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SYMPLECTIC TOPOLOGY ON ALGEBRAIC 3-FOLDS

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Dedicated to the memory of Andreas Floer

1. Introduction

In [18], the author introduced two Donaldson-type invariants, Φ and $\tilde{\Phi}$, based on Gromov's remarkable theory of pseudoholomorphic curves in a symplectic manifold V. Roughly speaking, ϕ is based on counting the number of holomorphic spheres in V, and $\tilde{\phi}$ is based on counting the perturbed holomorphic maps from S^2 to V. A major difference between the two invariants is that $\tilde{\Phi}$ takes into account multiple cover maps [8], but Φ does not. It turns out that $\tilde{\Phi}$ is the invariant used in topological σ models in mathematical physics. There is a remarkable mirror symmetry phenomenon among Calabi-Yau 3-folds relating this invariant to the variation of Hodge structures of its mirror. But we shall not say anything more about this phenomenon here. Instead, we refer the reader to [24], [14]. Here we deal with a different type of application, primarily for Φ. We should point out that various simple forms of these two invariants have been used by Gromov [5] and McDuff [10], [12].

Before we give the definition of Φ, recall that a symplectic manifold (V, ω) is semipositive if for any A in the image of Hurewicz map $\pi_2(V) \to H_2(V, \mathbb{Z})$, $\omega(A) > 0$ implies that $c_1(V)A \geq 0$. Now we give the definition of Φ, following the notation in [18].

Let $\Omega(V)$ be the oriented bordism group of V.

Definition. Let (V, ω) be a symplectic manifold and $A \in H_2(V, \mathbb{Z})$ with $c_1(V)A > 0$. Furthermore, if $\dim V \geq 8$, suppose that (V, ω) is semipositive. Choose a generic tamed almost complex structure J on V. For any $\alpha_1, \cdots, \alpha_k \in \Omega(V)$ such that $\deg \alpha_i \leq 2n-2$ and $\sum_i (2n - \deg \alpha_i - 2) = 2c_1(V)A + 2n - 6$, choose a representative of α_i, still denoted by α_i. We can define an integer $\Phi_{(A, J, \omega)}(\alpha_1, \cdots, \alpha_k)$ as follows. First notice the following:

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(1) There are only finitely many unparameterized, nonmultiple cover J-spheres in $\mathcal{M}^*_{(A,J)}$ intersecting $\alpha_1, \cdots, \alpha_k$.

(2) For each such J-sphere f, we can associate a multiplicity $m(f)$. Then, we define

$$\Phi_{(A,J,\omega)}(\alpha_1, \cdots, \alpha_k) = \sum m(f).$$

$\Phi_{(A,J,\omega)}(\alpha_1, \cdots, \alpha_k)$ is independent of the choice of representative of α_i, multilinear with respect to the group structure of $\Omega(V)$, and trivial with respect to the module structure of $\Omega(V)$ over the Thom bordism group. Furthermore, it is independent of J, hence an invariant of (V, ω, A), and so we denote it simply by $\Phi_{(A,\omega)}$. Furthermore, if ω_t is a path of symplectic structures, satisfying the semipositive hypothesis when $\dim V \geq 8$, then $\Phi_{(A,\omega_0)} = \Phi_{(A,\omega_1)}$.

To put this invariant into perspective, consider the classification question of symplectic structures. Let $\overline{\mathcal{W}}$ be the space of all symplectic structures on V. Then $\overline{\mathcal{W}}$ is an open set in the space of closed 2-forms on V, and the orientation preserving diffeomorphism group $\mathrm{Diff}_+ V$ acts on $\overline{\mathcal{W}}$. By considering the moduli of symplectic structures $\mathcal{W} = \overline{\mathcal{W}}/\mathrm{Diff}_+ V$, the classification of symplectic manifolds can be split into two problems: (1) distinguish the different components and (2) study the structure of each component. So far all the works on classification had concentrated on problem (2). The period defines a map P from $\mathcal{W}_0 = \overline{\mathcal{W}}/\mathrm{Diff}_0$ to $H^2(V, \mathbb{R})$, where Diff_0 is the group of diffeomorphisms inducing the identity in $H^2(V, \mathbb{R})$. Moser's theorem implies that the preimage of a point $P^{-1}(A)$ is discrete. McDuff's examples [8] demonstrated that $P^{-1}(A)$ can be infinite even when restricted to a single component of $\mathcal{W}_0$. So far there has been no work on problem (1). This is the main topic of this paper. Two symplectic structures are said to be *deformation equivalent* if and only if they are in the same component of $\mathcal{W}$. From this point of view, the period is not very relevant since it could change within a component. One possible way to use the period map is to study its image in $H_2(v, \mathbb{R})$. Then one can try to show that up to automorphism of $H_2(V, \mathbb{R})$, the image has two disconnected components. Indeed, Ono [15] gave an alternative proof of a pair of the author's examples, namely (Barlow surface) $\times S^2$ and (the blowing up R_8 of $\mathbb{P}^2$ at 8-points) $\times S^2$, by following this route. But one must be very lucky to be able to use this approach. It involves studying all the symplectic forms, a task which is usually impossible. This elementary method probably works best if we have a pair of sharply contrasting examples like a semipositive symplectic manifold versus a seminegative one.

