J. DIFFERENTIAL GEOMETRY
38 (1993) 263-313

# SPECTRAL DEGENERATION OF HYPERBOLIC RIEMANN SURFACES

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**Abstract**

Given a degenerating family $S_l$ $(1 \geq 0)$ of Riemann surfaces with their canonical hyperbolic metrics, we work out in detail the spectral degeneration of the collars around the pinching geodesics in $S_l$. Using the spectral degeneration of the pinching collars, we show that the eigenvalues of $S_l$ become dense at every point of the continuous spectrum $[\frac{1}{4}, +\infty)$ of $S_0$ and give upper and lower bounds for the rate of the clustering. Furthermore, we show that Eisenstein series, which are generalized eigenfunctions, of $S_0$ arise as limits of eigenfunctions of $S_l$ as $l \to 0$.

## 1. Introduction

Let $M_g$ $(g \geq 2)$ be the moduli space of compact Riemann surfaces of genus $g$, and $\overline{M_g}$ be the compatified moduli space of Riemann surfaces (see [11]). For any $S \in \overline{M_g}$, $S$ has a canonical hyperbolic metric (of constant curvature $-1$), induced from the uniformization. From now on, we call such a surface with its canonical hyperbolic metric a hyperbolic Riemann surface.

With respect to this metric on $S$, we have the Beltrami-Laplace operator $\Delta_S$, and its spectrum on $L^2(S)$ is denoted by $\operatorname{spec}(S)$. The spectrum is a very natural invariant of a manifold (see [19]). For generic $S \in M_g$, $\operatorname{spec}(S)$ uniquely determines $S$ (see [38]).

It is therefore a natural question to consider the dependence of $\operatorname{spec}(S)$ on $S \in \overline{M_g}$. For $S \in M_g$, $S$ is compact, and $\operatorname{spec}(S)$ is discrete. Furthermore, $\operatorname{spec}(S)$ changes real analytically in terms of suitable coordinates on the Teichmüller space, which is a covering space of $M_g$ (see [37]).

On the other hand, for $S_0 \in \overline{M_g} \setminus M_g$, $S_0$ is complete, noncompact, and has finite area and cusps as its ends (see §2). Furthermore $\operatorname{spec}(S) =$ discrete part $\cup$ continuous spectrum $[\frac{1}{4}, +\infty)$ (see Proposition 2.5). The discrete part may be finite, and the continuous part has

Received January 24, 1991 and, in revised form, December 2, 1991.

multiplicity equal to the total number of the cusps of $S_0$. Furthermore, the generalized eigenfunctions of the continuous spectrum are given by Eisenstein series. Any information about the discrete part of the spectrum will be quite useful in the theory of the Selberg trace formula and automorphic representations. Currently not much is known, especially about the embedded eigenvalues in $[\frac{1}{4}, +\infty)$. Actually there are two opposite conjectures about the embedded eigenvalues. A weak version of a conjecture of A. Selberg states that for any $S_0 \in \overline{M_g} \setminus M_g$, $S_0$ always has infinitely many embedded eigenvalues. On the other hand, R. Phillips and P. Sarnak conjectured that for a generic $S_0 \in \overline{M_g} \setminus M_g$, $S_0$ has at most finitely many embedded eigenvalues (see [15], [32], [33]).

In this paper, we want to study spectral degeneration of Riemann surfaces, that is, the behavior of the Beltrami-Laplace $\Delta_s$ of $S$ when $S$ approaches $\overline{M_g} \setminus M_g$. In this case, $S$ acquires cusps and becomes noncompact. More precisely, let $S_l$ $(l \geq 0)$ be a degenerating family of Riemann surfaces with $m \geq 1$ disjoint simple closed pinching geodesics $\gamma_1(l), \cdots, \gamma_m(l)$ on $S_l$. Let $l_i = |\gamma_i(l)|$ be the length of $\gamma_i(l)$ $(1 \leq i \leq m)$. Then $l_i \to 0$ as $l \to 0$ for $1 \leq i \leq m$. The limit surface $S_0$ is noncompact, while for $l > 0$, $S_l$ is compact. The behavior of eigenvalues $< \frac{1}{4}$ and their eigenfunctions of $S_l$ is understood well. Actually, they converge to the small eigenvalues $(< \frac{1}{4})$ and their eigenfunctions of $S_0$ (see [17, Theorems 6.6 and 7.2] and [8]). We would like to study the spectral degeneration related to the continuous spectrum of $S_0$. In particular, we will explain the occurrence of the continuous part $[\frac{1}{4}, +\infty)$ of $\operatorname{spec}(S_0)$, and their associated generalized eigenfunctions, which are Eisenstein series, during degeneration. Furthermore, we would also like to characterize the existence of embedded eigenvalues of $S_0$ in $[\frac{1}{4}, +\infty)$ through degeneration.

The main results of this paper are as follows:

**Theorem 1.1** (Clustering of Eigenvalues). 1. *For a degenerating family of hyperbolic Riemann surfaces $S_l$ as above, the eigenvalues of $S_l$ cluster at every point of the continuous spectrum $[\frac{1}{4}, +\infty)$ of $S_0$ as $l \to 0$.*

2. *For any $s > \frac{1}{4}$, let $N_l(x) = |\{\lambda \in \operatorname{spec}(S_l) | \frac{1}{4} \leq \lambda \leq x\}|$ be the spectral counting function of $S_l$. Then for small $l > 0$ with $l_i < \frac{1}{2}$, and any $x > \frac{1}{4}$,*

$$\frac{2}{\pi}\sum_{i=1}^{m}\Theta\left[\tau(l_i)\sqrt{x-\frac{1}{4}}\right] - \frac{2}{\pi}\sum_{i=1}^{m}\log\tau(l_i)\sqrt{x-\frac{1}{4}} - 4g + 2 - 2m \tag{1.1}$$

$$\leq N_l(x) \leq \frac{2}{\pi}\sum_{i=1}^{m}\tau(l_i)\sqrt{x-\frac{1}{4}} + B_1(x), \tag{1.2}$$

*where the function* $\Theta$ *is defined by* $\Theta[t] = \max\{0, t - t^{-1}\}$ *for* $t > 0$, $\tau(l_i) = \operatorname{arcsinh}(\frac{1}{2}\operatorname{csch}(l_i/2))$ ($\sim \log(2/l_i)$ *as* $l \to 0$) *is the half of width of the standard collar embedded around the pinching geodesic* $\gamma_i(l)$ *with length* $l_i$ *(see* §2 *for details),* $m$ *is the total number of the pinching geodesics on* $S_l$, *and* $B_1(x)$ *is some constant independent of* $l$. *In particular, as* $l \to 0$,

$$N_l(x) \sim \frac{2}{\pi}\sum_{i=1}^{m}\log\left(\frac{2}{l_i}\right)\sqrt{x - \frac{1}{4}}. \tag{1.3}$$

The clustering of the eigenvalues and the asymptotic behavior of the rate of the clustering $N_l(x)$ (1.3) are due to S. Wolpert [39, §4.5] and D. Hejhal [16, 17, Theorem 9.5]. Actually, D. Hejhal also gave bounds on $N_l(x)$ which are weaker than ours. The methods which we use here are more geometric and direct. Our contribution here is the asymptotically sharp lower and upper bounds ((1.1) and (1.2)). From these bounds, we can get bounds for eigenvalues of $S_l$ belonging to any finite subinterval of $[\frac{1}{4}, +\infty)$ in terms of $l$ and the length of the subinterval. The lower bound for $N_l(x)$ in (1.1) is not optimal, and better bounds will be proved in §3. There is a conjectural lower bound for $N_l(x)$ which is $\frac{2}{\pi}\sum_{i=1}^{m}\tau(l_i)\sqrt{x-\frac{1}{4}} - B^*(x)$, where $B^*(x)$ is some constant independent of $l$ (it is suggested by the referees). See §3 for discussions about this conjecture.

Theorem 1.1 deals with the asymptotic behavior of the eigenvalues of $S_l$ as $l \to 0$ and the appearance of the continuous spectrum $[\frac{1}{4}, +\infty)$ of $S_0$. A refinement of the problem about eigenvalues concerns the behavior of eigenfunctions of $S_l$, especially the occurrence of generalized eigenfunctions of $S_0$, which are Eisenstein series. To compare functions on $S_0$ and $S_l$, we use the harmonic map of infinite energy $\pi_l\colon S_0 \to S_l$ constructed by M. Wolf [37] to pull functions on $S_l$ back to $S_0$. The map $\pi_l$ is a homeomorphism from $S_0$ to $S_l\setminus\{\text{pinching geodesics } \gamma_i(l)\}$, and intuitively opens up each pair of cusps of $S_0$ into a pinching geodesic $\gamma_i(l)$ on $S_l$.

**Theorem 1.2** (Compactness of Eigenfunctions). *Let* $\varphi(l)$ *be an eigenfunction with eigenvalue* $\lambda(l)$ *on* $S_l$ *which has* $L^2$*-norm* 1. *Assume that* $\lambda(l)$ *converges as* $l \to 0$, *and denote the limit by* $\lambda(0)$.

1. *If* $\pi_l^*(\varphi(l)) \not\to 0$ *uniformly over some compact subsets of* $S_0$ *as* $l \to 0$, *then there exists a sequence* $l_j \to 0$ *such that as* $j \to +\infty$, $\pi_{l_j}^*(\varphi(l_j))$ *converges uniformly over all compact subsets of* $S_0$ *to a nonzero* $L^2$*-eigenfunction* $\varphi(0)$ *on* $S_0$ *with eigenvalue* $\lambda(0)$.

2. *If* $\pi_l^*(\varphi(l)) \to 0$ *uniformly over all compact subsets of* $S_0$ *as* $l \to 0$, *then the following hold*:

(a) *The limit* $\lambda(0) = \frac{1}{4} + t^2 \geq \frac{1}{4}$ *for some* $t \geq 0$.

(b) *There exist some constants* $K_l \to \infty$ *and a sequence* $l_j \to 0$ *such that* $K_{l_j}\pi_{l_j}^*(\varphi(l_j))$ *converges uniformly over all compact subsets of* $S_0$ *to some nonzero function* $\psi(0)$ *on* $S_0$ *as* $j \to +\infty$.

(c) *The function* $\psi(0)$ *satisfies* $\Delta_0\psi(0) + (\frac{1}{4} + t^2)\psi(0) = 0$, *where* $\Delta_0$ *is the Laplacian of* $S_0$.

(d) *There exist an* $L^2$*-function* $\varphi$ *(which could be zero) on* $S_0$ *with* $\Delta_0\varphi + (\frac{1}{4} + t^2)\varphi = 0$, *and constants* $a_1, \cdots, a_{2m}$ *such that*

$$\psi(0) = \sum_{i=1}^{2m} a_i E_i\left(\cdot\,; \frac{1}{2} + \sqrt{-1}t\right) + \varphi, \tag{1.4}$$

*where* $E_i(\cdot\,; \frac{1}{2} + \sqrt{-1}t)$ $(1 \leq i \leq 2m)$ *is the Eisenstein series associated to the* $i$ *th cusp of* $S_0$ *(see* §5 *for details), and* $2m$ *is the total number of cusps of* $S_0$.

(e) *If* $\lambda(0) = \frac{1}{4} + t^2$ *is not an eigenvalue of* $S_0$, *then*

$$\psi(0) = \sum_{i=1}^{2m} a_i E_i\left(\cdot\,; \frac{1}{2} + \sqrt{-1}t\right) \neq 0,$$

*where* $E_i(\cdot\,; \frac{1}{2} + \sqrt{-1}t)$ $(1 \leq i \leq 2m)$ *is the Eisenstein series associated to the* $i$ *th cusp of* $S_0$.

**Remarks.** 1. If $\lambda(0) = \frac{1}{4}$ or $t = 0$, then some derivatives $(d^k/d^k s) \cdot E_t(\cdot\,; s)|_{s=1/2}$ may enter the summation ((1.4) for $\psi(0)$).

2. After writing up a preliminary version of this paper, we received a preprint of S. Wolpert [41, Theorem 3.4]. By using elegant methods, he proved the above theorem among other results. Our methods here are more elementary.

3. From 2(d) above, eigenfunctions on $S_l$ can only limit in linear combinations of eigenfunctions and generalized eigenfunctions of $S_0$, while in Theorem 2(e), if $\lambda(0)$ is not an eigenvalue of $S_0$, we give a partial explanation for the occurrence of Eisenstein series of $S_0$ through degeneration.

In the above Theorem 1.2, we use the harmonic map $\pi_l\colon S_0 \to S_l$ to compare functions on $S_0$ and $S_l$. The reason is that $\pi_l$ is global and canonical (unlike the local cut-paste procedure, which depends on a choice of local coordinates). Let $ds^2(l)$ denote the hyperbolic metric of $S_l$. Then $\pi_l^* ds^2(l)$ converges to $ds^2(0)$ smoothly over all compact subsets of $S_0$. Furthermore, if the degenerating family $S_l$ is a real analytical family in $l$

at $l = 0$, then $\pi_l^* ds^2(l)$ is also real analytic in $l$ near $l = 0$ [37, Theorem 5.3]. It is conceivable that by exploiting the harmonicity of $\pi_l$, we can understand the behavior of the eigenfunctions on $S_l$ better.

The general philosophy of this paper is as follows. According to the Decomposition Principle [13, Proposition 2.1] (see Proposition 2.4 below), the continuous spectrum of a noncompact, complete Riemannian manifold is determined by the geometry of its ends. By the collar theorem of L. Keen [21], J. P. Matelski [27, the main Lemma and Remark 6.6], B. Randol [29], for a degenerating family of hyperbolic Riemann surfaces $S_l$, the noncompactness of $S_0$ is caused by the degeneration of the collars $\bigcup C_l$ around the pinching geodesics $\{\gamma_i(l)\}$ on $S_l$ (see §2). This philosophy that the degeneration localizes into the pinching collars has been used successfully by S. Wolpert [39], [40], and M. Wolf [37]. The proofs of Theorems 1.1 and 1.2 show that the formation of the continuous spectrum $[\frac{1}{4}, +\infty)$ and their associated generalized eigenfunctions, which are Eisenstein series, of $S_0$ can be understood through careful studies of the degeneration of the pinching collars $\bigcup C_l$.

To study the degeneration of the pinching collars $\bigcup C_l$, it is essential to choose a good set of coordinates on $\bigcup C_l$. For simplicity, we assume that there is only one pinching geodesic $\gamma(l)$ which has length $l$, and for $0 < l \le \frac{1}{2}$, we denote the standard collar of $\gamma(l)$ by $C_l$ which has width $2\tau(l) = 2\operatorname{arcsinh}(\frac{1}{2}\operatorname{csch}(\frac{l}{2}))$ (see §2). Let $(r, \theta)$ with $-\tau(l) \le r \le \tau$, $0 \le \theta \le 1$ be the Fermi coordinates on $C_l$ with respect to the core geodesic $\gamma(l)$. Since the collar $C_l$ is rotationally symmetric in $\theta$, the Hilbert space $L^2(C_l)$ decomposes according to phases, that is, $L^2(C_l) = \bigoplus_{n\in\mathbb{Z}} L_n^2(C_l)$, where for $n \in \mathbb{Z}$,

$$L_n^2(C_l) = \{f(r)e^{2\pi\sqrt{-1}n\theta} | f(r)e^{2r\sqrt{-1}n\theta} \in L^2(C_l)\}.$$

Correspondingly, the Laplacian $\Delta_l$ of $C_l$ decomposes into $\Delta_l(n)$ $(n \in \mathbb{Z})$, where $\Delta_l(n)$ acts on the Hilbert subspace $L_n^2(C_l)$.

For any $x > \frac{1}{4}$, let $DN_l^0(x)$ be the spectral counting function for the Dirichlet problem of $(\Delta_l(0), L_0^2(C_l))$, that is,

$$DN_l^0(x) = |\{\lambda \in \operatorname{Spec}(\Delta_l(0), L_0^2(C_l)) \text{ with Dirichlet condition } | \tfrac{1}{4} \le \lambda \le x\}|.$$

Then using the variational characterization for eigenvalues, we prove

**Theorem 1.3** (Dirichlet Problem on $C_l$; $n = 0$). 1. *With respect to the Dirichlet boundary condition,* $\operatorname{Spec}(\Delta_l(0), L_0^2(C_l))$ *cluster at every point of*

$[\frac{1}{4}, +\infty)$ *as* $l \to 0$. *More precisely, for any* $x > \frac{1}{4}$, $l \le \frac{1}{2}$, *and* $0 < \delta < 1$,

$$\frac{2}{\pi}(1-\delta)\tau(l)\sqrt{x - \frac{1}{4} - \frac{1}{4}\operatorname{sech}^2(\delta\tau(l))} - 2 \le DN_l^0(x) \le \frac{2}{\pi}\tau(l)\sqrt{x - \frac{1}{4}},$$

*where* $\tau(l) = \operatorname{arcsinh}(\frac{1}{2}\operatorname{csch}(\frac{l}{2})) \sim \log\frac{2}{l}$ *as* $l \to 0$, *and is half of the width of the collar* $C_l$. *In particular, let* $\delta = \frac{\log\tau(l)}{\tau(l)}$, *we get*

$$DN_l^0(x) \ge \frac{2}{\pi}\Theta\left[\tau(l)\sqrt{x - \frac{1}{4}}\right] - \frac{2}{\pi}\log\tau(l)\sqrt{x - \frac{1}{4}} - 2,$$

*where* $\Theta[t] = \max\{0, t - t^{-1}\}$ *for* $t > 0$.

2. *For any* $\lambda(l) \in \operatorname{Spec}(\Delta_l(0), L_0^2(C_l))$, *its normalized eigenfunction* $\varphi(l)$ *satisfies the following relations*:

$$\Delta_l(0)\varphi(l) + \lambda(l)\varphi(l) = 0, \qquad \|\varphi(l)\|_{L^2(s_l)} = 1.$$

*If* $\lambda(l)$ *converges to a limit* $\lambda(0)$, *then some multiple* $K_l\varphi(l)(r - \tau(l))$ *of* $\varphi(l)$ *converges uniformly over compact subsets of* $[0, +\infty)$ *to the generalized eigenfunction* $\widetilde{E}(\cdot, \frac{1}{2} + \sqrt{-1}t) = e^{r/2}\sin(rt)$ *of the standard cusp* $C_0$ (*see Definition* 2.1 *and Proposition* 2.3), *where* $t \ge 0$ *satisfies* $\lambda(0) = \frac{1}{4} + t^2$.

It is this special case which suggests Theorems 1.1 and 1.2 above. In order to prove Theorem 1.1, we have to study the Neumann problem on $C_l$ first. For any $x > \frac{1}{4}$, let $NN_l(x)$ be the spectral counting function for the Neumann problem of $(\Delta_l, L^2(C_l))$,

$$NN_l(x) = |\{\mu \in \operatorname{Spec}(\Delta_l, L^2(C_l)) \text{ with Neumann condition } |\tfrac{1}{4} \le \mu \le x\}|.$$

The first application of Theorem 1.3 is the following:

**Theorem 1.4** (Neumann Problem on $C_l$; all $n$). *With respect to the Neumann boundary condition,* $\operatorname{Spec}(\Delta_l, L^2(C_l))$ *cluster at every point of* $[\frac{1}{4}, +\infty)$ *as* $l \to 0$. *More precisely, for any* $x > \frac{1}{4}$, $l \le \frac{1}{2}$,

$$\frac{2}{\pi}\Theta\left[\tau(l)\sqrt{x - \frac{1}{4}}\right] - \frac{2}{\pi}\log\tau(l)\sqrt{x - \frac{1}{4}} - 4 \le NN_l(x) \le \frac{2}{\pi}\tau(l)\sqrt{x - \frac{1}{4}} + B(x),$$

*where* $B(x)$ *is some constant independent of* $l$, *the function* $\Theta[t] = \max\{0, t - t^{-1}\}$ *for* $t > 0$, *and* $\tau(l)$ *is half of the width of the collar as above.*

Applying the monotonicity for eigenvalues and the above Theorems 1.3 and 1.4, we prove Theorem 1.1 on the clustering of the eigenvalues of $S_l$.

