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COMPLETENESS OF LORENTZ MANIFOLDS OF CONSTANT CURVATURE ADMITTING KILLING VECTOR FIELDS

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Dedicated to Professor Akio Hattori on his sixtieth birthday

Introduction

A Lorentz manifold M of dimension n is a smooth manifold together with a Lorentz metric g. A Lorentz metric g on M is a smooth field $\{g_x\}_{x\in M}$ of nondegenerate symmetric bilinear forms g_x of type $(1, n-1)$ on the tangent space T_xM. Namely let $\mathbf{R}^{1,n-1}$ denote the real vector space of dimension n equipped with the bilinear form

$$Q(x, y) = -x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

A nondegenerate symmetric bilinear form g_x is of type $(1, n-1)$ if the pair (T_xM, g_x) is isometric to $(\mathbf{R}^{1,n-1}, Q)$ (cf. [31], [34]).

A pseudo-Riemannian manifold has a unique connection (Levi-Cività connection) on its frame bundle. Henceforth geodesics, curvature, completeness refer to the Levi-Cività connection. It is notorious that compactness does not necessarily imply completeness in *pseudo-Riemannian geometry.* In this paper we consider this problem for Lorentz manifolds of constant curvature which admit Killing vector fields of certain type. This leads to some precise classification results.

Theorem A. *Let M be a compact Lorentz manifold of constant curvature k. Suppose that M admits a timelike Killing vector field. Then M is complete, $k \le 0$ and the following hold:*

(1) *M is affinely diffeomorphic to a euclidean space form with nonzero first Betti number if $k = 0$;*

(2) *some finite covering of M is a circle bundle over a negatively curved manifold if k is a negative constant.*

This will be proved in Corollary 3.2, Theorem 2.15, and Theorem 2.17. A compact Lorentz manifold of $k = 0$ is called a Lorentz flat manifold. It is known that a compact Lorentz flat manifold is complete by the result

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of Carrière [3]. We notice that $\dim M$ is odd for a compact Lorentz manifold of nonzero constant curvature k. For this, it is known that a smooth manifold admits a Lorentz metric if and only if there exists a nonzero vector field (cf. [31]). Thus the Euler characteristic $\chi(M)$ is zero. On the other hand, the generalized Gauss-Bonnet formula can be applied to a compact pseudo-Riemannian manifold (cf. [1], [4], [24]). If $\dim M$ is even and $k \neq 0$, then the Gauss-Bonnet formula certainly implies that $\chi(M) \neq 0$.

It is a famous result that if M is a Riemannian manifold, then the group of all isometries acts properly on M. In particular the stabilizer at any point of M is compact. If $\mathrm{Iso}(M)$ is the group of all isometries of a Lorentz manifold M, then it is emphasized that in *pseudo-Riemannian geometry* $\mathrm{Iso}(M)$ need not act properly and hence its stabilizer fails to be compact. This fact causes difficulties in understanding the topology of Lorentz manifolds (cf. [20], [22]).

Let $b(x, y) = -x_1y_1 - x_2y_2 + \cdots + x_{2n+2}y_{2n+2}$ be the bilinear form on $\mathbf{R}^{2n+2}$. The quadric $\mathbf{H}^{1,2n} = \{x \in \mathbf{R}^{2n+2} | b(x, x) = -1\}$ supports a complete Lorentz metric of constant negative curvature, and moreover if $O(2, 2n)$ is the orthogonal group of $\mathrm{GL}(2n+2, \mathbf{R})$ preserving the form b, then $\mathrm{Iso}(\mathbf{H}^{1,2n}) = O(2, 2n)$. There is the canonical exact sequence

$$1 \to \mathscr{Z} \to O(2, 2n)^\sim \xrightarrow{P} O(2, 2n) \to 1,$$

associated with the covering projection $\widetilde{\mathbf{H}}^{1,2n} \to \mathbf{H}^{1,2n}$ (cf. §1). Thus we can find a Lie group $U(1, n)^\sim$ of $O(2, 2n)^\sim$ for which $U(1, n)^\sim$ acts transitively and $U(n)\backslash U(1, n)^\sim \approx \widetilde{\mathbf{H}}^{1,2n}$. Since there exists a torsion free discrete cocompact subgroup Γ in $U(1, n)^\sim$, the compact complete Lorentz manifold of negative curvature $\widetilde{\mathbf{H}}^{1,2n}/\Gamma$ is called a (complete) standard space form $U(n)\backslash U(1, n)^\sim/\Gamma$. It is a Seifert fiber space, namely it admits a circle action which induces a timelike Killing vector field. (See Proposition 2.19, also cf. [24], [25].) We shall give a necessary and sufficient condition for a compact Lorentz manifold of constant negative curvature admitting a Killing vector field to become a standard space form.

Theorem B. *Let M be a compact Lorentz manifold of constant negative curvature in dimension $2n+1$. Suppose that M admits a nontrivial Killing vector field. Let $\{\varphi_t\}_{|t|<\infty}$ be a one-parameter group of Lorentz transformations of M generated by the Killing vector field, and $\{\tilde{\varphi}_t\}_{|t|<\infty}$ its lift to the universal covering space $\widetilde{M}$. Denote by $\widetilde{H}$ the holonomy image of $\{\tilde{\varphi}_t\}_{|t|<\infty}$ in $O(2, 2n)^\sim$. Then M is a standard space form $U(n)\backslash U(1, n)^\sim/\Gamma$ if and only if $P(\widetilde{H})$ is compact where $P\colon O(2, 2n)^\sim \to$*

$O(2, 2n)$ is the covering map. In particular the Killing vector field is timelike.

A related work for a complete Lorentz 3-manifold of constant negative curvature to be standard has been found in Proposition 7.5 [25]. We remark that a compact complete Lorentz manifold of constant negative curvature is not always a standard one. In fact there is a three-dimensional nonstandard Lorentz space form (i.e., there is a proper action of a subgroup of $O(2, 2)$ not lying in the closed subgroup $\mathrm{PSL}_2(\mathbf{R})$). Kulkarni, Raymond and Goldman ([24], [25], [12]) classified three-dimensional complete Lorentz manifolds of constant negative curvature. It has been shown that if a complete Lorentz manifold of constant negative curvature is compact, then it is finitely covered by a circle bundle over a closed surface of genus ≥ 2 with nonzero Euler class. One of the crucial results used to prove this fact is that complete Lorentz 3-manifolds of constant negative curvature with abelian fundamental groups are not compact. We generalize this result without completeness.

Theorem C. *Let M be a Lorentz 3-manifold of constant negative curvature. If the holonomy group of M is virtually abelian, then M is not compact.*

Using this theorem, we have

Theorem D. *Let M be a Lorentz 3-manifold of constant negative curvature. Suppose that the universal covering space $\widetilde{M}$ of M admits a nontrivial complete Killing vector field and the developing map is injective. If M is compact, then M is geodesically complete.*

We relate Lorentz causal character of Killing vector fields to Lorentz 3-manifolds of constant curvature.

Theorem E. (a) *There exists no compact Lorentz 3-manifold of constant positive curvature which admits a spacelike Killing vector field or a lightlike Killing vector field.*

(b) *If a compact Lorentz flat 3-manifold admits a lightlike Killing vector field, then it is an infranilmanifold.*

(c) *If a compact Lorentz flat 3-manifold admits a spacelike Killing vector field and is not a Euclidean space form, then it is an infrasolvmanifold but not an infranilmanifold.*

(d) *A compact Lorentz 3-manifold of constant negative curvature admitting a timelike Killing vector field is a standard space form.*

(e) *There exists no lightlike Killing vector field on a compact Lorentz 3-manifold of constant negative curvature.*

(f) *If a compact Lorentz 3-manifold M of constant negative curvature admits a spacelike Killing vector field and the developing map is injective,* *then a finite covering of M is either a homogeneous standard space form or a nonstandard space form.*

For the current development of compact Lorentz flat manifolds, the reader should refer to [9], [14], [17], [28], [33] and for the three-dimensional Lorentz manifolds of negative curvature and related topics to [7], [8], [10], [15], [29], [30].

This paper is organized as follows: In §1 we define Lorentz causal character of vector fields and collect some elementary facts about Lorentz structure. §2 is devoted to Lorentz manifolds of nonpositive curvature. The above classification theorems are proved in §§3 and 4. Lorentz 3-manifolds of constant curvature are discussed in §4.

I. Preliminaries

1.1. Let M be a Lorentz manifold with metric g. A tangent vector v $(\neq 0)$ to M falls into the following types:

$$\begin{array}{ll} \textit{timelike} & \textit{if}\ g(v\,,\,v) < 0\,, \\ \textit{lightlike} & \textit{if}\ g(v\,,\,v) = 0\,, \\ \textit{spacelike} & \textit{if}\ g(v\,,\,v) > 0. \end{array}$$

A vector field V on M is timelike if all of the vectors $V_p \in T_pM$ are timelike; similarly for lightlike and spacelike vector fields.

1.2. Consider the following quadrics:

$$\mathbf{S}^{1,n} = \{p = (x_1\,,\,y_1\,,\,\cdots\,,\,y_{n+1}) \in \mathbf{R}^{1,n+1} | -x_1^2 + y_1^2 + \cdots + y_{n+1}^2 = 1\}\,,$$
$$\mathbf{H}^{1,n} = \{p = (x_1\,,\,x_2\,,\,y_1\,,\cdots,\,y_n) \in \mathbf{R}^{2,n} | -x_1^2 - x_2^2 + y_1^2 + \cdots + y_n^2 = -1\}.$$

Note that $\mathbf{S}^{1,n} \approx \mathbf{R}^1 \times \mathbf{S}^n$, $\mathbf{H}^{1,n} \approx \mathbf{S}^1 \times \mathbf{R}^n$. Then $\mathbf{S}^{1,n}$ and $\mathbf{H}^{1,n}$ are complete Lorentz $(n+1)$-dimensional manifolds of constant curvature 1 and -1 respectively. The groups $O(1\,,\,n+1)$ and $O(2\,,\,n)$ are the orthogonal subgroups of $\mathrm{GL}(n+2\,,\,\mathbf{R})$ that preserve the quadratic forms

$$Q^+(x_1\,,\,y_1\,,\,\cdots\,,\,y_{n+1}) = -x_1^2 + y_1^2 + \cdots + y_{n+1}^2\,,$$
$$Q^-(x_1\,,\,x_2\,,\,y_1\,,\,\cdots\,,\,y_n) = -x_1^2 - x_2^2 + y_1^2 + \cdots + y_n^2.$$

Thus it follows that $O(1\,,\,n+1) = \mathrm{Iso}(\mathbf{S}^{1,n})$ and $O(2\,,\,n) = \mathrm{Iso}(\mathbf{H}^{1,n})$ (cf. [24], [34]). Let $\tilde{\mathbf{S}}^{1,n}$ be the universal covering space of $\mathbf{S}^{1,n}$. Denote by $O(1\,,\,n+1)^{\sim}$ the corresponding lift of $O(1\,,\,n+1)$ to a group acting on $\tilde{\mathbf{S}}^{1,n}$. Similarly let $O(2\,,\,n)^{\sim}$ be the corresponding lift of $O(2\,,\,n)$ to the universal covering space $\tilde{\mathbf{H}}^{1,n}$. In this case there is the canonical exact sequence $1 \to \mathscr{Z} \to O(2, n)^{\sim} \xrightarrow{P} O(2, n) \to 1$, where $\mathscr{Z}$ is an infinite cyclic central subgroup. We note the following. (Compare [24, §7].)

Lemma 1.3. *The groups* $O(1, n+1)^{\sim}$ *and* $O(2, n)^{\sim}$ *are the full groups of isometries of* $\widetilde{\mathbf{S}}^{1,n}$ *and* $\widetilde{\mathbf{H}}^{1,n}$ *respectively* $(n \geq 2)$.

Proof. Since $\mathbf{S}^{1,n}$ is simply connected for $n \geq 2$, it follows that $\widetilde{\mathbf{S}}^{1,n} = \mathbf{S}^{1,n}$ and $O(1, n+1)^{\sim} = O(1, n+1)$. Recall that $O(1, n)\backslash O(2, n) = \mathbf{H}^{1,n}$ where $O(1, n)$ is isomorphic to the stabilizer of $O(2, n)$ at the point $p = (1, 0, \cdots, 0)$. If $\tilde{p}$ is a lift of the point p to $\widetilde{\mathbf{H}}^{1,n}$, then from the covering theory it follows that $O(1, n)^{\sim}$, the stabilizer of $O(2, n)^{\sim}$ at $\tilde{p}$, maps isomorphically onto $O(1, n)$ and $O(1, n)^{\sim}\backslash O(2, n)^{\sim} = \widetilde{\mathbf{H}}^{1,n}$. Let $\mathrm{Iso}(\widetilde{\mathbf{H}}^{1,n})$ be the group of all isometries of $\widetilde{\mathbf{H}}^{1,n}$. As $\mathrm{Iso}(\widetilde{\mathbf{H}}^{1,n})$ acts transitively on $\widetilde{\mathbf{H}}^{1,n}$, it is sufficient to prove that $\mathrm{Iso}(\widetilde{\mathbf{H}}^{1,n})_{\tilde{p}} = O(1, n)^{\sim}$. For this, note that $T_{\tilde{p}}\widetilde{\mathbf{H}}^{1,n}$ is isometric to $\mathbf{R}^{1,n}$. Taking the differentials, we have a homomorphism: $\mathrm{Iso}(\widetilde{\mathbf{H}}^{1,n})_{\tilde{p}} \to O(1, n)$. Obviously it is a monomorphism and so $\mathrm{Iso}(\widetilde{\mathbf{H}}^{1,n})_{\tilde{p}} = O(1, n)^{\sim}$ because $O(1, n)^{\sim} \approx O(1, n)$.

1.4. Models for complete Lorentz manifold. The vector space $\mathbf{R}^{1,n}$ (cf. Introduction) is a complete connected simply connected Lorentz manifold of zero curvature. The Lorentz metric is obtained by Euclidean parallel translation of the above form Q (cf. [34], [31]). We simply denote it by $\mathbf{R}^{n+1}$. The group of isometries of $\mathbf{R}^{n+1}$ is isomorphic to the semidirect product $\mathbf{R}^{n+1} \rtimes O(1, n)$.