This is what happens for the Barlow surface and the rational surface R_8. The author has been told that when we blow up the Barlow surface and R_8 at two points, this elementary method no longer works. Later on we shall see that blowing up does not affect the present invariants.

For the question of deformation equivalence, the most important classical invariant is the homotopy class of tamed almost complex structures, particularly its Chern classes. Throughout this paper, we will ignore the period. The *classical invariants* only mean the differentiable structure of the underlying manifold and the homotopy class of a tamed almost complex structure. The classical invariants are invariants of the deformation class of symplectic structures. Until now it was not known if they were complete invariants. Although semipositivity is not necessarily preserved in an arbitrary symplectic deformation, our new invariant *is* an invariant of the deformation class of symplectic structures when $\dim V = 4$ or 6. We shall calculate it for some examples, and we show that the classical invariants fail to classify the deformation class of symplectic structures and that the new invariant indeed goes beyond the classical one.

Our primary interest is algebraic 3-folds, which has real dimension 6. There are two reasons. So far, the most work has concentrated on the 4-dimension. But the results are not very satisfactory. Only uniqueness results are obtained [5], [10]. In order to find nontrivial examples, we have to be able to decide the classical invariants first. In the 4-dimension, the question surrounding the classical invariants are delicate. In fact, they are exactly the questions Donaldson theory tries to answer. But in dimension 6, the classical invariants appear to be trivial. For the differential structure, there is a series of classification theorems [6], [21], [22], [27]. For the homotopy class of almost complex structures, Wall has shown that it is uniquely decided by the first Chern class of the almost complex structure.

Another deeper reason for studying algebraic 3-folds is their relation with Mori's theory of 3-folds. Recall that in the definition of Φ, we have to fix a homology class $A \in H_2(V, \mathbb{Z})$. To use this invariant effectively, it is often a delicate issue to choose an appropriate A. Mori theory suggests that one should choose A in an extremal ray. Hence it is particularly important to establish a symplectic version of Mori's extremal ray theory. This will be done in [19] and further applications of these ideas will be explored in future work. We shall not pursue the matter here.

Our next task is to calculate the invariants which we have defined. If the symplectic manifold is Kähler, then this is just a question of classical algebraic geometry. Note that because the set of all Kähler forms on a given complex manifold V is convex, all such forms are deformation equivalent.

Thus, for our purpose, it does not matter which form we consider, and we will frequently talk about a Kähler manifold as a symplectic manifold without explicitly mentioning the form involved, since we always assume that this form is Kähler. Let V be a nonminimal algebraic surface and W be a minimal one. Then V, W, $V \times \Sigma$, $W \times \Sigma$ are Kähler manifolds, where Σ is any Riemann surface. We shall show

Main Theorem. *If W is not a rational ruled surface and $p_1(W_k) \neq 0$, then $V_k \times \Sigma$ is not deformation equivalent to $W_k \times \Sigma$, where V_k, W_k are obtained by blowing-up k-different points of V, W for $k = 0, 1, 2, \cdots$.*

Combined with the results about the geography of simply connected minimal surface of general type, one can use this theorem to get a vast number of interesting examples. For instance, we can prove the following corollary.

