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THE FRÖHLICHER SPECTRAL SEQUENCE ON A TWISTOR SPACE

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Abstract

Precise results are obtained about the degeneration of the Fröhlicher spectral sequence on the twistor space of a compact self-dual 4-manifold and several examples are studied. One of these shows that, for compact complex 3-manifolds, the property of *nondegeneration* of Fröhlicher is *unstable* under deformations of complex structure. Another consequence of the analysis is the discovery of a period mapping for (Riemannian) conformal structures on a compact 4-manifold.

1. Introduction

Associated to any compact self-dual 4-manifold M is a compact complex three-dimensional manifold Z known as its twistor space [1], [20]. Twistor spaces provide a source of interesting complex three-manifolds (cf. [16]). The purpose of this article is to investigate the Fröhlicher spectral sequence [8]

$$E_1^{p,q} = H^q(Z, \Omega^p) \Rightarrow H^{p+q}(Z, \mathbf{C}),$$

where Ω^p denotes the sheaf of holomorphic p-forms on Z. The Penrose transform [2], [3], [4], [6], [11] interprets the Dolbeault cohomology $H^q(Z, \Omega^p)$ in terms of differential equations on M. In this way, the Fröhlicher spectral sequence has differential-geometric consequences on M, and vice versa.

We shall explain this interpretation and its consequences. For example, we shall show that the spectral sequence is *degenerate* (i.e., $E_1 = E_\infty$) if and only if a certain conformally invariant system of linear differential equations has only constant solutions. The classical case in which $E_1 = E_\infty$ is when Z admits a Kähler metric. Hitchin [12] has shown that there are only two such twistor spaces, namely $\mathbf{CP}_3$ and the space of flags in $\mathbf{C}^3$. However, we shall construct other twistor spaces with $E_1 = E_\infty$. We shall

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show that if the spectral sequence is *nondegenerate* (i.e., $E_1 \neq E_\infty$), then $E_2 = E_\infty$ and that this possibility does occur. In fact we are able to give an example (cf. Theorem 5.4) to show that the property of *nondegeneration* of Fröhlicher is *unstable under deformations of complex structure* for compact complex 3-manifolds.

Our original motivation comes from the second author's work [13], [23] on four-dimensional conformal field theory for which it is desirable to have analogues, for twistor spaces of conformally flat M, of the constructions of classical Riemann surface theory. Another achievement of this article is the introduction, in §4, of a period map

$$\Phi\colon \{\text{conformal structure on } M\} \to G_M\,,$$

where G_M is the Grassmannian of Lagrangian subspaces of the symplectic vector space $V = H^1(M, \mathbf{C}) \oplus H^3(M, \mathbf{C})$. (Recall that the symplectic form on V is induced by Poincaré duality and that $W \subset V$ is Lagrangian if $\omega|W$ is identically zero and $\dim W = \dim V/2$.) This is precisely analogous to the period mapping for conformal structures on a compact (real) surface Σ,

$$\{\text{conformal structures } J \text{ on } \Sigma\} \to G_\Sigma$$

given by $J \mapsto H^{1,0}(\Sigma, J)$, where G_Σ is the Grassmannian of Lagrangian subspaces of $H^1(\Sigma, \mathbf{C})$, and $H^{1,0}(\Sigma, J) = \{J\text{-holomorphic 1-forms on } \Sigma\}$. This analogy is clear from its twistor description; from this point of view, Φ maps a complex structure J on Z to a Lagrangian subspace K_J of the middle-dimensional (de Rham) cohomology $H^3(Z, \mathbf{C})$. A full treatment of the properties of Φ will appear elsewhere.

For notation, we follow [21] save for a change of sign so that our Riemann tensor agrees with standard usage. (The opposite sign of [21] is more natural in the Lorentzian setting.) In particular, we shall make free use of Penrose's *abstract indices*—such indices do *not* imply a choice of local coördinates or frame though they do indicate the form that a tensor would take with such choices.

Except for a digression in §4, M will be a connected smooth compact oriented 4-manifold with a self-dual Riemannian metric. Let $\nu \to M$ denote the unit sphere bundle of the bundle of anti-self-dual 2-forms. The space Z inherits a natural complex structure [1] and is called the *twistor space* of M. We shall assume that the reader is familiar with this basic geometry as explained in [1], [20]. Though Z only depends on the conformal structure of M, it is convenient to work with a fixed metric on M.

Following [21], lowercase *Roman* superscripts $a, b, \cdots$ refer to the tangent bundle of M. Thus, X^a represents a vector field while ω_a represents a 1-form. The metric connection on M is denoted by ∇_a, and the Hodge Laplacian $dd^* + d^*d$ by Δ (and so equals $-\nabla^a\nabla_a$ on functions). The equation

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c = R_{abc}{}^d\omega_d$$

fixes the sign of the curvature tensor. The Ricci and scalar curvatures are

$$R_{ab} = R_{acb}{}^c \quad \text{and} \quad R = R_a{}^a$$

respectively. We use the notation $g = \dim H^1(M, \mathbf{C})$ and $b_\pm = \dim H^2_\pm(M, \mathbf{C})$ for the Betti numbers of M (noting that $\dim H^3(M, \mathbf{C}) = g$ by Poincaré duality).