The complete connected simply connected Lorentz $n+1$ dimensional manifolds of constant curvature k, with their groups of isometries are:

$$\begin{array}{ll} (O(1, n+1)^{\sim}, \widetilde{\mathbf{S}}^{1,n}) & \text{if } k = 1, \\ (\mathbf{R}^{n+1} \rtimes O(1, n), \mathbf{R}^{n+1}) & \text{if } k = 0, \\ (O(2, n)^{\sim}, \widetilde{\mathbf{H}}^{1,n}) & \text{if } k = -1. \end{array}$$

Notice that we may reduce the case of general k to those three cases by scaling the metric. By (G, X) we shall mean one of the above geometries. We say that a *Lorentz spherical structure* (resp. *Lorentz flat structure, and Lorentz hyperbolic structure*) on an $(n+1)$-dimensional manifold M is a *geometric structure modelled on* X *whose coordinate changes lie in* G where (G, X) represents one of the above geometries for $k = 1, 0$, and -1. A Lorentz spherical (resp. flat and hyperbolic) manifold M is a smooth manifold equipped with a Lorentz spherical (resp. flat and hyperbolic) structure. By the usual monodromy argument (cf. [23], [11], [34, Theorem 2.3.12], for example), given a Lorentz manifold M there exist an immersion $\mathrm{dev}\colon \widetilde{M} \to X$ preserving the Lorentz structure and a homomorphism $\rho\colon \pi_1(M) \to G$, where $\widetilde{M}$ is the universal covering space. Moreover the holonomy map ρ extends to a homomorphism of $\mathrm{Iso}(\widetilde{M})$ into G. Therefore we have the developing pair $(\rho, \mathrm{dev})\colon (\mathrm{Iso}(\widetilde{M}), \widetilde{M}) \to (G, X)$ such that $\pi_1(M) \subset \mathrm{Iso}(\widetilde{M})$.

By a Lorentz space form we shall mean a (geodesically) complete Lorentz manifold of constant curvature. It is noted that *a Lorentz manifold is (geodesically) complete if the developing map is a covering map onto* X. Then the Lorentz space form problem states that *a Lorentz space form is isometric (up to rescaling the metric by a constant) to a quotient* X/Γ *where* Γ *is a subgroup of* G, *that acts properly discontinuously and freely.* (Compare [34].)

2. Lorentz manifolds of nonpositive curvature

In this section we examine the structure of Lorentz manifolds of constant curvature k where $k = 0$ or $k = -1$.

2.1. Definition. Let $\{\varphi_t\}_{|t|<\infty}$ be a one-parameter group of Lorentz isometries on a Lorentz manifold M. The group $\{\varphi_t\}_{|t|<\infty}$ induces the vector field X on M. The vector X_p is tangent to the orbit $\{\varphi_t(p)\}_{|t|<\infty}$ at p for each point $p \in M$. Then the group $\{\varphi_t\}_{|t|<\infty}$ is said to be timelike if X is timelike; similarly for lightlike and spacelike (cf. 1.1).

Proposition 2.2. *Suppose that* H *is a one-parameter group of* $\mathbf{R}^{n+1} \rtimes O(1, n)$. *If* H *is either timelike or lightlike, then the closure* $\overline{H}$ *is noncompact.*

Proof. If $\overline{H}$ is compact, then it is conjugate to a subgroup of $SO(n)$. Choose the point $p = (0, 1, 0, \cdots, 0) \in \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{n+1}$. Thus the orbit $\overline{H}p$ sits in $\{0\} \times \mathbf{R}^n$, and any vector field V tangent to the orbit satisfies $g(V, V) > 0$, which is impossible. Hence $\overline{H}$ is noncompact. q.e.d.

Let $1 \to \mathscr{Z} \to O(2, n)^\sim \xrightarrow{P} O(2, n) \to 1$ be the exact sequence associated with the projection $P\colon \widetilde{\mathbf{H}}^{1,n} \to \mathbf{H}^{1,n}$ where $\mathscr{Z}$ is an infinite central cyclic subgroup.

Proposition 2.3. *Let* H *be a one-parameter group of* $O(2, n)^\sim$.

(1) *If* H *is either timelike or lightlike, then the closure* $\overline{H}$ *is noncompact.*

(2) *If* H *is noncompact and* $P(H)$ *is compact, then* H *is timelike.*

Proof. (1) The above exact sequence induces the exact sequence $1 \to \mathscr{Z} \to \mathbf{R} \times O(n)^\sim \to SO(2) \times SO(n) \to 1$. Suppose that H is either timelike or lightlike. If $\overline{H}$ is compact, then it is conjugate to a subgroup of the maximal compact subgroup $O(n)^{\sim}$. (Compare [18].) It follows that $P(\overline{H}) \subset \{1\} \times SO(n)$ up to conjugation. We can assume that $P(H)$ belongs to the maximal torus such that

$$P(H) = \begin{pmatrix} 1 & & & & & \\ & \cos t & -\sin t & & & \\ & \sin t & \cos t & & & \\ & & & * & & \\ & & & & \ddots & \\ & & & & & * \end{pmatrix}.$$

Taking $y = (\sqrt{2}, 0, 1, 0, \cdots, 0) \in \mathbf{H}^{1,n}$, we have

$$P(H)y = \{(\sqrt{2}, 0, \cos t, \sin t, 0, \cdots, 0)\} \approx \{\sqrt{2}\} \times \mathbf{S}^1.$$

Under the correspondence $\mathbf{H}^{1,n} \approx \mathbf{S}^1 \times \mathbf{R}^n$, the orbit $P(H)y$ is mapped onto the set $\{(1, 0; \cos t, \sin t, 0, \cdots, 0)\} = \mathbf{S}^1$. Thus any vector field V tangent to the orbit satisfies $g(V, V) > 0$. This contradicts the hypothesis on H. Therefore $\overline{H}$ is noncompact in $O(2, n)^{\sim}$.

(2) Suppose that $P(H)$ is compact. Then H is conjugate to a subgroup of $\mathbf{R} \times O(n)^{\sim}$. Since H is noncompact by the hypothesis, $P(H)$ has the following form in $SO(2) \times SO(n)$:

$$\begin{pmatrix} \cos\theta & -\sin\theta & & & & & \\ \sin\theta & \cos\theta & & & & & \\ & & \cos\lambda\theta & -\sin\lambda\theta & & & \\ & & \sin\lambda\theta & \cos\lambda\theta & & & \\ & & & & * & & \\ & & & & & \ddots & \\ & & & & & & * \end{pmatrix}.$$

So the orbit $P(H)(x_1, x_2, y_1, \cdots, y_n)$ consists of the set

$$\{(x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, \cdots)|_{\theta\in\mathbf{R}}\}.$$

Any vector field V tangent to the orbit $P(H)(x_1, x_2, y_1, \cdots, y_n)$ satisfies $g(V, V) = -x_1^2 - x_2^2 + y_1^2 + \cdots + y_n^2 = -1$. Therefore $P(H)$ (and so H) is timelike.

2.4. Timelike Killing vector fields and geodesically completeness. Let (G, X) be one of the following geometries:

$$\begin{aligned} &(\mathbf{R}^{n+1} \rtimes O(1, n), \mathbf{R}^{n+1}) && \text{if } k = 0, \\ &(O(2, n)^{\sim}, \widetilde{\mathbf{H}}^{1,n}) && \text{if } k = -1. \end{aligned}$$

Let M be a Lorentz flat (or hyperbolic) manifold of dimension $n+1$. Then for any developing pair $(\rho, \text{dev})\colon (\text{Iso}(\widetilde{M}), \widetilde{M}) \to (G, X)$, we have $\pi_1(M) \subset \text{Iso}(\widetilde{M})$. Put $\pi = \pi_1(M)$ and $\Gamma = \rho(\pi)$.

Proposition 2.5. *If a compact Lorentz flat (or hyperbolic) manifold M admits a timelike Killing vector field, then M is geodesically complete. In particular, M is a Lorentz space form X/Γ.*

Proof. Since M is compact, the timelike Killing vector field generates a one-parameter group of Lorentz transformations $\{\varphi_t\}_{|t|<\infty}$ on M. Let $\{\tilde{\varphi}_t\}_{|t|<\infty}$ be its lift to the universal covering space $\widetilde{M}$. Put $\rho(\{\tilde{\varphi}_t\}_{|t|<\infty}) = H$.

Let g be the Lorentz metric of X such that $\text{Iso}(X, g) = G$. Since $\{\tilde{\varphi}_t\}_{|t|<\infty} \subset \text{Iso}(\widetilde{M})$, note that $H \subset G$ and H is a timelike one-parameter group. Let ξ be the unit vector field associated with the H-action. Note that $g(\xi, \xi) = -1$. Let $\xi_x^{\perp}$ be the orthogonal complement of ξ_x in T_xX for each $x \in X$. Since g is nondegenerate on the vector space spanned by ξ, the tangent bundle TX decomposes into the orthogonal sum $\xi \oplus \xi^{\perp}$. Then we define a Riemannian metric h on X by setting

$$h(X, Y) = g(X, Y) + 2g(\xi, X) \cdot g(\xi, Y).$$

Since g is nondegenerate and positive definite on $\xi^{\perp}$, h is precisely a Riemannian metric on X. Let $\mathscr{C}_G(H)$ be the centralizer of H in G. If we note that $\alpha_* \xi = \xi$ for each $\alpha \in \mathscr{C}_G(H)$, then the Riemannian metric h is invariant under the group $\mathscr{C}_G(H)$. In particular $\mathscr{C}_G(H) \subset \text{Iso}(X, h)$. Since $\Gamma \subset \mathscr{C}_G(H)$, the pullback of h by the map dev defines a π-invariant Riemannian metric on $\widetilde{M}$. As M is compact, it follows that $\text{dev}\colon \widetilde{M} \to X$ is a covering map. In particular, since X is simply connected, dev is a homeomorphism and so $M \approx X/\Gamma$. q.e.d.

2.6. Consider the exact sequence (compare [2] for example):

$$1 \to \mathscr{C}(\Gamma) \to \mathscr{C}_{\text{Diff}(X)}(\Gamma) \xrightarrow{\eta} \text{Diff}(X/\Gamma)^0 \to 1,$$

where $\mathscr{C}(\Gamma)$ is the center of Γ, and $\mathscr{C}_{\text{Diff}(X)}(\Gamma)$ is the centralizer of Γ in $\text{Diff}(X)$. Let g^* be the induced Lorentz metric on X/Γ from g. Then the Riemannian metric h is invariant under Γ, and induces a Riemannian metric h^* on X/Γ. We consider the subgroups $\text{Iso}(X/\Gamma, g^*)^0$ and $\text{Iso}(X/\Gamma, h^*)^0$ of $\text{Diff}(X/\Gamma)^0$. The above exact sequence restricted to these groups induces the following exact sequences:

$$1 \to \mathscr{C}(\Gamma) \to \mathscr{C}_G(\Gamma) \xrightarrow{\nu} \text{Iso}(X/\Gamma, g^*)^0 \to 1,$$

$$1 \to \mathscr{C}(\Gamma) \to \mathscr{C}_{\mathrm{Iso}(X,h)}(\Gamma) \xrightarrow{\nu'} \mathrm{Iso}(X/\Gamma, h^*)^0 \to 1.$$

Let H be a timelike one-parameter group as in Proposition 2.5. Note that H is closed in G and $H \subset \mathscr{C}_G(\Gamma)$. It is not necessarily true that $\nu(H)$ is compact (i.e., isomorphic to S^1) in $\mathrm{Iso}(X/\Gamma, g^*)^0$. However we prove the following.

Lemma 2.7. *Under the assumption of Proposition* 2.5, *there is a timelike one-parameter group* H' *in* $\mathscr{C}_G(\Gamma)$ *(also in* $\mathscr{C}_{\mathrm{Iso}(X,h)}(\Gamma)$*) such that* $\nu(H')$ *is compact.*

Proof. Since $\mathscr{C}_G(H) \subset \mathrm{Iso}(X, h)$, we obtain that $H \subset \mathscr{C}_{\mathrm{Iso}(X,h)}(\Gamma)$. Put $\eta(H) = H^*$, and note that $\nu(H) = \nu'(H) = H^*$. Let $\overline{H^*}$ be its closure in $\mathrm{Diff}(X/\Gamma)^0$. Then $\overline{H^*}$ sits in both $\mathrm{Iso}(X/\Gamma, g^*)^0$ and $\mathrm{Iso}(X/\Gamma, h^*)^0$. Since $\mathrm{Iso}(X/\Gamma, h^*)^0$ is compact relative to the Riemannian metric h^*, it follows that $\overline{H^*}$ is compact.

Let S be the identity component of the inverse image $\nu^{-1}(\overline{H^*})$. It is easy to see that $\nu^{-1}(\overline{H^*}) = \nu'^{-1}(\overline{H^*})$ so that $S \subset \mathscr{C}_{\mathrm{Iso}(X,h)}(\Gamma)$. The above exact sequence induces the exact sequence of covering groups $1 \to \mathscr{C}(\Gamma) \cap S \to S \xrightarrow{\nu} \overline{H^*} \to 1$. By Propositions 2.2 and 2.3, $\overline{H}$ $(= H)$ is noncompact. Thus S is noncompact, and $\mathscr{C}(\Gamma) \cap S$ is nontrivial. Passing to the universal covering group if necessary, we assume that S is simply connected. Then S is isomorphic to a vector space, and H is isomorphic to a straight line through the origin in the vector space. We can choose a sequence of one-parameter groups $\{H_i'\}$ in S such that

(i) the sequence H_i' converges to H.

(ii) $\nu(H_i')$ is compact, i.e., $1 \to \mathscr{C}(\Gamma) \cap H_i' \to H_i' \to \mathbf{S}^1 \to 1$ is an exact sequence.

It suffices to show that some H_i' is timelike. Let V^i be a unit vector field induced by H_i' for each i. If $P\colon X \to X/\Gamma$ is the canonical projection, then $W^i = P_*(V^i)$ is a unit vector field induced by $\nu(H_i')$. Since $\{\nu(H_i')\}$ converges to $\nu(H)$ by (i), $\{W^i\}$ converges to a timelike vector field W. Suppose that all H_i' are not timelike. Then there exists a sequence $\{x_i\}$ in X such that $g(V^i_{x_i}, V^i_{x_i}) \geq 0$. Note $g(V^i_{x_i}, V^i_{x_i}) = g^*(W^i_{P(x_i)}, W^i_{P(x_i)})$. Since $\{P(x_i)\}$ has an accumulation point x in X/Γ, $\{W^i_{P(x_i)}\}$ converges to W_x and therefore $g^*(W_x, W_x) \geq 0$. This contradicts that W is timelike.

2.8. A Seifert fiber space is a (locally trivial) fiber space over a (smooth) orbifold whose typical fiber is S^1, and exceptional fiber is homeomorphic to a circle (i.e., an orbit space S^1/F by a cyclic group F). See [6], [27] for higher-dimensional Seifert fiber spaces.