Corollary. *For any n, there is a simply connected minimal algebraic surface of general type W such that $W \times \Sigma$ admits more than n many distinct deformation classes of symplectic structures whose tamed almost complex structures are homotopy equivalent to each other. Furthermore, for any blowing-up W_k of $W_k \times \Sigma$ also admits more than n many distinct deformation classes of symplectic structures whose tamed almost complex structures are homotopy equivalent to each other.*

A particularly interesting example is the Barlow surface B which is the only known example of a simply connected minimal surface of general type with geometric genus $p_g = 0$. It is homeomorphic to a rational surface R_8 obtained by blowing-up $\mathbb{P}^2$ at 8 points in generic position. $R_8 \times \Sigma$ and $B \times \Sigma$ have the same classical invariants, but W is minimal and R_8 is, of course, nonminimal. The theorem implies that $R_8 \otimes \Sigma$ and $B \times \Sigma$ are not deformation equivalent. R_8 and the Barlow surface have been extensively studied by gauge theorists [7], and are examples of homeomorphic 4-manifolds carrying different differential structures. After stabilizing by taking the product with Σ, they become diffeomorphic, but their exoticness can still be detected by symplectic topology. We get more such examples by choosing W to be an elliptic surface. This fascinating phenomenon will be taken up again in [20]. Note that it is still a conjecture in gauge theory that a minimal algebraic surface is not diffeomorphic to a nonminimal algebraic surface unless it is a rational ruled surface.

This paper is organized as follows: In §2, we give topological information about 6-manifolds, the proof of the main theorem is in §3, and the examples are in §4.

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2. 6-Manifolds

The classification of smooth 4-manifolds is complicated, as is demonstrated by the complexity of Donaldson theory. But it is well known that the classification of higher dimensional manifolds is much easier. There is extensive literature on either topological classification or smooth classification of simply connected 6-manifolds by Wall and other [21], [6], [27]. Here we list one such classification which is the most useful for our application.

Theorem [6]. *Two closed, 1-connected, smooth 6-manifolds X, Y with torsion-free homology are diffeomorphic if and only if there is an algebraic homomorphism $p: H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ which preserves the cup product structure $\mu: H^2 \otimes H^2 \otimes H^2 \to \mathbb{Z}$, second Stiefel-Whiteney class w_2, and first Pontryagin class p_1. Furthermore, the diffeomorphism which realizes this algebraic isomorphism is orientation preserving.*

So, the smooth classification is the same as the homotopy classification plus characteristic classes. One should note that the assumption of H^* being torsion free is essential. Otherwise, there are new invariants to be considered. We refer to [27] for more detail.

Next we consider the homotopy class of almost complex structure. There is again a satisfactory classification due to Wall.

Wall's Theorem [21]. *Let M be a smooth oriented 6-manifold. Then M has an almost complex structure iff $w_3(M) = 0$. When this is so, there is just one homotopy class of almost complex structure for each $c_1 \in H^2(M)$ whose $\bmod 2$ reduction is $w_2(M)$.*

Remark. In the case of dimension 4, homotopy classification of almost complex structures is much more complicated. For example, recently Donaldson used his Yang-Mills invariants to show that on $K3$, there is a homotopy class of nonintegrable almost complex structures with $c_1 = 0$.

3. Main Theorem

The ultimate goal for our invariant is to distinguish different symplectic manifolds which have the same classical invariants. The only known examples, as far as the author knows, are the examples constructed by McDuff [8], which are actually deformation equivalent to each other. Her examples rely on Gromov's result on the symplectic diffeomorphism group of $S^2 \times S^2$. In this sense, the invariant we define is more general. Later, we shall see that our invariant is particularly suitable for algebraic geometric techniques when we deal with examples from Kähler manifolds.

Recall that Φ is defined by choosing a generic tamed almost complex structure J. Roughly speaking, J is generic iff all the relevant moduli spaces are smooth and have the correct dimension for any homology class in $H_2(V, \mathbb{Z})$. (See the precise definition in [16].) In practice, it is impossible to verify that a given almost complex structure is generic. We have to relax the genericity condition. In proving the existence of ϕ, one discovers that we need only two conditions on J: (1) for any $f \in \widetilde{\mathcal{M}}^*_{(a,J)}$, $H^{0,1}(T_f V) = 0$; and (2) the space of cusp A-spheres are of codimension at least 2. We say that J is *A-good* if J satisfies both conditions. In general, it is harder to prove that a particular almost complex structure is A-good than to calculate the invariant itself. Usually our argument includes three parts: show (1), (2), and calculate the invariant.