2. Preliminary observations

The fibers of $\nu: Z \to M$ are projective lines. If L is such a fiber, then

$$\Omega^1|_L \cong \mathscr{O}(-2) \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-1),$$

where $\mathscr{O}(1)$ is the hyperplane section bundle. In particular, Ω^1 has no sections over L. Since Z is fibered by such projective lines, we may conclude that

$$H^0(Z, \Omega^1) = 0.$$

Similarly,

$$\Omega^2|_L \cong \mathscr{O}(-3) \oplus \mathscr{O}(-3) \oplus \mathscr{O}(-2) \quad \text{and} \quad \Omega^3|_L \cong \mathscr{O}(-4),$$

so

$$H^0(Z, \Omega^2) = 0 \quad \text{and} \quad H^0(Z, \Omega^3) = 0.$$

Now, by Serre duality [22]

$$H^q(Z, \Omega^p) = H^{3-q}(Z, \Omega^{3-p})^*,$$

so

$$H^3(Z, \Omega^0) = 0, \quad H^3(Z, \Omega^1) = 0, \quad H^3(Z, \Omega^2) = 0.$$

Finally, since Z is compact, $H^0(Z, \Omega^0) = \mathbf{C}$. As part of the Fröhlicher spectral sequence, we should write this as

$$H^0(Z, \Omega^0) = H^0(M, \mathbf{C}),$$

and by Serre duality on Z and Poincaré duality on M we have

$$H^3(Z, \Omega^3) = H^4(M, \mathbf{C}).$$

3. The Penrose transform

The Penrose transform relates analytic cohomology on Z to 'fields' (solutions of a differential equation) on M. There are two ways to proceed, either by Riemannian methods or by integral geometry in the complexification of M. The former method was used by Hitchin [11] for the cohomology of Ω^0, Ω^3, and other line bundles and could be adapted to compute the more general Penrose transforms required for this work. The latter method is more in the spirit of Penrose's original construction of Z [20] and is explained in [2], [3], [4], [6]. We shall state the results of this transform below. The proofs may be gleaned from the above references. Let Λ^r denote the smooth complex-valued r-forms on M, and $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ be the splitting of the 2-forms into self-dual and anti-self-dual parts.

Proposition 3.1. *For each* $q = 0, 1, 2, 3$ *the space* $H^q(Z, \Omega^0)$ *is canonically isomorphic to the qth cohomology of the complex* $\Lambda^0 \to \Lambda^1 \to \Lambda^2_- \to 0$. *For each* $q = 0, 1, 2, 3$, *the space* $H^q(Z, \Omega^3)$ *is canonically isomorphic to the qth cohomology of the complex* $0 \to \Lambda^2_- \to \Lambda^3 \to \Lambda^4$.

This proposition is also valid locally. Notice that we obtain the vanishing of $H^0(Z, \Omega^3)$ and $H^3(Z, \Omega^0)$ together with $H^0(Z, \Omega^0) = H^0(M, \mathbf{C})$ and $H^3(Z, \Omega^3) = H^4(M, \mathbf{C})$ as in §2. Since M is compact we may proceed further.

Corollary 3.2.

$$H^1(Z, \Omega^0) = H^1(M, \mathbf{C}), \qquad H^2(Z, \Omega^0) = H^2_-(M, \mathbf{C}),$$
$$H^1(Z, \Omega^3) = H^2_-(M, \mathbf{C}), \qquad H^2(Z, \Omega^3) = H^3(M, \mathbf{C}).$$

Proof. These all follow from the standard elliptic theory (e.g., [9]) applied to the complexes of Proposition 3.1. For example, to show that $H^1(Z, \mathcal{O}) = H^1(M, \mathbf{C})$ we must show that if $\omega \in \Lambda^1$ and $d_-\omega = 0$, then $d\omega = 0$. Let $\eta = d\omega$. Then $*\eta = \eta$ and

$$\|\eta\|^2 = \int \eta \wedge *\eta = \int \eta \wedge \eta = \int d\omega \wedge \eta = \int \omega \wedge d\eta = 0,$$

so $\eta = 0$ as required.

Remark. Notice that this corollary computes the holomorphic Euler characteristics of Ω^0 (and Ω^3 by duality):

$$\chi(Z, \Omega^0) = \sum (-1)^q \dim H^q(Z, \Omega^0) = 1 - g + b_-.$$

This agrees with the answer obtained by Hitchin in [12] using the Hirzebruch-Riemann-Roch theorem.

The Penrose transform of the bundle of holomorphic 1-forms and 2-forms appears in [4, §9.3] (in the flat case—the curved version is essentially unchanged). An alternative method using *local twistors* is due to Mason [17]. By combining these two methods (or by a direct argument of Bailey and the first author), we obtain:

Proposition 3.3. *The spaces* $H^0(Z, \Omega^1)$, $H^3(Z, \Omega^1)$, $H^0(Z, \Omega^2)$, *and* $H^3(Z, \Omega^2)$ *all vanish. The Penrose transform gives the following commutative diagram with exact rows:*

$$
\begin{array}{ccccccccc}
 & & & & (f, \rho) & \mapsto & \mathscr{D}df - d\rho & & \\
 & & & & \cap & & \cap & & \\
0 & \to & H^1(Z, \Omega^1) & \to & \Lambda^0 \oplus \Lambda^2_+ & \to & \Lambda^3 & \to & H^2(Z, \Omega^1) \to 0 \\
 & & \Big\downarrow{\scriptstyle d} & & \cup & & \cup & & \Big\downarrow{\scriptstyle d} \\
 & & & & (f, \rho) & & \xi & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & df & & d\xi & & \\
 & & & & \cap & & \cap & & \\
0 & \to & H^1(Z, \Omega^2) & \to & \Lambda^1 & \to & \Lambda^2_+ \oplus \Lambda^4 & \to & H^2(Z, \Omega^2) \to 0 \\
 & & & & \cup & & \cup & & \\
 & & & & \omega & \mapsto & (d_+\omega, d\mathscr{D}\omega) & &
\end{array}
$$

where $\mathscr{D}: \Lambda^1 \to \Lambda^3$ *is given by* $\omega_b \mapsto (\nabla^a\nabla^b + 2R^{ab} - \frac{2}{3}Rg^{ab})\omega_b$ *(and we have used the volume form to identify* 3*-forms with vector fields).*

Our description is in terms of a particular choice of metric. The equation

$$\mathscr{D}df = d\rho \tag{3.1}$$

itself is not preserved under conformal change—if the metric g_{ab} is replaced by $\hat{g}_{ab} = \lambda^2 g_{ab}$, then

$$\widehat{\mathscr{D}}df = \mathscr{D}df + d * \sigma,$$

where $\sigma = 12\Upsilon \wedge df$ and $\Upsilon = \lambda^{-1}d\lambda$. However, under such a conformal change of metric

$$\hat{\rho} = \rho + 2\sigma_+,$$

and, noting that $d\sigma = 0$, we see that (3.1) is now preserved. This change may be derived by using *local twistors* as in [17]. Notice that if f satisfies (3.1), then

$$d\mathscr{D}df = \nabla_a(\nabla^a\nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab})\nabla_b f = 0.$$

This is the conformally invariant fourth-order equation of [7].

4. The Fröhlicher spectral sequence

As in [12], it is easy to show that there is a closed $(1, 1)$-form h on Z whose restriction to each fiber of ν generates $H^2(S^2, \mathbf{C})$. By the Leray-Hirsch theorem (e.g., [14]) we may identify the de Rham cohomology of Z by means of isomorphisms

$$H^{r-2}(M, \mathbf{C}) \oplus H^r(M, \mathbf{C}) \xrightarrow{\cong} H^r(Z, \mathbf{C}) \quad \text{for } 0 \leq r \leq 6$$

induced by $(\alpha, \beta) \mapsto h \wedge \nu^*\alpha + \nu^*\beta$. Combining this with the results of the previous section almost completely identifies the E_1-level of the Fröhlicher spectral sequence:

$$\begin{array}{ccccccc}
\uparrow q & & & & & & \\
0 & & 0 & & 0 & & H^4(M, \mathbf{C}) \\
| & & & & & & \\
H^2_-(M, \mathbf{C}) & & H^2(Z, \Omega^1) & \xrightarrow{d} & H^2(Z, \Omega^2) & & H^3(M, \mathbf{C}) \\
| & & & & & & \\
H^1(M, \mathbf{C}) & & H^1(Z, \Omega^1) & \xrightarrow{d} & H^1(Z, \Omega^2) & & H^2_-(M, \mathbf{C}) \\
| & & & & & & \\
H^0(M, \mathbf{C}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \quad \xrightarrow[p]{}
\end{array}$$

where all the differentials except those indicated by arrows are zero. These two remaining differentials are dual under Serre duality.

The kernel of $H^1(Z, \Omega^1) \to H^1(Z, \Omega^2)$ is canonically identified by Proposition 3.3 with $H^0(M, \mathbf{C}) \oplus H^2_-(M, \mathbf{C})$ so, in all cases, the spectral sequence stabilizes at the E_2 level:

$$\begin{array}{ccccccc}
\uparrow q & & & & & & \\
0 & & 0 & & 0 & & H^4(M, \mathbf{C}) \\
| & & & & & & \\
H^2_-(M, \mathbf{C}) & & H & & H^2_+(M, \mathbf{C}) \oplus H^4(M, \mathbf{C}) & & H^3(M, \mathbf{C}) \\
| & & & & & & \\
H^1(M, \mathbf{C}) & & H^0(M, \mathbf{C}) \oplus H^2_+(M, \mathbf{C}) & & K & & H^2_-(M, \mathbf{C}) \\
| & & & & & & \\
H^0(M, \mathbf{C}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \xrightarrow[p]{}
\end{array}$$

where

$$H = \frac{\{\xi \in \Lambda^3 \text{ such that } d\xi = d\mathscr{D}\eta \text{ for some } \eta \in \Lambda^1 \text{ with } d\eta = 0\}}{\{\xi = \mathscr{D}df - d\rho \text{ for some } f \in \Lambda^0 \text{ and } \rho \in \Lambda^2\}}$$

and

$$K = \frac{\{\omega \in \Lambda^1 \text{ such that } d\omega = 0 \text{ and } d\mathscr{D}\omega = 0\}}{\{\omega = df \text{ and } \mathscr{D}\omega = d\rho \text{ for some } \rho \in \Lambda^2\}}.$$