Theorem 2.9. *Let M be a compact Lorentz flat (or hyperbolic) manifold. Suppose that M admits a timelike Killing vector field. Then M admits an isometric action of a timelike one-parameter group of a circle S^1, and further is a Seifert fiber space over a nonpositively curved orbifold.*

Proof. Since H' is a closed subgroup of $\mathrm{Iso}(X, h)$ by Lemma 2.7, H' acts properly and freely on X. It induces a principal bundle $H' \to X \xrightarrow{\eta} W$ where $W = X/H'$. Suppose that H' induces a vector field ξ'. Then the Lorentz metric g satisfies $g(\xi', \xi') < 0$. Since $\eta_*: \xi'^{\perp} \to TW$ is an isomorphism, the restriction of g to $\xi'^{\perp}$ defines a Riemannian metric $\hat{g}$ on W. It is easy to see that η maps the group $\mathscr{C}_G(H')$ into $\mathrm{Iso}(W, \hat{g})$. We obtain the equivariant principal bundle

$$H' \to (\mathscr{C}_G(H'), X) \xrightarrow{\eta} (\mathrm{Iso}(W, \hat{g}), W).$$

The intersection $\Gamma \cap H'$ is an infinite cyclic group by (ii) of Lemma 2.7. Corresponding to the above bundle, there is an exact sequence $1 \to \Gamma \cap H' \to \Gamma \to Q \to 1$.

Since Γ acts properly discontinuously and H' acts freely, Q acts properly discontinuously on W. In particular Q is discrete in $\mathrm{Iso}(W, \hat{g})$. Therefore we have a Seifert fiber space

$$\mathbf{S}^1 \to X/\Gamma \to W/Q,$$

where $\mathbf{S}^1 = H'/\Gamma \cap H'$. Since H' is timelike, $\mathbf{S}^1$ acts as Lorentz isometries of a timelike one-parameter group on X/Γ with respect to g^*. Finally we prove that W/Q is a nonpositively curved orbifold. Let $\overline{Y}$, $\overline{Z}$ be orthonormal vectors of a plane in $\xi_x'^{\perp}$ such that $\eta_*(\overline{Y}) = Y$, and $\eta_*(\overline{Z}) = Z$, which span a plane section of $T_{\eta(x)}W$. Applying O'Neill's formula [31] to the above principal fibration yields that $4k(Y, Z) = c + \frac{3}{4}g([\overline{Y}, \overline{Z}]^{\mathscr{V}}, [\overline{Y}, \overline{Z}]^{\mathscr{V}})$ where k is the sectional curvature of W with respect to $\hat{g}$, c is the constant sectional curvature of X, and $\mathscr{V}$ stands for the vertical component. Since $g([\overline{Y}, \overline{Z}]^{\mathscr{V}}, [\overline{Y}, \overline{Z}]^{\mathscr{V}}) \leq 0$ and $c \leq 0$, we have $k \leq 0$.

2.10. Structure of (Q, W). Let k be the sectional curvature of W as above. It satisfies that $k \leq 0$ or $k \leq -\frac{1}{4}$ according as $c = 0$ or $c = -1$.

Proposition 2.11. (i) *Let $k \leq 0$. Suppose that Q is virtually polycyclic. Then W is necessarily isometric to the Euclidean space (i.e., $k = 0$), and Q is a virtually free abelian group.*

(ii) *Let $k \leq -\frac{1}{4}$. Then Q has no normal solvable subgroup.*

Proof. W/Q is a compact nonpositively curved orbifold. (i) is the special case of Corollary 3 of Gromoll and Wolf [16] (cf. also [26]). In fact suppose that a normal solvable subgroup of Q contains an element of infinite order. Then W is isometric to the product $E \times D$ where E is a Euclidean space such that $0 < \dim E =$ rank of a normal free abelian subgroup of Q.

For (ii) we need some lemmas. Let ρ be the distance function on W induced from $\hat{g}$. For each $\alpha \in \mathrm{Iso}(W, \hat{g})$ we have the displacement function $\delta_\alpha(w) = \rho(w, \alpha w)$. Put $C_\alpha = \{w \in W | \delta_\alpha(w) = 0\}$ (i.e., the fixed point set of α). It is known that C_α is convex.

Lemma 2.12 [16]. *If $C \neq \varnothing$ is closed, convex and invariant under an element $\alpha \in Q$ then $C \cap C_\alpha \neq \varnothing$.*

Using this lemma we can prove the following (cf. [16, Theorem 1]).

Lemma 2.13. *Let T be a torsion solvable subgroup of Q. Then T has a fixed point in W. In particular T is a finite group.*

Proof of (ii). If we note $k \leq -1$, then a normal solvable subgroup of Q has no element of infinite order by the proof of (i). Let T be a normal solvable subgroup of Q. Then T is a finite group by Lemma 2.13, and so $C_T = \bigcap_{\alpha \in T} C_\alpha$ is nonempty, convex by Lemma 2.12.

Since T is normal in Q, C_T is invariant under Q. Let $H' \to \widetilde{\mathbf{H}}^{1,n} \xrightarrow{\eta} W$ be the principal bundle as before. Then $Y = \eta^{-1}(C_T)$ is a Γ-invariant contractible submanifold of $\widetilde{\mathbf{H}}^{1,n}$, and thus $\mathrm{cd}\,\Gamma \leq \dim Y$. Since $\mathrm{cd}\,\Gamma = \dim \widetilde{\mathbf{H}}^{1,n}$, it follows that $Y = \widetilde{\mathbf{H}}^{1,n}$ or $W = C_T$. As T acts as isometries on W, we obtain that $T = \{1\}$.

2.14. Lorentz flat structure. A Lorentz flat manifold is an affinely flat manifold (cf. [10], [11]). We can classify compact Lorentz flat manifolds more clearly (cf. [17], [33]).

Theorem 2.15. *Let M be a compact Lorentz flat $(n+1)$-manifold $(n \geq 0)$. If M supports a timelike Killing vector field, then M is affinely diffeomorphic to a Euclidean space form with nonvanishing first Betti number.*

Proof. We have shown $M \approx \mathbf{R}^{n+1}/\Gamma$ which admits a Seifert fibration: $\mathbf{S}^1 \to \mathbf{R}^{n+1}/\Gamma \xrightarrow{P} W/Q$. Here W/Q is a compact Riemannian orbifold for which the sectional curvature k of W is nonpositive by Theorem 2.9. Since the fundamental group Γ is virtually polycyclic by the result of [13], Q is also virtually polycyclic. Proposition 2.11 implies that $W = \mathbf{R}^n$ (i.e., $k = 0$) and Q is virtually free abelian. We prove that Γ is also virtually free abelian. Passing to a subgroup of finite index if necessary, Q is a free abelian group in which W/Q is an n-torus T^n. Now the above fibration is a principal circle bundle over T^n. It is sufficient to show that the Euler class of this bundle vanishes. Let c^* and k^* be the induced sectional curvatures on $\mathbf{R}^{n+1}/\Gamma$ and T^n respectively. In our case we have $c^* = k^* = 0$. Let ξ be a unit vector field induced by the circle S^1. If we apply O'Neill's formula to the principal bundle, then $[\overline{X}, \overline{Y}]^{\mathscr{V}} = 0$ for $\overline{X}, \overline{Y} \in \xi^{\perp}$. (Compare the proof of Theorem 2.9.) Let ω be a real-valued 1-form on $\mathbf{R}^{n+1}/\Gamma$ defined by $\omega(\xi) = 1$ and $\omega(\xi^{\perp}) = 0$. Since the Lorentz metric g^* (cf. 2.6) and $\xi^{\perp}$ are invariant under $\mathbf{S}^1$, ω is a connection form in $\mathbf{R}^{n+1}/\Gamma$. There is a unique 2-form Ω on T^n such that $d\omega = P^*\Omega$ and the characteristic class $[\Omega]$ defines the Euler class of the above bundle (cf. [22]). Since $\xi^{\perp}$ consists of horizontal vectors for ω, the above fact implies $d\omega(\overline{X}, \overline{Y}) = 0$. Thus $\Omega \equiv 0$ on T^n, and the Euler class of the above bundle is zero.

If a compact complete affinely flat manifold has a virtually free abelian group as the fundamental group, then it is affinely diffeomorphic to a Euclidean space form. (Compare [14], [10], [19] for example.) Moreover, a compact Euclidean space form M admits a maximal T^k action if and only if $\operatorname{rank} H_1(M, \mathbf{Z}) = k$ (cf. [5], [34]). And so our Euclidean space form has the nonzero first Betti number.

2.16. Lorentz hyperbolic structure. When M is a compact Lorentz hyperbolic manifold, we can prove

Theorem 2.17. *If a compact Lorentz hyperbolic manifold admits a time-like Killing vector field, then some finite covering is diffeomorphic to a circle bundle over a negatively curved manifold.*

Proof. We have $M \approx \widetilde{\mathbf{H}}^{1,n}/\Gamma$. By Theorem 2.9 and its proof there exist the principal fibration $H' \to \widetilde{\mathbf{H}}^{n+1} \to W$ and the exact sequences:

$$\begin{array}{lccccccccc} & 1 & \longrightarrow & H' & \longrightarrow & \mathscr{C}_G(H') & \xrightarrow{\nu} & \operatorname{Iso}(W, \hat{g}) & & \\ (1) & & & \uparrow & & \uparrow & & \uparrow & & \\ & 1 & \longrightarrow & \mathbf{Z} & \longrightarrow & \Gamma & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

Note $k \leq -\frac{1}{4}$ for the sectional curvature k of W. If we can find a torsion free normal subgroup Q' of finite index in Q, then a finite covering of $\widetilde{\mathbf{H}}^{1,n}/\Gamma$ is a circle bundle over a Riemannian manifold W/Q' of the sectional curvature $k^* \leq -\frac{1}{4}$. The rest of proof is devoted to find such a group Q'.

Let $1 \to \mathscr{Z} \to G \xrightarrow{P} O(2, n) \to 1$ be the exact sequence where $G = O(2, n)^{\sim}$. This induces the exact sequence

$$1 \to \mathscr{Z} \to \mathscr{C}_G(H') \xrightarrow{P} \mathscr{C}_{O(2,n)}(P(H')) \to 1.$$

Put $\Gamma' = P(\Gamma)$. As $O(2, n) \subset \mathrm{GL}(n+2, \mathbf{R})$, we consider the real algebraic closure of $\mathscr{C}_{O(2,n)}(P(H'))$. If $\mathscr{A}$ is its identity component, then

$\mathscr{A}$ centralizes $P(H')$ because $\mathscr{C}_{O(2,n)}(P(H'))$ is the centralizer of $P(H')$. Let $\nu' : \mathscr{A} \to \mathscr{A}/P(H')$ be the quotient map. Passing to a subgroup of finite index if necessary, we assume $\Gamma' \subset \mathscr{A}$. Put $Q' = \nu'(\Gamma')$. Combining these with (1) yields the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{Z} \cap \Gamma & \longrightarrow & \Gamma & \xrightarrow{P} & \Gamma' & \longrightarrow & 1 \\ & & \downarrow & & {\scriptstyle \nu}\downarrow & & {\scriptstyle \nu'}\downarrow & & \\ 1 & \longrightarrow & \nu(\mathscr{Z}) \cap Q & \longrightarrow & Q & \longrightarrow & Q' & \longrightarrow & 1. \end{array} \tag{2}$$

Then we note $Q \approx Q'$ by Proposition 2.11.

On the other hand, if $\mathscr{R}$ is the radical of $\mathscr{A}$, i.e., a unique maximal connected solvable algebraic group, then there exists a complementary semisimple algebraic subgroup $\mathscr{S} \subset \mathscr{A}$. $\mathscr{S}$ maps onto $\mathscr{A}/\mathscr{R}$. The canonical projection of $\mathscr{A}/P(H')$ onto $\mathscr{A}/\mathscr{R}$ maps Q' onto a subgroup Q'' of $\mathscr{A}/\mathscr{R}$. Since the kernel of this projection is a solvable Lie group $\mathscr{R}/P(H')$, Q is isomorphic to Q''.

Consider the following exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{S} \cap \mathscr{R} & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{A}/\mathscr{R} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathscr{S} \cap \mathscr{R} & \longrightarrow & \Psi & \longrightarrow & Q'' & \longrightarrow & 1, \end{array} \tag{3}$$

where Ψ is the inverse image of Q''. Since both $\mathscr{S}$ and $\mathscr{R}$ are algebraic, $\mathscr{S} \cap \mathscr{R}$ is a finite central subgroup and so Ψ is a finitely generated subgroup lying in $\mathrm{GL}(n+2, \mathbf{R})$. Applying Selberg's lemma shows that Ψ contains a torsion free normal subgroup of finite index. Such a group maps isomorphically onto a torsion free normal subgroup of Q''. Therefore there exists a torsion free normal subgroup of finite index in Q. Thus the theorem is proved.

2.18. Examples of standard space forms of dimension $2n+1$ $(n \geq 1)$. It is difficult to determine the topology of the orbit space W/Q. We shall give examples of compact Lorentz hyperbolic space forms with timelike circle actions in higher dimensions (cf. 4.6). In the next theorem we consider the case where a compact Lorentz hyperbolic manifold with timelike Killing vector field becomes a standard space form.

Let $Q(z, w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+1} w_{n+1}$ be the Hermitian form on $\mathbf{C}^{n+1}$. The group $U(1, n)$ is the subgroup of $\mathrm{GL}(n+1, \mathbf{C})$ preserving the form Q. There is the natural embedding of $U(1, n)$ into $O(2, 2n)$. Then $U(1, n)$ acts transitively on $\mathbf{H}^{1,2n}$ whose stabilizer is isomorphic to the unitary group $U(n)$. Here $\mathbf{H}^{1,2n}$ is identified with the set $\{z \in \mathbf{C}^{n+1} | Q(z, z) = -1\}$. Let $U(1, n)^{\sim}$ be the lift of $U(1, n)$ corresponding to the universal covering space $\widetilde{\mathbf{H}}^{1,2n}$. If $\widetilde{\Gamma}$ is a discrete cocompact subgroup of $U(1, n)^{\sim}$, then we have a compact Lorentz hyperbolic space form $\widetilde{\mathbf{H}}^{1,2n}/\widetilde{\Gamma} \approx U(n)^{\sim}\backslash U(1, n)^{\sim}/\widetilde{\Gamma}$. (Note $U(n) \approx U(n)^{\sim}$.) Such a Lorentz manifold is called a standard space form following Kulkarni [24]. Let $\mathscr{Z}(1, n)$ be the kernel of the canonical projection of $U(1, n)$ onto the group $PU(1, n)$ consisting of biholomorphic transformations of complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n}$. The center $\mathscr{Z}(1, n)$ is isomorphic to $\mathbf{S}^1$. If $\widetilde{\mathscr{Z}(1, n)}$ is the lift of $\mathscr{Z}(1, n)$ to $U(1, n)^{\sim}$, then $\widetilde{\mathscr{Z}(1, n)}$ is isomorphic to $\mathbf{R}^1$ and is timelike by Proposition 2.3.