Let V be a nonminimal algebraic surface and W be a minimal one. We shall prove that

Main Theorem. *If W is not a rational ruled surface and $p_1(W_k) \neq 0$, then $V_k \times \Sigma$ is not deformation equivalent to $W_k \times \Sigma$, where V_k, W_k are obtained by blowing up k-distinct points of V, w, $k = 0, 1, 2, \cdots$.*

We divide the proof into a series of lemmas.

Lemma 1. *If $A \in H_2(V_k, \mathbb{Z})$ (or $H_2(W_k, \mathbb{Z})$) is represented by an exceptional rational curve E, then $\Phi_A([E]) = -1$.*

Proof. Fix a symplectic form ω which is Kähler with respect to a complex structure. If we deform the complex structure a little bit, as in moving the blow-up points, the new complex structure is at least ω-tamed, since the tamed condition is an open condition. Now all the tamed symplectic forms on a complex manifold form a convex set. We can replace ω by a Kähler form ω' of the new complex structure. Therefore, we can assume that the blow-up points are not on the exceptional divisors. Let J be such a complex structure on V_k and J_0 be a complex structure on Σ. We claim that $J \times J_0$ is A-good.

Clearly

$$\mathcal{M}_A = \{E \times \{a\};\ \text{for } a \in \Sigma\}.$$

E has normal bundle $\mathcal{O}(-1)$ in V_k. $E \times \{a\}$ has normal bundle $N_E = \mathcal{O} + \mathcal{O}(-1)$. $H^1(\mathcal{O} + \mathcal{O}(-1)) = 0$. By the exact sequence

$$0 \to T\mathbb{P}^1 \to T(V_k \times \Sigma)\_E \to N_E \to 0,$$

$H^1(T\mathbb{P}^1) = 0$ implies that $H^1(V_k \times \Sigma)|_f) = 0$. Hence (1) is satisfied.

Next we check that there are no cusp curves and thus $J \otimes J_0$ is A-good. If $f_1+f_2+\cdots+f_k$ is a cusp A-rational curve, then $[f_1]+[f_2]+\cdots+[f_k]=[E]$. If $f\colon \mathbb{P}^1 \to V_k \times \Sigma$ is an A-rational curve, let $f_i = f_i' \times \overline{f}_i$, where $f_i'\colon \mathbb{P}^1 \to V_k$ and $\overline{f}_i\colon \mathbb{P}^1 \to \Sigma$. If $\Sigma \neq \mathbb{P}^1$, $\overline{f}_i$ must be a constant map. If $\Sigma = \mathbb{P}^1$, then $[\overline{f}_i] = n_i[\mathbb{P}^1]$ for some $n_i \geq 0$. But $\sum[\overline{f}_i] = 0$. Thus, $[\overline{f}_i] = 0$ and $\overline{f}_i$ is a constant map. So $f_i \subset V_k \times \mathrm{pt}$ for some $\mathrm{pt} \in \Sigma$. Since $f = \sum_i f_i$ is connected, $f \subset V_k \times \mathrm{pt}$ and we can view f as a cusp A-rational curve in V_k. If E appears r times in this set for $r \geq 1$, then we get a set of rational curves $g_1, \cdots, g_{k-r}$ such that $[g_1]+\cdots+[g_{k-r}] = -(r-1)[E]$. This is impossible since $\omega(g_i) > 0$ for each g_i and $\omega(E) > 0$ where ω is the Kähler form. Suppose $f_i \neq E$ for any i. Blow down V_k at f to define another Kähler manifold V'. Let $\tilde{f}_i$ be the image of f_i. Then we get cusp rational curves $\tilde{f}_1+\tilde{f}_2+\cdots+\tilde{f}_k$ representing the zero homology class, which is impossible as we just showed. Hence $J \times J_0$ is A-good.

$\mathcal{M}_A = \Sigma$ is parameterized by $a \in \Sigma$. Choose the bordism class corresponding to E. It can be represented by an embedded 2-sphere $X \times \{b\}$, where $X \subset V_k$ is transverse to E. Clearly there is only one rational curve $E \times \{b\}$ intersecting $X \times \{b\}$ with multiplicity -1. Thus $\Phi_A([E]; V \times \Sigma) = -1$.