In Theorem 1.3, we have only studied the piece $(\Delta_l(0), L_0^2(C_l))$ in the decomposition $(\Delta_l, L^2(C_l)) = \bigoplus_{n \in \mathbb{Z}}(\Delta_l(n), L_n^2(C_l))$. In order to study the eigenfunctions on $S_l$ and prove Theorem 1.2, we need to study Dirichlet problems on the nonrotationally invariant pieces, that is, $(\Delta_l(n), L^2_n(C_l))$ $(n \neq 0)$.

Let $C_0$ be the standard cusp (see Definition 2.1), $\Delta_0(n)$ and $L^n_0(C_0)$ be the phase components of $\Delta_0$ and $L^2(C_0)$ respectively on $C_0$ as in the case of $C_l$. For any $n \neq 0$, let $0 \leq \lambda^n_1(l) \leq \lambda^n_2(l) \leq \cdots$ be the Dirichlet eigenvalues of $(\Delta_l(n), L^2_n(C_l))$, and $0 \leq \lambda^n_1(0) \leq \lambda^n_2(0) \leq \cdots$ the Dirichlet eigenvalues of $\Delta_0(n)$ acting $L^n_0(C_0) = L^2([0, +\infty), e^r\, dr)$. Intrinsically, the limit of $C_l$ as $l \to 0$ should consist of a pair of standard cusps. Because of the reasons that will be explained in §2, we only take one standard cusp $C_0$ here. Furthermore, let $\{\varphi^n_k(l)\}^\infty_{k=1}$ be a complete system of orthonormal Dirichlet eigenfunctions of $\{\lambda^n_k(l)\}^\infty_{k=1}$. Then they satisfy the following symmetry relations (Lemma 4.4):

$$\varphi^n_k(l)(-r) = \pm\varphi^n_k(l)(r) \tag{1.5}$$

(recall that $\varphi^n_k(l)(r)$ is defined for $r \in [-\tau(l), \tau(l)]$). Similarly let $\{\varphi^n_k(0)\}^\infty_{k=1}$ be the corresponding orthonormal Dirichlet eigenfunctions of $\{\lambda^n_k(0)\}^\infty_{k=1}$. Then we have

**Theorem 1.5** (Dirichlet Problem on $C_l$; $n \neq 0$). *With the notation as above, for any $n \neq 0$, and $k \geq 1$,*

$$\lim_{l\to 0} \lambda^n_{2k-1}(l) = \lim_{l\longrightarrow 0} \lambda^n_{2k}(l) = \lambda_k(0),$$

$$\lim_{l\to 0} \varphi^n_{2k-1}(l)^2(r - \tau(l)) = \lim_{l\to 0} \varphi^n_{2k-1}(l)^2(\tau(l) - r) = \frac{1}{2}\varphi^n_k(0)^2(r),$$

$$\lim_{l\to 0} \varphi^n_{2k}(l)^2(r - \tau(l)) = \lim_{l\to 0} \varphi^n_{2k}(l)^2(\tau(l) - r) = \frac{1}{2}\varphi^n_k(0)^2(r)$$

*uniformly for $r$ in compact subsets of* $[0, +\infty)$.

Using an analogue of Maass-Selberg relation (Lemma 5.3) and a characterization of Eisenstein series in terms of its growth in all the cusps of $S_0$ (Lemma 5.2), we derive Theorem 1.2 from Theorem 1.5.

The organization of this paper is as follows. In §2, we recall the collar theorem for hyperbolic Riemann surfaces with short geodesics (of lengths $\leq \frac{1}{2}$). In §3, using the monotonicity for eigenvalues with respect to potentials and domains, we prove Theorem 1.3. Then using the regular perturbation theory, we prove Theorems 1.4 and 1.1. We also give the conjectural lower bound for $N_l(x)$ mentioned earlier and heuristic arguments for this conjecture. In §4, we use the Feynman-Kac formula to prove Theorem 1.5. Then in §5, we prove Theorem 1.2.

Finally, in §6, we speculate on some questions related to the spectral degeneration for $S_l$, in particular, a characterization of embedded eigenvalues of $S_0$ through degeneration. One of the motivations of this paper is to understand $\mathrm{Spec}(S_0)$ through degeneration. By very simple arguments, we can prove known facts about small eigenvalues ($< \frac{1}{4}$) for noncompact surfaces from the corresponding results for compact surfaces (see [5] and [12]), thus justifying partially this point of view. Besides this, the spectral degeneration of $S_l$ is an interesting and subtle singular perturbation problem involving continuous spectrum and embedded eigenvalues.

**Acknowledgments**

I would like to thank Professor S. T. Yau for the suggestion of the problem and his encouragement and deep insights. I would also like to thank Professor M. Goresky for a lot of encouragement and numerous help in all aspects.

I am grateful to Professors D. Hejhal and S. Wolpert for their most generous help with early versions of this paper. I would also thank Professor J. Bernstein for showing me the proof of Lemma 5.2. Especially I would like to thank the referees for their careful reading of the manuscript, their inspiring comments, and pointing out a lot of small mistakes. They suggested improvements in the lower bounds for the counting function $N_l(x)$ in Theorem 1.1 in §3, and inspired the author to modify the conjecture in §6 and to elaborate on various questions.

I wish to thank my friends P. Z. Ong and Bich-Van Nguyen for a lot of conversations and their help during the preparation of this paper.

## 2. Standard collars around the pinching geodesics

In this section, we shall recall the Collar Theorem (Theorem 2.2) on structures of disjoint standard collars around short geodesics and standard cusps near punctures of any hyperbolic Riemann surfaces. Then in terms of the Fermi coordinates with respect to the pinching geodesics, we write down the phase decomposition for the Laplacians of the standard collars and the cusps (equation (2.5)). In order to express the fact that the collar around a pinching geodesic converges (intrinsically) to a pair of cusps in terms of explicit coordinates, we have to shift the Fermi coordinate $r$ (equation (2.1)). Finally we study the spectral analysis on the standard cusps to prepare for the proof of Theorems 1.1, 1.3 and 1.5.

At first, we set

**Definition 2.1.** 1. A hyperbolic cylinder with core geodesic length $l$ and width $2w$ is a cylinder $\{(r, \theta) | -w \leq r \leq w, \ 0 \leq \theta \leq 1\}/(r, 0) \sim (r, 1)$ endowed with the hyperbolic metric $dr^2 + l^2 \cosh^2(r) d^2\theta$.

2. The standard cusp $C_0$ is a half infinite cylinder $C_0 = \{(r, \theta) | 0 \leq r < +\infty, \ 0 \leq \theta \leq 1\}/(r, 0) \sim (r, 1)$ with the hyperbolic metric $dr^2 + e^{-2r} d\theta^2$.

**Remark.** The definition of the cusp here is easily seen to be the same as the usual definition $C_0 = \{z | \operatorname{Im}(z) \geq 1\}/\{z \sim z + 1\}$ with the hyperbolic metric $y^{-2}(dx^2 + dy^2)$. Then we have the following collar theorem for hyperbolic Riemann surfaces, which is due to L. Keen [21], J. P. Matelski [27, the main Lemma and Remark 6.6], and B. Randol [29].

**Theorem 2.2** (The Collar Theorem). 1. *There is a universal constant $\alpha > 0$ (for simplicity we take $\alpha = \frac{1}{2}$ from now on ) such that for any simple closed geodesic $\gamma$ on any hyperbolic Riemann surface $S$ with $|\gamma| \leq \frac{1}{2}(= \alpha)$, there is a hyperbolic cylinder with $\gamma$ as its core geodesic and of width $2\tau(l) = 2 \operatorname{arcsinh}(\frac{1}{2} \operatorname{csch} \frac{|\gamma|}{2})$ ($\sim 2 \log \frac{2}{|\gamma|}$ as $|\gamma| \to 0$) embedded isometrically in $S$, which is called the standard collar of $\gamma$ and denoted by $C_\gamma$.*

2. *For each of the punctures of any hyperbolic Riemann surface $S$, the standard cusp $C_0$ is embedded isometrically near the puncture.*

3. *For any surface $S$ as above, all the standard collars around short geodesics (with lengths $\leq \frac{1}{2}$) and the standard cusps around the punctures are disjoint.*

For a general degenerating family of hyperbolic Riemann surfaces $S_l$, we have more than one pinching geodesics. It is important that all the standard collars around different pinching geodesics are disjoint (Theorem 2.2(3)). For simplicity, we assume that there is only one pinching geodesic $\gamma(l)$ of length $l$, and the standard collar for $\gamma(l)$ is denoted by $C_l$. Intrinsically, as $l \to 0$, the collar $C_l$ should converge to a pair of the standard cusps $C_0 \cup C_0$. But in terms of the Fermi coordinates $(r, \theta)$ with respect to the pinching geodesic $\gamma(l)$ on $C_l$, $C_l$ does not converge to $C_0 \cup C_0$, since $l \cosh r \to 0 \neq e^{-2r}$ as $l \to 0$. An important observation here is that we shift and fix the left boundary of $C_l$ at $r = 0$. Then as $l \to 0$, the core pinching geodesic $\gamma(l)$ moves to infinity towards right, so to speak, and $C_l$ converges to $C_0$ (see equation (2.1)). It seems that the right cusp in the limit is missing. Actually the right cusp is at infinity in terms of the shifted coordinate $r$. Since the right cusp is isometric to the left cusp, we can concentrate on studying the left cusp which is at finite place, taking the symmetry into consideration. Examples of this symmetric consideration include the proofs of Theorem 1.4 (in particular, (3.20)) and Lemma 4.5, and the statements of Theorem 1.5 and Lemma 4.4.

More precisely, the collar $C_l$ around the pinching geodesic $\gamma(l)$ of length $l$ $(\leq \frac{1}{2})$ can be represented as $\{(r, \theta)|0 \leq r \leq 2\tau(l), 0 \leq \theta \leq 1\}/(r, 0) \sim (r, 1)$ with the hyperbolic metric $dr^2 + l^2 \cosh^2(r - r(l))d\theta^2$, where $\tau(l) = \operatorname{arcsinh}(\frac{1}{2} \operatorname{csch} \frac{1}{2}) \sim \log(\frac{2}{l})$ as $l \to 0$. Then in these shifted coordinates, the geometric convergence of $C_l$ to $C_0$ at finite $r$ follows from

$$\lim_{l \to 0} l \cosh(r - \tau(l)) = e^{-r}. \tag{2.1}$$

We are now going to express the Laplacian $\Delta_l$ of the standard collar $C_l$ in terms of the coordinates $(r, \theta)$. Recall that the Beltrami-Laplace operator $\Delta$ for any Riemannian manifold $(M, g_{ij}\, dx_i\, dx_j)$ is given by

$$\Delta = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j} \frac{\partial}{\partial x_i} \left( \sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial x_j} \right),$$

where $(g^{ij}) = (g_{ij})^{-1}$. For $l \geq 0$, let $\Delta_l$ be the Beltrami-Laplace operator for $C_l$. Then after direct computations, we have that for $l > 0$,

$$\Delta_l = \frac{\partial^2}{\partial r^2} - \tanh(r - \tau(l)) \frac{\partial}{\partial r} + \frac{1}{l^2 \cosh^2(r - \tau(l))} \frac{\partial^2}{\partial \theta^2}, \tag{2.2}$$

where $r \in [0, 2\tau(l)]$, $\theta \in [0, 1]$. Since $C_l$ is rotationally symmetric, that is, $S^1$ acts on $C_l$ and then on $L^2(C_l)$, we have the following decomposition according to phases,

$$L^2(C_l) = \bigoplus_{n \in \mathbb{Z}} L^2_n(C_l), \tag{2.3}$$

where $L^2_n(C_l) = \{f(r)e^{2\pi\sqrt{-1}n\theta} | f(r)e^{2\pi\sqrt{-1}n\theta} \in L^2(C_l)\}$ with the induced norm from $L^2(C_l)$, that is,

$$L^2_n(C_l) \simeq L^2([0, 2\tau(l)], l \cosh(r - \tau(l))dr). \tag{2.4}$$

Since $\Delta_l$ commutes with the rotational $S^1$ action, we can decompose $(\Delta_l, L^2(C_l))$ correspondingly,

$$\begin{aligned} (\Delta_l, L^2(C_l)) &\simeq \bigoplus_{n \in \mathbb{Z}} (\Delta_l(n), L^2_n(C_l)) \\ &\simeq \bigoplus_{n \in \mathbb{Z}} (\Delta_l(n), L^2([0, 2\tau(l)], l \cosh(r - \tau(l))dr)), \end{aligned} \tag{2.5}$$

where the operator $\Delta_l(n)$ acting on the Hilbert subspace $L^2_n(C_l)$ is given by

$$\Delta_l(n) = \frac{d^2}{dr^2} - \tanh(r - \tau(l))\frac{d}{dr} - \frac{4\pi^2 n^2}{l^2\cosh^2(r - \tau(l))}. \tag{2.6}$$

By conjugating the operator $\Delta_l(n)$, we find a unitarily equivalent one $\tilde{\Delta}_l(n)$ which acts on $L^2([0, 2\tau(l)], dr)$, the standard $L^2$ space over the interval $[0, 2\tau(l)]$. More precisely, define

$$\begin{aligned}\widetilde{\Delta}_l(n) &= \cosh^{1/2}(r - \tau(l))\Delta_l(n)\cosh^{-1/2}(r - \tau(l)) \\ &= \frac{d^2}{dr^2} - \left(\frac{1}{2} - \frac{1}{4}\tanh^2(r - \tau(l)) + \frac{4\pi^2 n^2}{l^2\cosh^2(r - \tau(l))}\right) \\ &= \frac{d^2}{dr^2} - \left(\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r - \tau(l)) + \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r - \tau(l))\right).\end{aligned} \tag{2.7}$$

The conjugated operator $\widetilde{\Delta}_l(n)$ is still in divergence form. It is important to note that only the Dirichlet boundary condition is preserved under the above conjugation. Then with respect to the Dirichlet boundary condition, for any $n \in \mathbf{Z}$ we have $(\Delta_l(n), L^2_n(C_l)) \simeq (\widetilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr))$ unitarily. Thus for the Dirichlet boundary condition,

$$(\Delta_l, L^2(C_l)) \simeq \bigoplus_{n\in\mathbb{Z}}(\widetilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr)). \tag{2.8}$$

Similarly, the Beltrami-Laplacian $\Delta_0$ of the standard cusp $C_0$ is given by

$$\Delta_0 = \frac{\partial}{\partial r^2} - \frac{\partial}{\partial r} + e^{2r}\frac{\partial^2}{\partial\theta^2}, \tag{2.9}$$

and is decomposed into $(\Delta_0, L^2(C_0)) = \bigoplus_{n\in\mathbb{Z}}(\Delta_0(n), L^2_n(C_0))$, where we have the Hilbert space $L^2_n(C_0) = \{f(r)e^{2\pi\sqrt{-1}n\theta} | f(r) \in L^2([0, +\infty), e^{-r}\,dr)\}$ and the operator $\Delta_0(n)$ acting on it:

$$\Delta_0(n) = \frac{d^2}{dr^2} - \frac{d}{dr} - 4\pi^2 n^2 e^{2r}. \tag{2.10}$$

The conjugated operator $\widetilde{\Delta}_0(n)$ is given by

$$\widetilde{\Delta}_0(n) = \frac{d^2}{dr^2} - \frac{1}{4} - 4\pi^2 n^2 e^{2r}, \tag{2.11}$$

and acts on $L^2([0, \infty), dr)$. Finally with respect to the Dirichlet boundary condition,

$$(2.12) \qquad (\Delta_0, L^2(C_0)) \simeq \bigoplus_{n\in\mathbb{Z}} (\widetilde{\Delta}_0(n), L^2([0, +\infty), dr).$$

We now recall the spectral analysis on $C_0$.

**Proposition 2.3** (Spectrum of $C_0$). *With respect to the Dirichlet boundary condition, $(\Delta_0, L^2(C_0))$ has continuous spectrum $[\frac{1}{4}, +\infty)$, which comes from the rotationally invariant piece $(\Delta_0(0), L_0^2(C_0))$. There are also infinitely many embedded eigenvalues from $(\Delta_0(n), L_n^2(C_0))$ for all $n \neq 0$. The continuous part of the spectral decomposition of $\Delta_0$ is given by the generalized eigenfunctions $\widetilde{E}(r, \theta; \frac{1}{2} + \sqrt{-1}t) = e^{r/2}\sin(rt)$ for all $t \geq 0$.*

**Remark.** We use the notation $\widetilde{E}(\cdot; \frac{1}{2} + \sqrt{-1}t)$ to resemble the Eisenstein series $E(\cdot; \frac{1}{2} + \sqrt{-1}t)$ which are the generalized eigenfunctions of general noncompact hyperbolic Riemann surfaces (see §5 for details).

*Proof.* With respect to the Dirichlet boundary condition, $\Delta_0(n)$ is unitarily equivalent to $\widetilde{\Delta}_0(n)$ for all $n \in \mathbb{Z}$, where $\widetilde{\Delta}_0(n)$ is given by (2.11).

First we assume that $n = 0$. As is well known, the following Dirichlet eigenvalue problem,

$$(2.13) \qquad \begin{cases} (\widetilde{\Delta}_0(n) + \lambda)f = \frac{d^2}{dr^2}f - \frac{1}{4}f + \lambda f = 0, \\ f(0) = 0, \qquad f \in L^2([0, +\infty), dr), \end{cases}$$

has continuous spectrum $[\frac{1}{4}, +\infty)$, and the generalized eigenfunctions are given by $\sin(tr)$ with all $t \geq 0$. Recalling the unitary isomorphism between $\Delta_0(n)$ and $\widetilde{\Delta}_0(n)$, we see that $\widetilde{E}(\cdot; \frac{1}{2} + \sqrt{-1}t) = e^{r/2}\sin(rt)$ are the generalized eigenfunctions of $(\Delta_0, L^2(C_0))$ for all $t \geq 0$.

Next we assume that $n \neq 0$. Recall from (2.11),

$$\widetilde{\Delta}_0(n) = \frac{d^2}{dr^2} - \left(\frac{1}{4} + r\pi^2 n^2 e^{2r}\right).$$

Since $\frac{1}{4} + r\pi^2 n^2 e^{2r} \to +\infty$ as $r \to +\infty$, by a Theorem of Titchmarsh-Weyl (see [34, §5.5, 5.9] or [29, Property B on p. 300 and Lemma 3]), the following Dirichlet eigenvalue problem,

$$(2.14) \qquad \begin{cases} \widetilde{\Delta}_0(n)f + \lambda f = 0, \\ f(0) = 0, \qquad f \in L^2([0, +\infty), dr), \end{cases}$$

has only discrete spectrum which does not accumulate at any finite point. By the Max-Min principle (Lemma 3.1), all the Dirichlet eigenvalues are strictly larger that $\frac{1}{4}$, and are embedded in the continuous spectrum $[\frac{1}{4}, +\infty)$ of $(\Delta_0, L^2(C_0))$.

The following is the Decomposition Principle [13, Proposition 2.1] mentioned earlier in the introduction.

**Proposition 2.4** (Decomposition Principle). *Let $M$ be a complete Riemannian manifold, and let $K \subset M$ be any compact submanifold. Then the continuous spectrum of $M$ is the same as the continuous spectrum of $M \setminus K$ with the Dirichlet boundary condition on $\partial K$.*

From Proposition 2.3, we have the following well-known fact.

**Corollary 2.5** (The Continuous Spectrum of $S_0$). *Let $S_0$ be any noncompact finite volume hyperbolic Riemann surface; then it has continuous spectrum $[\frac{1}{4}, +\infty)$.*

*Proof.* Let $K = S_0 \setminus \bigcup C_0$ be the complement of all the standard cusps in $S_0$; then $K$ is compact, and the conclusion follows from Propositions 2.3 and 2.4.

## 3. Bounds on the clustering of eigenvalues

In this section, we will first state the variational characterizations of eigenvalues (Lemmas 3.1 and 3.2) and the monotonicity properties of eigenvalues (Lemmas 3.3, 3.4 and 3.5). Applying the results in §2, we prove Theorems 1.3, 1.4 and 1.1. Then we discuss improvements for the lower bound for $N_l(x)$ in Theorem 1.1 and a conjectural lower bound.

