J. DIFFERENTIAL GEOMETRY
37 (1993) 505-522

ON THE MODULI SPACE OF VECTOR BUNDLES ON THE FIBERS OF THE UNIVERSAL CURVE

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Dedicated to the memory of S. K. Pichorides

Abstract

In this paper we describe the Picard group of the variety $\mathscr{U}(r, d)$ which parametrizes semistable vector bundles of rank r and degree d on the fibers of the universal curve $\mathscr{C}_g$. The bundle $\mathscr{U}(r, d)$ lies over the moduli space $\mathscr{M}_g^0$ of smooth curves of genus g $(g \geq 3)$ without automorphisms.

1. Introduction

We denote by $\mathscr{M}_g^0$ the moduli space of smooth curves of genus g $(g \geq 3)$ without automorphisms. To this space we can associate various varieties: The universal curve $\pi: \mathscr{C}_g \to \mathscr{M}_g^0$ which is a bundle with fiber the curve C over the point $[C] \in \mathscr{M}_g^0$; the variety $q: \mathscr{U}(r, d) \to \mathscr{M}_g^0$ with fiber over $[C]$ the space $U_C(r, d)$, which parametrizes semistable vector bundles of rank r and degree d on C—for the definition see [9]. In the special case when $r = 1$, this becomes the Jacobian variety $p: \mathscr{J}^d \to \mathscr{M}_g^0$ of degree d with fiber $J^d(C)$ over the point $[C]$, which parametrizes line bundles of degree d on C.

The Picard groups of $\mathscr{M}_g^0$ and $\mathscr{C}_g$ have been described by Harer, Arbarello and Cornalba (see [1]). The $\operatorname{Pic}\mathscr{M}_g^0$ is generated by the determinant λ of the Hodge bundle. On the other hand, the restriction of a line bundle on $\mathscr{C}_g$ to the fibers of π is something "canonical", namely a multiple of the canonical bundle (Franchetta's problem, see [1]). Therefore the relative Picard group $\operatorname{Pic}(\mathscr{C}_g/\mathscr{M}_g^0)$ is generated by the relative dualizing sheaf ω_π of the family π and the $\operatorname{Pic}\mathscr{C}_g$ is the free abelian group with generators ω_π and $\pi^*\lambda$.

In this paper we prove that a similar phenomenon holds for line bundles on $\mathscr{U}(r, d)$. The restriction of a line bundle on $\mathscr{U}(r, d)$ to a fiber

Received January 13, 1992.

$U_C(r, d)$ is again something "canonical" in the sense that we explain in §3. Before we continue, let us note that we have a natural isomorphism $\mathscr{U}(r, d) \cong \mathscr{U}(r, d+r(2g-2))$ given by $E \mapsto E \otimes K$, where K the canonical bundle. Using this, it is enough to describe the $\operatorname{Pic} \mathscr{U}(r, d)$ for large values of the degree d.

2. Some properties of θ divisors

We state here some technical lemmas concerning properties of θ divisors on the Jacobian of a smooth curve. First a notation. By fixing a line bundle $L \in J^{-d+g-1}(C)$, the locus of $\{M \in J^d(C)$ such that $h^0(M \otimes L) \geq 1\}$ is of codimension one in $J^d(C)$. We denote by θ_L the line bundle on $J^d(C)$ corresponding to this divisor (or sometimes the divisor itself).

Lemma 1. *Let $\mathscr{A}$ be an abelian variety, and $\mathscr{L}$ a principal polarization on $\mathscr{A}$. Then the map $\phi_{\mathscr{L}}: \mathscr{A} \to \operatorname{Pic}(\mathscr{A})$ which sends $A \mapsto T_A^* \mathscr{L} \otimes \mathscr{L}^{-1}$ is a group homomorphism.*

Proof. See [8, p. 59, Corollary 4].

Lemma 2. *$J^0(C)$ is naturally isomorphic to the variety $\operatorname{Pic}^0 J^d(C)$ which parametrizes the line bundle of class 0 on $J^d(C)$.*

Proof. Fix a principal polarization θ_M on $J^d(C)$, where $M \in J^{-d+g-1}(C)$ following the above notation. Consider the map $J^0(C) \to \operatorname{Pic}^0 J^d(C)$ which sends $L \mapsto \theta_{M \otimes L} \otimes \theta_M^{-1}$. As it turns out this does not depend on the choice of M and is an isomorphism (see [8] for details).