Lemma 2. *Suppose that W is neither* $\mathbb{P}62$ *nor a ruled surface. If $\Phi_A \neq 0$ for $A \in H_2(W_k, \mathbb{Z})$ and if $A^2 = -1$, $c_1(W_k)A = 1$, then A must be an exceptional class; i.e., A is represented by one of the exceptional rational curves.*

Proof. Let $l_1, \cdots, l_k$ be the exceptional curves in W_k. Suppose that $A \neq [l_i]$ where l_i is an exceptional rational curve. Then for a generic tamed almost complex structure J' on W_k, $\mathcal{M}^*_{(A,J')}(W_k) \neq \varnothing$. Otherwise, $J' \times J_0$ is A-good on $W_k \times \Sigma$ and $\phi_A = 0$. Since $\Phi_{[l_i]} \neq 0$, $[l_i]$ is also represented by a J'-rational curve for the same reason. By McDuff's theorem [10], $A[l_i] > 0$.

Since W is minimal, by the classification theorem [13] one of the following occurs:

(i) $c_1(W)(D) \leq 0$ for any effective divisor D.
(ii) $W = \mathbb{P}^2$.
(iii) W is a ruled surface.

By the hypothesis, W has the property (i). Now we claim that there is no A-rational curve and no A-cusp rational curve which contradicts the hypotheses $\Phi_A \neq 0$. If f is an A-rational curve, blow down each l_i to obtain a rational curve $\overline{f}$ on W with $c_1(W)(\overline{f}) \geq c_1(W_k)(f) = 1$. But since W is neither $\mathbb{P}^2$ nor a ruled surface, $c_1(W)(\overline{f}) \leq 0$. This is a contradiction. Let $f = f_1 + f_2 + \cdots f_s$ be a cusp A-rational curve. Then $[f_1]+[f_2]+\cdots+[f_s] = A$. Some of the components may appear more than one time corresponding to multiply covered components and components which have the same image. We can rewrite $f = \mu_1 l_1 + \cdots + \mu_k l_k + a_1 g_1 + \cdots + a_t g_t$ where g_i is not an exceptional curve and $g_i \neq g_j$ if $i \neq j$. Moreover, $\mu_i \geq 0$ and $a_i \geq 0$. Let $\overline{g}_i$ be the image of g_i in W. Then homologically $[g_i] = [\overline{g}_i] - r_{ij}[l_j]$, where $r_{ij} = [g_i][l_j] > 0$. Note that $A[l_j] > 0$. Hence $\sum_i a_i r_{ij} - \mu_j > 0$. Then

$$c_1(W_k)(A) = c_1(W_k)(\mu_1 l_1 + \cdots + \mu_k l_k + a_1 g_1 + \cdots + a_t g_t) = 1,$$

which implies that

$$\sum_i a_i c_1(W)(\overline{g}_i) + \sum_{i,j}(\mu_j - a_i r_{ij}) = 1.$$

But $\mu_j - \sum_i a_i r_{ij} < 0$ for each j. Then $\sum_i a_i c_1(\overline{g}_i) \geq 1$. Note that $a_i \geq 0$ and $c_1(W)(\overline{g}_i) \leq 0$. This is a contradiction.

Lemma 3. *Suppose that W is an irrational ruled surface. There are at most k distinct classes $A_1, \cdots, A_k \in H_2(W_k, \mathbb{Z})$ with $A_i^2 = -1$, $c_1(W_k)A_i = 1$, $A_i A_j = 0$, and $\Phi_{A_i} \neq 0$.*

Proof. Suppose that $A \neq [l_i]$. If f is an A-rational curve, blow down l_i to get a rational curve $\overline{f}$ on W. The only rational curves on W are the fibers, and so $\overline{f}$ is a fiber. Thus f is the strict transform of a fiber $\overline{f}$. In order to have that $f^2 = -1$, one can only blow up one point on each fiber $\overline{f}$. In fact, by choosing a generic complex structure, we can assume that this is always the case.

If f is a cusp A-rational curve, we can repeat the argument in Lemma 3. We find that

$$\sum_i a_i c_1(W)(\overline{g}_i) + \sum_{i,j}(\mu_j - a_i r_{ij}) = 1.$$

Again, each $\overline{g}_i$ has to be a fiber. Note that $\overline{g}_i$ is a smooth $\mathbb{P}^1$ and hence $r_{ij} = 1$. Furthermore, $f = \mu_1 l_1 + \cdots + \mu_k l_k + a_1 g_1 + \cdots + a_t g_t$ is connected. Then its image $a_1 \overline{g}_1 + \cdots + a_t \overline{g}_t$ in W is also connected, and there can be only one g_i. Hence, $A = [C] - [l_i]$.