Now we recall some basic materials on operators of divergence form (see [6, Chapter 1] and [10, Chapter VI]). Associated to each operator of divergence form

$$L = \frac{d}{dr}\left(a(r)\frac{d}{dr}\right) - V(r), \qquad m \le r \le M, \tag{3.1}$$

where $a(r) > 0$ for $r \in [m, M]$ and $V(r)$ is the potential, we have a quadratic form

$$Q(u, v) = \int_m^M a(r)\frac{du(r)}{dr}\frac{dv(r)}{dr} + V(r)u(r)v(r)\,dr,$$

where $u, v$ are two $C^1$ functions over $[m, M]$. The Green's formula for the operator $L$ is given by (see [10, p. 278])

$$\int_m^M vL(u)dr + Q(u, v) = a(r)\frac{du(r)}{dr}v(r)|_m^M, \tag{3.2}$$

which follows from integration by parts. Let $b(r)$ be a positive function on $[m, M]$. Then the following Dirichlet eigenvalue problem

$$\begin{cases} Lu(r) + \lambda b(r)u(r) = 0, \\ u(m) = u(M) = 0 \end{cases} \tag{3.3}$$

can be described through the *Rayleigh quotient*

$$\frac{Q(u, u)}{\int_m^M b(r)u^2(r)\,dr}.$$

That is, we have

**Lemma 3.1** (The Max-Min Principle). *Let $\{\lambda_k\}_1^\infty$ be all the Dirichlet eigenvalues of the problem* (3.3) *counted with multiplicity. Then for any $k \geq 1$,*

$$\lambda_k = \max_{\{\varphi_1, \cdots, \varphi_{k-1}\}} \min_{\{f \mid \int_m^M b(r)\varphi_i(r)f(r)\,dr = 0,\ 1 \leq i \leq k-1\}} \frac{Q(r, r)}{\int_m^M b(r)f^2(r)\,dr}, \tag{3.4}$$

*where $\varphi_i \in L^2([m, M])$ for $1 \leq i \leq k-1$ and $f \in H_0^1([m, M])$. The space $H_0^1([m, M])$ is the Sobolev space over $[m, M]$ with vanishing boundary values. Note that for $k = 1$, there is no other restriction on $f$ besides $f \in H_0^1([m, M])$ in* (3.4).

**Lemma 3.2** (Courant-Weyl Principle). *With the same notation as in Lemma* 3.1, *for any $k \geq 1$,*

$$\lambda_k = \max_{\{l_1, \cdots, l_{k-1}\}} \min_{\{f \mid l_i(f) = 0,\ 1 \leq i \leq k-1\}} \frac{Q(f, f)}{\int_m^M b(r)f^2(r)\,dr}, \tag{3.5}$$

*where $l_i$ $(i = 1, \cdots, k-1)$ are linearly independent functionals on $H_0^1([m, M])$, and $f \in H_0^1([m, M])$, the Sobolev space as above.*

**Remarks.** 1. If the end points of the interval $[m, M]$ are not finite, then the corresponding vanishing boundary conditions are dropped in Lemmas 3.1 and 3.2.

2. There is a similar variational characterization for the Neumann eigenvalues. The only difference is that in this case the admissible Hilbert space $H_0^1([m, M])$ is replaced by the Sobolev space $H^1([m, M])$ without vanishing boundary conditions (see [6, pp. 14–17]).

From the above Max-Min principle, we get immediately various monotonicity properties of eigenvalues (see [6, Chapter 1, §5]).

**Lemma 3.3** (Potential Monotonicity). *Let $L_1 = d^2/dr^2 - V_1(r)$ and $L_2 = d^2/dr^2 - V_2(r)$ be two operators over $[m, M]$ with $V_1(r) \geq V_2(r)$ for all $r \in [m, M]$. Then the eigenvalues of $L_1$ are larger than the corresponding eigenvalues of $L_2$ with respect to either the Dirichlet or the Neumann boundary conditions.*

**Lemma 3.4** (Domain Monotonicity). *Let $L$ be an operator of divergence form as in* (3.1) *and* $-\infty \le a < b < c \le +\infty$ *be three real numbers. Then the following hold*:

1. *Let all the Dirichlet eigenvalues of* $(L, L^2([a, c]))$ *be* $\{\lambda_i\}_1^\infty$ *counted with multiplicity, and combine all the Dirichlet eigenvalues of* $(L, L^2([a, b]))$ *and* $(L, L^2([b, c]))$ *into an increasing sequence* $\{\tilde{\lambda}_i\}_1^\infty$. *Then* $\lambda_i \le \tilde{\lambda}_i$ *for all* $i \ge 1$. *In particular, the Dirichlet eigenvalues of* $(L, L^2([a, c]))$ *are smaller than the corresponding Dirichlet eigenvalues of* $(L, L^2([a, b]))$.

2. *Let all the Neumann eigenvalues of* $(L, L^2([a, c]))$ *be* $\{\mu_i\}_1^\infty$ *counted with multiplicity, and combine all the Neumann eigenvalues of* $(L, L^2([a, b]))$ *with the Neumann eigenvalues of* $(L, L^2([b, c]))$ *into an increasing sequence* $\{\tilde{\mu}_i\}_1^\infty$. *Then* $\mu_i \ge \tilde{\mu}_i$ *for all* $i \ge 1$.

Similarly, using the Courant-Weyl principle, we have [36, Theorem 9.1 and Corollary 1]

**Lemma 3.5** (Boundary Condition Monotonicity). *Let $L$ be an operator of divergence form as in* (3.1) *over* $[m, M]$, *and* $b(r)$ *be a continuous positive function on* $[m, M]$. *Furthermore, let* $\{\lambda_i\}_1^\infty$ *and* $\{\mu_i\}_1^\infty$ *be respectively the Dirichlet and Neumann eigenvalues of* $Lu(r)+\lambda b(r)u(r) = 0$, $r \in [m, M]$. *Then for all* $i \ge 1$

$$\mu_i \le \lambda_i \le \mu_{i+2}.$$

After these preparations, we now start to prove Theorems 1.3, 1.4 and 1.1.

*Proof of Theorem* 1.3. Recall from §2 that $\Delta_l$ is the Beltrami-Laplace operator of the standard collar $C_l$ and can be decomposed according to phases as: $(\Delta_l, L^2(C_l)) = \bigoplus_{n\in\mathbb{Z}}(\Delta_l(n), L_n^2(C_l))$ (equation (2.5)). With respect to the Dirichlet boundary condition, we have a unitary equivalence

$$(\Delta_l(0), L_0^2(C_l)) \simeq (\widetilde{\Delta}_l(0), L_0^2([0, 2\tau(l)], dr)), \tag{3.6}$$

where the conjugated operator $\widetilde{\Delta}_l(n)$ is given by (equation (2.7))

$$\widetilde{\Delta}_l(0) = \frac{d^2}{dr^2} - \left(\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r - \tau(l))\right). \tag{3.7}$$

We would like to bound the Dirichlet eigenvalues of $\widetilde{\Delta}_l(0)$ (or $\Delta_l(0)$) by explicitly computable eigenvalue problems.

By the potential monotonicity (Lemma 3.3), the Dirichlet eigenvalues of

$$\begin{cases} \widetilde{\Delta}_l(0)u(r) + \lambda u(r) = 0, \\ u(0) = u(2\tau(l)) = 0 \end{cases} \tag{3.8}$$

are clearly larger than the corresponding Dirichlet eigenvalues of the following problem

$$(3.9)\qquad \begin{cases} \frac{d^2}{dr^2}u(r) - \frac{1}{4}u(r) + \lambda u(r) = 0, \\ u(0) = u(2\tau(l)) = 0. \end{cases}$$

On the other hand, by the domain monotonicity (Lemma 3.4), the Dirichlet eigenvalues of the problem (3.8) are smaller than the corresponding Dirichlet eigenvalues of the following problem:

$$(3.10)\qquad \begin{cases} \widetilde{\Delta}_l(n)u(r) + \lambda u(r) = 0, \\ r \in [0, (1-\delta)\tau(l)] \cup [(1+\delta)\tau(l), 2\tau(l)], \\ u(0) = u((1-\delta)\tau(l)) = u((1+\delta)\tau(l)) = u(2\tau(l)) = 0, \end{cases}$$

where $\delta$ is any constant satisfying $0 < \delta < 1$. Note that for any $r \in [0, (1-\delta)\tau(l)] \cup [(1+\delta)\tau(l), 2\tau(l)]$,

$$\frac{1}{4}\operatorname{sech}^2(r - \tau(l)) \le \frac{1}{4}\operatorname{sech}^2(\delta\tau(l)).$$

Thus by the potential monotonicity (Lemma 3.3) again, the eigenvalues of the Dirichlet problem (3.10) are smaller than the corresponding Dirichlet eigenvalues of

$$(3.11)\qquad \begin{cases} \frac{d^2}{dr^2}u(r) - (\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(\delta\tau(l)))u(r) + \lambda u(r) = 0, \\ r \in [0, (1-\delta)\tau(l)] \cup [(1+\delta)\tau(l), 2\tau(l)], \\ u(0) = u((1-\delta)\tau(l)) = u((1+\delta)\tau(l)) = u(2\tau(l)) = 0. \end{cases}$$

Therefore the Dirichlet eigenvalues of the problem (3.8) are bounded from below and above respectively by the corresponding Dirichlet eigenvalues of the problems (3.9) and (3.11). Clearly the Dirichlet eigenvalues of these two problems accumulate at every point of $[\frac{1}{4}, +\infty)$ as $l \to 0$ and $\delta \to 0$. Then it follows that the Dirichlet eigenvalues of the problem (3.8) cluster at every point of $[\frac{1}{4}, +\infty)$ as $l \to 0$. By the unitary equivalence (equation (3.6)), the Dirichlet eigenvalues of $(\Delta_l(0), L_0^2(C_l))$ cluster at every point of $[\frac{1}{4}, +\infty)$ as $l \to 0$.

We are now going to bound the rate of the clustering of the eigenvalues. From the previous paragraph, the Dirichlet eigenvalues of $(\Delta_l(0), L_0^2(C_l))$ are bounded by the corresponding Dirichlet eigenvalues of the problems (3.9) and (3.11) respectively from below and above. The asymptotics of these two problems are for the classical harmonic oscillators and can be evaluated explicitly. More precisely, for $l < \frac{1}{2}$ and any $x > \frac{1}{4}$,

$$\begin{aligned}
&|\{\lambda|\lambda,\ \text{an eigenvalue of } (3.9),\ \tfrac{1}{4} \le \lambda \le x\}| \\
&\quad = \left[\frac{1}{\pi} 2\tau(l)\sqrt{x - \frac{1}{4}}\right] \\
&\quad \le \frac{2}{\pi}\tau(l)\sqrt{x - \frac{1}{4}},
\end{aligned}$$

$$\begin{aligned}
&|\{\lambda|\lambda,\ \text{an eigenvalue of } (3.11),\ \tfrac{1}{4} \le \lambda \le x\}| \\
&\quad = 2\left[\frac{1}{\pi}(1-\delta)\tau(l)\sqrt{\left[x - \frac{1}{4} - \frac{1}{4}\operatorname{sech}^2(\delta\tau(l))\right]^+}\right] \\
&\quad \ge \frac{2}{\pi}(1-\delta)\tau(l)\sqrt{\left[x - \frac{1}{4} - \frac{1}{4}\operatorname{sech}^2(\delta\tau(l))\right]^+} - 2,
\end{aligned} \tag{3.12}$$

where the function $[t]$ equals the integral part of $t$ and the function $[t]^+ = \max\{t, 0\}$ for $t \in \mathbb{R}$. Recall the definition of $DN_l^0(x)$,

$$DN_l^0(x) = |\{\lambda|\lambda \text{ is an eigenvalue of } (3.8), \tfrac{1}{4} \le \lambda \le x\}|.$$

Then it is clear that for $l < \frac{1}{2}$ and any $x > \frac{1}{4}$,

$$\begin{aligned}
\frac{2}{\pi}(1-\delta)\tau(l)\sqrt{\left[x - \frac{1}{4} - \frac{1}{4}\operatorname{sech}^2(\delta\tau(l))\right]^+} - 2 &\le DN_l^0(x) \\
&\le \frac{2}{\pi}\tau(l)\sqrt{x - \frac{1}{4}}.
\end{aligned}$$

Noticing that for $M > m > 0$, $\sqrt{M - m} \ge \sqrt{M} - m/\sqrt{M}$, and $|\operatorname{sech}^2(y)| \le 4e^{-2y}$ for $y > 0$, we get immediately

$$DN_l^0(x) \ge \frac{2}{\pi}\tau(l)\left[\sqrt{x - \frac{1}{4}} - \frac{e^{-2\delta\tau(l)}}{\sqrt{x - \frac{1}{4}}}\right]^+ - \frac{2}{\pi}\delta\tau(l)\sqrt{x - \frac{1}{4}} - 2.$$

Let $0 < \delta = \frac{\log\tau(l)}{\tau(l)} \le e^{-1} < 1$; it then follows that

$$\begin{aligned}
DN_l^0(x) &\ge \frac{2}{\pi}\left[\tau(l)\sqrt{x - \frac{1}{4}} - \left(\tau(l)\sqrt{x - \frac{1}{4}}\right)^{-1}\right]^+ \\
&\quad - \frac{2}{\pi}\log(\tau(l))\sqrt{x - \frac{1}{4}} - 2 \\
&= \frac{2}{\pi}\Theta\left[\tau(l)\sqrt{x - \frac{1}{4}}\right] - \frac{2}{\pi}\log(\tau(l))\sqrt{x - \frac{1}{4}} - 2,
\end{aligned} \tag{3.13}$$

where the function $\Theta[t] = \max\{0, t - t^{-1}\}$ for $t > 0$. This proves the part 1 of the lemma.

Next we prove the part 2. By the Max-Min principle (Lemma 3.1) and (3.7), any Dirichlet eigenvalue $\lambda(l)$ of $(\Delta_l(0), L_0^2(C_l))$ satisfies

$$\lambda(l) \geq \frac{1}{4}. \tag{3.14}$$

Then $\lambda(0) = \lim_{l\to 0}\lambda(l) \geq \frac{1}{4}$. Write $\lambda(0) = \frac{1}{4} + t^2$ for some $t \geq 0$. Let $K_l$ be a constant satisfying $K_l \frac{d}{dr}\tilde{\varphi}(l)(\tau)|_{r=0} = t$, where $\tilde{\varphi}(l)(r) = \varphi(l)(r)(l\cosh(r-\tau(l)))^{1/2}$. Since $\Delta_l(0)\varphi(l) + \lambda(l)\varphi(l) = 0$, by (2.7) we have

$$\frac{d^2}{dr^2}K_l\tilde{\varphi}(l) - \left(\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r-\tau(l)) - \lambda(l)\right)K_l\tilde{\varphi}(l) = 0.$$

Notice that $\lim_{l\to 0}\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r-\tau(l)) - \lambda(l) = t^2$ for any finite $r > 0$, $K_l\tilde{\varphi}(l)(0) = 0$ and $K_l\frac{d}{dr}\tilde{\varphi}(l)(r)|_{r=0} = t$. Then from the stability of initial value problems for ordinary differential equations with respect to coefficients it follows that $\lim_{l\to 0}K_l\tilde{\varphi}(l)(r) = \sin(rt)$ uniformly for $r$ in compact subsets of $[0, +\infty)$, because $(d^2/dr^2 - t^2)\sin(rt) = 0$, $\sin(rt)|_{r=0} = 0$ and $\frac{d}{dr}\sin(rt)|_{r=0} = t$. Therefore,

$$\lim_{l\to 0}K_l\varphi(l)(r) = \widetilde{E}\left(r; \frac{1}{2} + \sqrt{-1}t\right) = e^{r/2}\sin(rt)$$

uniformly for $r$ in compact subsets of $[0, +\infty)$. This completes the proof of part 2 and Theorem 1.3. q.e.d.

The lower bound of $DN_l^0(x)$ in (3.13) is not optimal and can be improved greatly by choosing splittings of the interval $[0, 2\tau(l)]$ finer than $[0, (1-\delta)\tau(l)]\cup[(1+\delta)\tau(l), 2\tau(l)]$ in the eigenvalue problem (3.10). There is a conjectural lower bound for $DN_l^0(x)$, which is $\frac{2}{\pi}\tau(l)\sqrt{x - \frac{1}{4}} - b(x)$, where $b(x)$ is some constant independent of $l < \frac{1}{2}$. Correspondingly, the lower bound for $N_l(x)$ in Theorem 1.1 (equation (1.1)) can be improved. We will postpone these discussions after the proofs of Theorems 1.4 and 1.1.

*Proof of Theorem* 1.4. For $0 < l < \frac{1}{2}$ and $n \in \mathbb{Z}$, denote the Neumann and Dirichlet eigenvalues of $(\Delta_l(n), L_n^2(C_l))$ by $\{\mu_i^n(l)\}_1^\infty$ and $\{\lambda_i^n(l)\}_1^\infty$ respectively. We are going to prove Theorem 1.3 in two steps:

1. $(n = 0)$. The eigenvalues $\{\mu_i^0(l)\}_1^\infty$ cluster at every point of $[\frac{1}{4}, +\infty)$ as $l \to 0$ at the rate stated in Theorem 1.4.
2. $(n \neq 0)$. All other eigenvalues $\bigcup_{n\neq 0}\{\mu_i^n(l)\}_1^\infty$ do not accumulate at any point of $[\frac{1}{4}, +\infty)$ and thus do not contribute to the clustering.

*Step* 1 $(n = 0)$. Since $\Delta_l(0)$ is of divergence form with a zero potential, by the Green's formula (3.2) or the Max-Min principle (Lemma 3.1), it is clear that $\mu_1^0(l) = 0$ and $\mu_i^0(l) \geq 0$ for all $i \geq 2$. Note that the Neumann boundary condition is not preserved by the unitary transformation from $\Delta_l(0)$ to $\tilde{\Delta}_l(0)$. Thus we cannot bound the Neumann eigenvalues of $\Delta_l(0)$ in terms of the Neumann eigenvalues of $\tilde{\Delta}_l(0)$ as in the case of the proof of Theorem 1.3 for the Dirichlet problem. Instead, we compare the Neumann eigenvalues with the Dirichlet eigenvalues. According to Lemma 3.5, for all $i \geq 1$,

$$\mu_i^0(l) \leq \lambda_i^0(l) \leq \mu_{i+2}^0(l). \tag{3.15}$$

From (3.14), $\lambda_1^0(l) \geq \frac{1}{4}$. Thus $\mu_i^0(l) \geq \frac{1}{4}$ for all $i \geq 3$. It follows then for any $x > \frac{1}{4}$

$$DN_l^0(x) - 2 \leq NN_l^0(x) \leq DN_l^0(x) + 2,$$

where $DN_l^0(x)$ and $NN_l^0(x)$ are respectively the Neumann and Dirichlet counting functions of $\Delta_l(0)$. We have discarded the first two Neumann eigenvalues, and $-2$ appears on the left. On the other hand, $+2$ appears on the right because of the shifting of indices by 2 in (3.15). From the bounds of $DN_l^0(x)$ in Theorem 1.3, we get immediately that for $l < \frac{1}{2}$ and any $x > \frac{1}{4}$,

$$\begin{aligned}\frac{2}{\pi}\Theta\left[\tau(l)\sqrt{x - \frac{1}{4}}\right] - \frac{2}{\pi}\log(\tau(l))\sqrt{x - \frac{1}{4}} - r \leq NN_l^0(x)& \\ \leq \frac{2}{\pi}\tau(l)\sqrt{x - \frac{1}{4}} + 2.&\end{aligned}$$

*Step* 2 $(n \neq 0)$. Now we are going to show that $\bigcup_{n\neq 0}\{\mu_i^n(l)\}_1^\infty$ do not accumulate at any finite point as $l \to 0$. Let $\varphi$ be any Neumann eigenfunction of $\Delta_l(n)$ with eigenvalue $\mu_i^n(l)$ satisfying

$$\begin{aligned}&\frac{1}{l}\operatorname{sech}(r - \tau(l))\frac{d}{dr}\left(l\cosh(r - \tau(l))\frac{d\varphi}{dr}\right) \\ &\quad - \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r - \tau(l))\varphi + \mu_i^n(l)\varphi \\ &= 0.\end{aligned}$$

Multiply both sides of the above equation by $l\cosh(r-\tau(l))\varphi$ and integrate over $[0, 2\tau(l)]$. Using the Neumann boundary condition and integration by parts, we get

$$\begin{aligned}(3.16)\qquad &\int_0^{2\tau(l)} l\cosh(r-\tau(l))\left(\frac{d\varphi}{dr}\right)^2 dr\\ &\quad + 4\pi^2 n^2 \int_0^{2\tau(l)} \frac{1}{l^2}\operatorname{sech}^2(r-\tau(l))l\cosh(r-\tau(l))\varphi^2\,dr\\ &= \mu_i^n(l)\int_0^{2\tau(l)} l\cosh(r-\tau(l))\varphi^2\,dr.\end{aligned}$$

Note that $\tau(l) = \operatorname{arcsinh}(\frac{1}{2}\operatorname{csch}\frac{l}{2}) \sim \log\frac{2}{l}$ as $l \to 0$ and

$$\underline{\lim}_{l\to 0}\frac{1}{l^2}\operatorname{sech}^2(r-\tau(l)) \geq \underline{\lim}_{l\to 0}\frac{1}{l^2}\operatorname{sech}^2(\tau(l)) = 1.$$

Then there is some constant $c_0 > 0$ such that for $l < \frac{1}{2}$ and any $r \in [0, 2\tau(l)]$, $(1/l^2)\operatorname{sech}^2(r-\tau(l)) \geq c_0$. Substituting this inequality into (3.16), we obtain immediately

$$c_0 4\pi^2 n^2 \int_0^{2\tau(l)} l\cosh(r-\tau(l))\varphi^2 dr \leq \mu_i^n(l)\int_0^{2\tau(l)} l\cosh(r-\tau(l))\varphi^2\,dr.$$

That is, for all $i \geq 1$ and any $n$,

$$\mu_i^n(l) \geq c_0 4\pi^2 n^2. \tag{3.17}$$

For any $x > \frac{1}{4}$, let $n_0(x) = [\sqrt{x/c_0 4\pi^2}] + 1$. Then for any $n \geq n_0(x)$ and $i \geq 1$, $\mu_i^n(l) > x$. Therefore, $\bigcup_{n\geq n_0(x)}\{\mu_i^n(l)\}_1^\infty$ has no intersection point with $[\frac{1}{4}, x]$.

