$\begin{aligned} \partial T \llcorner \mathscr{U} &= \pi_{\mathscr{U}\#}(\partial[\widetilde{\mathscr{U}}] \llcorner (2\pi)^{-1}\, d\theta) \\ &= \Phi_\#^{-1} \pi_{\mathbb{C}\times\mathbb{C}^{k-1}\#}(\partial[\widetilde{\mathbb{C}} \times \mathbb{C}^{k-1}] \llcorner (2\pi)^{-1}\, d\theta) \\ &= \Phi_\#^{-1}([0] \times [\mathbb{C}^{k-1}]) \\ &= [\mathscr{U} \cap \varphi^{-1}(0)]. \end{aligned}$$

1. The conormal cycle of a subanalytic set

Let M be an oriented real analytic manifold of dimension m. Let T^*M denote the cotangent bundle of M, with projection $\pi: T^*M \to M$; let S^*M denote the cotangent ray space S^*M: T^*M-(zero − section)/ $\sim$ where $\xi \sim \eta$ if $\xi = t\eta$ for some $t > 0$. The space S^*M is a contact manifold in a natural way, with contact ideal $A = (\sigma^*\alpha)$ where $\sigma: S^*M \to T^*M$ is a section and α is the canonical 1-form of T^*M.

Let $X \subset M$ be a closed subanalytic set. There is then a naturally associated subanalytic integral current $N^*(X) \in I_{m-1}(S^*X)$ that is *closed* ($\partial N^*(X) = 0$) and *Legendrian* ($N^*(X)(\varphi) = 0$ for $\varphi \in A$). The details of the construction and uniqueness of $N^*(X)$ are given in [8]. The present section is devoted to describing, without proof, those properties of $N^*(X)$ relevant to the construction of the Chern-MacPherson homology classes of a complex analytic variety X.

1.1. Let $f: M \to \mathbb{R}$ be a locally Lipschitzian subanalytic function. According to Fu [7], there is a unique integral current $[df] \in I_m(T^*M)$ satisfying the conditions:

$$\partial[df] = 0 = [df] \llcorner \omega,$$

where ω is the symplectic 2-form of T^*M,

$$\pi|\text{spt}[df] \text{ is proper},$$

(1.1a) and

$$[df](\varphi \wedge \pi^*\varepsilon) = \int_M (\varphi \circ df) \cdot \varepsilon$$

for any compactly supported m-form ε on M and any smooth function $\varphi: T^*M \to \mathbb{R}$.

Here π is the projection of the cotangent bundle T^*M. The support of $[df]$ satisfies

(1.1b) $$\text{spt}[df] \subset \text{graph}\, \partial f,$$

where ∂f is the generalized differential of Clarke [5]. Furthermore,

$$\partial([df] \llcorner \pi^* f) = -[df] \llcorner \alpha.$$

Let ν denote the natural projection from T^*M-(zero section) $\to S^*M$. From the support relation above, the general facts about subanalytic sets, it follows that for any compact $K \subset M$, the set $f(\text{spt}[df] \cap (\text{zero-section}) \cap \pi^{-1}(K))$ is *finite*. In particular, $\nu_\#\langle [df], \pi^* f, r\rangle$ is well-defined for all but a countable set of r.

Theorem. *Let $f, g: M \to \mathbb{R}$ be locally Lipschitzian, subanalytic and nonnegative, with $f^{-1}(0) = g^{-1}(0) = X$. Then*

$$\lim_{r\to 0} \nu_\#\langle [df], \pi^* f, r\rangle = \lim_{r\to 0} \nu_\#\langle [dg], \pi^* g, r\rangle,$$

where the limits are taken in the flat metric topology.

Thus we may put $N^*(X)$ equal to any such limiting current. The closedness and Legendrian character of $N^*(X)$ follow from the properties of $[df]$ above. Observe that if X is a real analytic subvariety of M, then this construction has a particularly appealing form. Working locally we may assume that X is contained in an open subset U of R^m, with

$$X = \bigcap_i f_i^{-1}(0),$$

where the f_i are real analytic functions on U. Then, putting $g := \sum |f_i|^2$, we have $g \geq 0$, $X = g^{-1}(0)$, and

$$N^*(X) = \lim_{r \to 0} N^*(g^{-1}[0, r]),$$

where the conormal cycle on the right is integration over the usual conormal manifold to the smooth body $g^{-1}[0, r]$.

The operation $X \mapsto N^*(X)$ is additive in the sense that

$$N^*(X \cup Y) = N^*(X) + N^*(Y) - N^*(X \cap Y)$$

for any closed subanalytic $X, Y \subset M$. It follows that if we identify X with its characteristic function 1_X, then N^* may be extended to a homomorphism of abelian groups

$$N^*: \mathrm{Constr}(M) \to I_{m-1}(S^*M),$$

where the group $\mathrm{Constr}(M)$ is the group of "subanalytic constructible functions" generated by the characteristic functions of closed subanalytic subsets of M.

1.2. Now suppose that M is endowed with a smooth Riemannian metric, and let Ω denote the Chern-Gauss-Bonnet form of the metric. We say that an $(m-1)$-form Π on S^*M is a *transgression* if

$$d\Pi = -\pi^*\Omega, \qquad \int_{\pi^{-1}(x)} \Pi = 1,$$

where $\pi^{-1}(X)$ is any fiber of $S^*M \to M$, oriented canonically following the orientation of M.

Gauss-Bonnet Theorem. *If $X \subset M$ is subanalytic and compact, then*

$$N^*(X)(\Pi) + \int_X \Omega = \chi(X),$$

the Euler characteristic of X.

1.3. Given a smooth submanifold $S \subset M$, we let $\nu^*S \subset S^*M$ denote the image of the usual conormal bundle of S.

Proposition A. *There is a stratification $\mathcal{S}$ of X such that*

$$\mathrm{spt}\, N^*(X) \subset \bigcup\{\nu^*S: S \in \mathcal{S}\}.$$

If M is a complex manifold, and X is a holomorphic subvariety, then $\mathcal{S}$ may be chosen so that each $\overline{S}$, $S \in \mathcal{S}$, is again a holomorphic subvariety.

For the rest of this section we let $\mathcal{S}$ be such a stratification.

Now let $\varphi: M \to \mathbb{R}$ be analytic, and suppose that r is a regular value of $\varphi|S$ for each $S \in \mathcal{S}$. The next two propositions imply that $N^*(X \cap \varphi^{-1}(-\infty, r])$ is determined by its restriction to $\pi^{-1}\varphi^{-1}(-\infty, r)$.

Given a cone $C \subset T_xM$, put

$$\operatorname{Dual} C := \{\xi \in S_x^*M : \langle \xi, v\rangle \leq 0 \text{ for all } v \in C\}.$$

Proposition B. *If $\varphi(x) = r$ and $x \in S \in \mathscr{S}$, then*

$$\operatorname{spt} N^*(X \cap \varphi^{-1}(-\infty, r]) \cap \pi^{-1}(x) \subset \operatorname{Dual} \operatorname{Tan}(S \cap \varphi^{-1}(-\infty, r], x)).$$

Proposition B has the following converse.

Proposition C. *Suppose that* $\dim X < m$. *Then there is no closed nonzero flat chain of dimension* $(m-1)$ *with support in*

$$\bigcup_{S \in \mathscr{S}} \bigcup_{x \in S \cap \varphi^{-1}(r)} \operatorname{Dual} \operatorname{Tan}(S \cap \varphi^{-1}(-\infty, r], x).$$

Corollary. $N^*(X \cap \varphi^{-1}(-\infty, r])$ *is the unique cycle Q such that*

$$\operatorname{spt}(Q - (N^*(X) \llcorner \varphi^{-1}(-\infty, r))) \subset \bigcup_{S \in \mathscr{S}} \bigcup_{x \in S \cap \varphi^{-1}(r)} \operatorname{Dual} \operatorname{Tan}(S \cap \varphi^{-1}(-\infty, r], x).$$

It will be convenient in the following pages to define $N^*(X)$ for any locally closed subanalytic subset $X \subset M$. For this it is sufficient to write; when U is open and $X \cap U$ is closed in U,

$$N^*(X) \llcorner \pi^{-1}(U) = N^*(X \cap U),$$

where X is considered as a subset of U on the right-hand side; and $N^*(X) \llcorner \pi^{-1}(K) = 0$ whenever $K \cap X = \varnothing$.

1.3½. Theorem. *Let $X \subset M$ be subanalytic, and let $\varphi: M \to \mathbb{R}$ be a subanalytic function. Then for any $r \in \mathbb{R}$,*

$$\lim_{s \downarrow r} N^*(X \cap \varphi^{-1}(-\infty, s]) = N^*(X \cap \varphi^{-1}(-\infty, r]).$$

By Gauss-Bonnet we have therefore:

Corollary. *If X is compact, then*

$$\lim_{s \downarrow r} \chi(X \cap \varphi^{-1}(-\infty, s]) = \chi(X \cap \varphi^{-1}(-\infty, r]).$$

1.4. Now we specialize to the case where M is a complex analytic manifold of complex dimension m, and $X \subset M$ is a holomorphic subvariety. Using Proposition 1.3A, the conormal cycle $N^*(X) \in \mathbf{I}_{2m-1}(S^*M)$ can be reduced to a projectivized cycle $\mathbb{P}N^*(X) \in \mathbf{I}_{2m-2}(\mathbb{P}^*M)$ (where $\mathbb{P}^*M$ is the projectivized cotangent bundle of m) in the following way.