It is easy to check that in any cases, there are at most k-many classes A_i with $A_i A_j = 0$ if $i \neq j$.

Proof of the theorem. Now we prove that $V_k \times \Sigma$ is not deformation equivalent to $W_k \times \Sigma$. First of all, we can assume that W is not $\mathbb{P}^2$. Otherwise, W and V have different Betti numbers since $b_2(V) \geq 2$.

Thus $V_k \times \Sigma$ is not even diffeomorphic to $W_k \times \Sigma$. Suppose that $V_k \times \Sigma$ is deformation equivalent to $W_k \times \Sigma$. There is a diffeomorphism $F\colon V_k \times \Sigma \to W_k \times \Sigma$ such that $F^* c_i(W_k \times \Sigma) = c_i(V_k \times \Sigma)$, $F^* p_1(W_k \times \Sigma) = p_1(V_k \times \Sigma)$, and $\Phi_{A_i}(A_i; V_k \times \Sigma) = \Phi_{F_*(A_i)}(F_*(A_i); W_k \times \Sigma) = -1$ for $(k+1)$-many $A_i \in H_2(V_k, \mathbb{Z})$ with $c_1(A_i) = 1$, $A_i^2 = -1$, $A_i A_j = 0$.

We claim that for any $A \in H_2(V_k, \mathbb{Z})$, $F^*(A) \in H_2(W_k, \mathbb{Z})$. Let $\alpha \in H^2(\Sigma, \mathbb{Z})$ be the positive generator. Suppose that $F^*(\alpha) = n\alpha + \beta$ for $\beta \in H^2(V_k, \mathbb{Z})$. Note that the first Pontryagin class $p_1(V_k \times \Sigma) = p_1(V_k) \neq 0$ and $p_1(W_k \times \Sigma) = p_1(W_k) \neq 0$. Let $\gamma(V_k) \in H^4(V_k, \mathbb{Z})$ be such that $\gamma(V_k)[V_k] = 1$. Define $\gamma(W_k)$ in the same way. Then $p_1(V_k)$ is a nonzero multiple of $\gamma(V_k)$, and $p_1(W_k)$ is a nonzero multiple of $\gamma(W_k)$. Thus, $F^*\gamma(W_k) = \gamma(V_k)$, which implies that

$$\begin{aligned} 1 &= (\gamma(W_k) \cup \alpha)[W_k \times \Sigma] = F^*(\gamma(W_k) \cup \alpha)[V_k \times \Sigma] \\ &= \gamma(V_k) \cup (n\alpha + \beta)_[V_k \times \sigma] = n. \end{aligned}$$

Hence $n = 1$. Furthermore, $F^*(\alpha^2) = 0$. Then $(\alpha + \beta)^2 = 2\alpha\beta + \beta^2 = 0$. Therefore $2\alpha\beta = 0$ and $\beta^2 = 0$, which imply that $\beta = 0$. Now let $[\Sigma]$ be the orientation class of Σ. For any $A \in H_2(V_k, \mathbb{Z})$, if $F_*(A) = k[\Sigma] + C$, then $k = \alpha(F_*(A)) = f^*(\alpha)(A) = 0$. Hence $F_*(A) \in H_2(W_k, \mathbb{Z})$. So, each $F_*(A_i) \in H_2(W_k, \mathbb{Z})$. Clearly $c_1(W_k)(F_*(A_i)) = 1$, $F_*(A_i)^2 = -1$, and $F_*(A_i)F_*(A_j) = 0$ as homology classes of W_k. But we just showed that there are at most k many such classes with nonzero invariants. This is a contradiction.

Remark. Symplectic rational ruled rational surfaces have been classifed by [10], to which we refer the reader for more details.

4. Examples

Of course in many cases $V \times \Sigma$ and $W \times \Sigma$ are not diffeomorphic. If one can find V and W such that there is a homotopy equivalence between V and W preserving the first Chern class, then by smooth classification of 6-manifolds [21], [6], there is a diffeomorphism between $V \times \Sigma$ and $W \times \Sigma$ preserving the first Chern class. By Wall's theorem, two complex structures are homotopy equivalent to each other under this diffeomorphism. There are no classical invariants to distinguish them. But by the main theorem, V_k and W_k are not deformation equivalent.