Proposition 2.19. $\widetilde{\mathbf{H}}^{1,2n}/\widetilde{\Gamma}$ *is a Seifert fiber space over a complex* (*Kähler*) *hyperbolic orbifold* $\mathbf{H}_{\mathbf{C}}^{n}/\Gamma$, *where the circle acts as a timelike one-parameter group of Lorentz transformations.*

Proof. Put $\Delta = \widetilde{\mathscr{Z}(1, n)} \cap \widetilde{\Gamma}$ and consider the exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{\mathscr{Z}(1, n)} & \longrightarrow & U(1, n)^{\sim} & \xrightarrow{\widetilde{P}} & PU(1, n) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1.
\end{array}$$

Then Δ is infinite cyclic if and only if Γ is discrete. If we prove that Γ is discrete, then the result follows from the following diagram:

$$\begin{array}{ccccc}
\widetilde{\mathscr{Z}(1, n)} & \longrightarrow & U(n)^{\sim}\backslash U(1, n)^{\sim} = \widetilde{\mathbf{H}}^{1,2n} & \longrightarrow & U(n)\backslash PU(1, n) = \mathbf{H}_{\mathbf{C}}^{n} \\
\downarrow /\Delta & & \downarrow /\widetilde{\Gamma} & & \downarrow /\Gamma \\
\mathbf{S}^1 & \longrightarrow & \widetilde{\mathbf{H}}^{1,2n}/\widetilde{\Gamma} & \longrightarrow & \mathbf{H}_{\mathbf{C}}^{n}/\Gamma.
\end{array}$$

Suppose that Γ is not discrete. Then we will show that it contradicts the cohomological dimension $\mathrm{ch}\,\widetilde{\Gamma} = 2n + 1$. Let $\overline{\Gamma}^0$ be the identity component of the closure of Γ in $PU(1, n)$. Then it is known that $\overline{\Gamma}^0$ is solvable (cf. [32, Lemma 8.24]).

Case A. If $\overline{\Gamma}^0$ is compact, then the fixed point set of $\overline{\Gamma}^0$ is the totally geodesic subspace $\mathbf{H}_{\mathbf{C}}^{k}$ of $\mathbf{H}_{\mathbf{C}}^{n}$ $(n > k)$, and Γ leaves $\mathbf{H}_{\mathbf{C}}^{k}$ invariant. Moreover, Γ lies in the subgroup $P(U(1, k) \times U(n - k))$, and thus we obtain $\widetilde{\Gamma} \subset U(1, k)\widetilde{\times U}(n - k)$. On the other hand, since $U(1, k)\widetilde{\times U}(n-k)$ acts transitively on $\widetilde{\mathbf{H}}^{1,2k}$, $\widetilde{\Gamma}$ acts properly discontinuously on $\widetilde{\mathbf{H}}^{1,2k}$ so that $\mathrm{ch}\,\widetilde{\Gamma} \leq 2k+1$. This contradicts the cohomological dimension of $\widetilde{\Gamma}$.

Case B. Suppose that $\overline{\Gamma}^0$ is noncompact. Then its normalizer $N(\overline{\Gamma}^0)$ is conjugate to a subgroup of the maximal amenable Lie subgroup $\mathscr{N} \rtimes (U(n - 1) \times \mathbf{R}^+)$ of $PU(1, n)$. Here $\mathscr{N}$ is the $(2n - 1)$-dimensional Heisenberg Lie group. (See for example [21], [30].) Since $\overline{\Gamma}^0$ is solvable, we may assume that $\overline{\Gamma}^0 = \mathscr{N} \rtimes (T^{n-1} \times \mathbf{R}^+)$. Then it is easy to see that $N(\overline{\Gamma}^0) = \mathscr{N} \rtimes (N(T^{n-1}) \times \mathbf{R}^+)$ where $N(T^{n-1})$ is the normalizer of the maximal torus in $U(n-1)$. Note that $N(T^{n-1})/T^{n-1}$ is finite. Now $\Gamma \subset N(\overline{\Gamma}^0)$, passing to a subgroup of finite index, we can assume $\Gamma \subset \mathscr{N} \rtimes (T^{n-1} \times \mathbf{R}^+)$. It follows from the above exact sequence that $\tilde{\Gamma} \subset \mathscr{N} \rtimes H$ where $H = T^{n-1} \times \widetilde{\mathscr{Z}(1, n)} \times \mathbf{R}^+$.

Let $\psi: \mathscr{N} \rtimes H \to H$ be the natural projection. If $\psi(\tilde{\Gamma}) \subset T^{n-1} \times \widetilde{\mathscr{Z}(1, n)}$, then $\operatorname{ch} \tilde{\Gamma} = \dim \mathscr{N} + \dim \widetilde{\mathscr{Z}(1, n)} = 2n$, which is impossible. On the other hand if $\psi(\tilde{\Gamma})$ has nontrivial $\mathbf{R}^+$-summand, then the intersection $\mathscr{N} \cap \tilde{\Gamma}$ is trivial. For this, $\mathbf{R}^+$ acts as left multiplication on $\mathscr{N}$, but $\mathscr{N} \cap \tilde{\Gamma}$ is a lattice of $\mathscr{N}$. Now $\tilde{\Gamma}$ must be a free abelian group, i.e., isomorphic to a subgroup of H. If we note that $P = \mathscr{N} \times \widetilde{\mathscr{Z}(1, n)}$ is the nilradical of $\mathscr{N} \rtimes H$, then the intersection $\tilde{\Gamma} \cap P$ is uniform in P (cf. [32, Theorem 3.3]). But this is impossible because $\tilde{\Gamma} \cap P$ is abelian. Hence the proof is complete.

Theorem 2.20. *Let M be a $(2n+1)$-dimensional compact Lorentz hyperbolic manifold which admits a one-parameter group of Lorentz transformations $\{\phi_t\}_{|t|<\infty}$. Let $(\rho, \mathrm{dev}): (\pi, \{\tilde{\phi}_t\}_{|t|<\infty}, \tilde{M}) \to (\Gamma, \tilde{H}, \tilde{\mathbf{H}}^{1,2n})$ be the developing pair and $1 \to \mathscr{Z} \to O(2, 2n)^\sim \xrightarrow{P} O(2, 2n) \to 1$ be the exact sequence associated with the projection $P: \tilde{\mathbf{H}}^{1,2n} \to \mathbf{H}^{1,2n}$. Then $P(\tilde{H})$ is compact in $O(2, 2n)$ if and only if M is a standard space form $U(n)^\sim \backslash U(1, n)^\sim / \Gamma$. In particular $\{\phi_t\}_{|t|<\infty}$ is a timelike one-parameter group.*

Proof. The sufficient condition follows from the fact that $\tilde{H} = \widetilde{\mathscr{Z}(1, n)}$ and $P(\tilde{H}) = \mathscr{Z}(1, n)$ is a circle (cf. 2.18).

Put $H = P(\tilde{H})$. Suppose that H is compact in $O(2, 2n)$. Then H is a circle embedded into the maximal connected compact subgroup $SO(2) \times SO(2n)$ of $O(2, 2n)^0$. Consider the extreme case where

$$H = \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \cdots \times \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right).$$

By direct calculation from the Lie algebra theory it follows that the centralizer $\mathscr{C}_{O(2,2n)}(H) = U(1, n)$.

The above projection P induces the exact sequence:

$$1 \to \mathscr{Z} \to \mathscr{C}_{O(2,2n)^\sim}(\tilde{H}) \to \mathscr{C}_{O(2,2n)}(H) \to 1.$$

Then it follows $\mathscr{C}_{O(2,2n)^\sim}(\tilde{H}) = U(1, n)^\sim$. Furthermore since $\tilde{H}$ centralizes the holonomy group Γ, we obtain $\Gamma \subset U(1, n)^\sim$. As $U(1, n)^\sim$ acts properly on $\tilde{\mathbf{H}}^{1,2n}$, there is a $U(1, n)^\sim$-invariant Riemannian metric on

$\widetilde{\mathbf{H}}^{1,2n}$. M is compact and so we obtain a π-invariant complete Riemannian metric on $\widetilde{M}$ by the pullback of dev. Therefore dev is a homeomorphism of $\widetilde{M}$ onto $\widetilde{\mathbf{H}}^{1,2n}$ and hence $M \approx \widetilde{\mathbf{H}}^{1,2n}/\Gamma \approx U(n)^{\sim}\backslash U(1,n)^{\sim}/\Gamma$. This proves the extreme case.

In general H has the following form:

$$\begin{pmatrix} \cos a_1\theta & -\sin a_1\theta \\ \sin a_1\theta & \cos a_1\theta \end{pmatrix} \times \begin{pmatrix} \cos a_2\theta & -\sin a_2\theta \\ \sin a_2\theta & \cos a_2\theta \end{pmatrix} \times \cdots \\ \times \begin{pmatrix} \cos a_k\theta & -\sin a_k\theta \\ \sin a_k\theta & \cos a_k\theta \end{pmatrix} \times (1),$$

for nonzero numbers $a_1, a_2, \cdots, a_k$. Then it turns out that $\mathscr{C}_{O(2,2n)}(H)$ becomes a smaller subgroup than that of the extreme case. In fact $\mathscr{C}_{O(2,2n)}(H)$ belongs to the group G with the following possibilities:

$$\begin{aligned} &G = \{1\} \times SO(2n), \\ &G = U(1,k) \times SO(2n-2k) \qquad (1 \leq k < n), \\ &G = SO(2) \times SO(2n). \end{aligned}$$

Let $\widetilde{G}$ be its lift to $O(2,2n)^{\sim}$. Then we notice that $\widetilde{G}$ acts properly on $\widetilde{\mathbf{H}}^{1,2n}$. Since $\Gamma \subset \widetilde{G}$, we can apply the same argument as above. If $\widetilde{G} = SO(2n)$, then Γ must be finite. If $\widetilde{G} = U(1,k)^{\sim} \times SO(2n-2k)$, then $\operatorname{cd}\Gamma \leq 2k+1 < 2n+1$. If $\widetilde{G} = \mathbf{R}^1 \times SO(2n)$, then $\operatorname{cd}\Gamma = 1$. Since $\operatorname{cd}\Gamma = 2n+1$, these are impossible. Hence the theorem is proved.

3. Lorentz spherical structure

Lemma 3.1. *Let H be a timelike, lightlike, or spacelike one-parameter group of $O(1, n+1)$. Then the closure $\overline{H}$ is compact, and every one-parameter group of $\overline{H}$ is spacelike.*

Proof. Put $\mathscr{R} = \overline{H}$. Suppose that $\mathscr{R}$ is noncompact in $O(1, n+1)$. Since $\mathscr{R}$ is an abelian subgroup of $O(1, n+1)$, the group $\mathscr{R}$ is conjugate to a subgroup of the maximal amenable group $\operatorname{Sim}(\mathbf{R}^n) = \mathbf{R}^n \rtimes (O(n) \times \mathbf{R}^+)$ (cf. [14]). Consider the following cases.

Case 1. $\mathscr{R} \subset \mathbf{R}^n \rtimes (O(n) \times \mathbf{R}^+)$ for which the projection onto $\mathbf{R}^n$ is nontrivial. The orbit $\mathscr{R}p$ at the point $p = (0, 0, 1, 0, \cdots, 0) \in \mathbf{S}^n$ of $\mathbf{S}^{1,n}$ is homeomorphic to a horosphere, and in fact the orbit is asymptotic to a straight line lying on the light cone in $\mathbf{R}^{1,n+1}$. The orbit Hp will be a horocycle. So there are vector fields V, W such that $g(V, V) < 0$ and $g(W, W) > 0$. This contradicts the hypothesis on H.

Case 2. $\mathscr{R} \subset O(n) \times \mathbf{R}^+$. Suppose that the projection onto the first summand, $\mathrm{P}_1(\mathscr{R})$, is nontrivial. Then $\mathrm{P}_1(\mathscr{R})$ lies in the maximal torus in $O(n)$. Choosing the point $p = (0, 0, 1, 0, \cdots, 0) \in \mathbf{S}^{1,n}$ shows that the orbit of $\mathscr{R}$ at p is contained in the sphere $\mathbf{S}^n \subset \mathbf{S}^{1,n}$. Any vector field V tangent to the orbit satisfies $g(V, V) > 0$, while choosing the point $p' = (1, \sqrt{2}, 0, \cdots, 0)$, we see that any vector field W tangent to the orbit at p' satisfies $g(W, W) < 0$. Hence H cannot be timelike, lightlike, or spacelike. On the other hand if $\mathscr{R} \subset \mathbf{R}^+$, i.e., $\mathscr{R} = \mathbf{R}^+$, then the orbit $\mathscr{R}p$ at the point $p = (1, 0, \sqrt{2}, 0, \cdots, 0)$ is the subset $\{(\cosh\theta, \sinh\theta, \sqrt{2}, 0, \cdots, 0)|,\ \theta \in \mathbf{R}^1\}$. Therefore it is easy to find vector fields V, W tangent to the orbit $\mathscr{R}p$ which satisfy $g(V, V) > 0$ and $g(W, W) < 0$. This yields a contradiction.

Now, $\overline{H}$ is conjugate to a subgroup of $O(n+1)$. If a one-parameter group of $\overline{H}$ induces a vector field V, then we can readily see that $g(V, V) > 0$.

Corollary 3.2. *There exists neither timelike nor lightlike Killing vector field on a Lorentz spherical manifold.*

There is a Lorentz spherical $(n+1)$-manifold which admits a spacelike one-parameter group of Lorentz transformations; however we have the following.