The spaces T^*M and $\mathbb{P}^*M$ each carry a natural complex structure, in such a way that if $V \subset M$ is a holomorphic subvariety of M, then the conormal bundle $\vec{\nu}^*(V) \subset T^*V$ is a holomorphic subvariety of T^*V, and such that the projection $T^*V - (\text{zero-section}) \to \mathbb{P}^*V$ is holomorphic. Given a smooth Hermitian norm $\| \ \|$ on T^*M, we put

$$h\colon T^*V \to \mathbb{R}, \qquad h(\xi) := \|\xi\|^2.$$

Then $(2\pi\sqrt{-1})^{-1} d^c \log h =: \beta'$ is a real-valued 1-form on T^*M-(zero-section), and it is not hard to see that there is a 1-form β on S^*M such that $\beta' = \pi^*_{S^*M}\beta$.

Let $\mathscr{H}$ denote the Hopf bundle $S^*M \to \mathbb{P}^*M$, with fibers S^1 which are canonically oriented by the requirement that $\theta \mapsto e^{i\theta}\xi$ be an orientation preserving map $\mathbb{P} \to \pi_{\mathscr{H}}^{-1}(\pi_{\mathscr{H}}(\xi))$ for each $\xi \in S^*M$. With this orientation we have $\int_{\pi_{\mathscr{H}}^{-1}(\eta)} \beta = 1$ for $\eta \in \mathbb{P}^*M$.

Proposition 1.3A, together with the constancy Theorem [6, 4.1.7], implies that

$$\text{(1.4a)} \qquad \mathbb{P}N^*(X) := \pi_{\mathscr{H}\#}(N^*(X) \llcorner \beta)$$

is an integral current of dimension $2m - 2$ in $\mathbb{P}^*M$. An inverse formula is

$$\text{(1.4b)} \qquad N^*(X) = \mathbb{P}N^*(X) \times_{\mathscr{H}} [S^1].$$

Since $0 = \partial N^*(X) = \partial \mathbb{P}N^*(X) \times_{\mathscr{H}} [S^1]$, it follows that $\mathbb{P}N^*(X)$ is *closed*.

1.5. Proposition 1.3A and the constancy theorem imply that there are integers $n(S)$, $S \in \mathscr{S}$, such that

$$\text{(1.5a)} \qquad \mathbb{P}N^*(X) = \sum_{S \in \mathscr{S}} n(S)\mathbb{P}N^*(S).$$

Since the supports of the $\mathbb{P}N^*(S)$ are holomorphic subvarieties of $\mathbb{P}^*M$, this is a *holomorphic* cycle. Furthermore, $\mathbb{P}N^*(X)$ is *legendrian*, in the following sense. Let σ be a local section of T^*M-(zero) $\to \mathbb{P}^*M$, and consider the ideal $A_{\mathbb{C}}$ of differential forms on $\mathbb{P}^*M$ generated by α', α'', where

$$\langle \alpha'(\eta), v\rangle = \operatorname{Re}\langle \sigma(\eta), \pi_{M^*} v\rangle,$$
$$\langle \alpha''(\eta), v\rangle = \operatorname{Im}\langle \sigma(\eta), \pi_{M^*} v\rangle$$

for any $v \in T_\eta(\mathbb{P}^*M)$. Clearly this ideal does not depend on the choice of σ. We have

$$\mathbb{P}N^*(X)(\varphi) = \sum n(S)\mathbb{P}N^*(S)(\varphi) = 0$$

for any $\varphi \in A_{\mathbb{C}}$.

The preceding discussion has a converse.

Proposition A. *Given any holomorphic, Legendrian cycle* $T \in I_{2m-2}(\mathbb{P}^*M)$, *there are locally finite family* $\{Y_1, Y_2, \cdots\}$ *of holomorphic subvarieties of* M, *and integers* $n_1, n_2, \cdots$ *such that* $T = \sum n_i \mathbb{P}N^*(Y_i)$.

Proof. Let $V_1, V_2, \cdots$ be the irreducible components of $\operatorname{spt} T$, and put $Y_i := \pi_M(V_i)$ to be the image variety in M, with regular locus $Y_i^0 := Y_i - \operatorname{sing} Y_i$. Since the V_i are Legendrian and irreducible, we have $V_i = \overline{\mathbb{P}\nu^*(Y_i^0)}$, $i = 1, 2, \cdots$, and so by the constancy theorem there are integers $m_1, m_2, \cdots$ such that

$$T = \sum m_i \mathbb{P}N^*(Y_i^0) = \sum \pm m_i [V_i].$$

Clearly each $\mathbb{P}N^*(Y_i^0)$ is a cycle: for $\operatorname{spt} \partial \mathbb{P}N^*(Y_i^0) \subset \pi^{-1}(\operatorname{sing} Y_i) \cap V_i$, which is a proper subvariety of V_i and so has real codimension ≥ 2 in V_i; therefore $\partial \mathbb{P}N^*(Y_i^0) = 0$ by [6, 4.1.20]. Let $Y_{i_1}, Y_{i_2}, \cdots$ be the varieties of maximal dimension k among the Y_i. Then by Proposition 1.3A,

$$\pi_M \operatorname{spt}\left(T - \sum_j m_{i_j} \mathbb{P}N^*(Y_{i_j})\right) \subset \bigcup_{\dim Y_i < k} Y_i \cup \bigcup_j \operatorname{sing}(Y_{i_j}),$$

since $\mathbb{P}N^*(Y_i^0) = \mathbb{P}N^*(Y_i) \llcorner \pi_M^{-1}(Y_i^0)$. Our assertion now follows by induction on k.

1.6. Finally, we note that the coefficients in the expansion (1.5a) are independent of the embedding $X \hookrightarrow M$. This is most readily seen via the following result from [8]. Let $\sigma\colon S^*M \to T^*M$ be a section, and put $\sum\colon S^*M \times [0, \infty) \to T^*M$ by $\sum(\xi, t) := t\sigma(\xi)$. Define $\vec{N}^*(x) := \sum_\#(N^*(x) \times [[0, \infty)])$, and note that this does not depend on the choice of σ.

Proposition. *If* X *and* Y *are subanalytic subsets of the real analytic manifold* M *and* N, *respectively, then*

$$\vec{N}^*(X \times Y) = \vec{N}^*(X) \times \vec{N}^*(Y),$$

where we identify $T^*(M \times N) \cong T^*M \times T^*N$.

Corollary. *If* $f\colon ; X \to N$ *is an analytic isomorphism into a complex manifold* N *with* $\dim X < \dim N$, *and* (1.5a) *holds, then*

$$\mathbb{P}N^*(f(X)) = \sum_{S \in \mathcal{S}} n(S) \mathbb{P}N^*(f(S)).$$

Proof. Given $p \in X$, there is a neighborhood $U \subset M$ of P and an analytic mapping $\overline{f}\colon U \to N$ such that $\overline{f}|X = f|U$ (this can be taken as the definition of an analytic morphism $X \to N$; cf. Shafarevich [18, I.2.3]). Choosing local coordinates in $\mathbb{C}^{m+n}$ for $M \times N$ about the point $(p, f(p))$ so that graph $\overline{f}$ becomes $\mathbb{C}^m \times \{0\}$, we may apply the Proposition above (with $Y =$ point) to obtain the identity of currents in $\mathbb{P}^*(M \times N)$:

$$\mathbb{P}N^*(\mathrm{graph}(f)) = \sum_{S \in \mathcal{S}} n(S)\mathbb{P}N^*(\mathrm{graph}(f|S)).$$

Applying the same process to $f^{-1}: f(X) \to X$ and the stratification $\{f(S): S \in \mathcal{S}\}$ now gives the desired result.

2. Canonical forms on S^*M and $\mathbb{P}^*M$

The definition of the characteristic homology classes of $X \subset M$ will be of the form

$$\hat{c}_k(X) = \pi_{M\#}(\mathbb{P}N^*(X) \llcorner \gamma_k),$$

where the γ_k are certain "universal" differential forms on $\mathbb{P}^*M$. The proof that the homology classes of these closed currents satisfy the desired transformation law will depend on the algebraic properties of these forms.

2.1. We have introduced the 1-form β on S^*M in the previous section. Since the tautological line bundle $\mathcal{O}_{T^*M}(-1)$ over $\mathbb{P}^*M$ is holomorphic, it follows from Chern [4, §6] and the definition of β that there is a closed 2-form ζ on $\mathbb{P}^*M$ representing the Chern class $-c_1(\mathcal{O}_{T^*M}(-1)) = c_1(\mathcal{O}_{T^*M}(1))$, such that

$$d\beta = \pi_{\mathcal{H}}^*\zeta.$$

A fundamental identity from the theory of Chern classes states that if E is a complex vector bundle of rank r over a space X and $\zeta = c_1(\mathcal{O}_E(1)) \in H^2(\mathbb{P}E)$, then the Chern classes $c_1(E) \in H^2(X)$ satisfy the relation

$$\sum_{i=0}^{r} \zeta^{r-1} \cup c_i(E) = 0 \tag{2.1y}$$

in $H^*(\mathbb{P}E)$ (cf. Grothendieck [11] for a treatment of Chern classes based on this identity, or Fulton [9, 3.1.24]). This implies that

$$c_*(E) = \left[\pi_*\left(\sum_{k\geq 0} \zeta^k\right)\right]^{-1}, \tag{2.1z}$$

where $\pi_*: H^*(\mathbb{P}E) \to H^{*-2(r-1)}(X)$ is integration along the fiber. In particular, if we let $c_1(T^*M), \cdots, c_m(T^*M)$ denote fixed choices of differential forms on M representing the Chern cohomology classes of the cotangent bundle T^*M, then we have the fundamental relation

$$\zeta^m + c_1(T^*M) \wedge \zeta^{m-1} + \cdots + c_m(T^*M) = d\tau \tag{2.1a}$$

for some $(2m-1)$-form τ on $\mathbb{P}^*M$.