One can easily verify that

$$\begin{array}{ccccccccc} & & \omega & \longmapsto & (\omega, \mathscr{D}\omega) & & & & \\ & & \cap\!\!\!| & & \cap\!\!\!| & & & & \\ (4.1) \quad 0 & \to & K & \xrightarrow{p} & H^1(M, \mathbf{C}) \oplus H^3(M, \mathbf{C}) & \to & H & \longrightarrow & 0 \\ & & & & \Psi & & \Psi & & \\ & & & & (\eta, \xi) & \longmapsto & \xi - \mathscr{D}\eta & & \end{array}$$

is an exact sequence as anticipated by the Fröhlicher spectral sequence. Also, by Serre duality in the Fröhlicher spectral sequence, H, and K are dual vector spaces. In particular, in conjunction with (4.1), this shows that $\dim H = \dim K = g$.

Remark. Combining this conclusion with our discussion of the E_1 and E_2 levels of the Fröhlicher spectral sequence, shows that

$$\chi(Z, \Omega^1) = -1 + g - b_+.$$

This answer is also easily obtained using the Hirzebruch-Riemann-Roch theorem.

The sequence (4.1) is easily checked to be exact without assuming M to be self-dual. The duality of H and K is also true in general but this requires more work:

Proposition 4.1. *On a general compact oriented conformal* 4*-manifold* M *(not necessarily self-dual), let* H *and* K *be defined as above. Then their definition is independent of choice of metric in the conformal class and*

$$\begin{array}{ccc} H \otimes K & \to & \mathbf{C} \\ \Psi & & \Psi \\ (\xi, \omega) & \mapsto & \int \xi \wedge \omega \end{array}$$

is a well-defined perfect pairing of finite-dimensional vector spaces.

Proof. If the metric g_{ab} is replaced by $\hat{g}_{ab} = \lambda^2 g_{ab}$, then

$$\widehat{\mathscr{D}}\eta = \mathscr{D}\eta - 12\Upsilon \wedge *d\eta + 12d * (\Upsilon \wedge \eta),$$

where $\Upsilon = \lambda^{-1} d\lambda$ from which the conformal invariance of H and K easily follows. It is straightforward to check that the pairing is well-defined. For example, if $\omega = df$ and $D\omega = d\rho$ for some $\rho \in \Lambda^2$, and ξ represents an element of H, then

$$\begin{aligned} \int \xi \wedge \omega &= \int \xi \wedge df = \int f\, d\xi = \int f\, d\mathscr{D}\eta = \int \mathscr{D}\eta \wedge df \\ &= \int \mathscr{D}\eta \wedge \omega = -\int \eta \wedge \mathscr{D}\omega = -\int \eta \wedge d\rho = -\int d\eta \wedge \rho = 0. \end{aligned}$$

The exact sequence (4.1) shows that H and K are finite dimensional. To show that the pairing is perfect, first suppose $\omega \in \Lambda^1$ such that $d\omega = 0$, $d\mathscr{D}\omega = 0$, and

$$\int \xi \wedge \omega = 0 \quad \text{for all } \xi \in H.$$

Then, in particular, we can take ξ to be any closed 3-form whence, by Poincaré duality, we can write $\omega = df$ for some smooth function f. We also claim that $\mathscr{D}\omega$ is exact so that ω represents zero in K. Again, by Poincaré duality, it suffices to show that $\int \mathscr{D}\omega \wedge \eta = 0$ for any closed 1-form η. This is true since we can take $\xi = \mathscr{D}\eta$ whence

$$0 = \int \xi \wedge \omega = \int \mathscr{D}\eta \wedge \omega = \int \mathscr{D}\omega \wedge \eta$$

as required.

Now suppose $\xi \in \Lambda^3$ represents an element of H and

$$\int \xi \wedge \omega = 0 \quad \text{for all } \omega \in K.$$

By Hodge theory,

$$\Delta d_+\omega = 0 \Rightarrow dd_+\omega = 0 \Rightarrow d * d\omega = 0 \Rightarrow d\omega = 0,$$

so, if $\Delta d_+\omega = 0$ and $d\mathscr{D}\omega = 0$, then $\int \xi \wedge \omega = 0$. Therefore, ξ is orthogonal to the kernel of the adjoint of

$$\begin{array}{ccc} \Lambda^0 \oplus \Lambda^2_+ & \to & \Lambda^3 \\ \Psi & & \Psi \\ (f, \sigma) & \mapsto & \mathscr{D}df - d\Delta\sigma. \end{array}$$

It is easily verified that this is an elliptic operator whence, by the Fredholm alternative (e.g., [9]) ξ is in its range. Taking $\rho = \Delta\sigma$ shows that ξ represents zero in H. The idea of using the operator $\mathscr{D}df - d\Delta\sigma$ rather than the more obvious $\mathscr{D}df - d\rho$ was suggested to us by Nick Buchdahl.

Corollary 4.2. *For any compact conformal four-manifold,* $\dim H = \dim K = g$.

Proof. The duality of the proposition implies that $\dim H = \dim K$ and the conclusion is immediate from the exact sequence (4.1).

Remark. As discussed in the Introduction, the mapping Φ which assigns to a conformal structure on M the homomorphism p in (4.1) very much resembles the period mapping which assigns to a conformal structure on a compact surface X the homomorphism $H^0(X, \Omega^1) \to H^1(X, \mathbf{C})$.