Theorem 3.3. *There exists no compact Lorentz spherical* 3-*manifold admitting a spacelike Killing vector field.*

Proof. Since M is compact, a spacelike Killing vector field generates a spacelike one-parameter group $\{\phi_t\}_{|t|<\infty}$ of Lorentz transformations on M. We will show that the existence of such a one-parameter group contradicts the cohomological dimension of $\pi = \pi_1(M)$. Let $(\pi, \{\tilde{\phi}_t\}_{|t|<\infty}, \widetilde{M}) \overset{(\rho,\mathrm{dev})}{\rightarrow} (\Gamma, H, \mathbf{S}^{1,2})$ be the developing pair where $H \subset O(1, 3)$. By Lemma 3.1 the closure $\overline{H}$ is compact in $O(1, 3)$. It implies $\overline{H} = SO(2)$ up to conjugation and so H is closed. If we recall $\mathbf{S}^{1,2} = \{(x_1, y_1, y_2, y_3) \in \mathbf{R}^{1,3} | -x_1^2 + y_1^2 + y_2^2 + y_3^2 = 1\}$, then the fixed point set of H is $\mathbf{S}^{1,0} = \{(x_1, y_1, 0, 0) | -x_1^2 + y_1^2 = 1\}$. Since the holonomy group Γ leaves $\mathbf{S}^{1,0}$ invariant, it follows $\Gamma \subset O(1, 1) \times SO(2)$ in which $H = \{1\} \times SO(2)$. Passing to a subgroup of finite index we may assume that $\Gamma \subset O(1, 1)^0 \times SO(2)$. We note the following lemma.

Lemma 3.4. *The identity component of* $O(1, 1)$, $O(1, 1)^0$, *does not act properly on any* $O(1, 1)^0 \times SO(2)$-*invariant domain* Ω *of* $\mathbf{S}^{1,2}$, *that contains the set* $\{-x_1^2 + y_1^2 = 0,\ x_1 \neq 0,\ y_1 \neq 0\}$. *In particular any discrete infinite subgroup of* $O(1, 1)^0 \times SO(2)$ *does not act properly discontinuously on* Ω.

Proof. Consider the following sets in $\mathbf{S}^{1,2}$:

$$l_+ = \{(x_1, y_1, 1, 0) | x_1 = y_1, x_1 < 0, y_1 \neq 0\},$$
$$l_- = \{(x_1, y_1, 1, 0) | x_1 = -y_1, x_1 < 0, y_1 \neq 0\}.$$

Each half-line is invariant under $O(1, 1)^0$. Choose points $p \in l_+$, $q \in l_-$. Let $\{p_i\}$ be the sequence of points lying in the component with $x_1 < 0$, and suppose $\lim p_i = p$. We note that each orbit $O(1, 1)^0 \cdot p_i$ is asymptotic to the half-line l_- (also l_+). So there exists a sequence $\{g_i\} \in O(1, 1)^0$ such that $\lim g_i \cdot p_i = q$. On the other hand, since l_- is invariant under $O(1, 1)^0$ and $l_- \cap l_+ = \varnothing$, the sequence $\{g_i\}$ does not converge in $O(1, 1)^0$. Therefore $O(1, 1)^0$ does not act properly. If Γ is an infinite discrete subgroup of $O(1, 1)^0 \times SO(2)$, then $O(1, 1)^0 \times SO(2)/\Gamma$ is compact. Thus there exists a compact set $K \subset O(1, 1)^0 \times SO(2)$ such that $O(1, 1)^0 \subset \Gamma \cdot K$, so that Γ cannot act properly discontinuously on Ω.

Notice that $O(1, 1)^0$ has the fixed point set $S^1 = \{(0, 0, y_2, y_3) | y_2^2 + y_3^2 = 1\}$ in $\mathbf{S}^{1,2}$.

We continue the proof of the theorem. By the above observation, $O(1, 1)^0$ acts properly on the domain X of $\mathbf{S}^{1,2}$ which satisfies $-x_1^2 + y_1^2 \neq 0$. If we put $Y = \{(x_1, y_1, y_2, y_3) \in \mathbf{S}^{1,2} | -x_1^2 + y_1^2 = 0$ and $y_2^2 + y_3^2 = 1\}$, then $\mathbf{S}^{1,2} - Y = X$. Since Y is invariant under $SO(2)$, $O(1, 1)^0 \times SO(2)$ acts properly on X. X consists of 4 components; 2 copies A, A' of a 3-ball and 2 copies B, B' of a circle $\times$ 2-ball. Furthermore $O(1, 1)^0$ acts freely on X, and $O(1, 1)^0 \times SO(2)$ acts freely on $X - \mathbf{S}^{1,0}$. Thus we can construct an $O(1, 1)^0 \times SO(2)$-invariant complete Riemannian metric on X. Since M is compact, from the result of [13] it follows that $\text{dev}: \widetilde{M} - \text{dev}^{-1}(Y) \to X$ is a covering map on each component. Let L be a component of $\widetilde{M} - \text{dev}^{-1}(Y)$. We dissect the argument into two cases.

Case A. dev: $L \to A$ *is a covering map.* Since A is simply connected, dev: $L \to A$ is a homeomorphism. In particular $\rho: \{\tilde{\phi}_t\}_{|t|<\infty} \to SO(2)$ is an isomorphism. L has the boundary component in $\widetilde{M}$. For this, if $\partial L = \varnothing$, then $\widetilde{M} = L$ which implies that $\pi \approx \Gamma$ is discrete in $O(1, 1)^0 \times SO(2)$ and $\text{cd}\,\Gamma \leq 1$. This is impossible because M is aspherical in this case so that $\text{cd}\,\pi = 3$. Since $\partial L \neq \varnothing$, there is another component N adjacent to L such that dev: $N \to B$ (or B') is a covering map. The group $\{\tilde{\phi}_t\}_{|t|<\infty}$ acts freely on N because so does $SO(2)$ on B. Then the map dev induces a map $\widehat{\text{dev}}: N/\{\tilde{\phi}_t\} \to B/SO(2)$ which is also a covering map. Since $B/SO(2)$ is simply connected, $\widehat{\mathrm{dev}}$ is a homeomorphism. Thus $\mathrm{dev}\colon N \to B$ is a homeomorphism. We can continue in this way whenever the boundary component is nonempty.

Let $A \cup B$ be the manifold obtained from A and B glued along the common boundary part; we can define similarly for the manifold $A \cup B \cup B'$, etc. The following possibilities occur from the construction of X:

(1) $\mathrm{dev}\colon \widetilde{M} \to A \cup B$ is a homeomorphism.

(2) $\mathrm{dev}\colon \widetilde{M} \to A \cup B \cup B'$ is a homeomorphism.

(3) $\mathrm{dev}\colon \widetilde{M} \to A \cup B \cup A'$ is a homeomorphism.

(4) $\mathrm{dev}\colon \widetilde{M} \to A \cup B \cup A' \cup B'$ $(= \mathbf{S}^{1,2} - \mathbf{S}^1)$ is a homeomorphism.

For (1), (2), they are homeomorphic to 3-balls. Since they are aspherical, (1), (2) do not occur by the same argument as above. For (3), (4), $\pi \approx \Gamma \subset O(1,1)^0 \times SO(2)$ as above, and Γ acts properly discontinuously and freely on $A \cup B \cup A'$ (resp. $A \cup B \cup A' \cup B'$). But these noncompact domains clearly contain the lines $\{-x_1^2 + y_1^2 = 0\}$ with the origin removed. By the above lemma, the holonomy group Γ must be finite. Since $\widetilde{M}$ is noncompact, it is impossible.

Case B. $\mathrm{dev}\colon L \to B$ *is a covering map.* If $\partial L = \varnothing$, then $\mathrm{dev}\colon \widetilde{M} \to B$ is a covering map, and so there is a covering homeomorphism $\widehat{\mathrm{dev}}\colon \widetilde{M} \approx \widetilde{B}$, where $\mathbf{Z} \to (O(1,1)^0 \times \mathbf{R}, \widetilde{B}) \to (O(1,1)^0 \times SO(2), B)$ is the covering projection. Therefore the image π' of π under $\widehat{\mathrm{dev}}$ is discrete in $O(1,1)^0 \times \mathbf{R}$. In particular we have $\mathrm{cd}\,\pi' \leq 2$. Since M is aspherical and $\pi \approx \pi'$, this is impossible. If $\partial L \neq \varnothing$, then there is another component L' such that $\mathrm{dev}\colon L' \to A$ is covering and hence a homeomorphism. This goes back to Case A, and so it does not occur. Therefore there exists no spacelike one-parameter group of Lorentz transformations on M. This completes the proof of the theorem.

4. Lorentz 3-manifolds with killing vector fields and their examples

In this section we give examples of Lorentz manifolds admitting timelike Killing vector fields, and examine the structure of Lorentz 3-manifolds which support lightlike Killing vector fields; similarly for spacelike Killing vector fields.

4.1. Compact Lorentz flat 3-manifolds. We shall give examples of compact Lorentz flat space forms. First of all, a 3-torus is an example of compact Lorentz flat manifolds. (See [34].) As the nontrivial ones we prove that there exists a complete Lorentz flat structure on 3-dimensional nilmanifolds and solvmanifolds. (See [10], [14] for the related work.)

Recently Margulis and Grunewald [17] gave a list of classification of compact complete Lorentz flat manifolds.

4.2. Examples.

Example (1). Let N denote the semidirect product $\mathbf{R}^2 \rtimes \mathbf{R}$ with the group law:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \theta\right)\left(\begin{pmatrix} x' \\ y' \end{pmatrix}, \theta'\right) = \left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}, \theta + \theta'\right).$$

Then N is isomorphic to the 3-dimensional 1-connected nilpotent nonabelian Lie group. We construct a continuous homomorphism $\rho: N \to \mathbf{R}^3 \rtimes O(1, 2)$. Let $\{e_1, e_2, e_3\}$ be the orthogonal basis such that $\langle e_1, e_1\rangle = -1$, $\langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$. Define a map ρ, with respect to the basis $\{(e_1 + e_3)/\sqrt{2}, e_2, (e_1 - e_3)/\sqrt{2}\}$, to be

$$\rho\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

and

$$\rho(\theta) = \left(\begin{pmatrix} \theta^3/6 \\ \theta^2/2 \\ \theta \end{pmatrix}, \begin{pmatrix} 1 & \theta & \theta^2/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}\right).$$

It is easy to see that ρ is a continuous homomorphism. Moreover ρ acts simply transitively on $\mathbf{R}^3$. Choose a discrete cocompact subgroup Δ in N, we obtain a compact Lorentz flat nilmanifold N/Δ.

Example (2). Let $\mathbf{S}$ denote the semidirect product $\mathbf{R}^2 \rtimes \mathbf{R}$ with the group law:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \theta\right)\left(\begin{pmatrix} x' \\ y' \end{pmatrix}, \theta'\right) = \left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}, \theta + \theta'\right).$$

Then $\mathbf{S}$ is a 3-dimensional solvable Lie group. For a nonzero real number a we define a homomorphism $\rho_a: \mathbf{S} \to \mathbf{R}^3 \rtimes O(1, 2)$ to be

$$\rho\left(\begin{pmatrix} x \\ y \end{pmatrix}, \theta\right) = \left(\begin{pmatrix} a\theta \\ x \\ y \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix}\right).$$

Then ρ acts simply transitively on $\mathbf{R}^3$. We can find a discrete cocompact subgroup Δ of $\mathbf{S}$, so that $\mathbf{S}/\Delta \approx \mathbf{R}^3/\rho(\Delta)$ is a compact Lorentz flat solvmanifold.

We know that T^3 (more generally, a compact Euclidean space form whose linear holonomy lies in $\mathbf{Z}/2 \times O(2)$) is a Lorentz flat manifold admitting a Killing vector field from Theorem 2.11. It is easy to see that T^3 also admits a spacelike Killing vector field and a lightlike Killing vector field. We shall examine how the above examples will be characterized by those Killing vector fields.

Lemma 4.3. *Let $\mathscr{R}$ be a closed one-parameter subgroup of $\mathbf{R}^3 \rtimes O(1, 2)$ isomorphic to $\mathbf{R}^1$. Let $\varphi: \mathbf{R}^3 \rtimes O(1, 2) \to O(1, 2)$ be the lienar holonomy map. If $\mathscr{R}$ is timelike, lightlike, or spacelike, then $\varphi(\mathscr{R})$ is trivial.*

Proof. Suppose not. First if $\varphi(\mathscr{R})$ is compact, then we have $\varphi(\mathscr{R}) = SO(2)$ up to a conjugation. In this case it follows that

$$\mathscr{R} = \left(\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \right).$$

Thus the orbit $\mathscr{R}p$ at the point $p = (0, a, 0)$ is the set $\{(t, a\cos t, a\sin t)\}$, and the vector field V tangent to the orbit satisfies $g(V, V) = -1 + a^2$. The sign of g varies as a varies. This contradicts the hypothesis of $\mathscr{R}$. Now if $\varphi(\mathscr{R})$ is noncompact, then it is conjugate to either the parabolic subgroup $\mathbf{R}^1$ or the loxodromic subgroup $\mathbf{R}^+$ of the similarity group $\mathrm{Sim}(\mathbf{R}^1)$.

(1) $\varphi(\mathscr{R}) = \mathbf{R}^1$. Then it is isomorphic to

$$\left\{ \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, t \in \mathbf{R}^1 \right\}$$

with respect to the basis $\{(e_1+e_3)/\sqrt{2}, e_2, (e_1-e_3)/\sqrt{2}\}$, where $\{e_1, e_2, e_3\}$ is the orthogonal basis such that $g(e_1, e_1) = -1$, $g(e_2, e_2) = g(e_3, e_3) = 1$. Then it is easy to see that

$$\mathscr{R} = \left(\begin{pmatrix} ct^3/6 \\ ct^2/2 \\ ct \end{pmatrix}, \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right),$$

where c is a constant multiple. Thus the orbit at $(0, 0, 1)$ is

$$\mathscr{R} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ct^3/6 + t^2/2 \\ ct^2/2 + t \\ ct + 1 \end{pmatrix}.$$

The vector field W tangent to the orbit satisfies $g(W, W) = 1 > 0$. On the other hand if $c = 0$, then the orbit at $(0, 1, 0)$ is $\mathscr{R} \cdot (0, 1, 0) = \{(t, 1, 0)\}$. Since the vector field W_1 tangent to this orbit is generated by $\{e_1 + e_3\}$, it follows that $g(W_1, W_1) = 0$. When $c \neq 0$, the orbit at the origin is the set $\{(ct^3/6, ct^2/2, ct)\}$. The vector field W_2 tangent to the orbit satisfies $g(W_2, W_2) = 0$. These contradict the hypothesis that $\mathscr{R}$ is timelike, lightlike, or spacelike.