Suppose for the moment that $X \subset M$ is a *smooth* subvariety, with conormal bundle $\nu^*(X)$. Let $\pi_*: \mathscr{E}^*(\mathbb{P}^*M) \to \mathscr{E}^{*-2(m-\dim X-1)}(M)$ be the operation of integration along the fibers of the projectivized bundle $\mathbb{P}\nu^*(X) \subset \mathbb{P}^*M$. Now (2.1z) implies that the total Chern class of $\nu^*(X)$ is represented by the differential form

$$c_*(\nu^*(X)) = \left[\pi_*\left(\sum_{r\geq 0}\zeta^r\right)\right]^{-1}.$$

The Whitney sum formula yields at once a formula for the Chern class of the cotangent bundle of X:

$$c_*(T^*X) = \pi_*\left(\sum_{r\geq 0}\zeta^r\right) \wedge c_*(T^*M).$$

Thus we define the forms γ_k on $\mathbb{P}^*M$ by expanding

$$\left(\sum_{r\geq 0}\zeta^r\right) \wedge \pi^* c_*(T^*M) = \sum_k (-1)^{m-k}\gamma_k, \qquad \deg\gamma_k = 2(m-k-1).$$

(Observe that the index k may be negative.) Computing the Chern classes of X itself,

$$\begin{aligned} c_j(X) = c_j(TX) &= (-1)^j c_j(T^*X) \\ &= (-1)^j \pi_*((-1)^{m+j-\dim X}\gamma_{\dim X-j}) \\ &= (-1)^{m-\dim X}\pi_*\gamma_{\dim X-j}. \end{aligned}$$

Lemma. *If X is smooth, then*

$$\mathbb{P}N^*(X) = (-1)^{m-\dim X}[\mathbb{P}\nu^*(X)].$$

Proof. Applying a change of variable and Proposition 1.6, it is enough to see this for $X = \{0\} \subset M = \mathbb{C}$, which case is obvious.

Now we may express the currents dual to the forms $c_j(X)$ by

$$\begin{aligned} [X] \llcorner c_*(TX) &= (-1)^{m-\dim X}\pi_\#\left([\mathbb{P}\nu^*(X)] \llcorner \sum_k \gamma_k\right) \\ &= \pi_\#\left(\mathbb{P}N^*(X) \llcorner \sum_k \gamma_k\right). \end{aligned} \tag{2.1b}$$

Example. If $M = \mathbb{P}^m$, then formula (2.1b) reduces to a classical formula of Todd.

The projectivized contangent space $\mathbb{P}^*\mathbb{P}^m$ embeds canonically as a subvariety of $\mathbb{P}^m \times \mathbb{P}^{m_*}$. Let ω and $\tilde{\omega}$ denote the Kähler forms of $\mathbb{P}^m$ and $\mathbb{P}^{m_*}$ respectively. Then

$$\zeta = \omega + \tilde{\omega},$$
$$c_*(T^*\mathbb{P}^m) = \sum(-1)^k \binom{m+1}{k} \omega^k,$$
$$\gamma_k = \sum_{j=0}^{m-k-1} (-1)^k \binom{j+k+1}{j} \omega^j \wedge \tilde{\omega}^{m-j-k-1}.$$

In particular,

$$\gamma_{-1} = \sum_{j=0}^{m} (-1)^j \omega^j \wedge \tilde{\omega}^{m-j}.$$

That this form is exact in $\mathbb{P}^*\mathbb{P}^m$ may be seen as follows. If $\mathbb{P}^m$ is equipped with its invariant Hermitian metric, then this induces an anti-holomorphic diffeomorphism $\mathbb{P}^m \simeq \mathbb{P}^{m_*}$. Thus we may identify $\mathbb{P}^m \times \mathbb{P}^m$ with $\mathbb{P}^m \times \mathbb{P}^{m_*}$. The cohomology class of $\mathbb{P}^m \times \mathbb{P}^{m_*}$ that is Poincaré dual to the diagonal $\Delta \subset \mathbb{P}^m \times \mathbb{P}^m$ is easily seen to contain the form γ_{-1}. Since the image of $\mathbb{P}^*\mathbb{P}^m \hookrightarrow \mathbb{P}^m \times \mathbb{P}^{m_*} \to \mathbb{P}^m \times \mathbb{P}^m$ is obviously disjoint from Δ, the claim follows.

It is worthwhile to note that γ_{-1} is not identically zero if $m > 1$.

2.2. The currents of (2.1b) may also be written, using (1.4a), as $\pi_\#(N^*(X) \llcorner \beta \wedge \gamma_k)$. Thus it is useful to compute

$$\begin{aligned} d\left(\beta \wedge \sum(-1)^{m-k}\gamma_k\right) &= \zeta \wedge \sum(-1)^{m-k}\gamma_k \\ &= \zeta \wedge \left(\sum_{k\geq 0} \zeta^k\right) \wedge c_*(T^*M) \\ &= \left(\sum_{k\geq 0} \zeta^k\right) \wedge c_*(T^*M) - c_*(T^*M) \\ &= \sum(-1)^{m-k}\gamma_k - c_*(T^*M). \end{aligned}$$

Comparing elements of the same degrees, we obtain

$$d(\beta \wedge \gamma_k) = \zeta \wedge \gamma_k \equiv -\gamma_{k-1} \bmod \pi_M^*(\mathscr{E}^*(M)), \quad k = 1, \cdots, m-1, \tag{2.2a}$$

and

(2.2b) $$d(\beta \wedge \gamma_0) = -\gamma_{-1} + (-1)^{m+1} c_m(T^*M) = -\gamma_{-1} - c_m(TM).$$

Choosing $c_m(TM)$ to be the Gauss-Bonnet-Chern form Ω and noting that

$$\gamma_{-1} = (-1)^{m+1}(\zeta^m + c_1(T^*M) \wedge \zeta^{m+1} + \cdots + c_m(T^*M)) = (-1)^{m+1}\, d\tau$$

by (2.1a), we have

(2.2c) $$d(\beta \wedge \gamma_0) = \zeta \wedge \gamma_0 = -\Omega + (-1)^m\, d\tau.$$

It is important to recall here that τ is the pullback of a form from $\mathbb{P}^*M$. Finally, we use this last relation to identify the transgression

(2.2d) $$\Pi := \beta \wedge \gamma_0 + (-1)^{m+1}\tau.$$

2.3. Now we can state our main result.

Theorem. *Given a compact analytic variety X embedded in a smooth analytic manifold M, the Chern-MacPherson homology classes of X are represented by the currents $\pi_{M\#}(\mathbb{P}N^*(X) \llcorner \gamma_k)$, $k = 0, \cdots, m-1$.*

This theorem will be proved by direct verification of the axioms C1–3 of the Introduction. We have already shown in (2.1b) that C1 holds, and the final remark of 1.1 implies C2. To establish C3 we prove the following.

Let N be a second complex manifold, and $f\colon X \to N$ an analytic morphism with $\dim f(X) < \dim N$. We denote the canonical forms on $\mathbb{P}^*M$ and S^*M by Greek letters $(\beta, \gamma, \zeta, \cdots)$ as above, and use the corresponding Roman letters $(b, g, z, \cdots)$ for the corresponding forms on $\mathbb{P}^*N$ and S^*N.

Theorem. *There are holomorphic varieties $Y_1, \cdots, Y_r \subset N$ and integers $n_1, \cdots, n_r$ such that*

(2.3a) $$\chi f^{-1}(q) = \sum n_i 1_{Y_i}(q) \quad \textit{for each } q \in N,$$

and

(2.3b) *for each $k \geq 0$, the current*

$$f_\# \pi_{M\#}(\mathbb{P}N^*(X) \llcorner \gamma_k) - \sum n_i \pi_{N\#}(\mathbb{P}N^*(Y_i) \llcorner g_k)$$

is a boundary in $f(X)$.

Remark. Our expression for the Chern-MacPherson classes has the same form as those of [16] and [2], that is, as a weighted sum of Chern-Mather classes of strata. To see this, simply recall the decomposition (1.5a) and note that the Chern-Mather class of a variety X with top stratum S_0 is the homology class of $\sum_k \pi_\#(\mathbb{P}N^*(S_0) \llcorner \gamma_k)$.