An interesting pair of such examples consists of the blow-up R_8 of $\mathbb{P}^2$ at 8 points in general position and the Barlow surface B. B is, of course, nonminimal. There has been a lot of discussion [7] of the Barlow surface in gauge theory. We refer to [1], [7] for details. The Barlow surface is a minimal surface of general type with $K^2 = 1$, where $K = -c_1$ is the canonical class. It has the same homotopy type as V. By Freedman's theorem, B is homeomorphic to R_8. Note that $K_B^2 = 1$ too. D. Kotschick pointed out to me that Wall [22] proved that there is an isomorphism of $H^2(B, \mathbb{Z})$ to $H^2(R_8, \mathbb{Z})$ preserving the intersection form and K. So there are no classical invariants to distinguish the deformation class of $B_k \times \Sigma$ and $(R_8)_k \times \Sigma$. It is easy to check that $p_1(B_k) = -21 - 3k$ is not zero. By the main theorem, they are not deformation equivalent. this give infinitely many examples of smooth 6-manifolds which admit at least two deformation classes of symplectic structures which cannot be distinguished by classical invariants. When $k = 0$, Ono [15] gives an interesting elementary proof.

Note that both blow-up of $\mathbb{P}^2$ and the Barlow surface have geometric genus $p_g = 0$. The Barlow surface is the only known simply connected minimal surface of general type with $p_g = 0$. One may ask if there are any such examples with $p_g > 0$. It turns out that not only such examples do exist, but they exist in abundance. Let me describe a way to find such examples in the surfaces of general type. At the same time, we shall find examples with many deformation classes of symplectic structures.

One wants to find two minimal surfaces V, W of general type such that the blow-up V_k for some $k > 0$ is homotopy equivalent to W preserving the first Chern class c_1. We recall some basic facts about simply connected 4-manifolds. We are particularly interested in odd, indefinite manifolds. Their intersection forms always have the form $\lambda(1) \oplus \mu(-1)$. Regarding the uniqueness of c_1, Wall has proved the following theorem.

Theorem (Wall [23]). *If $\lambda, \mu \geq 2$, then the automorphism group of the intersection form is transitive on the primitive characteristic elements of fixed square.*

Of course, c_1 is characteristic. If c_1^2 is prime, then c_1 is primitive. Hence by Wall's theorem, c_1^2 uniquely determines c_1 up to the automorphism of the intersection form. Note that in blowing up, c_1^2 decreases by 1 and c_2 decreases by 1. Also $3\tau = c_1^2 - 2c_2$ decreases by 3 where τ is the signature. Conversely, if $c_1^2(V) - c_1^2(W) = c_2(W) - c_2(V) = k > 0$, it is easy to check that $\lambda(V) = \frac{1}{6}(c_1^2 + c_2) - 1 = \lambda(W)$ and $\mu(W) = \frac{5}{6}c_2 - \frac{1}{6}c_1^2 - 1 = \mu(V) + k$. By blowing up V k-times, we get a nonminimal algebraic surface V_k which has the same homotopy type as W. If we also assume that $c_1^2(W)$ is an odd prime and $\lambda(W), \mu(W) \geq 2$, then by Wall's theorem, there is an algebraic isomorphism $q: H^*(V_k) \to H^*(W)$ preserving the first Chern class. It follows that there is a diffeomorphism $\overline{q}: V_k \times S^2 \times W \times S^2$ preserving the characteristic classes. Furthermore, the tamed almost complex structures are homotopic to each other under this diffeomorphism.