(2) $\varphi(\mathscr{R}) = \mathbf{R}^+$. In this case it is isomorphic to

$$\varphi(\mathscr{R}) = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{R}^+ \right\}$$

with respect to the basis $\{(e_1 + e_3)/\sqrt{2}, e_2, (e_1 - e_3)/\sqrt{2}\}$. Then it follows that

$$\mathscr{R} = \left(\begin{pmatrix} 0 \\ b\log\lambda \\ 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \right).$$

When $b = 0$, consider the orbits at the points $(1, 0, -1)$, $(1, 0, +1)$. Then the vector fields V_1, V_2 tangent to these orbits satisfy $g(V_1, V_1) = -2\lambda^{-2} < 0$ and $g(V_2, V_2) = +2\lambda^{-2} > 0$ respectively. This is impossible by the hypothesis of $\mathscr{R}$. When $b \neq 0$, consider the orbits at $(b, 0, -b)$, $(b, 0, +b)$. The vector fields W_1, W_2 tangent to the orbits satisfy $g(W_1, W_1) = -b^2\lambda^{-2} < 0$ and $g(W_2, W_2) = +3b^2\lambda^{-2} > 0$ respectively. This yields also a contradiction. Therefore $\varphi(\mathscr{R})$ is trivial.

Proposition 4.4. *If a compact Lorentz flat* 3*-manifold admits a spacelike Killing vector field, then it is either a Euclidean space form or an infrasolvmanifold.*

Proof. A spacelike Killing vector field generates a spacelike one-parameter group H of Lorentz transformations on M. Let $(\pi, \widetilde{H}, \widetilde{M}^3) \xrightarrow{(\rho, \mathrm{dev})} (\Gamma, G, \mathbf{R}^3)$ be the developing pair where $G \subset \mathbf{R}^3 \rtimes O(1, 2)$. We prove first that G is closed. If the closure $\overline{G}$ of G contains a compact subgroup K, then $K = SO(2) \subset \{0\} \times O(1, 2)$ up to a conjugation. The subgroup of $O(1, 2)$ whose elements commute with $SO(2)$ is $\mathbf{Z}/2 \times SO(2)$. Since K centralizes the holonomy group Γ, each element γ of Γ has the form

$$\gamma = \left(\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 & \\ & B \end{pmatrix} \right),$$

where $a \in \mathbf{R}^1$, and $B \in SO(2)$. It follows that $\Gamma \subset \mathbf{R}^3 \rtimes O(3) = E(3)$. Hence M^3 is a Euclidean space form $\mathbf{R}^3/\Gamma$. In particular Γ is discrete. If we note that a subgroup of finite index in Γ consists of translations, then Γ has an infinite cyclic subgroup of finite index from the above form. This is impossible because M is compact. Therefore $\overline{G} = G$, which is a closed subgroup isomorphic to $\mathbf{R}^1$. Thus $G \subset \mathbf{R}^3$ by Lemma 4.3. Since

G is spacelike and centralizes Γ, we may obtain $\Gamma \subset G \oplus (\mathbf{R}^2 \rtimes O(1, 1))$ where $G = \mathbf{R}^1$.

On the other hand, $\mathbf{R}^2 \rtimes O(1, 1)^0$ is the solvable Lie group $\mathbf{S}$ of Example (2). Note from the result of Carrière [3] that $M \approx \mathbf{R}^3/\Gamma$, and so Γ is discrete. Thus M is finitely covered by $\mathbf{R}^3/\Gamma'$ where $\Gamma \subset G \oplus \mathbf{S}$. Put $\Delta = \mathbf{S} \cap \Gamma'$. Since Δ leaves $\mathbf{R}^2$ invariant, $\mathbf{R}^2/\Delta$ is compact in $\mathbf{R}^3/\Gamma'$ and therefore $\operatorname{rank}\Delta = 2$. Then it is easy to see that either $\Gamma' \subset \mathbf{R}^3$ or $\Gamma' \subset \mathbf{R}^2 \rtimes \rho_a(\mathbf{R}^1)$ $(= \rho_a(\mathbf{S}))$. Here ρ_a is a representation of Example (2). In this case $\mathbf{R}^3/\Gamma'$ is a 3-torus or a solvmanifold $\mathbf{S}/\Gamma''$.

When M is an infrasolvmanifold, notice that the spacelike one-parameter group G on $\mathbf{R}^3$ induces an action of a group G' on M for which G' is a closed spacelike one-parameter group of Lorentz transformations isomorphic to $\mathbf{R}^1$.

Proposition 4.5. *If a compact Lorentz flat 3-manifold admits a lightlike Killing vector field, then it is an infranilmanifold.*

Proof. Let $(\rho, \mathrm{dev})\colon (\pi, \widetilde{H}, \widetilde{M^3}) \to (\Gamma, G, \mathbf{R}^3)$ be the developing pair as before. First suppose that G is closed. Then by Lemma 4.3 we can assume that G is spanned by the vector $\{e_1 + e_3\}$ of $\mathbf{R}^3$. Let $\gamma = (a, A)$ be an element of Γ in $\mathbf{R}^3 \rtimes O(1, 2)$ with respect to the basis $\{(e_1 + e_3)/\sqrt{2}, e_2, (e_1 - e_3)/\sqrt{2}\}$ of $\mathbf{R}^3$, and $\varphi\colon \mathbf{R}^3 \rtimes O(1, 2) \to O(1, 2)$ be the linear holonomy map as in Lemma 4.3. Since G centralizes Γ and the linear holonomy group $\varphi(\Gamma)$ preserves the bilinear form $Q(x, y) = -x_1y_1 + x_2y_2 + x_3y_3$, we obtain

$$A = \begin{pmatrix} 1 & \theta & \theta^2/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\theta \in \mathbf{R}^1$. It follows that $\Gamma \subset \mathbf{R}^3 \rtimes \mathbf{R}^1$ (cf. Example (1)). The real algebraic closure $A(\Gamma)$ is a simply connected nilpotent Lie group because $\mathbf{R}^3 \rtimes \mathbf{R}^1$ is nilpotent. As Γ is discrete and rank Γ is 3, $A(\Gamma)$ is a 3-dimensional Lie group. Therefore $A(\Gamma)$ is isomorphic to either $\mathbf{R}^3$ or $\mathbf{R}^2 \rtimes \mathbf{R}^1$, and so M^3 is either a Euclidean space form or an infranilmanifold.

For the rest of proof (the case where G is not closed), $\overline{G}$ contains a connected compact subgroup K in $\mathbf{R}^3 \rtimes O(1, 2)$. Thus K is conjugate to $\{0\} \times SO(2)$. If we note that $\overline{G}$ centralizes Γ, then K commutes with the elements of the linear holonomy group $\varphi(\Gamma)$. It is easily seen that $\varphi(\Gamma) \subset O(1) \times O(2)$ and so $\Gamma \subset \mathbf{R}^3 \rtimes O(3)$. Therefore M^3 is a Euclidean space form.

4.6. Lorentz hyperbolic 3-manifolds. Let $O(2,2)^0$ be the identity component of $O(2,2)$. If we identify $\mathbf{H}^{1,2}$ with $\mathrm{SL}_2\mathbf{R}$, then it follows $O(2,2)^0 \approx \mathrm{SL}_2\mathbf{R} \times_{\mathbf{Z}_2} \mathrm{SL}_2\mathbf{R}$ in which the action of $\mathrm{SL}_2\mathbf{R} \times_{\mathbf{Z}_2} \mathrm{SL}_2\mathbf{R}$ on $\mathrm{SL}_2\mathbf{R}$ is given by $([A, B], X) = AXB^{-1}$. (Compare [25].) By recalling the exact sequence: $1 \to \mathscr{Z} \to O(2,2)^{0\sim} \xrightarrow{P} O(2,2)^0 \to 1$, we have $O(2,2)^{0\sim} = \widetilde{\mathrm{SL}\mathbf{R}} \times_{\mathbf{Z}} \widetilde{\mathrm{SL}_2\mathbf{R}}$ where $\widetilde{\mathrm{SL}_2\mathbf{R}}$ is the universal covering group of $\mathrm{PSL}_2\mathbf{R}$.

Examples. (1). **Standard space forms of dimensions** 3 (cf. 2.18, [25]). Consider the subgroup $\mathbf{J} = \mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{SL}_2\mathbf{R}} \subset \widetilde{\mathrm{SL}_2\mathbf{R}} \times_{\mathbf{Z}} \widetilde{\mathrm{SL}_2\mathbf{R}}$. Then it is easy to see that any discrete subgroup of $\mathbf{J}$ acts properly discontinuously on $\tilde{\mathbf{H}}^{1,2}$ where $\tilde{\mathbf{H}}^{1,2} \approx \widetilde{\mathrm{SL}_2\mathbf{R}}$. Let Γ be a torsion free discrete cocompact subgroup of $\mathbf{J}$. Then $\tilde{\mathbf{H}}^{1,2}/\Gamma$ is called a 3-dimensional *standard space form.*

(2). **Homogeneous standard space forms of dimension** 3. Let U, A be a parabolic one-parameter group and a hyperbolic one-parameter group of $\mathrm{PSL}_2\mathbf{R}$ respectively. Note $U \times \widetilde{\mathrm{SL}_2\mathbf{R}}$ $(= \mathbf{Z} \times U \times_{\mathbf{Z}} \widetilde{\mathrm{SL}_2\mathbf{R}}) \subset O(2,2)^{0\sim}$. Similarly for A. If Γ is a discrete torsion free cocompact subgroup of $\widetilde{\mathrm{SL}_2\mathbf{R}}$, we have a compact *homogeneous standard space form* $\widetilde{\mathrm{SL}_2\mathbf{R}}/\Gamma$ for which U (also A) acts as Lorentz isometries of a spacelike one-parameter group.

(3). **Nonstandard space forms of dimension** 3. There is a properly discontinuous action of a group $\Gamma \subset O(2,2)^0$ on $\mathbf{H}^{1,2}$ which is not conjugate to a subgroup of $S^1 \times_{\mathbf{Z}/2} \mathrm{SL}_2\mathbf{R}$ in $O(2,2)$, and so the orbit space $\mathbf{H}^{1,2}/\Gamma$ is not a standard space form. In fact the manifold $\mathbf{H}^{1,2}/\Gamma$ is obtained from a homogeneous standard space form by a small deformation of a holonomy representation. (See [12] for details.)

It is easy to check the following.

Lemma 4.7. *Let $O(2,2)^0 \approx \mathrm{SL}_2\mathbf{R} \times_{\mathbf{Z}_2} \mathrm{SL}_2\mathbf{R}$ be as above. Then closed connected noncompact abelian subgroups of $O(2,2)^0$ are of the following types up to a conjugacy and switching of factors*:

$$(1) \qquad \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \times 1 \,|\, t \in \mathbf{R} \right\}, \quad \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \times 1 \,|\, t \in \mathbf{R} \right\}.$$

$$(2) \qquad \left\{ \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & b\theta \\ 0 & 1 \end{pmatrix} |\, t, \theta \in \mathbf{R} \right\}, \qquad a, b \neq 0.$$

$$(3) \qquad \left\{ \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} \times \begin{pmatrix} e^{b\theta} & 0 \\ 0 & e^{-b\theta} \end{pmatrix} |\, t, \theta \in \mathbf{R} \right\}, \qquad a, b \neq 0.$$

(4) $$\left\{\begin{pmatrix}1 & at\\ 0 & 1\end{pmatrix}\times\begin{pmatrix}e^{b\theta} & 0\\ 0 & e^{-b\theta}\end{pmatrix} \middle| t,\theta\in\mathbf{R}\right\}, \qquad a, b\neq 0.$$

(5) $$\left\{\begin{pmatrix}1 & \theta\\ 0 & 1\end{pmatrix}\times\begin{pmatrix}\cos a\theta & -\sin a\theta\\ \sin a\theta & \cos a\theta\end{pmatrix} \middle| \theta\in\mathbf{R}\right\},$$

$$\left\{\begin{pmatrix}e^{\theta} & 0\\ 0 & e^{-\theta}\end{pmatrix}\times\begin{pmatrix}\cos a\theta & -\sin a\theta\\ \sin a\theta & \cos a\theta\end{pmatrix} \middle| \theta\in\mathbf{R}\right\}, \qquad a\neq 0.$$

Put

$$N=\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \middle| t\in\mathbf{R}\right\},\quad A=\left\{\begin{pmatrix}e^{\theta} & 0\\ 0 & e^{-\theta}\end{pmatrix} \middle| \theta\in\mathbf{R}\right\},$$

$$K=\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \middle| \theta\in\mathbf{R}\right\}.$$

Set

$$S_0=NA=\left\{\begin{pmatrix}\lambda & t\\ 0 & \lambda^{-1}\end{pmatrix} \middle| \lambda\in\mathbf{R}^+,\ t\in\mathbf{R}\right\}.$$

Let $S=S_0\cup\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)S_0$, $S_{\pi/2}=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)S$.

Lemma 4.8. (i) *The group* $N\times N$ *acts properly on* $\mathrm{SL}_2\mathbf{R}-S$.

(ii) *The group* $A\times A$ *acts properly on* $\mathrm{SL}_2\mathbf{R}-\{S\cup S_{\pi/2}\}$.

(iii) *The group* $N\times A$ *acts properly on* $\mathrm{SL}_2\mathbf{R}-S$.

(iv) *The groups* N, A, $N\times K$, $A\times K$, *and* $K\times_{\mathbf{Z}/2}K$ *act properly on* $\mathrm{SL}_2\mathbf{R}$.

Proof. Let $x=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2\mathbf{R}\approx\mathbf{H}^{1,2}$. Then,

$$\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & \theta\\ 0 & 1\end{pmatrix}\right)\cdot x=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & -\theta\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}a+ct & -ct\theta-a\theta+dt+b\\ c & -c\theta+d\end{pmatrix}.$$

Thus it is easy to see that $N\times N$ acts properly on the subset of $\mathrm{SL}_2\mathbf{R}$ with $c\neq 0$. We can prove similarly for (ii), (iii) and (iv).

Remark 4.9. The groups of types (i), (ii), and (iii) leave S or $S\cup S_{\pi/2}$ invariant, but do not act properly. This follows from a direct calculation. In particular a discrete cocompact subgroup does not act properly discontinuously on S or $S\cup S_{\pi/2}$.