Corollary 4.3. *The following are equivalent*:

$K \to H^1(M, \mathbf{C})$ *is injective.*

$K \to H^1(M, \mathbf{C})$ *is surjective.*
$H^3(M, \mathbf{C}) \to H$ *is injective.*
$H^3(M, \mathbf{C}) \to H$ *is surjective.*

Proof. By Proposition 4.1 and Poincaré duality, the homomorphisms $K \to H^1(M, \mathbf{C})$ and $H^3(M, \mathbf{C}) \to H$ are dual.

Definition. We shall say that M is *regular* if one of the equivalent conditions of Corollary 4.3 holds.

Proposition 4.4. *M is regular if and only if for any smooth function f with $d\mathscr{D}df = 0$ there is a two-form ρ such that $\mathscr{D}df = d\rho$.*

Proof. Suppose M is regular. Then $K \to H^1(M, \mathbf{C})$ is injective. Now df represents an element of K which maps to zero in $H^1(M, \mathbf{C})$. Therefore, there is a 2-form ρ with $\mathscr{D}df = d\rho$. This argument is reversible.

Corollary 4.5. *An Einstein manifold is regular.*

Proof. Suppose f is a smooth function with $d\mathscr{D}df = 0$. In order to show that $\mathscr{D}df$ is exact it suffices to show that $\int \mathscr{D}df \wedge \eta = 0$ for any closed 1-form η. If η is exact, then this is true by integration by parts. Thus, we may assume that η is harmonic and, in particular, that $\nabla^b \eta_b = 0$. Now,

$$\int \mathscr{D}df \wedge \eta = \int \mathscr{D}\eta \wedge df = \int f\, d\mathscr{D}\eta,$$

and, using the volume form to identity 4-forms with functions we have

$$d\mathscr{D}\eta = \nabla_a(2R^{ab} - \tfrac{2}{3}Rg^{ab})\eta_b = -\tfrac{1}{6}\nabla^b(R\eta_b) = 0,$$

as required. q.e.d.

We anticipate that, in some sense, a generic M will be regular. We shall present below several examples of regular self-dual M. We know of no irregular M. If $g = 0$, then M is trivially regular.

From now on we shall denote the operator

$$d\mathscr{D}d = \nabla_a(\nabla^a\nabla^b + 2R^{ab} - \tfrac{2}{3}Rg^{ab})\nabla_b \colon \Lambda^0 \to \Lambda^4$$

by L. We return to the case of M self-dual with twistor space Z.

Theorem 4.6. *There is an exact sequence*

$$0 \to H^2_+(M, \mathbf{C}) \to H^1(Z, \Omega^1) \to \{f \in \Lambda^0 \text{ such that } Lf = 0\},$$

and if M is regular, then the final mapping is surjective. If $Lf = 0$ implies that f is constant, then $E_1 = E_\infty$ in the Fröhlicher spectral sequence. If M is regular, then the converse is also true.

Proof. The exact sequence follows from Proposition 3.3, and surjectivity for regular M from Proposition 4.4. If $Lf = 0$ implies that f is constant, then certainly M is regular. Now from the exact sequence

$$E_1^{1,1} = H^1(Z, \Omega^1) = H^0(M, \mathbf{C}) \oplus H^2_+(M, \mathbf{C}) = E_2^{1,1}$$

and by counting dimensions we have $E_1^{p,q} = E_2^{p,q}$ for all p, q. Conversely, if M is regular and $Lf = 0$ has nonconstant solutions, then $\dim H^1(M, \Omega^1) > 1 + b_+$ so $E_1^{1,1} \neq E_2^{1,1}$.

5. Examples

The examples of this section owe a great deal to conversations with and ideals of Claude LeBrun (for the material on Kähler surfaces) and Paul Tod (for metrics on the connected sum $g(S^3 \times S^1)$).

Given the conjugate orientation, any Kähler surface with zero scalar curvature is self-dual (cf. [15]). On such a surface, one has the Lichnerowicz operator

$$\mathscr{L} = \Delta^2 + 2R^{ab}\nabla_a\nabla_b.$$

As shown in [5, Proposition 2.151], its kernel modulo constants can be identified (by means of $f \mapsto \nabla^a f$) with the space of holomorphic vector fields modulo those which are parallel.

On the other hand, when the scalar curvature is zero, the contracted Bianchi identity becomes $\nabla^a R_{ab} = 0$ and so $\mathscr{L} = L$. Hence, as an immediate consequence of Theorem 4.6, we obtain:

Theorem 5.1. *Let Z be the twistor space of a zero-scalar-curvature Kähler surface S (with the conjugate orientation). If S supports no nonparallel holomorphic vector fields, then $E_1 = E_\infty$ in the Fröhler spectral sequence on Z.*

Corollary 5.2. *On the twistor space of a K3 surface, $E_1 = E_\infty$ in the Fröhlicher spectral sequence.*

Theorem 5.3. *A zero-scalar-curvature Kähler surface S is regular and, if Z denotes its twistor space, then*

$$H^1(Z, \Omega^1) = H^0(S, \mathbf{C}) \oplus H^2_+(S, \mathbf{C}) \oplus \frac{\{\text{holomorphic vector fields on } S\}}{\{\text{parallel vector fields on } S\}}.$$

In particular, $E_1 = E_\infty$ in the Fröhlicher spectral sequence on Z if and only if S supports no nonparallel holomorphic vector fields.