Recall the famous geography of minimal surfaces of general type. Roughly speaking, one wants to know if there are minimal surfaces of general type with a given pair (c_1^2, c_2) of positive numbers. There are several restrictions in addition to positivity on the possible value of (c_1^2, c_2). They are that $c_1^2 + c_2$ is divisible by 12, $p_g \leq \frac{1}{2}c_1^2 + 2$, and $c_1^2 \leq 3c_2$. Let $x = \frac{1}{12}(c_1^2 + c_2)$, which is the holomorphic Euler characteristics, and let $y = c_1^2$. Then the last two conditions are $2x - 6 \leq y \leq 9x$. $\lambda \geq 2$ is the same as $x \geq \frac{3}{2}$ and $\mu \geq 2$ is the same as $y \leq 10x - 3$. The signature $\tau = 0$ is equivalent to $y = 8x$. The condition $c_1^2(V) - c_1^2(W) = c_2(W) - c_2(V)$ corresponds to the requirement that $(ax(V), y(V))$ and $(x(W), (W))$ be on the vertical line $x = b$. Now it is obvious how to find such minimal surfaces V and W. For large b, draw the vertical line $x = b$. Choose a minimal y such that y is prime and (b, y) is represented by a simply connected, minimal algebraic surface W of general type. Then we can try to find all the pairs $(b, y_1), (b, y_2), \cdots, (b, y_k)$ such that $y_i > y$ and (b, y_i) is represented by a simply connected minimal algebraic surface X_i of general type. From Figure 1 there are clearly at most finitely many such (b, y_i). Now blowing up X_i suitable times, we get $V_0 = W, V_1, \cdots, V_k$ with the same $c_1^2 = y$, a prime. By the argument above, the manifolds $V_i \times \Sigma$ are diffeomorphic to each other and have the same classical invariants. If $p_1(W) \neq 0$, the main theorem implies that these manifolds are mutually nondeformation equivalent to each other. Note that $p_1 = 0$ corresponds to $y = 8x$. Hence we only have to choose W below the line $y = 8x$ in the geography map. Continuing to blow up, p_1 remains nonzero, and we get infinitely many smooth 6-manifolds which admit $k+1$ many distinct deformation classes of symplectic structures.

Now the question is how to get such a pair (b, y). There are many results about the geography of minimal surfaces of general type. let me just mention two of them.

Persson's Theorem [17]. *If* (x, y) *satisfies*

$$2x - 6 \leq y \leq 8\left(x - \frac{9}{\sqrt[3]{12}}x^{2/3}\right),$$

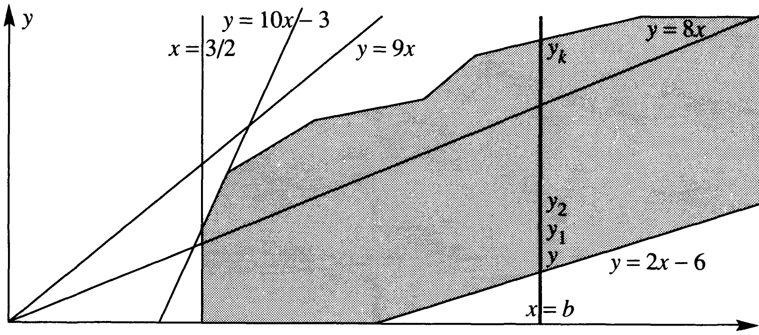

FIGURE 1

then (x, y) *is represented by a simply connected minimal algebraic surface of general type.*

Chen's Theorem [2]. *If* (x, y) *satisfies*

$$\left(\frac{352}{89}\right)x + 140.2x^{2/3} < y < \left(\frac{18644}{2129}\right)x - 365.7x^{2/3}$$

for large x, *then* (x, y) *is represented by a simply connected minimal surface of general type.*

Chen's examples are particularly interesting since they contain examples of positive signature. We will not go into any of the constructions here. Instead we refer the reader to the original papers [2], [17]. One thing worth mentioning is that genus 2 fibrations play a crucial role in the construction of these examples. We refer to Xiao's work [21] for further information.

Now it is clear how to get (b, y). Persson's and Chen's examples cover a large portion of the geography map of simply connected minimal surfaces of general type. Actually the only region not covered is the strip close to $y = 9x$. For any b, one only has to find the first prime number on the line $x = b$ over $y = 2x - 6$ which is in the region covered by Persson's and Chen's examples. For example, any odd prime number ρ can be expressed as $2b_\rho - 5$ for some b_ρ. Then $(b_\rho, 2b_\rho - 5)$ will be a satisfactory pair. A whole segment of the vertical line $x = b_\rho$ is in the Persson-Chen region. As b_ρ becomes larger and larger, the length of this segment will approach infinity. Hence we have shown

Corollary. *For any* n, *there is a simply connected minimal algebraic surface of general type* W *such that* $W \times \Sigma$ *admits more than* n *many distinct deformation classes of symplectic structures with the same classical invariants. Furthermore, for any blow-up* W_k *of* W, $W_k \times S^2$ *also admits more than* n *many distinct deformation classes of symplectic structures with the same classical invariants.*

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