Theorem 4.10. *Let* M *be a Lorentz hyperbolic* 3*-manifold. If the holonomy group is virtually abelian, then* M *is not compact.*

Proof. Let $(\rho,\mathrm{dev})\colon(\pi,\widetilde{M})\to(\Gamma,\widetilde{\mathbf{H}}^{1,2})$ be the developing pair, and $P\colon O(2,2)^{0\sim}\to O(2,2)^0$ the covering map. Passing to a subgroup of finite index, we assume that Γ is abelian and $\Gamma\subset O(2,2)^{0\sim}$. Then

$P(\Gamma)$ is an abelian subgroup of $O(2,2)^0$. If $A(P(\Gamma))$ is the real algebraic closure of $P(\Gamma)$ in $O(2,2)^0$, then it is an abelian Lie subgroup such that $P(\Gamma) \subset A(P(\Gamma))^0$. Thus the identity component is either one of the groups in Lemma 4.8. Suppose that $A(P(\Gamma))^0$ is one of the groups of (iv). Then $A(P(\Gamma))^0$ acts properly on $\mathbf{H}^{1,2} = \mathrm{SL}_2\mathbf{R}$. Since $\mathscr{Z} \to (O(2,2)^{0\sim}, \widetilde{\mathbf{H}}^{1,2}) \xrightarrow{P} (O(2,2)^0, \mathbf{H}^{1,2})$ is the covering map, the group $P^{-1}(A(P(\Gamma))^0)$ acts properly on $\widetilde{\mathbf{H}}^{1,2}$. There is a $P^{-1}(A(P(\Gamma))^0)$-invariant Riemannian metric on $\widetilde{\mathbf{H}}^{1,2}$ such that $\Gamma \subset P^{-1}(A(P(\Gamma))^0)$. The developing map dev induces a π-invariant Riemannian metric on $\widetilde{M}$. So if M is compact, then dev is a covering map, and thus M is geodesically complete. Thus the result follows from Theorem 6.1 of [25]. Indeed, since the abelian group $P^{-1}(A(P(\Gamma))^0)$ has dimension at most two, Γ is a free abelian group of rank ≤ 2. Hence $\widetilde{\mathbf{H}}^{1,2}/\Gamma$ cannot be compact.

Suppose that $A(P(\Gamma))^0$ is either one of the groups in (i), (ii) or (iii) of Lemma 4.8. Consider (i), i.e., $\Gamma \subset N \times N$. Note that each component Z of $\mathrm{SL}_2\mathbf{R} - S$ is invariant under $N \times N$, and $N \times N$ acts properly on Z by Lemma 4.8. Choose an $N \times N$-invariant *complete* Riemannian metric on Z. Let $P \circ \mathrm{dev}\colon \widetilde{M} \to \mathbf{H}^{1,2} = \mathrm{SL}_2\mathbf{R}$ be the immersion of Lorentz hyperbolic structure. If M is compact, then from Lemma B of [13] it follows that $P \circ \mathrm{dev}\colon Y \to Z$ is a covering map for each component Y of $(P \circ \mathrm{dev})^{-1}(Z)$. As Z is simply connected (homeomorphic to $\mathbf{R}^3$), we have $Y \approx Z$. On the other hand, we shall prove $\widetilde{M} = Y$. Let $\widetilde{S}$ be a lift of S to $\widetilde{H}^{1,2}$. Then it is sufficient to show $\mathrm{dev}^{-1}(\widetilde{S}) = \varnothing$. For this, $\mathrm{dev}^{-1}(\widetilde{S})$ is a π-invariant closed subset in $\widetilde{M}$, and so if $p\colon \widetilde{M} \to M$ is the covering map, then $p(\mathrm{dev}^{-1}(\widetilde{S}))$ is a closed subset consisting of a disjoint union of closed submanifolds in M. Let Q be a component of $\mathrm{dev}^{-1}(\widetilde{S})$, and suppose Q to be a boundary component of Y. Then there exists a component $\widetilde{S}_0$ of $\widetilde{S}$ such that $\mathrm{dev}\colon Q \to \widetilde{S}_0$ is a homeomorphism. Since P maps $\widetilde{S}_0$ onto a component S_0 of S, $P \circ \mathrm{dev}\colon Q \to S_0$ is a homeomorphism. If we note from the above remark that $p(Q)$ is a closed submanifold Q/π' in M for a subgroup $\pi' \subset \pi$, the corresponding holonomy group Γ' acts properly discontinuously and freely on $\widetilde{S}_0$ with compact quotient. Therefore $P(\Gamma')$ is a discrete cocompact subgroup of $N \times N$ acting properly discontinuously on S_0. This is impossible by Remark 4.9. Hence we obtain $\widetilde{M} = Y$ such that $P \circ \mathrm{dev}\colon \widetilde{M} \to Z$ is a homeomorphism. But this implies that $P(\Gamma)$ is a discrete subgroup of $N \times N$ consisting of a free abelian subgroup of rank ≤ 2. Hence $M \approx Z/P(\Gamma)$ cannot be compact. This proves (i). We can prove similarly for (ii), (iii).

Theorem 4.11. *Let M be a Lorentz hyperbolic 3-manifold. Suppose that $\widetilde{M}$ admits a nontrivial complete Killing vector field, and the developing map is injective. If M is compact, then M is geodesically complete.*

Proof. Since $\widetilde{M}$ admits a complete vector field, the identity component $\mathrm{Iso}(\widetilde{M})^0$ is a nontrivial closed connected subgroup normalized by the fundamental group π. As the developing map is injective, we assume that there is a smallest connected closed Lie subgroup G normalized by the holonomy group Γ in $O(2,2)^{0\sim}$ for which G acts on $\mathrm{dev}(\widetilde{M})$ and $\Gamma \subset O(2,2)^{0\sim}$. Let $N(G)$ be the normalizer of G in $O(2,2)^{0\sim}$.

Case I. G has the radical. If $N(G)$ is solvable, then from Lemma 4.7 it follows that $N(G)$ is an abelian Lie subgroup of dimension 1 or 2, or isomorphic to the solvable Lie subgroup S_0, $S_0 \times S_0$, or $S_0 \times \mathbf{R}$ of $O(2,2)^{0\sim} \approx \widetilde{\mathrm{SL}_2\mathbf{R}} \times_{\mathbf{Z}} \widetilde{\mathrm{SL}_2\mathbf{R}}$. Since Γ is discrete, Γ is a free abelian group in this case. By the above theorem, M cannot be compact. If $N(G)$ is not solvable, then $N(G)$ is conjugate to the subgroup $G \times \widetilde{\mathrm{SL}_2\mathbf{R}}$ (up to switching factors) where $G = N$, A, or to the subgroup $\mathbf{R} \times_{\mathbf{Z}} \widetilde{\mathrm{SL}_2\mathbf{R}}$. In the latter case, $P(N(G)) = K \times_{\mathbf{Z}/2} \mathrm{SL}_2\mathbf{R}$ which acts properly on $\mathbf{H}^{1,2} = \mathrm{SL}_2\mathbf{R}$. Thus $N(G)$ acts properly on $\widetilde{\mathbf{H}}^{1,2}$. This implies $\mathrm{dev}(\widetilde{M}) = \widetilde{\mathbf{H}}^{1,2}$. Moreover M is a standard space form by the Example (1) of 4.6. Let $N(G) = N \times \widetilde{\mathrm{SL}_2\mathbf{R}}$. First note that N (or A) is a spacelike one-parameter group, and so the action $(N \times \widetilde{\mathrm{SL}_2\mathbf{R}}, \widetilde{\mathbf{H}}^{1,2})$ induces the two-dimensional Lorentz hyperbolic geometry $(\widetilde{\mathrm{SL}_2\mathbf{R}}, \widetilde{\mathbf{H}}^{1,1})$. Here $N\backslash\mathbf{H}^{1,2} = \mathbf{H}^{1,1}$ on which $\mathrm{SL}_2\mathbf{R} = O(1,2)^0$ acts as isometries. Consider the exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & N \times \widetilde{\mathrm{SL}_2\mathbf{R}} & \longrightarrow & \widetilde{\mathrm{SL}_2\mathbf{R}} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & \Gamma_2 & \longrightarrow & 1.
\end{array}$$

Put $\mathrm{dev}(\widetilde{M})^* = N\backslash \mathrm{dev}(\widetilde{M})$. Then the action $(\Gamma_2, \mathrm{dev}(\widetilde{M})^*)$ is a Lorentz hyperbolic manifold of dimension two. As there exists no compact Lorentz hyperbolic manifold of dimension two (cf. Introduction), $\mathrm{dev}(\widetilde{M})^*$ is simply connected and noncompact. Thus $\mathrm{dev}(\widetilde{M})$ is contractible. In particular, $\mathrm{ch}\,\Gamma = 3$.

If Δ is nontrivial, then Γ_2 acts properly discontinuously on $\mathrm{dev}(\widetilde{M})^*$ with compact quotient, but it is impossible, and so $\Gamma \approx \Gamma_2$. Moreover Γ_2 is discrete in $\widetilde{\mathrm{SL}_2\mathbf{R}}$; otherwise Γ would be abelian as before. On the other hand, Γ_2 acts as right translations of $\widetilde{\mathrm{SL}_2\mathbf{R}}$ on the domain

$\operatorname{dev}(\widetilde{M})$ of $\widetilde{\mathbf{H}}^{1,2} = \widetilde{\mathrm{SL}_2\mathbf{R}}$. For this, let $\gamma_2 \in \Gamma_2$ and $\gamma = (u, \gamma_2) \in \Gamma$. Since N leaves $\operatorname{dev}(\widetilde{M})$ invariant, it follows $x \cdot \gamma_2 = u^{-1} \cdot \gamma \cdot x \in \operatorname{dev}(\widetilde{M})$ for $x \in \operatorname{dev}(\widetilde{M})$. As Γ_2 is discrete, it acts properly discontinuously on $\operatorname{dev}(\widetilde{M})$. If we note $\operatorname{ch}\Gamma_2 = 3$, $\operatorname{dev}(\widetilde{M})/\Gamma_2$ is compact in $\widetilde{\mathrm{SL}_2\mathbf{R}}/\Gamma_2$. Hence we have $\operatorname{dev}(\widetilde{M}) = \widetilde{\mathrm{SL}_2\mathbf{R}}$, and M is complete.

Case II. G is semisimple. Since $P(G)$ is semisimple in $O(2, 2)^0 = \mathrm{SL}_2\mathbf{R} \times_{\mathbf{Z}/2} \mathrm{SL}_2\mathbf{R}$, it follows that $P(G) = \mathrm{SL}_2\mathbf{R} \times_{\mathbf{Z}/2} \mathrm{SL}_2\mathbf{R}$, $\mathrm{SL}_2\mathbf{R} \times \{1\}$, or $P(G) = \{[g, aga^{-1}] | g \in \mathrm{SL}_2\mathbf{R}\}$ for some $a \in \mathrm{SL}_2\mathbf{R}$. G is transitive on $\widetilde{\mathbf{H}}^{1,2}$ for the first two cases. Hence $\operatorname{dev}(\widetilde{M}) = \widetilde{\mathbf{H}}^{1,2}$, and M is complete.

We shall prove that the last case does not occur when M is compact. As the developing map is unique up to a conjugation by elements of $O(2, 2)^{0\sim}$, we assume $P(G) = \{[g, g] | g \in \mathrm{SL}_2\mathbf{R}\}$ $(\approx \mathrm{SL}_2\mathbf{R})$. Since $P(\Gamma)$ normalizes $P(G)$, we have $P(\Gamma) \subset P(G)$. If we note $G \approx P(G)$ in this case, it follows $\Gamma \subset P^{-1}(P(G)) = \mathscr{Z} \times G$. Consider the covering:

$$(\Delta, \mathscr{Z}) \to (\Gamma, \widetilde{\mathbf{H}}^{1,2}) \to (P(\Gamma), \mathbf{H}^{1,2}),$$

where $\Delta = \mathscr{Z} \cap \Gamma$. As before $P(\Gamma)$ is discrete and not abelian. Moreover we may assume that M is orientable.

Subcase A. Suppose that Δ is nontrivial. Then Γ contains an infinite normal cyclic subgroup. Thus $\operatorname{dev}(\widetilde{M})/\Gamma$ is prime and irreducible, and therefore is an aspherical manifold. Hence $\operatorname{ch}\Gamma = 3$.

Let $S_0 \subset \mathrm{SL}_2\mathbf{R} = \mathbf{H}^{1,2}$ be the solvable Lie subgroup as in Lemma 4.8. Note that P maps each component of $\widetilde{\mathrm{SL}_2\mathbf{R}} - \widetilde{S}_0$ homeomorphically onto $\mathrm{SL}_2\mathbf{R} - S_0$. If $P \circ \operatorname{dev}(\widetilde{M}) \cap S_0 = \varnothing$, then P: $\operatorname{dev}(\widetilde{M}) \to P \circ \operatorname{dev}(\widetilde{M})$ is homeomorphic, while $P \circ \operatorname{dev}(\widetilde{M}) = \operatorname{dev}(\widetilde{M})/\Delta$ is not simply connected. Thus $P \circ \operatorname{dev}(\widetilde{M}) \cap S_0 \neq \varnothing$. Note that each element of $S_0 - N$ has the form nan^{-1} for $a \in A$, $n \in N$. Let $x = nan^{-1} \in P \circ \operatorname{dev}(\widetilde{M}) \cap S_0$. Now $P(\Gamma)$ acts properly discontinuously on $P \circ \operatorname{dev}(\widetilde{M})$ and leaves $P(G) \cdot x$ invariant. On the other hand, the correspondence $g \to gxg^{-1}$ defines a homeomorphism of $\mathrm{PSL}_2(\mathbf{R})/nAn^{-1}$ $(\approx S^1 \times \mathbf{R}^1)$ onto $P(G) \cdot x$. So we obtain a two-dimensional manifold $S^1 \times \mathbf{R}^1/P(\Gamma)$. This implies that $\operatorname{ch} P(\Gamma) \leq 1$, so that $\operatorname{ch}\Gamma \leq 2$, which is a contradiction. Similarly for $x \in N$ because $\mathrm{PSL}_2(\mathbf{R})/N \approx S^1 \times \mathbf{R}^1$.