Proof. Once we show that S is regular, everything will follow from Theorem 4.6 and the discussion above. Regularity will certainly follow if we can show that

$$Lf = 0 \Rightarrow \mathscr{D}df = 0.$$

As in [5, Proposition 2.151], if $Lf = 0$, then $\nabla^a f$ is a holomorphic vector field. Now from [5, Proposition 2.140], $\mathscr{D}df = 0$.

Example. The Kähler surface $M = S^2 \times \Sigma_p$, where Σ_p is a compact Riemann surface of genus $p \geq 2$, has zero scalar curvature if we give S^2 and Σ_p their standard metrics of constant curvature with equal magnitude but opposite sign. It is thus self-dual (in fact, conformally flat). We may investigate $H^1(Z, \Omega^1)$ directly as follows. Consider equation (3.1) on M. Since $R = 0$,

$$\mathscr{D}df = (\nabla^a\nabla^b + 2R^{ab})\nabla_b f.$$

Let g_{ij} denote the metric on the unit 2-sphere. The Ricci curvature R_{ij} on this 2-sphere is equal to the metric. Thus, if f is a smooth function on $S^2 \times \Sigma_p$ depending only on the S^2 factor, then

$$\mathscr{D}df = (\nabla^i\nabla^j + 2g^{ij})\nabla_j f = \nabla^i(-\Delta + 2)f,$$

where Δ is the Laplacian on the sphere. However, 2 is an eigenvalue of Δ with eigenspace of dimension 3 (consisting of spherical harmonics). Together with constant f it follows that $\mathscr{D}df = 0$ has a solution space of dimension at least 4. Thus, in (3.1) we may take such an f and ρ to be any closed self-dual 2-form. Hence,

$$\dim H^1(Z, \Omega^1) \geq 4 + b_+. \tag{5.1}$$

(In fact $b_+ = 1$.) In particular, $E_1^{1,1} \neq E_2^{1,1}$ in the Fröhlicher spectral sequence. In fact, from Theorem 5.3 we may deduce equality in equation (5.1).

We now combine this discussion with Theorem 5.3 in the following:

Theorem 5.4. *For compact complex 3-manifolds, the property of non-degeneration of the Fröhlicher spectral sequence is unstable under deformations of complex structure.*

Proof. By Theorem 5.3 it is enough to perturb $S^3 \times \Sigma_p$ through Kähler surfaces of zero scalar curvature such that the perturbed surface has no nonzero holomorphic vector fields.

We claim that if $E \to \Sigma_p$ is a stable holomorphic rank 2 vector bundle, then the corresponding ruled surface $\mathbf{P}(E)$ admits a Kähler metric of zero scalar curvature. Indeed, according to a theorem of Narasimhan and Seshadri [19], any such bundle arises from a representation of $\pi_1(\Sigma_p)$ into SU_2. So $\mathbf{P}(E)$ is a quotient $(S^2 \times H)/\pi_1$ where H is the upper half-plane, and π_1 acts by isometries of the natural metric (the Riemannian product of the metric of curvature 1 on S^2 with that of curvature -1 on H). This natural metric is Kähler and of zero scalar curvature.

For any holomorphic vector bundle E on Σ_p we claim that

$$\Gamma(\mathbf{P}(E), \Theta)) \simeq \Gamma(\Sigma_p, \mathrm{End}_0(E)),$$

where $\mathrm{End}_0 =$ trace-free endomorphisms. Indeed, if ξ is a holomorphic vector field on $\mathbf{P}(E)$, the normal component to any fiber of $\mathbf{P}(E) \to \Sigma_p$ is constant along the fiber (because the normal bundle is trivial). So this normal component projects to a holomorphic vector field on Σ_p, which vanishes since $p \geq 2$. Hence ξ is vertical and the result follows. if E is also stable, then $\Gamma(\Sigma_p, \mathrm{End}_0(E)) = 0$ (as observed, for example, in [18]).

Therefore, it remains to be shown that the trivial rank 2 bundle on Σ_p can be perturbed to be stable. The following argument is a modification of reasoning found in [18]. The first step is to perturb the trivial bundle to a nontrivial extension of trivial line bundles. Such extensions are parametrized by $H^1(\Sigma_p, \mathcal{O})$ with the origin corresponding to the trivial rank 2 bundle. Fix a point $x \in \Sigma_p$ and let L_x denote the corresponding point bundle. The canonical section f of this bundle gives rise to a homomorphism in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] & E \arrow[r] & \mathcal{O} \arrow[r] & 0 \\ & & & L_x^{-1} \arrow[u, "f"'] & \end{tikzcd}$$

and this homomorphism lifts back to E if and only if the extension class maps to zero under $H^1(\Sigma_p, \mathcal{O}) \to H^1(\Sigma_p, L_x)$. By the Riemann-Roch theorem this is a surjective homomorphism between vector spaces of dimension p and $p-1$ respectively. Choose an extension which has this property. It is easy to see that this forces the lifting $L_x^{-1} \to E$ to be an inclusion of vector bundles. We conclude that E is an extension of L_x by L_x^{-1}, and now we investigate when such an extension is stable. This will require a further perturbation. For a general nontrivial extension E it is easy to see that E admits no line subbundles of positive degree. So suppose that L is a degree-zero line subbundle:

$$\begin{tikzcd} 0 \arrow[r] & L_x^{-1} \arrow[r] & E \arrow[r] & L_x \arrow[r] & 0 \\ & & L \arrow[u] & & \end{tikzcd}$$