3. Proof of (2.3b)

The graph $\Gamma \subseteq M \times N$ of f is a holomorphic subvariety, hence its projectivized conormal cycle $\mathbb{P}N^*(\Gamma) \in I_{2m+2n-2}(\mathbb{P}^*(M \times N))$ is a holomorphic current supported on the subvariety $\nu_\Gamma \subset \mathbb{P}^*(M \times N)$. Let $\mathcal{S}$ be a Whitney stratification of X as in 1.4, with $\mathbb{P}N^*(X) = \sum_{S \in \mathcal{S}} a(S)\mathbb{P}N^*(S)$, $a(S) \in \mathbb{Z}$. By the corollary of 1.6, we have also

$$\text{(3.1z)} \qquad \mathbb{P}N^*(\Gamma) = \sum_{S \in \mathcal{S}} a(S)\mathbb{P}N^*((\mathrm{id}, f)(S)).$$

Observe that there are naturally embedded copies of $M \times \mathbb{P}^*N$ and $\mathbb{P}^*M \times N$ within $\mathbb{P}^*(M \times N)$, and canonical "rational mappings"

$$\pi_{\mathbb{P}^*M}\colon \mathbb{P}^*(M \times N) - M \times \mathbb{P}^*N \to \mathbb{P}^*M,$$
$$\pi_{\mathbb{P}^*N}\colon \mathbb{P}^*(M \times N) - \mathbb{P}^*M \times N \to \mathbb{P}^*N.$$

Let $\widetilde{P}$ denote the blow-up of $\mathbb{P}^*(M \times N)$ over the union of the subvarieties $M \times \mathbb{P}^*N$ and $\mathbb{P}^*M \times N$. Then $\widetilde{P}$ is a smooth complex manifold, and there are projections of *all* of $\widetilde{P}$ to $\mathbb{P}^*M$ and $\mathbb{P}^*N$ such that the following diagram commutes, where σ is the blowing-down map:

$$\text{(3.1a)} \qquad \begin{array}{ccccc} & & \widetilde{P} & & \\ & \swarrow & \downarrow \sigma & \searrow & \\ \mathbb{P}^*M & \leftarrow - - & \mathbb{P}^*(M \times N) & - - \rightarrow & \mathbb{P}^*N \end{array}$$

The subvarieties $E_0 := \sigma^{-1}(\mathbb{P}^*M \times N)$, $E_1 := \sigma^{-1}(M \times \mathbb{P}^*N)$ are complex hypersurfaces in $\widetilde{P}$, each isomorphic to $\mathbb{P}^*M \times \mathbb{P}^*N$ via the projections of (3.1a).

The set $\sigma^{-1}(\nu_\Gamma)$ is a subvariety of $\widetilde{P}$. Thus we may choose coefficients for its irreducible components to construct a holomorphic current $\mathbb{P}\widetilde{N}^*(\Gamma) \in I_{2m+2n-2}(\widetilde{P})$ such that

$$\text{(3.1b)} \qquad \sigma_\# \mathbb{P}\widetilde{N}^*(\Gamma) = \mathbb{P}N^*(\Gamma), \qquad \mathbb{P}\widetilde{N}^*(\Gamma) \llcorner (E_0 \cup E_1) = 0.$$

It is clear that these conditions determine $\mathbb{P}\widetilde{N}^*(\Gamma)$; this current is called the *proper transform* of $\mathbb{P}N^*(\Gamma)$.

Observe finally that the pullbacks to $\widetilde{P}$ of the forms γ_i, g_j are everywhere-defined, smooth, and closed.

3.2. Let h_0, h_1 be Hermitian metric functions on T^*M, T^*N respectively (i.e., $h_i(\xi) = \|\xi\|^2$). Inside $T^*(M \times N) \cong T^*M \times T^*N$ there are canonically embedded copies of $T^*M \times N = h_1^{-1}(0)$ and $M \times T^*N = h_0^{-1}(0)$ corresponding to the inclusions $\mathbb{P}^*M \times N$, $M \times \mathbb{P}^*N \hookrightarrow \mathbb{P}^*(M \times N)$ above. It will be convenient to identify the deleted spaces

$$\begin{aligned}
\mathbb{P}^*(M \times N)^0 &:= \mathbb{P}^*(M \times N) - (\mathbb{P}^*M \times N \cup M \times \mathbb{P}^*N), \\
S^*(M \times N)^0 &:= \pi_{\mathbb{P}^*(M\times N)}^{-1}(\mathbb{P}^*(M \times N^0)) \\
&= S^*(M \times N) - (S^*M \times N \cup M \times S^*N), \\
T^*(M \times N)^0 &:= \pi_{S^*(M\times N)}^{-1}(S^*(M \times N)^0) \\
&= T^*(M \times N) - (T^*M \times N \cup M \times T^*N).
\end{aligned}$$

It is easily seen that the ratio $h_0 \cdot h_1^{-1}$ is the pullback to $T^*(M \times N)^0$ of a positive function $\rho\colon \mathbb{P}^*(M \times N)^0 \to \mathbb{R}$. Put

$$B := (2\pi\sqrt{-1})^{-1} d^c \log \rho,$$

and observe that the pullback to $S^*(M \times N)^0$ may be expressed (cf. 1.4)

$$B = \beta - b. \tag{3.2z}$$

Thus we have the equality of the forms on $\mathbb{P}^*(M \times N)^0$

$$dB = \zeta - z. \tag{3.2a}$$

Our immediate goal is to show that the expression $\mathbb{P}N^*(\Gamma) \llcorner B$ defines a normal current, in the sense of Federer and Fleming, despite the singularity of B along $\mathbb{P}^*M \times N \cup M \times \mathbb{P}^*N$. We want also to evaluate its boundary. To accomplish these we recall that

$$\begin{aligned}
\mathbb{P}N^*(\gamma) \llcorner B &= \sigma_\#(\mathbb{P}\widetilde{N}^*(\Gamma) \llcorner B), \\
\partial(\mathbb{P}N^*(\gamma) \llcorner B) &= \sigma_\#\partial(\mathbb{P}\widetilde{N}^*(\Gamma) \llcorner B)
\end{aligned} \tag{3.2b}$$

and work in the blow-up $\widetilde{P}$. We observe that although the pullback to $\widetilde{P}$ of B remains singular, the singularity is quite mild.

3.3. Lemma. *Given any point $p \in E_0$ (resp., E_1), there is a neighborhood U of p in $\widetilde{P}$ and a holomorphic function $\varphi\colon U \to \mathbb{C}$ such that*

(a) *$d\varphi \neq 0$ in U,*

(b) *$\varphi^{-1}(0) = E_0 \cap U$ (resp., $E_1 \cap U$), and*

(c) *$\rho|\varphi|^2$ (resp. $\rho|\varphi|^{-2}$) is smooth and nowhere zero in U.*

Proof. Since this is a local statement, we can assume that $M = \mathbb{C}^m$ and $N = \mathbb{C}^n$, so that

$$\mathbb{P}^*M = \mathbb{C}^m \times \mathbb{P}^{m-1^*},$$
$$\mathbb{P}^*N = \mathbb{C}^n \times \mathbb{P}^{n-1^*},$$
$$\mathbb{P}^*(M \times N) = \mathbb{C}^{m+n} \times \mathbb{P}^{m+n-1^*},$$

where the last projective factor is naturally identified with the complex join of the previous two:

$$\mathbb{P}^{m+n-1^*} = \mathbb{P}^{m-1^*} *_{\mathbb{C}} \mathbb{P}^{n-1^*}.$$

In other words, if $[\xi_1, \cdots, \xi_m]$ and $[\eta_1, \cdots, \eta_n]$ are holomorphic homogeneous coordinates for $\mathbb{P}^{m-1^*}$ and $\mathbb{P}^{n-1^*}$ respectively, then $[\xi_1, \cdots, \xi_m; \eta_1, \cdots, \eta_n]$ are holomorphic homogeneous coordinates for $\mathbb{P}^{m+n-1^*}$.

The space $\widetilde{P}$ may be realized as $\mathbb{C}^{m+n} \times Q$, where Q is the blow-up of $\mathbb{P}^{m+n-1^*}$ over the subvarieties $\mathbb{P}^{m-1^*}$, $\mathbb{P}^{n-1^*}$. Thus

$$\begin{aligned} Q = \{([\xi;\eta], [\xi'], [\eta']) \colon \xi, \xi' \in \mathbb{C}^{m^*};\ \eta, \eta' \in \mathbb{C}^{n^*}; \\ \xi' \neq 0 \neq \eta';\ (\xi, \eta) \neq 0;\ \xi \wedge \xi' = 0 = \eta \wedge \eta'\} \\ \subset \mathbb{P}^{m+n-1^*} \times \mathbb{P}^{m-1^*} \times \mathbb{P}^{n-1^*}. \end{aligned}$$

We may assume that the point $p \in E_0 = \mathbb{C}^{m+n} \times \{([\xi;\eta], [\xi'], [\eta']) \in Q \colon \eta = 0\}$ is $(0, [1, 0, \cdots, 0; 0, \cdots, 0], [1, 0, \cdots, 0], [1, 0, \cdots, 0]) \in \mathbb{C}^{m+n} \times Q$. Then we put

$$U := \mathbb{C}^{m+n} \times \{([\xi;\eta], [\xi'], [\eta']) \in Q \colon \xi_1 \neq 0,\ \eta'_1 \neq 0\},$$

and define $\varphi \colon U \to \mathbb{C}$ by

$$\varphi(q, [\xi;\eta], [\xi'], [\eta']) := \eta_1 \xi_1^{-1}.$$

Hence conclusions (a) and (b) are immediate. To prove (c), we express the function ρ in these coordinates as

$$\rho(q, [\xi;\eta], [\xi'], [\eta']) = \langle \xi, \xi \rangle_{\pi_1(q)} \cdot \langle \eta, \eta \rangle_{\pi_2(q)}^{-1},$$

where the π_i are the projections of $\mathbb{C}^{m+n}$ into $\mathbb{C}^m$ and $\mathbb{C}^n$, and multiply this expression by $|\varphi|^2$, to obtain

$$\begin{aligned} |\varphi|^2 \rho &= \langle \xi_1^{-1}\xi, \xi_1^{-1}\xi \rangle_{\pi_1(q)} \cdot \langle \eta_1^{-1}\eta, \eta_1^{-1}\eta \rangle_{\pi_2(q)}^{-1} \\ &= \langle \xi_1^{-1}\xi, \xi_1^{-1}\xi \rangle_{\pi_1(q)} \cdot \langle \eta_1'^{-1}\eta', \eta_1'^{-1}\eta' \rangle_{\pi_2(q)}^{-1}, \end{aligned}$$

which is clearly smooth and nonzero in U. The method for $p \in E_1$ is similar. q.e.d.