Subcase B. Suppose $\Delta = \{1\}$ so that $\Gamma \approx P(\Gamma)$. Choose $x = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2\mathbf{R}$. As above the orbit $P(G) \cdot x$ is homeomorphic to $\mathrm{PSL}_2\mathbf{R}/P(K)$ $(\approx \mathbf{R}^2)$. We note from 3.5 of [25] that $P(G) \cdot x$ is a closed subset in

$\mathrm{SL}_2\mathbf{R}$. So if $x \in P \circ \mathrm{dev}(\widetilde{M})$, then $P(G) \cdot x$ is a $P(\Gamma)$-invariant closed subset of $P \circ \mathrm{dev}(\widetilde{M})$. Choose $\tilde{x} \in \mathrm{dev}(\widetilde{M})$ such that $P(\tilde{x}) = x$. Put $W = P^{-1}(P(G) \cdot x) \cap \mathrm{dev}(\widetilde{M})$. The set W is a Γ-invariant closed subset consisting of a disjoint union of copies of $G \cdot \tilde{x}$. Let $\pi\colon \mathrm{dev}(\widetilde{M}) \to \mathrm{dev}(\widetilde{M})/\Gamma$ be the covering map. Then $\pi(W)$ is a finite number of closed surfaces. Thus there exists a subgroup Γ' in Γ for which $\pi(G \cdot \tilde{x}) = G \cdot \tilde{x}/\Gamma'$ is a closed surface in $\mathrm{dev}(\widetilde{M})/\Gamma$. Since $G \cdot \tilde{x}/\Gamma' \approx P(G) \cdot x/P(\Gamma')$ and $P(G) \cdot x/P(\Gamma')$ covers $P(G) \cdot x/P(\Gamma)$, $P(G) \cdot x/P(\Gamma)$ is compact, which is homeomorphic to $P(\Gamma)\backslash \mathrm{PSL}_2\mathbf{R}/P(K)$ $(\approx \mathbf{R}^2/P(\Gamma))$. Thus Γ is isomorphic to the fundamental group of a closed surface so that $\mathrm{ch}\,\Gamma = 2$, while it follows that $\mathrm{dev}(\widetilde{M})/\Gamma$ is a prime manifold. Hence $\mathrm{dev}(\widetilde{M})/\Gamma$ is aspherical or $\mathrm{ch}\,\Gamma = 3$, being a contradiction.

Now $x \notin P \circ \mathrm{dev}(\widetilde{M})$, i.e., $P \circ \mathrm{dev}(\widetilde{M}) \cap P(G) \cdot x = \varnothing$. Since $\mathrm{PSL}_2\mathbf{R} - P(G) \cdot x$ is connected and simply connected (cf. [25]), $P\colon \mathrm{dev}(\widetilde{M}) \to P \circ \mathrm{dev}(\widetilde{M})$ is homeomorphic and so $P(\Gamma)$ acts properly discontinuously on $P \circ \mathrm{dev}(\widetilde{M})$. As $P \circ \mathrm{dev}(\widetilde{M})$ is a domain of $\mathrm{SL}_2\mathbf{R}$, $P \circ \mathrm{dev}(\widetilde{M})$ contains a hyperbolic element hxh^{-1} $(x \in A)$ or an elliptic element hxh^{-1} $(x \in K)$ for some $h \in \mathrm{SL}_2\mathbf{R}$. The orbit $P(G) \cdot hxh^{-1}$ is either homeomorphic to $\mathrm{PSL}_2\mathbf{R}/hAh^{-1} \approx S^1 \times \mathbf{R}$ or $\mathrm{PSL}_2\mathbf{R}/P(hKh^{-1}) \approx \mathbf{R}^2$. On the other hand, we note that $P(G) \cdot hxh^{-1}$ is closed in $P \circ \mathrm{dev}(\widetilde{M})$. For this, if $\overline{P(G) \cdot hxh^{-1}}$ is the closure of $P(G) \cdot hxh^{-1}$ in $\mathrm{SL}_2\mathbf{R}$, then in each case we see that $\partial P(G) \cdot hxh^{-1}$ $(= \overline{P(G) \cdot hxh^{-1}} - P(G) \cdot hxh^{-1})$ is homeomorphic to a circle. (Compare [25].) So if $\partial P(G) \cdot hxh^{-1}$ is nonempty, then $P(\Gamma)$ leaves this set invariant. By properness, $P(\Gamma)$ will be finite. Then it would follow $M \approx \widetilde{\mathbf{H}}^{1,2}/\Gamma$, which cannot be compact. Hence

$$\overline{P(G) \cdot hxh^{-1}} \cap P \circ \mathrm{dev}(\widetilde{M}) = P(G) \cdot hxh^{-1} \cap P \circ \mathrm{dev}(\widetilde{M}).$$

Now, let $z \in \mathrm{dev}(\widetilde{M})$ such that $P(z) = hxh^{-1}$. Since $P\colon G \cdot z \approx P(G) \cdot hxh^{-1}$, $G \cdot z$ is a Γ-invariant closed subset of $\mathrm{dev}(\widetilde{M})$. Thus $\pi(G \cdot z) = G \cdot z/\Gamma$ is a closed surface in $\mathrm{dev}(\widetilde{M})/\Gamma$. Γ is isomorphic to the fundamental group of a closed surface of genus ≥ 2. It implies $\mathrm{ch}\,\Gamma = 2$, while $\mathrm{dev}(\widetilde{M})/\Gamma$ is prime and so aspherical. This yields a contradiction again. Hence the proof of Theorem 4.11 is complete.

4.12. We consider Lorentz hyperbolic 3-manifolds which admit spacelike Killing vector fields. Let $\eta\colon O(2,2)^0 \to \mathrm{PSL}_2\mathbf{R} \times \mathrm{PSL}_2\mathbf{R}$ be the two-fold covering map.

Corollary 4.13. *Let* $(\pi, \tilde{H}, \widetilde{M^3}) \overset{(\rho, \mathrm{dev})}{\rightarrow} (\Gamma, G, \tilde{\mathbf{H}}^{1,2})$ *be the developing pair of a compact Lorentz hyperbolic* 3*-manifold* M *which admits a spacelike one-parameter group* H *of Lorentz transformations. Then the group* $\eta(P(G))$ *is a closed noncompact subgroup.*

Proof. Let $\overline{\eta(P(G))}$ be the closure of $\eta(P(G))$ in $\mathrm{PSL}_2\mathbf{R} \times \mathrm{PSL}_2\mathbf{R}$. We show that $\overline{\eta(P(G))}$ is noncompact. Then it follows from Lemma 4.7 that $\eta(P(G))$ is closed. Put $B = \overline{\eta(P(G))}$. If B is compact, then it is conjugate to a subgroup of $SO(2) \times SO(2)$. Suppose $B \subset SO(2) \times SO(2)$. If $B = SO(2) \times \{1\}$ or $\{1\} \times SO(2)$, then a vector field tangent to the orbit $B \cdot 1$ at $1 \in \mathrm{PSL}_2\mathbf{R}$ is timelike on the induced Lorentz hyperbolic manifold $\mathrm{PSL}_2\mathbf{R}$ (cf. 4.16), which contradicts the hypothesis. Thus the centralizer of B in $\mathrm{PSL}_2\mathbf{R} \times \mathrm{PSL}_2\mathbf{R}$ is $SO(2) \times SO(2)$. Put $\Gamma' = \Gamma \cap O(2,2)^{0\sim}$ which is of finite index in Γ. Since G centralizes Γ, it follows that $\eta(P(\Gamma')) \subset SO(2) \times SO(2)$. So we have $\Gamma' \subset \mathbf{R} \times SO(2)$ in $O(2,2)^{0\sim}$. Hence Γ' is abelian, but it does not occur by Theorem 4.10.

Corollary 4.14. *If a compact Lorentz hyperbolic* 3*-manifold* M *admits a spacelike Killing vector field, and the developing map is injective, then some finite covering of* M *is either a homogeneous standard space form or a nonstandard space form.*

Proof. Let $(\rho, \mathrm{dev})\colon (\pi, \tilde{H}, \tilde{M}) \to (\Gamma, G, \tilde{\mathbf{H}}^{1,2})$ be the developing pair. It follows $M \approx \tilde{\mathbf{H}}^{1,2}/\Gamma$ by Theorem 4.11. Put $\Gamma' = \Gamma \cap O(2,2)^{0\sim}$. Then Γ' belongs to the centralizer $\mathscr{C}(G)$ in $O(2,2)^{0\sim}$. Thus as in the argument of Theorem 4.11, it follows $\mathscr{C}(G) = N \times \widetilde{\mathrm{SL}_2\mathbf{R}}$ or $A \times \widetilde{\mathrm{SL}_2\mathbf{R}}$. If $\Gamma' \subset \widetilde{\mathrm{SL}_2\mathbf{R}}$, then a finite covering of M is a homogeneous standard space form $\widetilde{\mathrm{SL}_2\mathbf{R}}/\Gamma'$. Otherwise, $\tilde{\mathbf{H}}^{1,2}/\Gamma'$ is a nonstandard space form.

Problem 1. Let M be a compact Lorentz hyperbolic 3-manifold admitting a spacelike Killing vector field. Is M (geodesically) complete?

4.15. We examine Lorentz hyperbolic 3-manifolds which admit lightlike or timelike Killing vcector fields.

Lemma 4.16. *If* H *is a closed connected noncompact abelian subgroup of* $O(2,2)$, *then no one-parameter subgroup of* H *is lightlike.*

Proof. Put $\eta(H) = G$ where $\eta\colon O(2,2)^0 \to \mathrm{PSL}_2\mathbf{R} \times \mathrm{PSL}_2\mathbf{R}$ is the two-fold covering map. It is sufficient to show that any one-parameter group of G is not lightlike. There is the principal circle bundle $\mathbf{S}^1 \to \mathrm{PSL}_2\mathbf{R} \to \mathbf{H}^2$. If B is the subbundle of the tangent bundle of $\mathrm{PSL}_2\mathbf{R}$ which maps isomorphically onto the tangent bundle $T(\mathbf{H}^2)$, then each B_x has the positive scalar product with respect to the Killing form (Lorentz metric of constant curvature) of $\mathrm{PSL}_2\mathbf{R}$. On the other hand, H is either one of the groups of Lemma 4.7. If H is of type 1, then G acts as left translations of $\mathrm{PSL}_2\mathbf{R}$. Thus G is spacelike. When H is of type 2, we choose the point $x = 1$, $\left(\begin{smallmatrix} e^1 & 0 \\ 0 & e^{-1}\end{smallmatrix}\right)$ in $\mathrm{PSL}_2\mathbf{R}$ according as $a \neq b$, $a = b$. If H is of type 3, then choose the point $x = 1$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ according as $a \neq b$, $a = b$. For type 4, we choose the point $x = 1$. In each case the vector field tangent to the orbit $G \cdot x$ belongs to the subbundle B. Thus G is neither timelike nor lightlike. If H is of type 5, then the orbit $G \cdot 1$ winds infinitely many times around the $\mathbf{S}^1$-direction in $\mathrm{PSL}_2\mathbf{R}$. Hence G is neither lightlike nor spacelike in this case.

Corollary 4.17. *There exists no lightlike Killing vector field on a compact Lorentz hyperbolic* 3*-manifold.*

Lemma 4.18. *Let* G *be a timelike one-parameter group in* $O(2,2)^{0\sim}$, *and* $1 \to \mathscr{Z} \to O(2,2)^{\sim} \xrightarrow{P} O(2,2) \to 1$ *be the exact sequence. Then the group* $P(G)$ *satisfies either one of the following:*

(i) $P(G) \approx \mathbf{S}^1$.

(ii) $P(G) \approx \mathbf{R}^1$ *which is dense in* $SO(2) \times_{\mathbf{Z}/2} SO(2)$.

(iii) $P(G) \approx \mathbf{R}^1$ *which is a closed subgroup of type* (5) *in* $O(2,2)^0$ *of Lemma* 4.7.

Proof. Let $\overline{P(G)}$ be the closure of $P(G)$ in $O(2,2)^0$. If $\overline{P(G)}$ is compact, then $\overline{P(G)}$ is conjugate to a subgroup of the maximal compact subgroup $SO(2) \times_{\mathbf{Z}/2} SO(2)$. Thus either (i) or (ii) follows. Suppose that $\overline{P(G)}$ is noncompact. Then $\overline{P(G)}$ is isomorphic to one of the groups of Lemma 4.7 in which two-dimensional Lie group is isomorphic to either $\mathbf{R}^2$ or $\mathbf{R} \times S^1$. Thus the group $P(G)$ is itself closed and is isomorphic to $\mathbf{R}^1$. Since $P(G)$ is timelike, $P(G)$ is of type 5.

Proposition 4.19. *If a compact Lorentz hyperbolic* 3*-manifold admits a timelike Killing vector field, then it is a standard space form.*

Proof. Let $(\rho, \mathrm{dev})\colon (\pi, \widetilde{M}) \to (\Gamma, \widetilde{\mathbf{H}}^{1,2})$ be the developing pair. Given a timelike one-parameter group $\widetilde{H}$ of Lorentz transformations in $O(2,2)^{\sim}$, we put $H = P(\widetilde{H})$. If H is compact in $O(2,2)$, then the result follows from Theorem 2.20. Otherwise, from (ii), (iii) of Lemma 4.18 it follows that $\overline{H} = SO(2) \times_{\mathbf{Z}/2} SO(2)$ or

$$H = \left\{ \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \cos a\theta & -\sin a\theta \\ \sin a\theta & \cos a\theta \end{pmatrix} | \theta \in \mathbf{R} \right\}$$

$$\left(\text{resp. } \left\{ \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} \times \begin{pmatrix} \cos a\theta & -\sin a\theta \\ \sin a\theta & \cos a\theta \end{pmatrix} | \theta \in \mathbf{R} \right\} \right).$$

Since H centralizes the group $P(\Gamma)$, the closure $\overline{H}$ also centralizes $P(\Gamma)$. When $\overline{H} = SO(2) \times_{\mathbf{Z}/2} SO(2)$, the subgroup of $O(2, 2)$, whose elements commute with $\overline{H}$, is $\overline{H}$ itself. Thus $P(\Gamma) \subset \overline{H}$. By pulling back into $O(2, 2)^{\sim}$, we obtain $\Gamma \subset \mathbf{R} \times SO(2)$. Similarly the subgroup of $O(2, 2)^0 \approx SL_2\mathbf{R} \times_{\mathbf{Z}/2} SL_2\mathbf{R}$ which commutes with H is $\{(\begin{smallmatrix}1 & \theta\\ 0 & 1\end{smallmatrix}) | \theta \in \mathbf{R}\} \times SO(2)$. By passing to a subgroup of finite index in Γ, we have $P(\Gamma) \subset \{(\begin{smallmatrix}1 & \theta\\ 0 & 1\end{smallmatrix}) | \theta \in \mathbf{R}\} \times SO(2)$, and therefore $\Gamma \subset \{(\begin{smallmatrix}1 & \theta\\ 0 & 1\end{smallmatrix}) | \theta \in \mathbf{R}\} \times \mathbf{R}$ (cf. 4.6), which is impossible since $M \approx \widetilde{\mathbf{H}}^{1,2}/\Gamma$ by Proposition 2.5.

Problem 2. Let M be a compact Lorentz hyperbolic $(2n+1)$-manifold $(n \geq 2)$ which admits a timelike Killing vector field. Is M always a standard space form?

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