Now we look at the case where $0 \neq |n| < n_0(x)$. Since

$$\lim_{l\to 0}\frac{1}{l^2}\operatorname{sech}^2(r-\tau(l)) = e^{2r}, \tag{3.18}$$

we can pick constants $l_0(x) > 0$ and $0 < r_0(x) < \tau(l_0(x))$ such that for $l \leq l_0(x)$ and $r \in [r_0(x), 2r - \tau(l)_0(x)]$,

$$\frac{1}{l^2}\operatorname{sech}^2(r-\tau(l)) > \frac{x}{4\pi^2}. \tag{3.19}$$

By a similar computation as in (3.16), and replacing the interval $[0, 2\tau(l)]$ by $[r_0(x), 2\tau(l) - r_0(x)]$, we show that any Neumann eigenvalue $\mu$ of $(\Delta_l(n), L^2([r_0(x), 2\tau(l) - r_0(x)], l\cosh(r-\tau(l))dr))$ $(n \neq 0)$ satisfies

$$\mu > \frac{x}{4\pi^2}4\pi^2 n^2 \geq x.$$

That is, when $n \neq 0$ and $l \leq l_0(x)$ there are no Neumann eigenvalues of $(\Delta_l(n), L^2([r_0(x), 2\tau(l) - r_0(x)], l\cosh(r-\tau(l))dr))$ belonging to $[\frac{1}{4}, x]$.

In order to bound the Neumann eigenvalues of $(\Delta_l(n), L^2([0, 2\tau(l)], l\cosh(r-\tau(l))dr))$ from below by the domain monotonicity (Lemma 3.4), we need to consider the Neumann eigenvalues of $(\Delta_l(n), L^2([0, r_0(x)] \cup [2\tau(l)-r_0(x), 2\tau(l)], l\cosh(r-\tau(l))dr))$. Notice that for $1 \leq |n| < n_0(x)$, the Neumann eigenvalues of $(\Delta_l(n), L^2[0, r_0(x)], l\cosh(r-\tau(l))dr))$ converge to the Neumann eigenvalues of $(\Delta_0(n), L^2([0, r_0(x)], e^r dr))$ as $l \to 0$, which has of course only discrete spectrum. It implies that for $l \leq l_0(x)$, there are at most finitely many Neumann eigenvalues of

$$\bigcup_{1\leq|n|<n_0(x)} (\Delta_l(n), L^2([0, r_0(x)], l\cosh(r-\tau(l))dr))$$

belonging to $[\frac{1}{4}, x]$. Via the substitution $r' = 2\tau(l) - r$, it is clear that

$$\begin{aligned}(\Delta_l(n), L^2([2\tau(l) - r_0(x), 2\tau(l)], l\cosh(r-\tau(l))dr)) \\ \simeq (\Delta_l(n), L^2([0, r_0(x)], l\cosh(r-\tau(l))dr)).\end{aligned} \tag{3.20}$$

We have considered the case $0 < l \leq l_0$. Notice further that for $0 < l_0(x) \leq l < \frac{1}{2}$, the Neumann problems

$$\bigcup_{1\leq|n|<n_0(x)} (\Delta_l(n), L^2([0, 2\tau(l)], l\cosh(r-\tau(l))dr))$$

form a smooth family in $l$. It then follows that there exists a constant $B(x)$ independent of $l$ such that for $l < \frac{1}{2}$, the number of the eigenvalues of $\bigcup_{1\leq|n|<n_0(x)}\{\mu_i^n(l)\}_1^\infty$ belonging to $[\frac{1}{4}, +\infty)$ is bounded from above by $B(x)$. In particular, $\bigcup_{1\leq|n|<n_0(x)}\{\mu_i^n(l)\}_1^\infty$ do not cluster at any point of $[\frac{1}{4}, x]$ as $l \to 0$.

Combining the above discussions about the two cases $|n| \geq n_0(x)$ and $1 \leq |n| < n_0(x)$, we see that $\bigcup_{n\neq 0}\{\mu_i^n(l)\}_1^\infty$ do not cluster at any point of $[\frac{1}{4}, x]$. Since $x > \frac{1}{4}$ is arbitrary, $\bigcup_{n\neq 0}\{\mu_i^n(l)\}_1^\infty$ do not cluster at any point of $[\frac{1}{4}, +\infty)$. This proves Step 2.

Finally, noticing that the Neumann eigenvalues of $(\Delta_l, L^2(C_l))$ consist of $\{\mu_i^0(l)\}_1^\infty$ and $\bigcup_{n\neq 0}\{\mu_i^n(l)\}_1^\infty$, we get the conclusions in Theorem 1.4 immediately. The proof of Theorem 1.4 is now complete.

*Proof of Theorem* 1.1. By the assumption, $S_l$ is a degenerating family of hyperbolic Riemann surfaces with $m$ $(m \geq 1)$ pinching geodesics $\{\gamma_i(l), \cdots, \gamma_m(l)\}$, and the pinching geodesics $\gamma_i(l)$ have length $l_i$ with $\lim_{l\to 0} l_i = 0$ for $1 \leq i \leq m$. Let $l_0 > 0$ be a constant such that $l_i < \frac{1}{2}$ $(i = 1, \cdots, m)$ if $l \leq l_0$. For $l \leq l_0$, let $\bigcup C_l$ be the union of all the standard collars around the pinching geodesics in $S_l$, and $\bigcup C_0$ be the union of all the standard cusps embedded in $S_0$. Then for $0 \le l \le l_0$, $S_l \setminus \bigcup C_l$ is a compact Riemannian manifold with boundary. Furthermore, $\{S_l \setminus \bigcup C_l\}_{0 \le l \le l_0}$ form a smooth family of compact manifolds with boundary (see [3, Property IV in §3]). Therefore, the eigenvalues of $S_l \setminus \bigcup C_l$ converge to the corresponding eigenvalues of $S_0 \setminus \bigcup C_0$ as $l \to 0$, with respect to both the Dirichlet and the Neumann boundary conditions. This follows from the general theory for regular perturbations in K. Kodaira and D. C. Spencer [22] or T. Kato [19]. In particular, the eigenvalues of $S_l \setminus \bigcup C_l$, with respect to either boundary condition, do not accumulate at any finite point as $l \to 0$. Applying the domain monotonicity (Lemma 3.4) to $S_l = (S_l \setminus \bigcup C_l) \bigcup (\bigcup C_l)$ and Theorems 1.3 and 1.4, we see immediately that the eigenvalues of $S_l$ cluster at any point of $[\frac{1}{4}, +\infty)$ as $l \to 0$.

We are now going to bound the rate of the clustering. By the domain monotonicity for the Dirichlet eigenvalues, $\lambda_i(S_l) \le \lambda_i(\bigcup C_l)$ for all $i \le 1$. Since there are at most $4g-2$ eigenvalues of $S_l$ less than $\frac{1}{4}$ (see [4]), the lower bound for $N_l(x)$ follows from the lower bound of $DN_l^0(x)$ in Theorem 1.3.

On the other hand, by the domain monotonicity for the Neumann eigenvalues, an upper bound for $N_l(x)$ is given by the summation of the total number of Neumann eigenvalues of $(\Delta_l, S_l \setminus \bigcup C_l)$ and $(\Delta_l, \bigcup C_l)$ belonging to $[\frac{1}{4}, x]$. By the discussion at the beginning of the proof, the number of the Neumann eigenvalues of $(\Delta_l, S_l \setminus \bigcup C_l)$ belonging to $[\frac{1}{4}, x]$ is bounded from above by a constant $B_2(x)$ independent of $0 \le l \le l_0$. Then the upper bound for $N_l(x)$ follows from the upper bound of $NN_l(x)$ in Theorem 1.4. q.e.d.

As mentioned earlier, the lower bound for $N_l(x)$ in Theorem 1.1 can be improved. It is clear from the proof of Theorem 1.1 that it suffices to improve the lower bounds for $DN_l^0(x)$. Recall from (3.8) that $DN_l^0(x)$ is the spectral counting function of the following Dirichlet problem:

$$
(3.21) \qquad \begin{cases} [d^2/dr^2 - (\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r - \tau(l))]u(r) + \lambda u(r) = 0, \\ r \in [0, 2\tau(l)], \\ u(0) = u(2\tau(l)) = 0. \end{cases}
$$

We need to bound the eigenvalues of (3.21) from above by some explicitly computable eigenvalue problems. The idea is to make better approximations than the problem (3.10). For any $1 > \delta > \varepsilon > 0$, consider the following subintervals:

$$[0, (1-\delta)\tau(l)] \cup [(1-\delta)\tau(l), (1-\varepsilon)\tau(l) \cup [(1+\varepsilon)\tau(l), (1+\delta)\tau(l)] \\ \cup [(1+\delta)\tau(l), 2\tau(l)] \subset [0, 2\tau(l)].$$

Notice that for $r \in [0, (1-\delta)\tau(l)] \cup [(1+\delta)\tau(l), 2\tau(l)]$,

$$\frac{1}{4}\operatorname{sech}^2(r-\tau(l)) \leq \frac{1}{4}\operatorname{sech}^2(\delta\tau(l)),$$

and for $r \in [(1-\delta)\tau(l), (1-\varepsilon)\tau(l)] \cup [(1+\varepsilon)\tau(l), (1+\delta)\tau(l)]$,

$$\frac{1}{4}\operatorname{sech}^2(r-\tau(l)) \leq \frac{1}{4}\operatorname{sech}^2(\varepsilon\tau(l)).$$

Then by the domain monotonicity (Lemma 3.4(1)) and the potential monotonicity (Lemma 3.3), the Dirichlet eigenvalues of the problem (3.21) are smaller than the corresponding ones in the combination of the following two Dirichlet problems:

$$(3.22) \qquad \begin{cases} [d^2/dr^2 - (\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(\delta\tau(l)))]u(r) + \lambda u(r) = 0, \\ r \in [0, (1-\delta)\tau(l)] \cup [(1+\delta)\tau(l), 2\tau(l)], \\ u(0) = u((1-\delta)\tau(l)) = u((1+\delta)\tau(l)) = u(2\tau(l)) = 0, \end{cases}$$

$$(3.23) \quad \begin{cases} [d^2/dr^2 - (\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(\varepsilon\tau(l)))]u(r) + \lambda u(r) = 0, \\ r \in [(1-\delta)\tau(l), (1-\varepsilon)\tau(l)] \cup [(1+\varepsilon)\tau(l), (1+\delta)\tau(l)], \\ u((1-\delta)\tau(l)) = u((1-\varepsilon)\tau(l)) = u((1+\varepsilon)\tau(l)) = u((1+\delta)\tau(l)) = 0. \end{cases}$$

For any $x > \frac{1}{4}$, let $N_{l,\delta}(x) = |\{\lambda \text{ an eigenvalue of (3.22) } | \frac{1}{4} \leq \lambda \leq x\}|$ and $N_{l,\varepsilon} = |\{\lambda, \text{ an eigenvalue of (3.23) } | \frac{1}{4} \leq \lambda \leq x\}|$ be the spectral counting functions of the the problems (3.22) and (3.23) respectively. Then for any $x > \frac{1}{4}$, $l > 0$ and $0 < \varepsilon < \delta < 1$,

$$N_l(x) \geq N_{l,\delta}(x) + N_{l,\varepsilon}.$$

Now we take $\delta = \frac{\log \tau(l)}{\tau(l)}$; then by (3.13),

$$N_{l,\delta}(x) \geq \frac{2}{\pi}\left[\tau(l)\sqrt{x-\frac{1}{4}} - \left(\tau(l)\sqrt{x-\frac{1}{4}}\right)^{-1}\right] - \frac{2}{\pi}\log\tau(l)\sqrt{x-\frac{1}{4}} - 2 \\ \geq \frac{2}{\pi}\tau(l)\sqrt{x-\frac{1}{4}} - \frac{2}{\pi}\log\tau(l)\sqrt{x-\frac{1}{4}} - 3,$$

when $l$ is small. For the problem (3.23),

$$\begin{aligned} N_{l,\varepsilon} &\geq \frac{2}{\pi}\tau(l)(\delta-\varepsilon)\sqrt{x-\frac{1}{4}-e^{-2\varepsilon\tau(l)}}-2 \\ &\geq \frac{2}{\pi}\tau(l)\delta\left[\sqrt{x-\frac{1}{4}}-\frac{e^{-2\varepsilon\tau(l)}}{\sqrt{x-\frac{1}{4}}}\right]-\frac{2}{\pi}\tau(l)\varepsilon\sqrt{x-\frac{1}{4}}-2 \\ &= \frac{2}{\pi}\log\tau(l)\left[\sqrt{x-\frac{1}{4}}-\frac{e^{-2\varepsilon\tau(l)}}{\sqrt{x-\frac{1}{4}}}\right]-\frac{2}{\pi}\tau(l)\varepsilon\sqrt{x-\frac{1}{4}}-2. \end{aligned}$$

Then for $x > \frac{1}{4}$ and $l$ small,

$$N_l(x) \geq \frac{2}{\pi}\tau(l)\sqrt{x-\frac{1}{4}}-\frac{2}{\pi}\tau(l)\varepsilon\sqrt{x-\frac{1}{4}}-\frac{2}{\pi}\log\tau(l)\frac{e^{-2\varepsilon\tau(l)}}{\sqrt{x-\frac{1}{4}}}-5.$$

For small $l$, take $\varepsilon = \log\circ\log\tau(l)/\tau(l)$. Then it follows that $0 < \varepsilon < \delta < 1$, and

$$N_l(x) \geq \frac{2}{\pi}\tau(l)\sqrt{x-\frac{1}{4}}-\frac{2}{\pi}\log\circ\log\tau(l)\sqrt{x-\frac{1}{4}}-c_2,$$

where $c_2$ is some constant independent of $l$. By taking a better approximation to $[0, 2\tau(l)]$ using $0 < \varepsilon' < \varepsilon < \delta < 1$, and a similar argument, we can get that for small $l$,

$$N_l(x) \geq \frac{2}{\pi}\tau(l)\sqrt{x-\frac{1}{4}}-\frac{2}{\pi}\log\circ\log\circ\log\tau(l)\sqrt{x-\frac{1}{4}}-c_3,$$

where $c_3$ is a constant. We can repeat this process and prove that for any $n \in \mathbb{N}$ and small $l$,

$$N_l(x) \geq \frac{2}{\pi}\tau(l)\sqrt{x-\frac{1}{4}}-\frac{2}{\pi}\log\circ\log\cdots\circ\log\tau(l)\sqrt{x-\frac{1}{4}}-c_n,$$

where $\log\circ\log\circ\cdots\circ\log$ is $n$ compositions of log, and $c_n$ is a constant independent of $l$. The above inequality strongly suggests the following conjectural lower bound for $N_l(x)$: for any $x > \frac{1}{4}$ and $l$,

$$N_l(x) \geq \frac{2}{\pi}\tau(l)\sqrt{x-\frac{1}{4}}-c, \tag{3.24}$$

where $c$ is a constant independent of $l$. Note that the constants $c_n \to +\infty$ as $n \to \infty$. Thus we cannot use the above arguments to prove the conjectural lower bound for $N_l(x)$.

## 4. Stability of embedded eigenvalues for the degenerating collars

In this section, we will recall the Feynman-Kac formula (Proposition 4.1), and use it to prove the convergence of the Dirichlet heat kernel of $(\Delta_l(n), L^2_n(C_l))$ as $l \to 0$ (Lemma 4.2) for $n \neq 0$. By bounding from below the Dirichlet eigenvalues of $\Delta_l(n)$ restricted to subcollars of $C_l$, we prove the convergence of the Dirichlet eigenvalues of $(\Delta_l(n), L^2_n(C_l))$ as $l \to 0$. This, combined with the convergence of the heat kernels, would finish the proof of Theorem 1.5.

For any operator $L = d^2/dr^2 - V(r)$ acting on $L^2([m, M])$, its (minimal) Dirichlet heat kernel is a function $P_L(x, y, t)$ in $C^\infty([m, M] \times [m, M] \times \mathbb{R}_+)$ which satisfies

$$\begin{cases} L_x P_L(x, y, t) = \frac{\partial}{\partial t} P_L(x, y, t), \\ P_L(m, y, t) = P_L(M, y, t) = 0, \\ \lim_{l \to 0} P_L(x, y, t) = \delta_y(x), \end{cases}$$

where $L_x$ is the operator $L$ acting on the $x$ variable, and the $\delta_y(\cdot)$ is the Dirac delta function at $y$. If the end points of the interval $[m, M]$ are not finite, then the corresponding vanishing boundary conditions are dropped. In this case, the minimal Dirichlet heat kernel is defined to be the limit of the Dirichlet heat kernels of an exhausting family of finite subintervals of $[m, M]$ (see [6, Chapter VIII]).

An important fact is that the Dirichlet heat kernel $P_L(x, y, t)$ can be expressed through the Brownian motion on $\mathbb{R}^1$. That is, we have [30, Equation (3)]

**Proposition 4.1** (Feynman-Kac Formula). *With the notation as above, the Dirichlet heat kernel $P_L(x, y, t)$ can be written as*

$$P_L(x, y, t) = P(x, y, t) E\{\exp(-\int_0^{2t} \frac{1}{2} V(r(s)) ds) | r(0) = x,\ r(2t) = y,\ r(s) \in [m, M] \text{ for } s \in [0, 2t]\},$$

*where $P(x, y, t) = \frac{1}{2\pi t} \exp(-\frac{(x-y)^2}{4t})$ is the heat kernel of the operator $d^2/dr^2$ on $\mathbb{R}^1$, and $E\{g(r(\cdot)) | r(\cdot) \in \Omega\}$ is the integration of $g$ over a measurable subset $\Omega$ of the Wiener space of $\mathbb{R}^1$ with respect to the Wiener measure.*

Let $n \neq 0$ be an integer, which will be fixed throughout this section. Let $P_l(x, y, t)$ be the Dirichlet heat kernel of $(\tilde{\Delta}_l(n), L^2([0, 2\tau(l)] dr))$, and $P_0(x, y, t)$ be the minimal Dirichlet heat kernel of $(\tilde{\Delta}_0(n), L^2([0, \infty), dr))$.