From the residue theorem (Lemma 0.1) we obtain at once

Corollary A. *Given a holomorphic current* $T \in I_*(\widetilde{P})$, *such that* $\operatorname{spt} T$ *has no irreducible components contained in* $E_0 \cup E_1$, *and the expression* $T \llcorner B$ *defines a normal current in* $\widetilde{P}$, *with*

$$\partial(T \llcorner B) = T \llcorner (\zeta - z) + T \cap E_1 - T \cap E_0, \tag{3.3d}$$

where $T \cap E_i$ *is a holomorphic current supported in* E_i, $i = 0, 1$.

Note that we use this statement to *define* the expressions $T \cap E_i$.

3.4. We turn now to characterizing the last two terms of (3.3d) for $T = \mathbb{P}\widetilde{N}^*(\Gamma)$. The very last term is easy, using

Lemma. *Let* M, N *be smooth manifolds, and* $X \subset M$ *a smooth submanifold. Let* $f: X \to N$ *be a smooth map. Then the conormal bundle* $\nu^*\text{graph}(f) \subset T^*(M \times N)$ *intersects* $T^*M \times N$ *transversally.*

Proof. We may assume that $N = \mathbb{R}^n$. The change of coordinates $(x, y) \mapsto (x, y - f(x))$ takes graph(f) to graph(zero) and preserves the vertical foliation and hence also its conormal manifold $T^*M \times N$. Thus we are reduced to the case $f = 0$, which is trivial. q.e.d.

We may express $\mathbb{P}\widetilde{N}^*(\Gamma) \cap E_0$ as follows. Let $\mathscr{B}$ denote the pullback of the bundle $\mathbb{P}^*N \to N$ to $\mathbb{P}^*M \times N$, so that the total space of $\mathscr{B}$ is $\mathbb{P}^*M \times \mathbb{P}^*N \cong E_0$. For each stratum $S \in \mathscr{S}$ of X, put

$$\tilde{S} := \sigma^{-1}(\mathbb{P}v^*(\text{graph}(f|S)));$$

thus the sets $\widetilde{S}$, $S \in \mathscr{S}$, partition $\sigma^{-1}(\nu_\Gamma)$. Under the identification $E_0 \cong \mathbb{P}^*M \times \mathbb{P}^*N$ we have

$$\widetilde{S} \cap E_0 = \pi_{\mathscr{B}}^{-1}(\text{id}_*, f)\mathbb{P}\nu^*S.$$

Applying the last lemma to the $f|S$, we find that $\nu^*\text{graph}(f|S)$ intersects $T^*M \times N$ transversally, and therefore $\mathbb{P}\nu^*\text{graph}(f|S)$ intersects $\mathbb{P}^*M \times N$ transversally. It follows that $\widetilde{S}$ intersects E_0 transversally. Furthermore, by the proof of Proposition 1.6, no irreducible component of the singular set of $\overline{\mathbb{P}\nu^*S}$ is contained within $\mathbb{P}^*M \times N$, and therefore no irreducible component of the singular set of $\overline{\widetilde{S}}$ is contained within E_0. Thus the $\overline{\widetilde{S}}$, $S \in \mathscr{S}$, and the locally defined functions φ of Lemma 3.3 satisfy the hypothesis of the second part of Lemma 0.1. That lemma now gives

$$\begin{aligned}\mathbb{P}\widetilde{N}^*(\Gamma) \cap E_0 &= \sum_{s \in \mathscr{S}} n_S[\pi_{\mathscr{B}}^{-1}(\text{id}_*, f)\mathbb{P}\nu^*S] \\ &= (\text{id}_*, f)_\# \mathbb{P}N^*(X) \times_{\mathscr{B}} [\mathbb{P}^{n-1^*}]\end{aligned} \tag{3.4a}$$

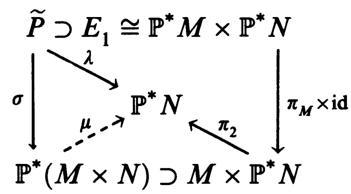
as a current in $\mathbb{P}^*M \times \mathbb{P}^*N \cong E_0$. In particular,

$$\pi_{\mathbb{P}^*M\#}((\mathbb{P}\widetilde{N}(\Gamma) \cap E_0) \llcorner g_0) = \mathbb{P}N^*(X), \tag{3.4b}$$

and

$$\pi_{\mathbb{P}^*M\#}[(\mathbb{P}\widetilde{N}^*(\Gamma) \cap E_0) \llcorner g_i] = 0, \qquad i > 0. \tag{3.4c}$$

The current $\mathbb{P}\widetilde{N}^*(\Gamma) \cap E_1$ is less clearly seen, but it will be enough to characterize it in a general way. Let $\tilde{\nu}_\Gamma \subset \widetilde{P}$ be the variety supporting $\mathbb{P}\widetilde{N}^*(\Gamma)$ and let $V_1', \cdots, V_s' \subset \widetilde{P}$ be the irreducible components of $\tilde{\nu}_\Gamma$. We consider the images of the V_i' under the maps of the commutative diagram



We claim that the images of the V_i' in $\mathbb{P}^*N$ annihilate the contact ideal A of $\mathbb{P}^*N$. For it is easily seen that

$$\pi_2^*A = \mu^*A|M \times \mathbb{P}^*N = \widehat{A}|M \times \mathbb{P}^*N,$$

where $\widehat{A}$ is the contact ideal of $\mathbb{P}^*(M \times N)$. Since each $\sigma(V_i')$ is contained in the Legendrian variety $\operatorname{spt} \mathbb{P}N^*(\Gamma) \subset \mathbb{P}^*(M \times N)$, we have

$$\mu^*(A|\lambda(V_i')) = \widehat{A}|\sigma(V_i') = 0,$$

which implies the claim.

In particular, each of these images has complex dimension $\leq n-1$, and those of dimension $n-1$ are, by definition, Legendrian subvarieties of $\mathbb{P}^*N$. Let $V_1, \cdots, V_q \subset \mathbb{P}^*N$ be these images of top dimension; we may change indices so that $V_i = \lambda(V_i')$, $i = 1, \cdots, q$. Let $\omega_i \in H_{2m-2}(\mathbb{P}^*M)$ be the homology class of the generic fiber of the map $V_i' \to V_i$, $i = 1, \cdots, q$. Pairing with the canonical form γ_0 pulled back from $\mathbb{P}^*M$, we compute

$$\begin{aligned}\pi_{\mathbb{P}^*N\#}([\mathbb{P}\widetilde{N}(\Gamma)\cap E_1]\llcorner\gamma_0) &= \pi_{\mathbb{P}^*N\#}\left[\sum_{i=1}^{s} m_i[V_i']\llcorner\gamma_0\right] \\ &\qquad (m_1, \cdots, m_s \in \mathbb{Z}) \\ &= \pi_{\mathbb{P}^*N\#}\left[\sum_{i=1}^{q} m_i[V_i']\llcorner\gamma_0\right] \\ &\qquad \text{(after comparing dimensions of the fibers with } \deg\gamma_0 = 2m-2) \\ &= \sum_{i=1}^{q} m_i\langle\omega_i, \gamma_0\rangle[V_i].\end{aligned}$$

By Proposition 1.5A, this Legendrian cycle may be written

$$\pi_{\mathbb{P}^*N\#}([\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_1]\llcorner\gamma_0) = \sum_{i=1}^{r} n_i\mathbb{P}N^*(Y_i) \tag{3.4d}$$

for some varieties $Y_1, \cdots, Y_r \subset N$ and integers $n_1, \cdots, n_r$. We have also

$$\pi_{\mathbb{P}^*N\#}[(\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_1)\llcorner\gamma_1] = 0, \qquad i>0. \tag{3.4e}$$

3.5. We are now in a position to prove (2.2b). We examine first the most straightforward case $k = 0$, which nonetheless exhibits all of the essential features of the computation:

$$\begin{aligned}\partial[\mathbb{P}N^*(\Gamma)\llcorner B\wedge\gamma_0\wedge g_0] &= \partial\sigma_\#[\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner B\wedge\gamma_0\wedge g_0] \\ &= \sigma_\#\partial[\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner B\wedge\gamma_0\wedge g_0] \\ &= \sigma_\#[\partial(\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner B)\wedge\gamma_0\wedge g_0] \\ &= \sigma_\#[\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner(\zeta - z)\wedge\gamma_0\wedge g_0] \\ &\quad + \sigma_\#((\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_1)\llcorner\gamma_0\wedge g_0) \\ &\quad - \sigma_\#((\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_0)\llcorner\gamma_0\wedge g_0).\end{aligned}$$