In particular, we obtain a nonzero homomorphism $L \to L_x$. Up to scale, this forces $L = L_x \otimes L_y^{-1}$ with homomorphism given by the canonical section of L_y for some $y \in \Sigma_p$. In particular, this extension class maps to zero under $H^1(\Sigma_p, L_x^{-2}) \to H^1(\Sigma_p, L_x^{-2} \otimes L_y)$, a surjective mapping of vector spaces of dimensions $p+1$ and p respectively. As $y \in \Sigma_p$

varies we obtain an analytic mapping $\Sigma_p \to \mathbf{P}(H^1(\Sigma_p, L_x^{-2}))$. The stable extensions are therefore the complement of the image variety, a curve in a p-dimensional projective space. The extension we first chose is the image of $x \in \Sigma_p$, but now we can perturb it to be stable. q.e.d.

This concludes our discussion of zero-scalar-curvature Kähler surfaces. There is another situation in which Theorem 4.6 may be applied.

Theorem 5.5. *Suppose that for some constant $\lambda \in (1, 3]$, the curvature tensor*

$$T_{ab} := Rg_{ab} - \lambda R_{ab}$$

is positive semidefinite at each point of M. Then $Lf = 0 \Rightarrow f$ is constant.

Proof. Fix a constant c and consider the manifestly positive operator

$$P := (\Delta g^{ab} + c(\nabla^a\nabla^b + \tfrac{1}{4}\Delta g^{ab}))(\Delta g_{ab} + c(\nabla_a\nabla_g + \tfrac{1}{4}\Delta g_{ab})).$$

We may compare this with L.

$$\begin{aligned} P &= (4 - \tfrac{1}{4}c^2)\Delta^2 + c^2\nabla_a\nabla^b\nabla^a\nabla_b \\ &= (4 + \tfrac{3}{4}c^2)\Delta^2 + c^2\nabla_a(\nabla^b\nabla^a - \nabla^a\nabla^b)\nabla_b \\ &= (4 + \tfrac{3}{4}c^2)\Delta^2 + c^2\nabla_a(R^{ab})\nabla_b \\ &= (4 + \tfrac{3}{4}c^2)[L + \tfrac{2}{3}\nabla_a(Rg^{ab} - \lambda R^{ab})\nabla_b], \end{aligned}$$

where

$$\lambda = 3\frac{16 + c^2}{16 + 3c^2}.$$

Note that λ may take on any value in the interval $(1, 3]$. Thus, we may choose c so as to obtain λ as in the statement of the theorem. Now, $-\nabla_a T^{ab}\nabla_b$ is a positive operator. Thus, $Lf = 0$ implies that $Pf = 0$. Integrating by parts shows that

$$(\Delta g^{ab} + c(\nabla^a\nabla^b + \tfrac{1}{4}\Delta g^{ab}))f = 0,$$

and taking that trace of this equation yields that f is harmonic and hence constant. q.e.d.

In our final example we construct conformally flat metrics on the connected sum $g(S^3 \times S^1)$ for which $\ker L = \mathbf{C}$ by virtue of Theorem 5.5. We start with a description of the operation required to equip $g(S^3 \times S^1)$ with a conformally flat metric.

Consider S^4 with the standard metric g_0. Pick a point p and a small ball B, center p. Then one can rescale g_0 in B so that the complement of p in a smaller ball B' is isometric to the cylinder $S^3(\delta) \times I$, the product of the 3-sphere of radius δ and some semi-infinite interval $I \subset \mathbf{R}$ (this for any sufficiently small positive δ). By performing this rescaling for $2g$ points p_j and balls B_j (assumed nonoverlapping) we can construct the connected sum $g(S^3 \times S^1)$ with a conformally flat metric by cutting each of these cylinders of at a finite distance and pairing off the exposed S^3's with orientation reversing isometries. Let us choose $\lambda = 2$ in Theorem 5.5. We shall show how to perform the rescaling in each of these balls so that $T_{ab} = Rg_{ab} - 2R_{ab}$ remains positive-definite. By Theorem 5.5 we will then have shown:

Theorem 5.6. *For each $g \geq 0$ there are conformally flat metrics on the connected sum $M = g(S^3 \times S^1)$ for which $Lf = 0$ on M implies that f is constant.*

Corollary 5.7. *Let Z be the twistor space of $g(S^3 \times S^1)$. There are flat conformal structures on $g(S^3 \times S^1)$ such that the complex structure induced on Z has $E_1 = E_\infty$ in the Fröhlicher spectral sequence.*

Remark. The idea of trying to keep curvature quantities positive while performing surgeries is standard: see [10] for a classic example.

Proof of Theorem 5.6. By the earlier discussion, all we need is the local result: given a small ball B, center p in S^4, there is a rescaling g of the standard metric g_0 which gives the cylinder near p such that $T_{ab}(g)$ is positive-semidefinite in B.