Recall from (2.7) and (2.11) that

$$\tilde{\Delta}_l(n) = \frac{d^2}{dr^2} - \left[\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r - \tau(l)) + \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r - \tau(l))\right],$$

$$\tilde{\Delta}_l(0) = \frac{d^2}{dr^2} - \left[\frac{1}{4} + r\pi^2 n^2 e^{2r}\right].$$

Then using the Feynman-Kac formula, we obtain

$$\begin{aligned} &P_l(x, y, t) \\ &\quad = P(x, y, t)\left\{\exp\left(-\int_0^{2t}\frac{1}{2}\left[\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r(s) - \tau(l))\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. + \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r(s) - \tau(l))\right] ds\right)\right| \\ &\qquad\qquad r(0) = x,\ r(2t) = y,\ r(s) \in [0, 2\tau(l)] \text{ for } s \in [0, 2t]\Big\}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &P_0(x, y, t) \\ &\quad = P(x, y, t)E\left\{\exp\left(-\int_0^{2t}\frac{1}{2}\left[\frac{1}{4} + 4\pi^2 n^2 e^{2r(s)}\right] ds\right)\right| r(0) = x, \\ &\qquad\qquad r(2t) = y,\ r(s) \in [0, \infty) \text{ for } s \in [0, 2t]\Big\}. \end{aligned} \tag{4.2}$$

We are going to use the above expressions of the heat kernels to prove

**Lemma 4.2.** *With the notation as above, for any* $x, y \geq 0$ *and* $t > 0$,

$$\lim_{l \to 0} P_l(x, y, t) = P_0(x, y, t).$$

*Proof.* Since

$$\exp\left(-\int_0^{2t}\frac{1}{2}\left[\frac{1}{4}+\frac{1}{4}\operatorname{sech}^2(r(s)-\tau(l))+\frac{r\pi^2 n^2}{l^2}\operatorname{sech}^2(r(s)-\tau(l))\right]ds\right) \leq 1,$$

$$\exp\left(-\int_0^{2t}\frac{1}{2}\left[\frac{1}{4} + r\pi^2 n^2 e^{2r(s)}\right] ds\right) \leq 1,$$

and $E\{1\} = 1$, by the Lebesgue dominated convergence theorem it suffices to prove the pointwise convergence. Each point in the domain of integration for $P_0(x, y, t)$ corresponds to a continuous path in $[0, \infty)$ with $x$

and $y$ as its end points, that is, a continuous map $r\colon [0, 2t] \to [0, \infty)$ with $r(0) = x$ and $r(2t) = y$. Since $\{r(x) | s \in [0, 2t]\}$ is a compact subset of $[0, \infty)$,

$$\lim_{l \to 0} \frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r(s) - \tau(l)) + \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r(s) - \tau(l)) = \frac{1}{4} + 4\pi^2 n^2 e^{2r(s)}$$

uniformly for $s \in [0, 2t]$. This implies the pointwise convergence, and thus proves Lemma 4.2. q.e.d.

Recall from (3.18) that $\lim_{l\to 0}(1/l^2)\operatorname{sech}^2(r - \tau(l)) = e^{2r}$. Furthermore, from (3.19), for any $x > \frac{1}{4}$, let $l_0(x) > 0$ and $0 < r_0(x) < \tau(l_0(x))$ be constants such that for any $l \le l_0(x)$ and $r \in [r_0(x), 2\tau(l) - r_0(x)]$,

$$\frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r - \tau(l)) > x.$$

**Lemma 4.3.** *With the notation as above, let* $\lambda_1([r_0(x), 2\tau(l) - r_0(x)])$ *be the first Dirichlet eigenvalue of* $(\tilde{\Delta}_l(n), L^2([r_0(x), 2\tau(l) - r_0(x)], dr))$. *Then for* $l \le l_0(x)$,

$$\lambda_1([r_0(x), 2\tau(l) - r_0(x)]) > x.$$

*Proof.* By the above assumptions on $l_0(x) > 0$ and $r_0(x) > 0$, the potential of $\tilde{\Delta}_l(n)$ is strictly larger than $x$ when $r$ is restricted to the subdomain $[r_o(x), 2\tau(l) - r_0(x)]$. Thus the conclusion follows from the Max-Min principle (Lemma 3.1). q.e.d.

For convenience, we set up the following:

**Lemma 4.4.** (1) *Any Dirichlet eigenvalue of* $(\tilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr))$ *is of multiplicity one.*

(2) *Let* $\varphi$ *be any Dirichlet eigenfunction of* $(\tilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr))$. *Then* $\varphi(2\tau(l) - r) = \pm\varphi(r)$.

*Proof.* Let $\lambda$ be a Dirichlet eigenvalue of $(\tilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr))$, and $\varphi$ be an eigenfunction of $\lambda$. Then $\varphi$ satisfies the following ordinary differential equation:

(4.3)
$$\frac{d^2}{dr^2}\varphi(r) - \left[\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r - \tau(l)) + \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r - \tau(l))\right]\varphi(r) + \lambda\varphi(r) = 0.$$

By the uniqueness of the initial value problems to ordinary differential equations, the eigenspace of $\lambda$ is one-dimensional, that is, $\lambda$ is of multiplicity one. This proves part (1). For part (2), notice that (4.3) is invariant under the substitution $r \to 2\tau(l) - r$, and thus $\varphi(2\tau(l) - r)$ also satisfies (4.3). Therefore, by part (1), $\varphi(2\tau(l) - r) = c\varphi(r)$, where $c$ is a constant. By iterating this equality, we obtain $\varphi(2\tau(l) - r) = c\varphi(r) = c^2\varphi(2\tau(l) - r)$, that is, $c^2 = 1$. It then follows that $c = \pm 1$ and $\varphi(2\tau(l) - r) = \pm\varphi(r)$. q.e.d.

Before proceeding to prove Theorem 1.5, we first need to establish the following Lemma 4.5 which is similar to Proposition 7.1 and Theorem 7.2 in [17]. However, for completeness, we include here the proof which also brings out the symmetric consideration pointed out at the beginning of §2.

**Lemma 4.5.** *For any integer $k_0 \geq 1$, let $\psi_1(l), \cdots, \psi_{k_0}(l)$ be any orthonormal Dirichlet eigenfunctions of $(\tilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr))$ with eigenvalues $\nu_1(l), \cdots, \nu_{k_0}(l)$ respectively. Assume that for a sequence $l_j \to 0$, for all $1 \leq k \leq k_0$, $\lim_{j\to\infty} \nu_k(l_j)$ exists, and $\psi_k(l_j)(r)$ converges to some function $\psi_k(0)$ uniformly for $r$ in compact subsets of $[0, \infty)$. Then $\psi_k(2\tau(l)-r)$ converges to some function $\psi_k^*(r)$ uniformly for $r$ in compact subsets of $[0, \infty)$ $(1 \leq k \leq k_0)$, the limit function $\psi_1(0), \cdots, \psi_{k_0}(0)$, $\psi_1^*(0), \cdots, \psi_{k_0}^*(0)$ are all Dirichlet eigenfunctions of $(\tilde{\Delta}_0(n)$, $L^2([0, \infty)], dr))$, and the $k_0$ pairs of functions $(\psi_1(0), \psi_1^*(0)), \cdots$, $(\psi_{k_0}(0), \psi_{k_0}^*(0))$ are linearly independent.*

*Proof.* By Lemma 4.4 we obtain $\psi_k(l)(2\tau(l) - r) = \pm\psi_k(l)(r)$ for $1 \leq k \leq k_0$. Since $\psi_k(l_j)$ converges to $\psi_k(0)(r)$ uniformly for $r$ in compact subsets of $[0, \infty)$, it is clear that $\psi_k(l_j)(2\tau(l_j)-r)$ converges to $\pm\psi_k(0)(r)$ uniformly for $r$ in compact subsets of $[0, \infty)$, by taking a subsequence if necessary, and the limit function is denoted by $\psi_k^*(0)$ $(1 \leq k \leq k_0)$.

If $(\psi_1(0), \psi_1^*(0)), \cdots, (\psi_{k_0}(0), \psi_{k_0}^*(0))$ are not linearly independent, then there exist constants $a_1, \cdots, a_{k_0}$ not all zero such that $\sum_1^{k_0} a_k(\psi_k(0)$, $\psi_k^*(0)) = 0$. In particular,

$$\lim_{l_j\to 0} \sum_1^{k_0} a_k \psi_k(l_j)(r) = \lim_{l_j\to 0} \sum_1^{k_0} a_k \psi_k(l_j)(2\tau(l_j) - r) = 0 \tag{4.4}$$

uniformly for $r$ in compact subsets of $[0, \infty)$. Let $\nu_k(0) = \lim_{l_j\to 0} \nu_k(l_j)$ for $1 \leq k \leq k_0$, $x = \max\{\nu_1(0), \cdots, \nu_{k_0}(0)\} + 1$, and $r_0(x)$ be the constant in Lemma 4.3. Furthermore, let $\eta(r)$ be a cut-off function with $\eta(r) = 1$ for $[r_0(x) + 1, 2\tau(l) - r_0(x) - 1]$, $\eta(r) = 0$ for $r \leq r_0(x)$ or $r \geq 2\tau(l) - r_0(x)$, $|\eta(r)| \leq 1$ and $|\frac{d\eta}{dr}| \leq 2$ for all $r$. Let $V_l(r) = \frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r - \tau(l)) + (4\pi^2 n^2/l^2)\operatorname{sech}^2(r - \tau(l))$ be the potential of $\tilde{\Delta}_l(n)$.

Then by the Max-Min principle (Lemma 3.1) and Lemma 4.3 we have

$$\begin{aligned}(4.5)\qquad &\frac{\int_0^{2\tau(l_j)}[(\frac{d}{dr}(\eta\sum_1^{k_0}a_k\psi_k(l_j)))^2+V_{l_j}(\eta\sum_1^{k_0}a_k\psi_k(l_j))^2]\,dr}{\int_0^{2\tau(l_j)}(\eta\sum_1^{k_0}a_k\psi_k(l_j))^2\,dr}\\ &\geq\lambda_1([r_0(x)2\tau(l)-r_0(x)])>x.\end{aligned}$$

On the other hand

(4.6)

$$\begin{aligned}&\int_0^{2\tau(l_j)}\left[\left(\frac{d}{dr}\left(\eta\sum_1^{k_0}a_k\psi_k(l_j)\right)\right)^2+V_{l_j}\left(\eta\sum_1^{k_0}a_k\psi_k(l_j)\right)^2\right]dr\\ &=\int_0^{2\tau(l_j)}-\eta\frac{d^2\eta}{dr^2}\left(\sum_1^{k_0}a_k\psi_k(l_j)\right)^2-2\eta\frac{d\eta}{dr}\sum_1^{k_0}a_k\psi_k(l_j)\frac{d}{dr}\sum_1^{k_0}a_k\psi_k(l_j)\\ &\quad-\eta^2\sum_1^{k_0}a_k\psi_k(l_j)\sum_1^{k_0}a_k\frac{d^2}{dr^2}\psi_k(l_j)+V_{l_j}\eta^2\left(\sum_1^{k_0}a_k\psi_k(l_j)\right)^2dr\\ &=\left(\int_0^{r_0(x)+1}+\int_{2\tau(l_j)-r_0(x_1)-1}^{2\tau(l_j)}\right)-\eta\frac{d^2\eta}{dr^2}\left(\sum_1^{k_0}a_k\psi_k(l_j)\right)^2\\ &\quad-2\eta\frac{d\eta}{dr}\sum_1^{k_0}a_k\psi_k(l_j)\frac{d}{dr}\sum_1^{k_0}a_k\psi_k(l_j)\\ &\quad+\int_0^{2\tau(l_j)}\eta^2\sum_1^{k_0}a_k\psi_k(l_j)\sum_1^{k_0}a_k\nu_k(l_k)\psi_k(l_j)\,dr.\end{aligned}$$

In the last equality, we have used $(d^2/dr^2)\psi_k(l)=\nu_l\psi_k(l)-\nu_k(l)\psi_k(l)$. Then from (4.6) and (4.4), it follows that for small $l_j$

$$\begin{aligned}&\int_0^{2\tau(l_j)}\left(\frac{d}{dr}\left(\eta\sum_1^{k_0}a_k\psi_k(l_j)\right)\right)^2+V_{l_j}\left(\eta\sum_1^{k_0}a_k\psi_k(l_j)\right)^2dr\\ (4.7)\qquad&\leq\varepsilon(l_j)+\int_0^{2\tau(l_j)}\sum_1^{k_0}a_k\psi_k(l_j)\sum_1^{k_0}a_k\nu_k(l_j)\psi_k(l_j)\\ &\leq\varepsilon'(l_j)+\sum_1^{k_0}a_k^2\nu_k(0)\leq\left(x-\frac{1}{2}\right)\sum_1^{k_0}a_k^2,\end{aligned}$$

where $\varepsilon(l_j)\to 0$ as $l_j\to 0$. Substituting the inequality (4.7) into the

Rayleigh quotient, we get that for small $l_j$,

$$\frac{\int_0^{2\tau(l_j)}[(\frac{d}{dr}(\eta\sum_1^{k_0} a_k\psi_k(l_j)))^2 + V_{l_j}(\eta\sum_1^{k_0} a_k\psi_k(l_j))^2]\,dr}{\int_0^{2\tau(l_j)}(\eta\sum_1^{k_0} a_k\psi_k(l_j))^2\,dr} \leq x - \frac{1}{4} < x.$$

This is a contradiction to the inequality (4.5). Therefore $(\psi_1(0), \psi_1^*(0))$, $\cdots$, $(\psi_{k_0}(0), \psi_{k_0}^*(0))$ are linearly independent. Finally, notice that $\psi_k(0)$ and $\psi_k^*(0)$ clearly satisfy

$$\tilde{\Delta}_0(n)u(r) + \nu_k(0)u(r) = 0, \qquad u(0) = 0.$$

That is, they are Dirichlet eigenfunctions of $(\tilde{\Delta}_0(n), L_n^2([0, \infty), dr))$. The proof of Lemma 4.5 is now complete.

*Proof of Theorem* 1.5. The proof is divided into two steps. First, we use Lemma 4.5 to prove $\lim_{l\to 0}\lambda_{2k-1}^n(l) = \lim_{l\to 0}\lambda_{2k}^n(l) = \lambda_k^n(0)$. Then we use Lemma 4.2 to prove

$$\lim_{l\to 0}\varphi_{2k-1}^n(l)^2(r - \tau(l)) = \lim_{l\to 0}\varphi_{2k}^n(l)^2(r - \tau(l)) = \tfrac{1}{2}\varphi_k^n(0)^2(r)$$

for $r \geq 0$.

*Step* 1. First all, intrinsically the limit of the pinching collar $C_l$ consists of two copies of the standard cusps, $C_0 \cup C_0$. More precisely, denote a function on $C_0 \cup C_0$ by $(\varphi(r), \varphi^*(r))$. Then a sequence of functions $\varphi_l(r)$ on $C_l$ converges to $(\varphi(r), \varphi^*(r))$ if and only if $\lim_{l\to 0}\varphi_l(r - \tau(l)) = \varphi(r)$ and $\lim_{l\to 0}\varphi_l(\tau(l) - r) = \varphi^*(r)$ for $r \in [0, \infty)$.

For $k \geq 1$, let $\lambda_k^n(C_0 \cup C_0)$ be the $k$th Dirichlet eigenvalue of $\tilde{\Delta}_0(n)$ on $C_0 \cup C_0$. Then it is clear that $\lambda_{2k-1}^n(C_0 \cup C_0) = \lambda_{2k}^n(C_0 \cup C_0) = \lambda_k^n(0)$, which is the $k$th Dirichlet eigenvalue of $\tilde{\Delta}_0(n)$ on $C_0$. Since all the eigenvalues $\{\lambda_k^n(C_0 \cup C_0)\}_1^\infty$ can be obtained from the Max-Min principle by restricting the variation process to the subspace $L_n^2(C_0)$, and $C_0 \cup C_0$ is complete, that for all $k \geq 1$, $\overline{\lim}_{l\to 0}\lambda_k^n(l) \leq \lambda_k^n(C_0 \cup C_0)$. That is, for any $k \geq 1$,

$$\overline{\lim}_{l\to 0}\lambda_{2k-1}^n(l) \leq \lambda_k^n(0), \qquad \overline{\lim}_{l\to 0}\lambda_{2k}^n(l) \leq \lambda_k^n(0). \tag{4.8}$$

We need to establish the reverse inequalities. Let $\{\tilde{\varphi}_k(l)\}_1^\infty$ be a complete system of orthonormal Dirichlet eigenfunctions of $(\tilde{\Delta}_l(n), L^2([0, 2\tau(l)], dr))$. Then by the stability of initial value problems to orinary differential equations, for any sequence $l_j \to 0$, there is a subsequence, still denoted by $l_j$, such that for all $k \geq 1$, $\tilde{\varphi}_k(l)$ converges to a function $(\tilde{\varphi}_k(0), \tilde{\varphi}_k^*(0))$ on $C_0 \cup C_0$. By Lemma 4.5, the limit functions $\{(\tilde{\varphi}_k(0), \tilde{\varphi}_k^*(0))\}_1^\infty$ are linearly independent Dirichlet eigenfunctions of $\tilde{\Delta}_0(n)$ on $C_0 \cup C_0$. Of course, this implies that $\lim_{l_j \to 0} \lambda_k^n(l_j) \geq \lambda_k^n(C_0 \cup C_0)$ for $k \geq 1$. Since $l_j \to 0$ is an arbitrary sequence, we have $\underline{\lim}_{l \to 0} \lambda_k^n(l) \geq \lambda_k^n(C_0 \cup C_0)$ for all $k \geq 1$. This, combined with the above inequalities (4.8), implies that for any $k \geq 1$.

$$\lim_{l \to 0} \lambda_{2k-1}^n(l) = \lim_{l \to 0} \lambda_{2k}^n(l) = \lambda_k^n(0).$$

*Step* 2. For $l > 0$, let $\{\varphi_k(l)\}_1^\infty$ be a complete system of orthonormal Dirichlet eigenfunctions of $(\Delta_l(n), L_n^2(C_l))$, where each (unshifted) eigenfunction $\varphi_k^n(l)$ is defined for $r \in [-\tau(l), \tau(l)]$. Let

$$\tilde{\varphi}_k(l)(r) = \varphi_k^n(l)(r - \tau(l))(l \cosh(r - \tau(l)))^{1/2} \tag{4.9}$$

for $r \in [0, 2\tau(l)]$. Then the functions $\{\tilde{\varphi}_k(l)\}_1^\infty$ form a complete system of orthonormal Dirichlet eigenfunctions of $(\tilde{\Delta}_0(n), L^2([0, 2\tau(l)], dr))$, and Dirichlet heat kernel $P_l(r, r, t)$ can be written as

$$P_l(r, r, t) = \sum_1^\infty e^{-\lambda_k^n(l)t} \tilde{\varphi}_k(l)^2(r).$$

Similarly, let $\{\varphi_k(0)\}_1^\infty$ be a complete system of orthonormal Dirichlet eigenfunctions of $(\tilde{\Delta}_0(n), L_n^2(C_0))$. Define

$$\tilde{\varphi}_k(0)(r) = \varphi_k(0)(r)e^{-r/2}, \qquad r \in [0, \infty). \tag{4.10}$$

Then for $r \in [0, \infty)$,

$$P_0(r, r, t) = \sum_1^\infty e^{-\lambda_k^n(0)t} \tilde{\varphi}_k(0)^2(r).$$

Set $x = y = r$ in Lemma 4.2; it follows that for any $r > 0$ and $t > 0$,

$$\lim_{l \to 0} \sum_l^\infty e^{-\lambda_k^n(l)t} \tilde{\varphi}_k(l)^2(r) = \sum_1^\infty e^{-\lambda_k^n(0)t} \tilde{\varphi}_k(0)^2(r).$$

Multiplying both sides of the above equation by $e^{\lambda_1^n(0)t}$, we get

(4.11)

$$\lim_{l \to 0} \tilde{\varphi}_1(l)^2(r) e^{(\lambda_1^n(0) - \lambda_1^n(l))t} + \tilde{\varphi}_2(l)^2(r) e^{\lambda_1^n(0) - \lambda_2^n(l))t} + \sum_3^\infty e^{-(\lambda_k^n(l) - \lambda_1(0))t} \tilde{\varphi}_k(l)^2(r)$$

$$= \tilde{\varphi}_1(0)^2(r) + \sum_2^\infty e^{-(\lambda_k^n(0) - \lambda_1(0))t} \tilde{\varphi}_k(0)^2(r).$$