To compute the first term, note that

$$\begin{aligned}(\zeta - z)\wedge\gamma_0\wedge g_0 &= (\zeta\wedge\gamma_0)\wedge g_0 - \gamma_0\wedge(z\wedge g_0) \\ &= (-\Omega + (-1)^m\,d\tau)\wedge g_0 - \gamma_0\wedge(-O + (-1)^n\,dt)\end{aligned}$$

by (2.2c). Since the projections of $\mathbb{P}\widetilde{N}^*(\Gamma)$ to M and N are proper subvarieties, it follows that

$$\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner\Omega = 0 = \mathbb{P}\widetilde{N}^*(\Gamma)\llcorner O;$$

since $\mathbb{P}\widetilde{N}^*(\Gamma)$ is a cycle, the first term above becomes

$$(-1)^m\partial(\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner \tau\wedge g_0)+(-1)^{n+1}\partial(\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner \gamma_0\wedge t)=\partial R.$$

Projecting the second and third terms to M and N yields

$$\begin{aligned}\pi_{M\#}\sigma_\#((\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_0)\llcorner \gamma_0\wedge g_0)&=\pi_{M\#}(\pi_{\mathbb{P}^*M\#}(\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner g_0)\llcorner \gamma_0)\\&=\pi_{M\#}(\mathbb{P}N^*(X)\llcorner \gamma_0)\end{aligned}$$

by (3.4b) and

$$\begin{aligned}\pi_{N\#}\sigma_\#((\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_1)\llcorner \gamma_0\wedge g_0)&=\pi_{N\#}(\pi_{\mathbb{P}^*N\#}(\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner \gamma_0)\llcorner g_0)\\&=\pi_{N\#}\left(\sum n_i\mathbb{P}N^*(Y_i)\llcorner g_0\right)\end{aligned}$$

by (3.4d). Thus we have shown that there is a normal 1-dimensional current $T=\pi_{M\times N\#}[\mathbb{P}N^*(\Gamma)\llcorner B\wedge\gamma_0\wedge g_0]$, supported on the graph Γ, such that $\partial T=S_1-S_0+\partial\pi_{M\times N\#}R$, where

$$\pi_{M\#}S_0=\pi_{M\#}(\mathbb{P}N^*(\Gamma)\llcorner \gamma_0),$$

and

$$\pi_{N\#}S_1=\pi_{N\#}\left(\sum_{i=1}^r n_i\mathbb{P}N^*(Y_i)\llcorner g_0\right).$$

This is equivalent to the case $k=0$ of (2.2b).

To prove (2.2b) for $k>0$, we will show that if we put

$$T^k:=\pi_{M\times N\#}\left(\mathbb{P}N^*(\Gamma)\llcorner B\wedge\sum_{i=0}^k\gamma_i\wedge g_{k-i}\right),$$

then $\partial T^k=S_1^k-S_0^k+\partial R^k$ for some current R^k, where

$$\pi_{M\#}S_0^k=\pi_{M\#}(\mathbb{P}N^*(X)\llcorner \gamma_k)$$

and

$$\pi_{N\#}S_1^k=\pi_{N\#}\left(\sum_{i=1}^r n_i\mathbb{P}N^*(Y_1)\llcorner g_k\right).$$

Proceeding as above, we compute

$$\begin{aligned}&\partial\left[\mathbb{P}N^*(\Gamma)\llcorner B\wedge\sum_{i=0}^k\gamma_i\wedge g_{k-i}\right]\\&\quad=pN^*(\Gamma)\llcorner(\zeta-z)\wedge\sum_{i=0}^k\Gamma_i\wedge g_{k-i}\\&\qquad+\sigma_\#\left((\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_1)\llcorner\sum\gamma_i\wedge g_{k-i}\right)\\&\qquad-\sigma_\#\left((\mathbb{P}\widetilde{N}^*(\Gamma)\cap E_0)\llcorner\sum\gamma_i\wedge g_{k-i}\right).\end{aligned}$$

In view of (3.4b, c, d, e) we may take S_1^k and S_0^k to be the projections to $M \times N$ of the second and third terms, and it only remains to check that the image of the first term under $\pi_{M\times N}$ is homologous to zero.

Using (2.2a), we find that

$$\begin{aligned}(\zeta - z)\wedge\sum_{i=0}^{k}\gamma_i\wedge g_{k-i} &= \sum_i(\zeta\wedge\gamma_i)\wedge g_{k-i} - \gamma_i(z\wedge g_{k-1}) \\ &= \sum_i -\gamma_{i-1}\wedge g_{k-i} + \gamma_i\wedge g_{k-i-1} \\ &\quad + \sum_i \varphi_i\wedge g_{k-i} - \gamma_i\wedge\psi_{k-i}\end{aligned}$$

for some "horizontal" forms $\varphi_i \in \mathscr{E}^*(M)$, $\psi_j \in \mathscr{E}^*(N)$. The first summation telescopes to

$$-\gamma_{-1}\wedge g_k + \gamma_k\wedge g_{-1} = (-1)^m d\tau\wedge g_k + (-1)^{n+1}\gamma_k\wedge dt,$$

which is exact. It remains to show that

$$\pi_{M\times N\#}[\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner\sum(\varphi_i\wedge g_{k-i} - \gamma_i\wedge\Psi_{k-i})] = 0.$$

Since the forms here are pullbacks from $\mathbb{P}^*M\times N$ or $M\times\mathbb{P}^*N$ via the projections $\lambda\colon\widetilde{P}\to\mathbb{P}^*M\times N$ and $\mu\colon\widetilde{P}\to M\times\mathbb{P}^*N$, it is enough to prove:

Lemma. $\lambda_\#\mathbb{P}\widetilde{N}^*(\Gamma) = 0$, $\mu_\#\mathbb{P}\widetilde{N}^*(\Gamma) = 0$.

Proof. The second relation follows from the dimension count:

$$\begin{aligned}\dim_{\mathbb{C}}\operatorname{spt}\mathbb{P}\widetilde{N}^*(\gamma) &= m+n-1, \\ \dim_{\mathbb{C}}\mu(\operatorname{spt}\mathbb{P}\widetilde{N}^*(\Gamma)) &\le \dim X + \dim\mathbb{P}^{n-1^*} < m+n-1.\end{aligned}$$

To prove the first relation, let $x\in X$ be given and let $x\in S\in\mathscr{S}$. Put $p := (x, f(x))\in\Gamma$. Then the conormal space $\nu_p^*\operatorname{graph}(f|S)$ at p intersects $\{0\}\times T_{f(x)}^*N$ nontrivially, namely in the subspace $\{0\}\times[f_*(T_xS)]^\perp$. It follows that the projection of $\nu_p^*\operatorname{graph}(f|S)$ to $T_x^*M\times\{0\}$ decreases dimension, and therefore so does the projection of $\mathbb{P}\nu^*\operatorname{graph}(f|S) - M\times\mathbb{P}^*N$ to $\mathbb{P}^*M\times N$. Applying this to each stratum, the support theorem gives

$$\begin{aligned}\lambda_\#\mathbb{P}\widetilde{N}(\Gamma) &= \lambda_\#(\mathbb{P}\widetilde{N}^*(\Gamma)\llcorner(\widetilde{P} - (E_0\cup E_1))) \\ &= \pi_{\mathbb{P}^*M\times N\#}(\mathbb{P}N^*(\Gamma)\llcorner\mathbb{P}^*(M\times N)^0) \\ &= 0.\end{aligned}$$

4. Proof of (2.3a)

4.1. The plan is as follows. Let $q \in N$ be any given point, and let $\delta : N \to [0, \infty)$ be the distance from q in N with respect to some analytic Hermitian metric. For $s > 0$, we consider the semianalytic set

$$\Gamma_s : = \{(x, y) \in M \times N : \delta(y) \leq s\} \cap \Gamma$$

and its conormal cycle $N^*(\Gamma_s) \in I_{2(m+n)-1}(S^*(M \times N))$. Let Π and P be the transgression forms

$$\Pi := \beta \wedge \gamma_0 + (-1)^{m+1}\tau,$$
$$P := b \wedge g_0 + (-1)^{n+1} t$$

on S^*M and S^*N respectively. Pulling these back to $S^*(M \times N)^0$ via the projections of this space to S^*M and S^*N, Π and P may be regarded as singular forms on $S^*(M \times N)$. We will prove that the expression

(4.1a) $$N^*(\Gamma_s) \llcorner P \wedge \Pi$$

defines a normal current of dimension 1 in $S^*(M \times N)$, with

(4.1b) $$\partial[N^*(\Gamma_s) \llcorner P \wedge \Pi] = \Sigma_1 - \Sigma_0,$$

where the 0-currents (i.e. measures) Σ_i satisfy

(4.1c) $$\begin{aligned} \pi_{M\#}\Sigma_0 &= \pi_{M\#}(N^*(X_s) \llcorner \Pi), \\ \pi_{N\#}\Sigma_1 &= \pi_{N\#}\left(\sum_{i=1}^{r} n_i N^*(Y_{i,s}) \llcorner P\right). \end{aligned}$$

Here we have defined

$$X_s := f^{-1}\delta^{-1}[0, s], \qquad Y_{i,s} := Y_i \cap \delta^{-1}[0, s].$$

In particular the signed measures of (4.1c) have the same total measure. Then Theorem $1.3\frac{1}{2}$ and the Gauss-Bonnet Theorem 1.2 give for small $s > 0$

$$\begin{aligned} \chi f^{-1}(q) = \chi(X_s) &= N^*(X_s)(\Pi) = \pi_{M\#}\Sigma_0(X_s) = \pi_{N\#}\Sigma_1(\delta^{-1}[0, s]) \\ &= \sum_{i=1}^{r} n_i N^*(Y_{i,s})(P) = \sum_{i=1}^{n} n_i \chi(Y_{i,s}) = \sum_{i=1}^{r} n_i 1_{Y_i}(q). \end{aligned}$$

Thus we are reduced to proving (4.1b) and (4.1c).