Let us write the spherically symmetric conformally flat metric in the form

$$ds^2 = du^2 + G(u)^2 dS_3^2,$$

where, for example $G = \sin u$ gives g_0 on S^4, and $G = \text{constant}$ gives the metric on the cylinder.

The curvature may be calculated using Cartan's formalism in the orthonormal frame,

$$\theta^i = G\sigma^i, \quad i = 1, 2, 3 \qquad \text{and} \qquad \theta^4 = du,$$

where the σ^i are standard forms on S^3 satisfying $d\sigma^1 = 2d\sigma^2 \wedge d\sigma^3$, etc. A straightforward calculation reveals that the Ricci tensor is diagonal in this frame, with

$$R_{11} = R_{22} = R_{33} = -\frac{G''}{G} - \frac{2}{G^2}(G'^2 - 1), \qquad R_{44} = -\frac{3G''}{G}.$$

It is easy to check that when $G(u) = \sin u$, all components take the value 3, just as they should for S^4.

The tensor of interest, T_{ab}, is also diagonal in this frame and

$$T_{11} = T_{22} = T_{33} = -4\frac{G''}{G} - \frac{2}{G^2}(G'^2 - 1), \qquad T_4 = -\frac{6}{G^2}(G'^2 - 1).$$

Note that if $G(u) = \sin u$, all these components take the value 6; if $G(u) = \delta$, then $T_{11} = 2/\delta^2$ and $T_{44} = 6/\delta^2$; so T_{ab} is positive-definite on S^4 and on the cylinder. We may take the center of our ball to be given by $u = \pi$; choosing $\varepsilon \in (0, \pi/2)$, our task is to construct a smooth positive G on $(0, \infty)$ such that

$$G(u) = \sin u, \qquad u < \pi - \varepsilon, \tag{5.2}$$

and

$$G(u) = \delta, \qquad u \gg \pi \tag{5.3}$$

for arbitrarily small $\delta > 0$ while satisfying the inequalities

$$2GG'' < 1 - G'^2 \tag{5.4}$$

and

$$0 < 1 - G'^2 \tag{5.5}$$

throughout.

Choose η with $0 < \eta < \sin^2\varepsilon$ and let $\delta = \sin\varepsilon([\sin^2\varepsilon - \eta]/[1 - n])$. Define u as a function of $G \in [\delta, \sin\varepsilon]$ by

$$u(G) = \int_G^{\sin\varepsilon} \sqrt{\frac{t}{(1-\eta)(t-\delta)}}\, dt + \pi - \varepsilon,$$

noting that the singularity at $t = \delta$ is integrable. Set $u_1 = u(\delta)$. Then $u\colon [\delta, \sin\varepsilon] \to [\pi - \varepsilon, u_1]$ is invertible to give $G\colon [\pi - \varepsilon, u_1] \to [\delta, \sin\varepsilon]$ continuous and smooth on $(\pi - \varepsilon, u_1)$ with

$$G' = -\sqrt{(1-\eta)(G-\delta)/G} \tag{5.6}$$

throughout this interval. Notice that $-1 < G' \le 0$ so that (5.5) holds. G is differentiable at the endpoints with

$$G'(\pi - \varepsilon) = -\cos\varepsilon \quad \text{and} \quad G'(u_1) = 0.$$

Moreover, differentiating (5.6) and simplifying show that

$$2GG'' = 1 - G'^2 - \eta \quad \text{on } (\pi - \varepsilon, u_1)$$

and (5.4) holds too. These arguments have shown the following: there is a function $G\colon [0, \infty) \to (0, \infty)$ which is smooth except at $\pi - \varepsilon$ and u_1 where it is C^1 and this function satisfies (5.2)–(5.5) except at these two points. For each discontinuity u_0 of G'' we may choose a constant C such that $G'' < C < \frac{1}{2}(1 - G'^2)/G$ in a neighborhood of u_0. The lemma below shows how this may be used to smooth G while maintaining (5.2)–(5.5). Notice that δ may be made as small as we wish.

Lemma 5.8. *Suppose* $G\colon [a, b] \to \mathbf{R}$ *is* C^1, *and* C^∞ *except at* $u_0 \in (a, b)$ *where* G'' *has a finite jump discontinuity. Suppose* $G'' < C$ *on* $[a, b]$. *Then, given* $\varepsilon > 0$, *there exists a* C^∞ *function* $H\colon [a, b] \to \mathbf{R}$ *with the following properties*:

$|H - G| < \varepsilon$ *on* $[a, b]$.

$|H' - G'| < \varepsilon$ *on* $[a, b]$.

$H = G$ *on* $[a, u_0 - \varepsilon] \cup [u_0 + \varepsilon, b]$.

$H'' < C$ *on* $[a, b]$.

Proof. Let β be a C^∞ function on $[a, b]$ supported in $[u_0 - \frac{1}{2}\varepsilon, u_0 + \frac{1}{2}\varepsilon]$ and identically equal to 1 near u_0. One can verify that, if δ is sufficiently small, then

$$H = (1 - \beta)G + \rho_\delta * (\beta G)$$

solves this problem. Here, $*$ denotes convolution, and ρ_δ is a C^∞ function with support in $(-\delta, \delta)$ and total integral 1.

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