Note that for $k \geq 2$, $\lim_{l \to 0} \lambda_{2k-1}^n(l) = \lim_{l \to 0} \lambda_{2k}^n(l) = \lambda_k^n(0) > \lambda_1^n(0)$ by step 1 and Lemma 4.4. Furthermore, by the upper bounds of heat kernels of the Gaussian type in [25, Theorem 3.1], $\sum_3^\infty e^{-(\lambda_k^n(l)-\lambda_1(0))t}\tilde{\varphi}_k(l)^2(r)$ are bounded independent of $l$ for $r$ in compact subsets of $[0, \infty)$ and for $t > 0$. More precisely, fix a constant $t_0 > 0$. Then for $t > t_0$,

$$\begin{aligned}
&\left|\sum_3^\infty e^{-(\lambda_k^n(l)-\lambda_1(0))t}\tilde{\varphi}_k(l)^2(r)\right| \\
(4.12)\qquad &\leq e^{-(\lambda_3^n(l)-\lambda_1^n(0))(t-t_0)}\sum_3^\infty e^{-(\lambda_k^n(l)-\lambda_1(0))t}\tilde{\varphi}_k(l)^2(r) \\
&\leq e^{-(\lambda_3^n(l)-\lambda_1^n(0))(t-t_0)}K(r, t_0),
\end{aligned}$$

where $K(r, t_0)$ is a constant independent of $l$. Substituting the inequality (4.12) into (4.11), we see immediately that for $r \geq 0$,

$$\lim_{l\to 0}\tilde{\varphi}_1(l)^2(r) + \tilde{\varphi}_2(l)^2(r) = \tilde{\varphi}_1(0)^2(r).$$

By induction on $k$, we can similarly prove that for all $k \geq 1$ and $r \geq 0$,

$$\lim_{l\to 0}\tilde{\varphi}_{2k-1}(l)^2(r) + \tilde{\varphi}_{2k}(l)^2(r) = \tilde{\varphi}_k(0)^2(r). \tag{4.13}$$

We need further to show that for all $k \geq 1$,

$$\lim_{l\to 0}\tilde{\varphi}_{2k-1}(l)^2(r) = \lim_{l\to 0}\tilde{\varphi}_2(l)^2(r) = \frac{1}{2}\tilde{\varphi}_1(0)^2(r).$$

Actually, by Lemma 4.4, we have

$$\int_0^{\tau(l)}\tilde{\varphi}_{2k-1}(l)^2(r)\,dr = \frac{1}{2}\int_0^{2\tau(l)}\tilde{\varphi}_{2k-1}^n(l)^2(r) = \frac{1}{2}.$$

Furthermore, $\tilde{\varphi}_{2k-1}(l)(0) = 0$, and

$$\begin{aligned}
\left[\frac{d^2}{dr^2} - \left(\frac{1}{4} + \frac{1}{4}\operatorname{sech}^2(r-\tau(l))\right.\right. \\
\left.\left. + \frac{4\pi^2 n^2}{l^2}\operatorname{sech}^2(r-\tau(l)) + \lambda_{2k-1}^n(l)\right)\right]\tilde{\varphi}_{2k-1}^n(l)(r) = 0.
\end{aligned}$$

By the stability of initial value problems for ordinary differential equations and a diagonal argument, for any sequence $l_j \to 0$, there exists a subsequence, still denoted by $l_j$, such that $\tilde{\varphi}_{2k-1}^n(l_j)(r)$ converges to a function $\tilde{\psi}_{2k-1}(0)$ uniformly for $r$ in compact subsets of $[0, \infty)$. Of course, the limit function $\tilde{\psi}_{2k-1}(0)$ satisfies

$$\int_0^\infty \tilde{\psi}_{2k-1}^2(r)dr \leq \overline{\lim_{l_j\to 0}}\int_0^{\tau(l)}\tilde{\varphi}_{2k-1}(l)^2(r) = \frac{1}{2},$$

$$\tilde{\Delta}_0(n)\tilde{\psi}_{2k-1}(l)(r) + \lambda_k^n(0)\tilde{\psi}_{2k-1}(l)(r) = 0, \qquad \tilde{\psi}_{2k-1}(l)(0) = 0.$$

By the uniqueness of solutions to ordinary differential equations, there exists a constant $c$ such that $\tilde{\psi}_{2k-1}(l) = c\tilde{\varphi}_k(0)$. Then clearly

$$c^2 = \int_0^\infty c^2\tilde{\varphi}_k(0)^2(r)\,dr = \int_0^\infty \tilde{\psi}_{2k-1}^2(r)\,dr \le \frac{1}{2}. \tag{4.14}$$

Similarly, there exist a subsequence of $l_j$, still denoted by $l_j$, and a constant $c'$ such that $\lim_{l_j\to 0}\tilde{\varphi}_{2k}(l_j) = c'\tilde{\varphi}_k(0)$ with

$$c'^2 \le \frac{1}{2}. \tag{4.15}$$

Substituting the inequalities (4.14) and (4.15) into (4.13), we get immediately that $c^2 = c'^2 = \frac{1}{2}$, so that

$$\lim_{l_j\to 0}\tilde{\varphi}_{2k-1}(l_j)^2(r) = \lim_{l_j\to 0}\tilde{\varphi}_{2k}(l_j)^2(r) = \frac{1}{2}\tilde{\varphi}_k(0)^2(r).$$

Since the sequence $l_j \to 0$ is arbitrary, we have, for any $k \ge 1$,

$$\lim_{l\to 0}\tilde{\varphi}_{2k-1}(l)^2(r) = \lim_{l\to 0}\tilde{\varphi}_{2k}(l)^2(r) = \frac{1}{2}\tilde{\varphi}_k(0)^2(r)$$

uniformly for $r$ in compact subsets of $[0, \infty)$. Finally, from the unitary transformations ((4.9) and (4.10)) and Lemma 4.4, it follows immediately that for any $k \ge 1$,

$$\lim_{l\to 0}\varphi_{2k-1}^n(l)^2(r-\tau(l)) = \lim_{l\to 0}\varphi_{2k-1}^n(l)^2(\tau(l)-r) = \frac{1}{2}\varphi_k^n(0)^2(r),$$
$$\lim_{l\to 0}\varphi_{2k}^n(l)^2(r-\tau(l)) = \lim_{l\to 0}\varphi_{2k}^n(l)^2(\tau(l)-r) = \frac{1}{2}\varphi_k^n(0)^2(r),$$

for $r$ in compact subsets of $[0, \infty)$. This completes the step 2 and the proof of Theorem 1.5.

## 5. Formation of Eisenstein series

In this section, we study the behavior of the eigenfunctions of $S_l$ as $l \to 0$, and prove Theorem 1.2. Specifically, we will recall some basic properties of Eisenstein series of noncompact surfaces and their characterization in terms of their growths in the cusps of the surfaces (Lemma 5.2). Then we formulate an analogue of the Maass-Selberg relation for compact hyperbolic Riemann surfaces with short geodesics (Lemma 5.3), and use this lemma and Theorem 1.5 to prove Theorem 1.2.

First, let us recall the definition of Eisenstein series for noncompact Riemann surfaces (see [15] for details). Let $S$ be a noncompact and complete hyperbolic Riemann surface of finite area. Assume that $S$ has $p$ cusps, which are denoted by $C_1, \cdots, C_p$. For every cusp $C_i$, we associate with it Eisenstein series $E_i(z; s)$, where $z \in S$ and $s \in \mathbb{C}$. The Eisenstein series are generalized eigenfunctions of $S$ and play an important role in the Selberg trace formula, analytic number theory and automorphic representations.

**Proposition 5.1.** *With the notation as above, the Eisenstein series satisfy the following properties:*

1. *For all* $1 \le i \le p$, $E_i(z; s)$ *is a meromorphic function of* $s \in \mathbb{C}$ *for every fixed* $z \in S$, *and its poles are independent of* $z$.

2. *If a cusp* $C_i$ *is represented as* $C_i = \{z | \operatorname{Im}(z) \ge 1\}/\{z \sim z+1\}$. *Then for* $z = x + \sqrt{-1}y \in C_i$ *and* $s \in \mathbb{C}$, $E_i(x + \sqrt{-1}y; s)$ *has the following Fourier expansion:*

$$E_i(x+\sqrt{-1}y; s) = y^s + \Phi_{ii}(s)y^{1-s} + \sum_{n \neq 0} a_n(i) y^{1/2} K_{s-1/2}(2\pi|n|y)e^{2\pi\sqrt{-1}nx},$$

*where* $\Phi_{ii}(s)$ *is a meromorphic function of* $s \in \mathbb{C}$, $a_n(i)$ $(n \in \mathbb{Z}\setminus\{0\})$ *are constants, and* $K_{s-1/2}(\cdot)$ *is the MacDonald Bessel function* [35, *equation* (6), *p.* 78]. *In particular, if* $s \neq \frac{1}{2}$, $E_i(\cdot; s) \notin L^2(S)$.

3. *Let* $C_j$ *be a different cusp from* $C_i$. *If* $C_j$ *and* $C_i$ *are not on one connected component of* $S$, *then we define* $E_i(z; s) = 0$ *for* $z \in C_j$. *Otherwise, represent* $C_j = \{|\operatorname{Im}(z) \ge 1\}/\{z \sim z+1\}$. *Then for* $z = x + \sqrt{-1}y \in C_j$ *and* $s \in \mathbb{C}$, $E_i(x + \sqrt{-1}y; s)$ *has the following Fourier expansion:*

$$E_i(x + \sqrt{-1}y; s) = \Phi_{ij}(s)y^{1-s} + \sum_{n \neq 0} a_n(ij) y^{1/2} K_{s-1/2}(2\pi|n|y)e^{2\pi\sqrt{-1}nx},$$

*where* $\Phi_{ij}(s)$ *is a meromorphic function of* $s \in \mathbb{C}$, *and* $a_n(ij)$ $(n \in \mathbb{Z}\setminus\{0\})$ *are constants, and* $K_{s-1/2}(\cdot)$ *is the MacDonald Bessel function as above.*

4. *If* $s \in \mathbb{C}$ *is not a pole of* $E_i(z; s)$, *then* $(\Delta + s(1-s))E_i(z; s) = 0$, *where* $\Delta$ *is the Laplacian of* $S$.

From parts 4 and 2 of Proposition 5.1, the Eisenstein series are generalized eigenfunctions of $S$. Actually, for $0 \le 0 < +\infty$ and $1 \le i \le p$, $E_i(z; \frac{1}{2} + \sqrt{-1}t)$ form the spectral measures corresponding to the continuous spectrum of $S$. Notice that $K_\nu(y) \sim \sqrt{\frac{\pi}{2y}}e^{-y}$ as $y \to +\infty$ for any $\nu \in \mathbb{C}$. It follows that the Eisenstein series have polynomial growths in every cusp of $S$. An important fact is that they can be characterized by their growths in all the cusps of $S$. More precisely [31, p. 297], for $t \in \mathbb{C}$, we define the space of automorphic forms $\mathscr{A}(S, t)$ on $S$ with characteristic $t$ consisting of functions $u(z)$ satisfying:

1. $\Delta u(z) + (\frac{1}{4} + t^2)u(z) = 0$ on $z \in S$;

2. for every cusp $C_i = \{z | \operatorname{Im}(z) \geq 1\}/\{z \sim z+1\}$ of $S$, $x + \sqrt{-1}y \in C_i$, the following Fourier expansion holds:

$$u(x+\sqrt{-1}y) = b_0(y) + \sum_{n\neq 0} b_n y^{1/2} K_{\sqrt{-1}t}(2\pi|n|y)e^{2\pi\sqrt{-1}nx}, \tag{5.1}$$

where $b_n$ $(n \neq 0)$ are constants and $K_{\sqrt{-1}t}(\cdot)$ is the MacDonald Bessel function.

Furthermore, we define the space of cusp forms $\mathscr{C}(S, t)$ on $S$ with characteristic $t$ to be a subspace of $\mathscr{A}(S, t)$, which consists of those functions $u(z)$ whose zero terms of their Fourier expansions in every cusp of $S$ vanish, that is, for $x + \sqrt{-1}y \in C_i$,

$$u(x+\sqrt{-1}y) = \sum_{n\neq 0} b_n y^{1/2} K_{\sqrt{-1}t}(2\pi|n|y)e^{2\pi\sqrt{-1}nx}. \tag{5.2}$$

Then we have

**Lemma 5.2** [26, Satz 10 and Satz 11]. *Let $\mathscr{A}(S, t)$ and $\mathscr{C}(S, t)$ be the space of automorphic forms and cusp forms on $S$ with characteristic $t$ respectively as above. Then*

1. $\dim \mathscr{A}(S, t)/\mathscr{C}(S, t) = p$, *where $p$ is the total number of the cusps of $S$.*

2. *If $t \neq 0$, then the images of the Eisenstein series $E_1(z; \frac{1}{2}+\sqrt{-1}t), \cdots, E_p(z; \frac{1}{2}+\sqrt{-1}t)$ form a basis of the quotient space $\mathscr{A}(S, t)/\mathscr{C}(S, t)$.*

3. *If $t = 0$, then some linear combinations of $E_i(z; \frac{1}{2}+\sqrt{-1}t)$ and their derivatives at $t = 0$ form a basis of $\mathscr{A}(S, t)/\mathscr{C}(S, t)$.*

**Remark.** For some (arithmetic) surfaces $E_i(z; \frac{1}{2}+\sqrt{-1}t)$ may have zero Fourier coefficients which vanish at $t = 0$; thus their images in the quotient space are zero. In this case, we should take their derivatives with respect to $t \in \mathbb{C}$ at $t = 0$. This is a standard method in the construction of the modified Bessel functions (see [35, §§3.5, 3.7]).

Next we establish a generalization of the Maass-Selberg relation to compact hyperbolic surfaces with short geodesic (length $< \frac{1}{2}$). Let $\gamma$ be a simple closed geodesic on a compact hyperbolic surface $S$ with length $|\gamma| < \frac{1}{2}$. Further, let $C_\gamma$ be the standard collar around $\gamma$ as in Theorem 2.2, that is, $C_\gamma = \{(r, \theta) | -\tau(|\gamma|) \leq r \leq \tau(|\gamma|), 0 \leq \theta \leq 1\}/\{(r, 0) \sim (r, 1)\}$ with the hyperbolic metric $dr^2 + |\gamma|^2\cosh^2(r)\,d\theta^2$. For any $0 < a < \tau(|\gamma|)$, let $C_\gamma(a) = \{(r, \theta) \in C_\gamma | -\tau(|\gamma|) + a \leq r \leq \tau(|\gamma|) - a\}$ be a subcollar of $C_\gamma$.

For any function $\varphi(r, \theta)$ on $C_\gamma$, let

$$\varphi(r, \theta) = f_0(r) + \sum_{n\neq 0} f_n(r)e^{2\pi\sqrt{-1}n\theta} \tag{5.3}$$

be the Fourier expansion of $\varphi$. For convenience, define $[\varphi]_0(r) = f_0(r)$ and $[\varphi]_1(r, \theta) = \sum_{n\neq 0} f_n(r) e^{2\pi\sqrt{-1}n\theta}$ to be the zero Fourier term and the remaining summation respectively. Motivated by the Maass-Selberg relation (see [23, pp. 18–20]), for any $a \geq 0$, we define a function $\varphi^a$ on $S$ by

$$\varphi^a = \begin{cases} [\varphi]_1 & \text{on } C_\gamma(a), \\ \varphi & \text{elsewhere}. \end{cases} \tag{5.4}$$

Then we have the following.

**Lemma 5.3.** *Let $\varphi$ be any function on $S$ satisfying $\Delta\varphi + (\frac{1}{4} + t^2)\varphi = 0$ with some $t \geq 0$. Then*

$$\int_S |\nabla(\varphi^a)|^2 d\mu = \left(\frac{1}{4} + t^2\right) \int_S |\varphi^a|^2 d\mu + \left(\frac{d[\varphi]_0(a)}{dr}[\varphi]_0(a) - \frac{d[\varphi]_0(-a)}{dr}[\varphi]_0(-a)\right) |\gamma| \cosh(a),$$

*where $[\varphi]_0$ is the zero Fourier coefficient of $\varphi$ on $C_\gamma$, $d\mu$ is the Riemannian measure of $S$, and $|\gamma|$ is the length of $\gamma$.*

*Proof.* By Green's formula (see [6, p. 7]), we have

$$\int_{S\setminus C_\gamma(a)} \Delta\varphi\varphi \, d\mu + \int_{S\setminus C_\gamma(a)} |\nabla(\varphi)|^2 d\mu = \int_0^1 \frac{\partial\varphi(a, \theta)}{\partial r}\varphi(a, \theta)|\gamma| \cosh(a)\, d\theta - \int_0^1 \frac{\partial\varphi(-a, \theta)}{\partial r}\varphi(-a, \theta)|\gamma| \cosh(a)\, d\theta,$$

noticing that $\partial(S \setminus C_\gamma(a)) = \{(a, \theta)|0 \leq \theta \leq 1\} \cup \{(-a, \theta)|0 \leq \theta \leq 1\}$. From (5.3) it follows that

$$\int_{S\setminus C_\gamma(a)} \Delta\varphi\varphi \, d\mu + \int_{S\setminus C_\gamma(a)} |\nabla(\varphi)|^2 d\mu = \left(\sum_{n\in\mathbf{Z}} \frac{df_n(a)}{dr} f_{-n}(a) - \frac{df_n(-a)}{dr} f_{-n}(-a)\right) |\gamma| \cosh(a). \tag{5.5}$$

Similarly,

(5.6)

$$\int_{C_\gamma(a)} \Delta[\varphi]_1 + |\nabla[\varphi]_1|^2 \, d\mu = -\int_0^1 \frac{\partial[p]_1(a, \theta)}{\partial r}[\varphi]_1(a\theta)|\gamma| \cosh(a)\, d\theta + \int_0^1 \frac{\partial[p]_1(-a, \theta)}{\partial r}[\varphi]_1(-a\theta)|\gamma| \cosh(a)\, d\theta$$
$$= \left(\sum_{n\neq 0} -\frac{df_n(a)}{dr} f_{-n}(a) + \frac{df_n(-a)}{dr} f_{-n}(-a)\right) |\gamma| \cosh(a).$$

Adding the above equations (5.5) and (5.6) together yields immediately

$$\begin{aligned}
&\int_S \Delta\varphi^a\varphi^a + |\nabla\varphi^a|^2 d\mu \\
(5.7)\qquad &= \left(\frac{df_0(a)}{dr}f_0(a) - \frac{df_0(-a)}{dr}f_0(-a)\right)|\gamma|\cosh(a) \\
&= \left(\frac{d[\varphi]_0(a)}{dr}[\varphi]_0(a) - \frac{d[\varphi]_0(-a)}{dr}[\varphi]_0(-a)\right)|\gamma|\cosh(a).
\end{aligned}$$

Furthermore, by the assumption $\Delta\varphi^1 + (\frac{1}{4} + t^2)\varphi^a = 0$ on $S \setminus \partial C_\gamma(a)$. Then Lemma 5.3 follows from (5.7). q.e.d.

After these preparations, we are ready to prove Theorem 1.2.

*Proof of Theorem* 1.2. The proof is divided into two parts, corresponding to parts 1 and 2 of Theorem 1.2, and the proof of the part 2 consists of four steps.

For simplicity, we assume that there is only one pinching geodesic $\gamma(l)$ of length $l$ on $S_l$. The standard collar around the pinching geodesic $\gamma(l)$ will be denoted by $C_l$. Further, the Riemannian measure of $S_l$ will be denoted by $d\mu_l$. In the following, we will argue intrinsically on the surfaces, that is, depending on the size of the injectivity radius. Hence, we do not shift the Fermi coordinates on the pinching collar $C_l$ and do not concentrate on studying one side of the pinching collar as in §§2 and 3. Therefore we argue simultaneously for both sides of the pinching collar.