4.2. That (4.1a) defines a normal current follows from a blowing up construction and generalities about compact semianalytic sets. Let $\widetilde{S}$ be the real oriented blowup of $S^*(M \times N)$ over the canonically embedded copies of $S^*M \times N$ and $M \times S^*N$. Then we have the commutative diagram:

$$\begin{array}{ccccc} & & \widetilde{S} & & \\ & \swarrow & \downarrow \bar{\sigma} & \searrow & \\ S^*M & \longleftarrow & S^*(M \times N) & \longrightarrow & S^*N \end{array} \tag{4.2a}$$

Locally, $S^*(M \times N)$ is just $\mathbb{R}^{2m+2n} \times S^{2m+2n-1}$, and $\widetilde{S} = \mathbb{R}^{2m+2n} \times (S^{2m-1} \times S^{2n-1} \times [0, 1])$, with the projection $\bar{\sigma}: \widetilde{S} \to S^*(M \times N)$ induced by the map

$$S^{2m-1} \times S^{2n-1} \times [0, 1] \to S^{2m+2n-1},$$

$$\begin{aligned}&(x_1, \cdots, x_{2m}; y, \cdots, y_{2n}; t)\\ &\quad \mapsto (t^2 + (1-t)^2)^{-1/2}(tx_1, \cdots, tx_{2m}; (1-t)y_1, \cdots, (1-t)y_{2n}).\end{aligned}$$

It is clear that $\widetilde{S}$ is a real analytic manifold with boundary, where $\partial\widetilde{S}$ is analytically diffeomorphic to two copies of $S^*M \times S^*N$ via the projections of (4.2a). These boundary components may be identified as

$$F_0 := \bar{\sigma}^{-1}(S^*M \times N), \qquad F_1 := \bar{\sigma}^{-1}(M \times S^*N).$$

By results of Hardt [13], the compact semianalytic subset $\bar{\sigma}^{-1}(\operatorname{spt} N^*(\Gamma_s)) \subset \widetilde{S}$ has finite $(2m+2n-1)$-dimensional Hausdorff measure (with respect to any Riemannian metric on $\widetilde{S}$). Assigning multiplicities appropriately, it follows that there is a unique semianalytic integral current $\widetilde{N}^*(\Gamma_s) \in I_{2m+2n-1}(\widetilde{S})$ such that

$$\bar{\sigma}_\# \widetilde{N}^*(\Gamma_s) = N^*(\Gamma_s), \qquad \widetilde{N}^*(\Gamma_s) \llcorner \partial\widetilde{S} = 0,$$

and

$$\operatorname{spt} \partial\widetilde{N}^*(\Gamma_s) \subset \partial\widetilde{S} = F_0 \cup F_1.$$

Since the pullbacks to $\widetilde{S}$ of Π of P are *smooth* forms, we find that

$$N^*(\Gamma_s) \llcorner \Pi \wedge P = \bar{\sigma}_\#(\widetilde{N}^*(\Gamma_s) \llcorner \Pi \wedge P) \tag{4.2b}$$

is a well-defined normal current. Moreover,

$$\begin{aligned}\partial(\widetilde{N}^*(\Gamma_s) \llcorner \Pi) &= \partial\widetilde{N}^*(\Gamma_s) \llcorner \Pi \pm \widetilde{N}^*(\Gamma_s) \llcorner \Omega \\ &= \partial\widetilde{N}^*(\Gamma_s) \llcorner \Pi\end{aligned} \tag{4.2c}$$

since, for any $\varphi \in \mathscr{D}^*(\widetilde{S})$,

$$\begin{aligned}(\widetilde{N}^*(\Gamma_s) \llcorner \Omega)(\varphi) &= \widetilde{N}^*(\Gamma_s)(\Omega \wedge \varphi) \\ &= \pm\, \pi_{M\#}(\widetilde{N}^*(\Gamma_s) \llcorner \varphi)(\Omega) \\ &= 0\end{aligned}$$

in view of the fact that $X = \pi_M(\Gamma)$ is a proper subvariety of M. Similarly,

$$\partial(\widetilde{N}^*(\Gamma_s) \llcorner P) = \partial \widetilde{N}^*(\Gamma_s) \llcorner P, \tag{4.2d}$$

and so putting, for $0 < s \leq \infty$,

$$\widetilde{N}^*(\Gamma_s) \cap F_1 := \partial \widetilde{N}^*(\Gamma_s) \llcorner F_1,$$
$$\widetilde{N}^*(\Gamma_s) \cap F_0 := \partial \widetilde{N}^*(\Gamma_s) \llcorner F_0$$

we have

$$\begin{aligned}\partial(\widetilde{N}^*(\Gamma_s) \llcorner \Pi \wedge P) &= [\widetilde{N}^*(\Gamma_s) \cap F_1 + \widetilde{N}^*(\Gamma_s) \cap F_0] \llcorner \Pi \wedge P \\ &= ((\widetilde{N}^*(\Gamma_s) \cap F_1) \llcorner \Pi) \llcorner P \\ &\quad - ((\widetilde{N}^*(\Gamma_s) \cap F_0) \llcorner P) \llcorner \Pi.\end{aligned} \tag{4.2e}$$

Since $(\widetilde{N}^*(\Gamma_s) \cap F_1) \llcorner \Omega = 0 = (\widetilde{N}^*(\Gamma_s) \cap F_0) \llcorner O$, the currents $(\widetilde{N}^*(\Gamma_s) \cap F_1) \llcorner \Pi$ and $(\widetilde{N}^*(\Gamma_s) \cap F_0 \llcorner P$ are closed. We will use the Proposition 1.3C, with $\varphi = \delta \circ \pi_N$, to establish for small s that

$$\begin{aligned}\pi_{S^*M\#}((\widetilde{N}^*(\Gamma_s) \cap F_0) \llcorner P) &= N^*(X_s), \\ \pi_{S^*N\#}((\widetilde{N}^*(\Gamma_s) \cap F_1) \llcorner \Pi) &= \sum_{i=1}^{r} n_i N^*(Y_{i,s}).\end{aligned} \tag{4.2f}$$

Taking

$$\Sigma_0 := \pi_{M \times N\#}((\widetilde{N}^*(\Gamma_s) \cap F_0) \llcorner P \wedge \Pi),$$
$$\Sigma_1 := \pi_{M \times N\#}((\widetilde{N}^*(\Gamma_s) \cap F_1) \llcorner \Pi \wedge P),$$

we thus prove (4.1b, c).

4.3. Put $\Gamma_s^0 := \pi_2^{-1}\delta^{-1}[0, s) \cap \Gamma$, and define X_s^0, $Y_{i,s}^0$ similarly. Since conormal cycles are locally determined, to show that the currents of (4.2f) agree over X_s^0 and $\bigcup_i Y_{i,s}^0$ it is enough to prove

$$\begin{aligned}\pi_{S^*M\#}[(\widetilde{N}^*(\Gamma) \cap F_0) \llcorner P] &= N^*(X), \\ \pi_{S^*N\#}[(\widetilde{N}^*(\Gamma) \cap F_1) \llcorner \Pi] &= \sum_{i=1}^{r} n_i N^*(Y_i).\end{aligned} \tag{4.3a}$$

To this end, let λ and μ denote the compositions

$$\lambda \colon \widetilde{S} \to S^*M \times S^*N \to \mathbb{P}^*M \times S^*N,$$
$$\mu \colon \widetilde{S} \to S^*M \times S^*N \to S^*M \times \mathbb{P}^*N.$$

Lemma. $\lambda_\# \widetilde{N}^*(\Gamma) = 0$, $\mu_\# \widetilde{N}^*(\Gamma) = 0$.

Proof. Given $x \in S \in \mathscr{S}$, observe that the full conormal space $\nu^*_{(x, f(x))}\mathrm{graph}(f|S)$ contains as a direct summand the complex vector space $\nu^*_x S \times \{0\}$. Then the projection of $S^*(M \times N)^0$ into $\mathbb{P}^*M \times S^*N$ decreases the dimension of the conormal ray bundle of $\mathrm{graph}(f|S)$. Applying this to each stratum and using the support theorem we obtain

$$\begin{aligned}\lambda_\# \widetilde{N}^*(\Gamma) &= \lambda_\#(\widetilde{N}^*(\Gamma) \llcorner (\widetilde{S} - (F_0 \cup F_1))) \\ &= \pi_{\mathbb{P}^*M \times S^*N\#}(N^*(\Gamma) \llcorner S^*(M \times N)^\circ) \\ &= 0.\end{aligned}$$

The second relation has a similar proof.

Since τ and t are pullback from $\mathbb{P}^*M$ and $\mathbb{P}^*N$ respectively, it follows that

$$\pi_{S^*M\#}(N^*(\Gamma) \llcorner t) = 0, \qquad \pi_{S^*N\#}(N^*(\Gamma) \llcorner \tau) = 0. \tag{4.3b}$$

Thus we are reduced to proving (4.3a) with P and Π replaced by $b \wedge g_0$ and $\beta \wedge \gamma_0$.