*Proof of Part* 1. By the assumption, $\varphi(l)$ is an eigenfunction on $S_l$ with eigenvalue $\lambda(l)$ and $L^2$-norm $1$, that is,

$$\Delta\varphi(l) + \lambda(l)\varphi(l) = 0, \qquad \int_{S_l}\varphi(l)^2\,d\mu_l = 1.$$

From integration by parts, it follows that

$$\int_{S_l}|\nabla\varphi(l)|^2\,d\mu_l = \lambda(l)\int_{S_l}\varphi(l)^2\,d\mu_l = \lambda(l).$$

Since $\lim_{l\to 0}\lambda(l) = \lambda(0) < +\infty$, by the regularity theory [14, Theorems 8.8 and 8.10], for any compact subset $K \subset S_0$ and $k \in \mathbb{N}$, there exists a constant $c = c(k, K)$ independent of $l$ such that

$$\|\pi_l^*\varphi(l)\|_{W^{k,2}(K)} \le c,$$

where $\|\cdot\|_{W^{k,2}(K)}$ is the Sobolev norm. Take an exhausting family of compact subsets of $S_0$. Then by the Sobolev embedding theorem [1, Theorem 5.4] and a diagonal argument, there is a sequence $l_j \to 0$ as $j \to \infty$ such that $\pi_{l_j}^*\varphi(l_j)$ converge to a function $\varphi$ on $S_0$ $C^k$-uniformly over compact subsets of $S_0$ as $j \to \infty$, for all $k \in \mathbb{N}$. The limit function $\varphi(0)$ clearly satisfies

$$\Delta_0 \varphi(0) + \lambda(0)\varphi(0) = 0, \qquad \int_{S_0} \varphi(0)^2 \, d\mu_0 \leq 1.$$

By the assumption again, we can choose the sequence $l_j \to 0$ such that $\pi_{l_j}^* \varphi(l_j)$ do not converge to zero over some compact subsets of $S_0$. Then, of course, $\varphi(0) \not\equiv 0$. Therefore, $\varphi(0)$ is an $L^2$-eigenfunction on $S_0$ with eigenvalue $\lambda(0)$.

*Proof of Part* 2. The proof of the part 2 is further divided into the following four steps:

1. $\lim_{l \to 0} \lambda(l) = \lambda(0) \geq \frac{1}{4}$.
2. Show that there are constants $K_l$ and a sequence $l_j \to 0$ as $j \to \infty$ such that $K_{l_j} \pi_{l_j}^* \varphi(l_j)$ converge to a function $\psi(0)$ uniformly over compact subsets of $S_0$.
3. Show that the limit function $\psi(0)$ is not identically zero on $S_0$.
4. Show that the function $\psi(0) \in \mathscr{A}(S_0, t)$, where $t = \sqrt{\lambda(0) - \frac{1}{4}}$.

*Step* 1. If $\lambda(0) < \frac{1}{4}$, then by Theorem 7.2 in [17], there is a sequence $l_j \to 0$ as $j \to \infty$ such that $\pi_{l_j}^* \varphi(l_j)$ converge to an eigenfunction on $S_0$ with eigenvalue $\lambda(0)$ and $L^2$-norm 1 as $j \to \infty$. In particular, $\pi_{l_j}^* \varphi(l_j)$ do not converge to zero uniformly over compact subsets of $S_0$. This is a contradiction to the assumption. Therefore, $\lambda(0) \geq \frac{1}{4}$.

*Step* 2. Let $[\varphi(l)]_0$ be the zero Fourier coefficient of $\varphi(l)$ on the collar $C_l$. For $a \geq 1$, let $\varphi^a(l)$ be the function obtained from $\varphi(l)$ by subtracting off the zero Fourier coefficient $[\varphi(l)]_0$ inside the subcollar $C_l(a)$ as in (5.4). Choose constants $K_l > 0$ for $l > 0$ such that

$$\int_{S_l} |K_l \varphi^a(l)|^2 \, d\mu_l = 1.$$

In particular,

$$\int_{C_l(a)} |K_l(\varphi^a(l)|^2 \, d\mu_l \leq 1. \tag{5.8}$$

Recall from (2.6) that the zero Fourier coefficient $K_l[\varphi(l)]_0$ satisfies the following ordinary differential equation:

$$\left(\frac{d^2}{dr^2} - \tanh(r)\frac{d}{dr} + \lambda(l)\right) K_l[\varphi(l)]_0(r) = 0, \qquad r \in [-\tau(l), \tau(l)]. \tag{5.9}$$

Using substitutions $r \to r \pm \tau(l)$, the stability of initial value problems of ordinary differential equations and the inequality (5.8), we see that

$K_l[\varphi(l)]_0(\pm a)$ and $\frac{d}{dr}K_l[\varphi(l)]_0(\pm a)$ are bounded independently of $l$. Applying Lemma 5.3 to the function $K_l\varphi(l)$ yields immediately

$$\int_{S_l} |\nabla K_l \varphi^a(l)|^2 \, d\mu_l \leq c' < +\infty, \tag{5.10}$$

where $c'$ is a constant independent of $l$. Note that $\Delta_l \varphi^a(l) + \lambda(l)\varphi^a(l) = 0$ on $S_l \setminus \partial C_l(a)$ and $\lim_{l\to 0} \lambda(l) = \lambda(0) < +\infty$. Then by the same argument as in the proof of the part 1, there exists a sequence $l_j \to 0$ as $j \to \infty$ such that $K_{l_j}\pi^*_{l_j}\varphi^a(l_j)$ converge to a function $\tilde{\psi}(0)$ on $S_0 \setminus \partial(C_0(a) \cup C_0(a))$ $C^k$-uniformly over compact subsets $S_0 \setminus \partial(C_0(a) \cup C_0(a))$ as $j \to \infty$, for all $k \in \mathbb{N}$. Remember that $\lim_{l\to 0} C_l(a) = C_0(a) \cup C_0(a)$ intrinsically, where $C_0(a) = \{(r, \theta) \in C_0 | r \geq a\}$ is a subcusp of the standard cusp $C_0$ (see §2). (See step 1 of the proof of Theorem 1.5 in §3 for the convergence of functions on $C_l$ to functions on $C_0 \cup C_0$.)

In order to prove the uniform convergence of $K_{l_j}\pi^*_{l_j}\varphi(l_j)$ over all compact subsets of $S_0$, we should study the behavior of $[\varphi(l_j)]_1$ and $[\varphi(l_j)]_0$ inside the collar $C_l$. We treat these two cases separately.

*Case* 1. *The nonzero Fourier terms.* By the assumption we have

$$(\Delta_{l_j} + \lambda(l_j))K_{l_j}[\varphi(l_j)]_1 = 0, \qquad \int_{C_{l_j}} |K_{l_j}[\varphi(l_j)]_1|^2 \, d\mu_{l_j} \leq 1.$$

Furthermore, from (5.10) it follows that

$$\int_{C_{l_j}} |\nabla K_{l_j}[\varphi(l_j)]_1|^2 \, d\mu_{l_j} \leq \int_{S_l} |\nabla K_{l_j} \varphi^a(l_j)|^2 \leq c' < +\infty,$$

where $c'$ is a constant independent of $l_j$. By the same argument as in the proof of the part 1 above, there exists a subsequence of $\{l_j\}$, which is still denoted by $\{l_j\}$ for simplicity, such that $K_{l_j}\pi^*_{l_j}[\varphi(l_j)]_1$ converge to a function $[\tilde{\psi}(0)]_1$ on $C_0 \cup C_0$ as $j \to \infty$, and the convergence is $C^k$-uniform over compact subsets of $C_0 \cup C_0$ for all $k \in \mathbb{N}$. The point here is that $K_{l_j}\pi^*_{l_j}[\varphi(l_j)]_1$ converge to $[\tilde{\psi}(0)]_1$ $C^k$-uniformly across the boundary $\partial(C_0(a) \cup C_0(a))$.

*Case* 2. *The zero Fourier term.* Since $\lim_{j\to\infty} K_{l_j}\pi^*_{l_j}\varphi(l_j)(z) = \tilde{\psi}(0)(z)$ uniformly for $z$ in compact subsets of $S_0 \setminus \partial(C_0(a) \cup C_0(a))$, and $\lim_{j\to\infty} K_{l_j}[\varphi(l_j)]_1(z) = [\tilde{\psi}(0)]_1(z)$ uniformly for $z$ in compact subsets of $\{C_0 \cup C_0\} \setminus \{C_0(a) \cup C_0(a)\}$, it follows that

$$\lim_{j\to\infty} \pi^*_{l_j} K_{l_j}[\varphi(l_j)]_0(z) = [\tilde{\psi}(0)]_0(z) \tag{5.11}$$

uniformly for $z$ in compact subsets of $\{C_0 \cup C_0\} \setminus \{C_0(a) \cup C_0(a)\}$. Now we are going to extend the domain of definition of $[\tilde{\psi}(0)]_0(z)$, which is currently defined for $z \in \{C_0 \cup C_0\} \setminus \{C_0(a) \cup C_0(a)\}$. Let $[\psi(0)]_0$ be the solution of the following differential equation

$$\left(\frac{d^2}{dr^2} - \frac{d}{dr} + \lambda(0)\right)[\psi(0)]_0 = 0$$

with the initial condition $[\psi(0)]_0 = [\tilde{\psi}(0)]_0$ on $\{C_0 \cup C_0\} \setminus \{C_0(a) \cup C_0(a)\}$. From (5.9), the stability of solutions to initial value problems of ordinary differential equations and (5.11), it follows that

$$\lim_{j\to\infty} \pi_{l_j}^* K_{l_j}[\varphi(l_j)]_0(z) = [\psi]_0(z)$$

$C^k$-uniformly for $z$ in compact subsets of both the cusps $C_0 \cup C_0$. In particular, the convergence is $C^k$-uniform across the boundary $\partial(C_0 \cup C_0)$ for all $k \in \mathbb{N}$. This finishes the case 2.

Now define a function $\psi(0)$ on $S_0$ by

$$\psi(0) = \begin{cases} \tilde{\psi}(0) & \text{on } S_0 \setminus \{C_0 \cup C_0\}, \\ [\tilde{\psi}(0)]_1 + [\psi(0)]_0 & \text{on } C_0 \cup C_0. \end{cases} \tag{5.12}$$

Then by the above discussions, $\psi(0)$ is a smooth function on $S_0$ and satisfies

$$\Delta_0 \psi(0) + \lambda(0)\psi(0) = 0, \qquad \lim_{j\to\infty} K_{l_j} \pi_{l_j}^* \varphi(l_j)(z) = \psi(0)(z),$$

where the convergence is $C^k$-uniform for $z$ in compact subsets of $S_0$ for all $k \in \mathbb{N}$.

*Step* 3. By the definition of $\psi(0)$ (5.12), it is clear that

$$\int_{S_0} |\psi(0)|^2 \, d\mu_0 \geq \int_{S_0} |\tilde{\psi}(0)|^2 \, d\mu_0 .$$

We are going to prove

$$\int_{S_0} |\psi(0)|^2 \, d\mu_0 \geq \int_{S_0} |\tilde{\psi}(0)|^2 \, d\mu_0 \geq 1 . \tag{5.13}$$

In particular, $\psi(0) \not\equiv 0$ on $S_0$. For any $\rho > a$, define

$$\varepsilon(\rho) = 1 - \underline{\lim}_{l\to 0} \int_{S_{l_j} \setminus C_{l_j}(\rho)} |K_{l_j} \varphi^a(l_j)|^2 \, d\mu_{l_j} = \overline{\lim_{l\to 0}} \int_{C_{l_j}(\rho)} |K_{l_j} \varphi^a(l_j)|^2 \, d\mu_{l_j},$$

which is intuitively the mass of $K_{l_j}\varphi(l_j)$ inside the subcollar $C_{l_j}(\rho)$, where $C_l(\rho) = \{(r, \theta) \in C_l | -\tau(l) + \rho \leq r \leq \tau(l) - \rho\}$. Since $\|K_{l_j}\varphi^a(l_j)\|_{L^2(S_{l_j})} = 1$ and $\lim_{l_j\to 0} K_{l_j} \pi_{l_j}^* \varphi^a(l_j) = \tilde{\psi}(0)$, (5.13) follows from

**Claim 5.4.** *With the notation as above,* $\lim_{\rho\to\infty} \varepsilon(\rho) = 0$.

This claim intuitively means that no mass of the function $K_{l_j}\varphi^a(l_j)$ is lost inside the pinching collar during the degeneration. The proof of this claim is the technical part of the proof of Theorem 1.2 and depends essentially on Theorem 1.5.

Now we prepare to prove Claim 5.4. Arrange all the nonrotationally invariant Dirichlet eigenvalues $\bigcup_{n\neq 0}\{\lambda_k^n(l)\}_{k=1}^{\infty}$ of the pinching collar $C_l$ into an increasing sequence $\{\tilde{\lambda}_i(l)\}_{i=1}^{\infty}$ with multiplicity. Further, let $\{u_i(l)\}_{i=1}^{\infty}$ be the corresponding complete system of orthonormal Dirichlet eigenfunctions on $C_l$. Similarly, let $\{\tilde{\lambda}_i(0)\}_{i=1}^{\infty}$ be all the nonrotationally invariant Dirichlet eigenvalues with multiplicity of $C_0 \cup C_0$ in the increasing order, and $\{u_i(0)\}_{i=1}^{\infty}$ be a corresponding complete system of orthonormal Dirichlet eigenfunctions on $C_0 \cup C_0$ which are the limits of the eigenfunctions $\{u_i(l)\}_1^{\infty}$ on $C_l$ as $l \to 0$, that is, for $i \geq 1$,

$$\lim_{l\to 0} \pi_l^* u_i(l)^2(z) = u_i(0)^2(z) \tag{5.14}$$

uniformly for $z$ in compact subsets of both the cusps $C_0 \cup C_0$, where $\pi_l$ is the restriction of the harmonic map $\pi_l \colon S_0 \to S_l$. The existence of such a complete system of orthonormal Dirichlet eigenfunctions on $C_0 \cup C_0$ follows from Theorem 1.5. (5.14) intuitively means that *none of the eigenfunctions* $u_i(l)$ *loses any mass inside the collar* $C_l$ *as* $l \to 0$. More precisely, for $i \geq 1$ and $\rho > 0$, define

$$\varepsilon_i^*(\rho) = \overline{\lim_{l\to 0}} \int_{C_l(\rho)} |u_i(l)|^2 \, d\mu_l .$$

Then (5.14) implies that, for $i \geq 1$,

$$\lim_{l\to 0} \varepsilon_i^*(\rho) = 0 . \tag{5.15}$$

Let $\xi_l$ be a cut-off function on $S_l$ with $\xi_l = 1$ on $C_l(a+1)$, $\xi_l = 0$ on $S_l \setminus C_l(a)$ $|\xi_l| \leq 1$ and $|\nabla \xi_l| \leq 2$. Note that $\xi$ cuts off both sides of the pinching collar $C_l$. Consider the function $\xi_{l_j} K_{l_j} \varphi^a(l_j)$ on $C_{l_j}$ which clearly satisfies the Dirichlet boundary condition. Let

$$\xi_{l_j} K_{l_j} \varphi^a(l_j) = \sum_{i=1}^{\infty} a_n(l_j) u_i(l_j) \tag{5.16}$$

be the Fourier expansion of $\xi_{l_j} K_{l_j} \varphi^a(l_j)$ in terms of the Dirichlet eigenfunctions $\{u_i(l_j)\}_1^{\infty}$ on $C_{l_j}$. It is clear that $a_i(l_j) = \langle \xi_{l_j} K_{l_j} \varphi^a(l_j) , u_i(l_j) \rangle$

for $i \geq 1$, and

$$\sum_{i=1}^{\infty} a_i(l_j)^2 = \langle \xi_{l_j} K_{l_j} \varphi^a(l_j), \xi_{l_j} K_{l_j} \varphi^a(l_j) \rangle .$$

For any $N > 1$, define $\delta(N)$ by

$$\delta(N) = \overline{\lim}_{l_j \to 0} \sum_{i \geq N}^{\infty} a_i(l_i)^2 .$$

It is clear that $\delta(N)$ is monotonically decreasing in $N$. Furthermore, we have the following.

**Claim 5.5.** *With notation as above,* $\lim_{N \to +\infty} \delta(N) = 0$.

Actually, if this claim does not hold, without loss of generality, we assume that $\delta(N) \geq c_0 > 0$ for $N \geq N_0$, where $N_0$ and $c_0$ are positive constants.

By the assumptions on $\varphi(l_j)$ and $K_{l_j}$, it is clear that $(\Delta_{l_j} + \lambda(l_j)) K_{l_j} \varphi^a(l_j) = 0$ on $S_{l_j} \setminus \partial C_{l_j}(a)$ and $\|K_{l_j} \varphi^a(l_j)\|_{L^2(S_{l_j})} = 1$. Further, $\lim_{l_j \to 0} \lambda(l_j) = \lambda(0) < +\infty$. Then by direct computations,

$$\overline{\lim}_{l_j \to 0} |\langle \Delta_{l_j} \xi_{l_j} K_{l_j} \varphi^a(l_j), \xi_{l_j} K_{l_j} \varphi^a(l_j) \rangle| \leq M < +\infty , \tag{5.17}$$

where $M$ is a finite constant.

On the other hand, by Theorem 1.5, the eigenvalues $\{\tilde{\lambda}_i(l)\}_{i=1}^{\infty}$ do not accumulate at any finite point as $l \to 0$. Thus, there exist constant $l_0 > 0$ and $N_1 \geq N_0 > 0$ such that for $i \geq N_1$ and $l \leq l_0$,

$$\tilde{\lambda}_i(l) \geq \frac{M+1}{c_0} .$$

For $N \geq N_1$ and $l_j \leq l_0$, from the Fourier expansion (5.16) of $\xi_{l_j} K_{l_j} \varphi^a(l_j)$ we get

$$\begin{aligned} \langle \Delta_{l_j} \xi_{l_j} K_{l_j} \varphi^a(l_j), \xi_{l_j} K_{l_j} \varphi^a(l_j) \rangle &= \sum_{i=1}^{\infty} \tilde{\lambda}_i(l_j) a_i(l_j)^2 \geq \sum_{i \geq N}^{\infty} \tilde{\lambda}_i(l_j) a_i(l_j)^2 \\ &\geq \frac{M+1}{c_0} \sum_{i \geq N}^{\infty} a_i(l_j)^2 \end{aligned}$$

and, by the assumption $\delta(N) \geq c_0$ for $N \geq N_0$,

$$\begin{aligned} \overline{\lim}_{l_j \to 0} \langle \Delta_{l_j} \xi_{l_j} K_{l_j} \varphi^a(l_j), \xi_{l_j} K_{l_j} \varphi^a(l_j) \rangle &\geq \frac{M+1}{c_0} \overline{\lim}_{l_j \to 0} \sum_{i \geq N}^{\infty} a_i(l_j)^2 \\ &\geq \frac{M+1}{c_0} \delta(N) \geq M + 1 . \end{aligned} \tag{5.18}$$

The above inequality (5.18) is a contradiction to the inequality (5.17). Therefore, the proof of Claim 5.5 is complete.

Now we use Claim 5.5 to prove Claim 5.4. The proof goes as follows. By Claim 5.4, only the first finitely many terms really contribute to the summation in (5.16). On the other hand, for these finitely many terms, their masses deep inside the collar $(\int_{C_{l_j}(\rho)} u_i(l_j)^2\, d\mu_{l_j})$ are negligible by (5.15). Thus Claim 5.4 follows.

More precisely, by Claim 5.5, for any $\varepsilon > 0$ there exists an integer $N_2 > 0$ such that

$$\overline{\lim_{l_j \to 0}} \sum_{i \geq N_2} a_i(l_j)^2 \leq \varepsilon .$$

Then for $\rho > a + 1$,

$$\begin{aligned}
\varepsilon(\rho) &= \overline{\lim_{l_j\to 0}} \int_{C_{l_j}(\rho)} |K_{l_j}\varphi^a(l_j)|^2\, d\mu_{l_j} = \overline{\lim_{l_j\to 0}} \int_{C_{l_j}(\rho)} |\xi_{l_j} K_{l_j}\varphi^a(l_j)|^2\, d\mu_{l_j} \\
&\leq \overline{\lim_{l_j\to 0}} \int_{C_{l_j}(\rho)} \left| \sum_1^{N_2-1} a_i(l_j) u_i(l_j) \right|^2 d\mu_{l_j} \\
&\quad + \overline{\lim_{l_j\to 0}} \int_{C_{l_j}(\rho)} \left| \xi_{l_j} K_{l_j} \varphi^a(l_j) - \sum_1^{N_2-1} a_i(l_j) u_i(l_j) \right|^2 d\mu_{l_j} \qquad (5.19) \\
&\leq \overline{\lim_{l_j\to 0}} \int_{C_{l_j}(\rho)} \left| \sum_1^{N_2-1} a_i(l_j) u_i(l_j) \right|^2 d\mu_{l_j} + \varepsilon .
\end{aligned}$$

By (5.15) we have

$$\lim_{\rho\to\infty} \overline{\lim}_{l_j\to 0} \int_{C_{l_j}(\rho)} \left| \sum_1^{N_2-1} a_i(l_j) u_i(l_j) \right|^2 d\mu_{l_j} = 0 .$$

Therefore, substituting this into the inequality (5.19) gives $\overline{\lim}_{\rho\to 0}\varepsilon(\rho) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{\rho\to 0} \varepsilon(\rho) = 0 ,$$

which completes the proof of Claim 5.4 and Step 3.

*Step* 4. We need to study the Fourier expansion of the limit function $\psi(0)$ in every cusp of $S_0$. Let $C_i = \{z | \operatorname{Im}(z) \geq 1\}/\{z + 1 \sim z\}$ be a cusp of $S_0$. Then for $z = x + \sqrt{-1}y \in C_i$, we have the following Fourier expansion:

$$\psi(0)(x+\sqrt{-1}y) = f_0(y) + \sum_{n\neq 0} f_n(y) e^{2\pi\sqrt{-1}nx}.$$

By Step 2 above, $\Delta_0\psi(0) + \lambda(0)\psi(0) = 0$. Then for any $n \in \mathbb{Z}$, $f_n(y)$ satisfies that

$$y^2 \frac{d^2}{dr^2} f_n(y) - (4\pi^2 n^2 y^2 - \lambda(0)) f_n(y) = 0. \tag{5.20}$$

By Step 1, $\lambda(0) \geq \frac{1}{4}$. Let $\sqrt{\lambda(0) - \frac{1}{4}} \geq 0$. Now for $n \neq 0$, the functions $y^{1/2} K_{\sqrt{-1}t}(2\pi|n|y)$ and $y^{1/2} I_{\sqrt{-1}t}(2\pi|n|y)$ form a basis of the solutions to the differential equation (5.20), where $K_{\sqrt{-1}t}(2\pi|n|y)$ and $I_{\sqrt{-1}t}(2\pi|n|y)$ are the MacDonald Bessel functions. Thus there exist constants $\alpha_n$ and $\beta_n$ such that

$$f_n(y) = \alpha_n y^{1/2} K_{\sqrt{-1}t}(2\pi|n|y) + \beta_n I_{\sqrt{-1}t}(2\pi|n|y).$$

Note the following asymptotics

$$K_{\sqrt{-1}t}(y) \sim \left(\frac{\pi}{2y}\right)^{1/2} e^{-y}, \qquad I_{\sqrt{-1}t}(y) - \sim (2\pi y)^{-1/2} e^{y} \tag{5.21}$$

as $y \to +\infty$ (see [35, pp. 202–203]). Then it is clear that $y^{1/2} I_{\sqrt{-1}t}(2\pi|n|y) \notin L^2([1, \infty), y^{-2}\,dy)$ for $n \neq 0$. By Step 3, $\int_{S_0} |\tilde{\psi}(0)|^2 \, d\mu_0 = 1$, and for $n \neq 0$,

$$\int_1^\infty f_n(y)^2 y^{-2}\, dy \leq \int_{C_i} |\tilde{\psi}(0)|^2 \, d\mu_0 = 1.$$

Thus, of course, $\beta_n = 0$, and $f_n(y) = \alpha_n y^{1/2} K_{\sqrt{-1}t}(2\pi|n|y)$ for any $n \neq 0$. It follows that for $z = x + \sqrt{-1}y \in C_i$ and $1 \leq i \leq p$,

$$\psi(0)(x+\sqrt{-1}y) - f_0(y) + \sum_{n\neq 0} a_n y^{1/2} k_{\sqrt{-1}t}(2\pi|n|y) e^{2\pi\sqrt{-1}nx}.$$

Therefore, $\psi(0) \in \mathscr{A}(S_0, t)$. Finally, by Lemma 5.2, if $t > 0$, there exist constants $a_1, \cdots, a_p$ and an $L^2$-function $\varphi$ (possibly zero) on $S_0$ with $\Delta_0\varphi + (\frac{1}{4} + t^2)\varphi = 0$ such that

$$\psi(0) = \sum_{i=1}^p a_i E_i(\cdot\,; \frac{1}{2} + \sqrt{-1}t) + \varphi.$$

In particular, if $\lambda(0) = \frac{1}{4} + t^2$ is not an embedded eigenvalue of $S_0$, then $\varphi \equiv 0$, and

$$\psi(0) = \sum_{i=1}^{p} a_i E_i(\cdot\,; \frac{1}{2} + \sqrt{-1}t).$$

If $t = 0$, there is a similar expression for $\psi(0)$ involving derivatives of Eisenstein series $E_i(\cdot\,; \frac{1}{2} + \sqrt{-1}t)$ with respect to $t$ at $t = 0$. This finishes Step 4. Therefore, the proof of Theorem 1.2 is finally complete.

## 6. Spectral degeneration

In the previous sections, we have discussed the behavior of the eigenvalues and eigenfunctions of $S_l$ as $l \to 0$. In particular, the eigenfunctions of $S_l$ can only limit in linear combinations of Eisenstein series and $L^2$-eigenfunctions of $S_0$. We would like to reverse this process and show that Eisenstein series and $L^2$-eigenfunctions of $S_0$ can be approximated by suitably chosen eigenfunctions of $S_l$ as $l \to 0$ (see Conjecture 6.1). If so, we would have a nice picture of the spectral degeneration for the family $S_l$ $(l \geq 0)$. This spectral degeneration picture corresponds to a picture of how the terms in the Selberg trace formula for $S_l$ split and degenerate to the corresponding ones in the Selberg trace formula for $S_0$. We also discuss how to determine intrinsically the constants $a_i$ in (1.4) from the eigenfunctions $\varphi(l)$ on $S_l$.

First, we discuss the embedded eigenvalues of $S_0$. As mentioned in the introduction, the existence of the embedded eigenvalues of $S_0$ is mysterious (see [28]). From the degeneration point of view, we believe that the following conjecture should be true.

**Conjecture 6.1.** *For any degenerating family of hyperbolic Riemann surfaces $S_l$ $(l \geq 0)$, let $\lambda(0) \geq \frac{1}{4}$ be an embedded eigenvalue of $S_0$. Then there exist a sequence $l_j \to 0$ as $j \to \infty$, and eigenfunctions $\varphi(l_j)$ on $S_{l_j}$ with eigenvalues $\lambda(l_j)$ such that $\lim_{l_j \to 0} \lambda(l_j) = \lambda(0)$ and $\pi_{l_j}^* \varphi(l_j)(z)$ converges uniformly for $z$ in compact subsets of $S_0$ to a nonzero $L^2$-eigenfunction on $S_0$ with eigenvalue $\lambda(0)$.*

The statement of this conjecture is similar to those of Theorems 1.3 and 1.5, since the pinching collars $C_l$ can be thought of as a special family of degenerating surfaces. Because of the symmetric consideration as in Lemma 6.2 below, it may not be possible to approximate any prechosen eigenfunction of $\lambda(0)$ on $S_0$.

For noncompact hyperbolic surfaces of finite areas, P. Lax and R. Phillips [24] and Y. Colin de Verdière [9] introduced the notion of pseudo-Laplacians which is used successfully by R. Phillips and P. Sarnak in [28]. Motivated by the fact that all the embedded eigenvalues of $S_0$ are part of the eigenvalues of the pseudo-Laplacian of $S_0$, in [18], we generalized the pseudo-Laplacians to compact surfaces with short geodesics (lengths $< \frac{1}{2}$) and proved the convergence of the spectral measures of the pseudo-Laplacian of $S_l$ to the spectral measures of the pseudo-Laplacian of $S_0$ as $l \to 0$. In particular, embedded eigenvalues and their eigenfunctions of $S_0$ can be approximated by pseudo-eigenvalues and pseudo-eigenfunctions of $S_l$ as $l \to 0$. This gives some evidence for the above conjecture. If we can understand more qualitatively the behavior of those eigenfunctions on $S_l$ which converge to Eisenstein series in Theorem 1.2, we would be able to prove the above conjecture.

Next we study the approximation to Eisenstein series. An optimistic guess would be that for each $E_i(z; \frac{1}{2} + \sqrt{-1}t)$ $(1 \leq i \leq p)$, there exist a sequence $l_j \to 0$ as $j \to \infty$ and eigenfunctions $\varphi(l_j)$ on $S_{l_j}$ such that $\pi_{l_j}^* \varphi(l_j)(z)$ converges to $E_i(z; \frac{1}{2} + \sqrt{-1}t)$ uniformly for $z$ in compact subsets of $S_0$. Actually, by the following Lemma 6.2, Eisenstein series appear in pairs during the degeneration if the pinching geodesics separate the surfaces. Before stating Lemma 6.2, let us recall some notation first.

For simplicity, we assume that $S_l$ has only one pinching geodesic $\gamma(l)$ of length $l$ in the following discussions. The standard collar of $\gamma(l)$ is denoted by $C_l$. More explicitly,

$$C_l = \{(r, \theta) | -\tau(l) \leq r \leq \tau(l), 0 \leq \theta \leq 1\}/\{(r, 0) \sim (r, 1)\}.$$

Further, for any $a > 0$, define the left band of $C_l$ by $B_l^-(a) = \{(r, \theta) \in C_l | -\tau(l) \leq r \leq -\tau(l) + a\}$, and the right band by $B_l^+(a) = \{(r, \theta) \in C_l | \tau(l) - a \leq r \leq \tau(l)\}$. Since $S_l$ has only one pinching geodesic, the limit surface $S_0$ has two cusps $C_1$ and $C_2$. We assume that $B_l^-(a)$ and $B_l^+(a)$ converge to bands $B_0^-(a)$ and $B_0^+(a)$ in $C_1$ and $C_2$ respectively as $l \to 0$. For any function $\varphi(l)$ on $C_l$, let $[\varphi(l)]_0$ be the zero Fourier coefficient of $\varphi(l)$ in $C_l$ (see (5.3)). Then we have the following.

**Lemma 6.1** [41, Remark 2.9]. *Let two constants $\alpha$ and $\beta$ satisfy $\frac{1}{4} < \alpha < \beta$. Then for any family of eigenfunctions $\varphi(l)$ on $S_l$ with eigenvalues $\lambda(l) \in [\alpha, \beta]$, the following inequalities hold for large enough $a$:*

$$\overline{\lim_{l \to 0}} \frac{\int_{B_l^-(a)} |[\varphi(l)]_0|^2 \, d\mu_l}{\int_{B_l^+(a)} |[\varphi(l)]_0|^2 \, d\mu_l} < +\infty,$$

$$\lim_{l\to 0} \frac{\int_{B_l^-(a)} |[\varphi(l)]_0|^2 \, d\mu_l}{\int_{B_l^+(a)} |[\varphi(l)]_0|^2 \, d\mu_l} > 0.$$

**Corollary 6.2.** *Assume that the pinching geodesic $\gamma(l)$ separates $S_l$. Then for $i = 1, 2$ and $t > 0$, there do not exist any sequence $\{l_j\}$ and eigenfunctions $\varphi(l_j)$ with eigenvalues $\lambda(l_j)$ such that $\lim_{l_j\to 0}\lambda(l_j) = \frac{1}{4} + t^2 > \frac{1}{4}$ and $\lim_{l_j\to 0}\pi_{l_j}^*\varphi(l_j)(z) = E_i(z; \frac{1}{2} + \sqrt{-1}t)$ uniformly for $z$ in compact subsets of $S_0$.*

*Proof.* Suppose that there exist a sequence of eigenfunctions $\varphi(l_j)$ on $S_{l_j}$ with eigenvalues $\lambda(l_j)$ converging to, say, $E_1(z; \frac{1}{2}+\sqrt{-1}t)$ as $l_j \to 0$, where $\lim_{l_j\to 0}\lambda(l_j) = \frac{1}{4} + t^2$. Then for any $a > 0$ and $z \in B_0^-(a) \subset C_1$,

$$\begin{aligned}\lim_{l_j\to 0} \pi_{l_j}^*[\varphi(l_j)(z)]_0 &= [E_1(z; \tfrac{1}{2} + \sqrt{-1}t)]_0 \\ &= \operatorname{Im}(z)^{1/2+\sqrt{-1}t} + \Phi_{11}(\tfrac{1}{2} + \sqrt{-1}t)\operatorname{Im}(z)^{1/2-\sqrt{-1}t} \not\equiv 0.\end{aligned}$$

On the other hand, for $z \in B_0^+(a) \subset C_2$,

$$\lim_{l_j\to 0} \pi_{l_j}^*[\varphi(l_j)(z)]_0 = [E_1(z; \tfrac{1}{2} + \sqrt{-1}t)]_0 = 0,$$

by the definition of Eisenstein series (Proposition 5.1.(3)), since $C_1$ and $C_2$ are not on one connected component. This clearly contradicts Lemma 6.2 above. Hence the proof is complete. q.e.d.

Corollary 6.2 shows that in (1.4) of Theorem 1.2, we cannot assign $a_1$ and $a_2$ arbitrarily. A natural question then is how to determine algebraic relations between $a_1$ and $a_2$ intrinsically from $\varphi(l_j)$. Since we can use different scaling constants $K_{l_j}$, we are interested in their ratio $a_1/a_2$.

**Proposition 6.3.** *Assume that $\gamma(l)$ separates $S_l$. Let $l_j \to 0$ be a sequence and suppose $\varphi(l_j)$ are eigenfunctions on $S_{l_j}$ whose multiples satisfy $\lim_{l_j\to 0} K_{l_j}\pi_{l_j}^*\varphi(l_j)(z) = a_1E_1(z; \frac{1}{2}+\sqrt{-1}t)+a_2E_2(z; \frac{1}{2}+\sqrt{-1}t)$ uniformly for $z$ in compact subsets of $S_0$, where $t \geq 0$ is a constant. Then the following double limit exists:*

$$\lim_{a\to\infty}\lim_{l_j\to 0} \frac{\int_{B_l^-(a)} |[\varphi(l)]_0|^2 \, d\mu_l}{\int_{B_l^+(a)} |[\varphi(l)]_0|^2 \, d\mu_l} < +\infty.$$

*Denote the limit by $m_{12}$. Then the constants $a_1$ and $a_2$ satisfy*

$$\left(\frac{a_1}{a_2}\right)^2 = m_{12}\frac{\Phi_{22}(\frac{1}{2} + \sqrt{-1}t)}{\Phi_{11}(\frac{1}{2} + \sqrt{-1}t)},$$

*where* $\Phi_{11}(\frac{1}{2}+\sqrt{-1}t)$ *and* $\Phi_{22}(\frac{1}{2}+\sqrt{-1}t)$ *are the scattering matrix coefficients of* $E_1(\cdot;\frac{1}{2}+\sqrt{-1}t)$ *and* $E_2(\cdot;\frac{1}{2}+\sqrt{-1}t)$ *respectively* (*see Proposition* 5.1, *part* 2).

*Proof.* By the assumption, it is clear that for any $a>0$

$$\lim_{l_j\to 0}\frac{\int_{B_j^-(a)}\|[\varphi(l)]_0\|^2\,d\mu_l}{\int_{B_l^+(a)}\|[\varphi(l)]_0\|^2\,d\mu_l}=\frac{\int_{B_0^-(a)}(a_1E_1(z;\frac{1}{2}+\sqrt{-1}t))^2\,d\mu_0}{\int_{B_0^+(a)}(a_2E_2(z;\frac{1}{2}+\sqrt{-1}t))^2 d\mu_0}.$$

Now for $z=x+\sqrt{-1}y\in C_1$, by Proposition 5.1,

$$E_1(x+\sqrt{-1}y;\frac{1}{2}+\sqrt{-1}t)=y^{\frac{1}{2}+\sqrt{-1}t}+\Phi_{11}\left(\frac{1}{2}+\sqrt{-1}t\right)y^{\frac{1}{2}-\sqrt{-1}t}+O(e^{-cy})$$

for $y>1$, where $c>0$ is a constant. The fact that the error term is of exponential decaying follows from (5.21) and that the constants $a_n(i)(n\neq 0)$ in Proposition 5.1, part 2 can be bounded by $b_1e^{b_2n}$, where $b_1$ and $b_2$ are constants. Thus,

$$\begin{aligned}&\int_{B_0^-(a)}\left(a_1E_1\left(z;\frac{1}{2}+\sqrt{-1}t\right)\right)^2 d\mu_0\\&=\int_1^a\left[\Phi_{11}\left(\frac{1}{2}+\sqrt{-1}t\right)a_1^2y+a_1^2y^{1+2\sqrt{-1}t}\right.\\&\qquad\left.+a_1^2\Phi_{11}^2\left(\frac{1}{2}+\sqrt{-1}t\right)y^{1-2\sqrt{-1}t}+O(e^{-c'y})\right]y^{-2}dy\\&\sim\Phi_{11}\left(\frac{1}{2}+\sqrt{-1}t\right)a_1^2\log a,\end{aligned}$$

as $a\to+\infty$, where $c'>0$ is a constant. Similarly, as $a\to+\infty$,

$$\int_{B_0^+(a)}\left(a_2E_2\left(z;\frac{1}{2}+\sqrt{-1}t\right)\right)^2 d\mu_0\sim\Phi_{22}\left(\frac{1}{2}+\sqrt{-1}t\right)a_2^2\log a.$$

Therefore it follows that

$$\lim_{l_j\to 0}\frac{\int_{B_0^-(a)}\left(a_2E_2\left(z;\frac{1}{2}+\sqrt{-1}t\right)\right)^2 d\mu_0}{\int_{B_0^+(a)}\left(a_2E_2\left(z;\frac{1}{2}+\sqrt{-1}t\right)\right)^2 d\mu_0}=\frac{\Phi_{11}(\frac{1}{2}+\sqrt{-1}t)a_1^2}{\Phi_{22}(\frac{1}{2}+\sqrt{-1}t)a_2^2}.$$

Since, of course, the double limit exists, the conclusion follows. q.e.d.

We have only considered the case where $\gamma(l)$ separates $S_l$. On the other hand, if $\gamma(l)$ does not separate $S_l$, we cannot exclude as above the possibility that $E_i(\cdot;\cdot)$ can be approximated by some eigenfunctions of $S_l$. Now suppose that $E_1(\cdot;\cdot)$ and $E_2(\cdot;\cdot)$ can be approximated by two sequences of eigenfunctions on $S_l$. We would like to understand the difference between these two families intrinsically. By a similar computation as in the proof of Proposition 6.4, we have

**Proposition 6.4.** *If there exists a sequence of eigenfunctions* $\varphi(l_j)$ *on* $S_{l_j}$ *whose multiples* $K_{l_j}\pi_{l_j}^*\varphi(l_j)(z)$ *converge to* $E_i(z; \frac{1}{2}+\sqrt{-1}t)$ *uniformly for* $z$ *in compact subsets of* $S_0$ *as* $l_j \to 0$, *where* $i = 1, 2$ *and* $t > 0$, *then the following double limit exists*:

$$\lim_{a\to\infty}\lim_{l_j\to 0}\frac{\int_{B_{l_j}^-(a)} l\varphi(l_j)^2\,d\mu_{l_j}}{\int_{B_{l_j}^+(a)} \varphi(l_j)^2\,d\mu_{l_j}}.$$

*Denote the limit by* $m(i; t)$. *If* $\Phi_{11}(\frac{1}{2}+\sqrt{-1}t) \neq 0$ *and* $i = 1$, *then* $m(l; t) = \infty$. *Similarly, if* $\Phi_{22}(\frac{1}{2}+\sqrt{-1}t) \neq 0$ *and* $i = 2$, *then* $m(2; t) = 0$. *The functions* $\Phi_{11}(\frac{1}{2}+\sqrt{-1}t)$ *and* $\Phi_{22}(\frac{1}{2}+\sqrt{-1}t)$ *are the scattering matrix coefficients of the Eisenstein series* (*see Proposition* 5.1).

Added in proof. After submitting the revised version of this paper, the author together with M. Zworski has proved the conjectured lower bound (3.24) in L. Ji & M. Zworski, *The remainder estimate in spectral accumulation for degenerating hyperbolic surfaces*, J. Functional Analysis, to appear.

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