Since β is smooth in the neighborhood of F_0, we have

$$\begin{aligned}\partial(N^*(\Gamma) \llcorner b \wedge g_0) \llcorner \overline{\sigma}(F_0) &= \partial(N^*(\Gamma) \llcorner (b - \beta) \wedge g_0) \llcorner \overline{\sigma}(F_0) \\ &= -\partial(N^*(\Gamma) \llcorner B \wedge g_0) \llcorner \overline{\sigma}(F_0) \quad \text{by (3.2z)} \\ &= -\partial((\mathbb{P}N^*(\Gamma) \times_{\mathscr{H}} [S^1]) \llcorner B \wedge g_0) \llcorner \overline{\sigma}(F_0) \\ &\quad (\text{where } \mathscr{H} \text{ is the Hopf bundle } S^*(M \times N) \to \mathbb{P}^*(M \times N)) \\ &= -[\partial(\mathbb{P}N^*(\Gamma) \llcorner B \wedge g_0) \times_{\mathscr{H}} [S^1]] \llcorner \overline{\sigma}(F_0) \\ &= -[\partial(\mathbb{P}N^*(\Gamma) \llcorner B \wedge g_0) \llcorner \sigma(E_0)] \times_{\mathscr{H}} [S^1].\end{aligned}$$

The restriction $\mathscr{H}$ to $\sigma(E_0) \cong \mathbb{P}^*M \times N$ is identical to the pullback of the Hopf bundle $S^*M \to \mathbb{P}^*M$, which we denote by $\mathscr{H}_0$. Therefore, using (3.3d) and (3.4b) we obtain the image of this current in S^*M:

$$\begin{aligned}&\pi_{S^*M\#}[\partial(N^*(\Gamma) \llcorner b \wedge g_0) \llcorner \overline{\sigma}(F_0) \\ &\qquad = -\{\pi_{\mathbb{P}^*M\#}[\partial(\mathbb{P}N^*(\Gamma) \llcorner B \wedge g_0) \llcorner \sigma(E_0)]\} \times_{\mathscr{H}_0} [S^1] \\ &\qquad = \mathbb{P}N^*(X) \times_{\mathscr{H}_0} [S^1] = N^*(X),\end{aligned}$$

which is the first relation of (4.3a). A similar computation by (3.4d) in place of (3.4b) gives the second relation.

4.4. To complete the proof, we may suppose that the stratification $\mathscr{S}$ of X is such that the family $\{f(S) : S \in \mathscr{S}\}$ constitutes a Whitney stratification of $f(x)$. In particular, each restriction $f|S$, $S \in \mathscr{S}$, has constant rank. By Proposition 1.3b, we have

$$\begin{aligned}&\operatorname{spt} \vec{N}^*(\Gamma_s) \cap \pi_2^{-1}\delta^{-1}(s)\\ &\quad \subset \bigcup_{S\in\mathscr{S}} \{(\xi, \eta) \in T^*(M \times N) \cong T^*M \times T^*N\colon \langle \xi, v\rangle + \langle \eta, f_* v\rangle \le 0\\ &\qquad\qquad \text{for all } v \in \operatorname{Tan}(S \cap f^{-1}(B(q, s)), \pi_M(\xi))\},\end{aligned}$$

whence

$$\begin{aligned}&\operatorname{spt} \pi_{S^*M\#}((\widetilde{N}^*(\Gamma_s) \cap F_0) \llcorner P) \cap f^{-1}\delta^{-1}(s)\\ &\quad \subset \pi_{S^*M}(\operatorname{spt} N^*(\Gamma_s) \cap (S^*M \times N)) \cap f^{-1}\delta^{-1}(s)\\ &\quad \subset \bigcup_{S\in\mathscr{S}} \{\xi \in S^*M\colon \pi_M(\xi) \in S \cap f^{-1}\delta^{-1}(s),\ \langle \xi, v\rangle \le 0\\ &\qquad\qquad \text{for all } v \in \operatorname{Tan}(S \cap f^{-1}\delta^{-1}[0, s],\ \pi_M(\xi))\},\end{aligned}$$

and the first part of (4.2f) follows from the Corollary to Proposition 1.3C. Meanwhile,

$$\begin{aligned}&\operatorname{spt} \pi_{S^*N}((\widetilde{N}^*(\Gamma_s) \cap F_1) \llcorner \Pi) \cap \delta^{-1}(s)\\ &\quad \subset \pi_{S^*N}(\operatorname{spt} N^*(\Gamma_s) \cap (M \times S^*N)) \cap \delta^{-1}(s)\\ &\quad \subset \bigcup_{S\in\mathscr{S}} \{\eta \in S^*N\colon \pi_N(\eta) \in \delta^{-1}(s) \cap f(S),\ \langle \eta, w\rangle \le 0\\ &\qquad\qquad \text{for all } w \in f_*\operatorname{Tan}(S \cap f^{-1}\delta^{-1}[0, s], x),\ x \in f^{-1}\pi_N(\eta)\}.\end{aligned}$$

Furthermore, the hypothesis on $\mathscr{S}$ implies that

$$f_*\operatorname{Tan}(S \cap f^{-1}\delta^{-1}[0, s], x) = \operatorname{Tan}(f(S) \cap \delta^{-1}[0, s],\ f(x))$$

for each $x \in S \cap f^{-1}\delta^{-1}(s)$ and each $S \in \mathscr{S}$. Hence the second part of (4.2f) follows from Proposition 1.3C.

5. Concluding remarks

The construction of MacPherson is founded on his definition of the "local Euler obstruction" $Eu_p(V)$ of a point p in a variety V. The approach of the present article suggests a geometric expression for $Eu_p(V)$ as follows. We may suppose that V is a subvariety of $\mathbb{C}^n$. Let κ_0 denote the pullback to $S^*\mathbb{C}^n \cong \mathbb{C}^n \times S^{2n-1}$ of the volume form on S^{2n-1} ($\operatorname{vol}(S^{2n-1}) = 1$). It then appears that

(5a) $$Eu_p(V) = \lim_{\varepsilon\to 0}(N^*(V \cap \overline{B}(p, \varepsilon)) \llcorner V^0 \cap S(p, \varepsilon))(\kappa_0),$$

where V^0 is the smooth locus of V, and $\overline{B}(p, \varepsilon)$, $S(p, \varepsilon)$ are, respectively, the closed ball of radius ε about p and its boundary sphere. In other words, $Eu_p(V)$ should in the limit be equal to the "Gauss curvature" of $V \cap \overline{B}(p, \varepsilon)$ within $V^\circ \cap S(p, \varepsilon)$. In what follows we give two loosely connected remarks supporting this assertion.

5.1. Let $k+1 := \dim V$. For each $\varepsilon > 0$, there is a complex k-plane bundle $\mathscr{B}_\varepsilon$ over $V^\circ \cap S(p, \varepsilon)$ with fiber over x equal to $T_x V \cap (p-x)^\perp$, where $\perp$ is computed via the Hermitian inner product. Of course there is a natural map of this bundle to the tautological bundle $\mathscr{T}$ over the Grassmannian $G(n, k)$; let $c_k(\mathscr{B}_\varepsilon)$ denote the pullback to $V^\circ \cap S(p, \varepsilon)$ of the kth Chern form $c_k(\mathscr{T})$. A formula of Loeser [15] asserts that

$$\text{(5b)} \qquad Eu_p(V) = \lim_{\varepsilon \to 0} \frac{1}{2\pi\sqrt{-1}} \int_{V^\circ \cap S(p, \varepsilon)} c_k(\mathscr{B}_\varepsilon) \wedge (d + d^c) \log|z-p|^2.$$

The connection with formula (5a) may be made via the following formula for $c_k(\mathscr{T})$. Let $\mathscr{S}$ denote the $(2n-2k-1)$-sphere bundle over $G(n, k)$ with fiber $S^{2n-1} \cap \lambda^\perp$ over $\lambda \in G(n, k)$ and the total space E_φ. Let $t: E_{\mathscr{S}} \to s^{2n-1}$ be the tautological map, and let π_* be fiber integration in $\mathscr{S}$. Then

$$\text{(5c)} \qquad c_k(\mathscr{T}) = \pi_* t^* \kappa_0.$$

5.2. The construction of MacPherson proceeds by defining an automorphism T of the free abelian group generated by the strata of the given singular variety X, and using the local Euler obstruction. The characteristic homology classes can then be obtained by weighting curvature integrals over the strata of X according to a recipe involving T^{-1} (cf. Shifrin [19]). The use of the inverse corresponds in our picture to the condition $(p \in X)$:

$$\begin{aligned}
1 &= \lim_{\varepsilon \to 0} \chi(X \cap \overline{B}(p, \varepsilon)) \\
&= \lim_{\varepsilon \to 0} N^*(X \cap \overline{B}(p, \varepsilon))(\kappa_0) \\
&= (N^*(X) \llcorner \pi^{-1}(p))(\kappa_0) \\
&\quad + \lim_{\varepsilon \to 0} (N^*(X \cap \overline{B}(p, \varepsilon)) \llcorner S(p, \varepsilon))(\kappa_0) \\
&= (N^*(X) \llcorner \pi^{-1}(p))(\kappa_0) \\
&\quad + \sum_{S \in \mathscr{S}} \lim_{\varepsilon \to 0} (N^*(X \cap \overline{B}(p, \varepsilon)) \llcorner (S(p, \varepsilon) \cap S))(\kappa_0) \\
&= n_p + \sum_{S \in \mathscr{S}} n_s Eu_p(\overline{S}),
\end{aligned}$$

if we use (5a), where $\mathscr{S}$ is a Whitney stratification of X containing the singleton $\{p\}$, and

$$N^*(X) = \sum n_s N^*(S).$$

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