Lemma 3. *If $A, B \in J^{-d+g-1}(C)$ with $A^n = B^n$, then $\theta_A^n = \theta_B^n$ on $J^d(C)$. More generally, if $A_i, B_j \in J^{-d+g-1}(C)$ with $\bigotimes_{i=1}^s A_i^{n_i} = \otimes_{j=1}^t B_j^{m_j}$, $\sum_{i=1}^s n_i = \sum_{j=1}^t m_j$, then $\bigotimes_{i=1}^s \theta_{A_i}^{n_i} = \bigotimes_{j=1}^t \theta_{B_j}^{m_j}$.*

Proof. We prove the general case. It is enough to prove the lemma for the case $d = 0$. Then using an identification of $J^d(C)$ with $J^0(C)$ it is true for all d. Fix a polarization θ_C on $J^d(C)$. Then $\theta_{A_i} = T^*_{A_i \otimes C^{-1}} \theta_C$. We want to prove that

$$\bigotimes_{i=1}^{s} \theta_{A_i}^{n_i} = \bigotimes_{j=1}^{t} \theta_{B_j}^{m_j},$$

or

$$\bigotimes_{i=1}^{s} (\theta_{A_i}^{n_i} \otimes \theta_C^{-n_i}) = \bigotimes_{j=1}^{t} (\theta_{B_j}^{m_j} \otimes \theta_C^{-m_j}),$$

or

$$\bigotimes_{i=1}^{s}(T^*_{A_i\otimes C^{-1}}\theta_C^{n_i}\otimes\theta_C^{-n_i}) = \bigotimes_{j=1}^{t}(T^*_{B_j\otimes C^{-1}}\theta_C^{m_j}\otimes\theta_C^{-m_j}),$$

or

$$\bigotimes_{i=1}^{s}\phi_{\theta_C}(A_i\otimes C^{-1})^{n_i} = \bigotimes_{j=1}^{t}\phi_{\theta_C}(B_j\otimes C^{-1})^{m_j},$$

which is true by Lemma 1.

3. The Picard group of $U_C(r, d)$

We review now the description of the Picard group of the variety $U_C(r, d)$ (resp. $U_C(r, L)$) which parametrizes the semistable vector bundles of rank r and degree d (resp. determinant $L \in J^d(C)$) on a smooth curve C. The reference is [3].

The smooth locus of $U_C(r, d)$ is the set of points $U_C^s(r, d)$ which correspond to stable vector bundles. Also $\operatorname{codim}_{U_C(r,d)}(U_C(r, d)\backslash U_C^s(r, d)) \geq 2$. The space $U_C(r, d)$ is locally factorial (see [3, Theorem A], and so any line bundle on $U_C^s(r, d)$ can be extended uniquely to a line bundle on $U_C(r, d)$. Similarly, one can see that the space $\mathscr{U}(r, d)$ is locally factorial too. The map det: $U_C(r, d) \to J^d(C)$ which sends $E \mapsto \det E$ has fiber over the point $[L] \in J^d(C)$ the variety $U_C(r, L)$. We have the following (see [3]):

(1) $\operatorname{Pic} U_C(r, L) = \mathbf{Z}$;

(2) $\operatorname{Pic} U_C(r, d) = \mathbf{Z} \oplus \det^* \operatorname{Pic} J^d(C)$.

A geometric description of the generators is given as follows. For a generic choice of a vector bundle F of rank $\frac{r}{n}$ and degree $\frac{-d+r(g-1)}{n}$ where $n =$ g.c.d. (r, d), the set of points $\{E \in U_C(r, d)$ (resp. $E \in U_C(r, L)$) such that $h^0(E \otimes F) \geq 1\}$ defines a divisor in $U_C(r, d)$ (resp. in $U_C(r, L)$). This has been proven in [5]. Note that F has the minimum possible rank for which there exists a degree such that the Euler characteristic $\chi(E\otimes F) = 0$. We denote the induced line bundle by Θ_F (resp. by $\Theta_{L,F}$). The basic facts about these line bundles are

1. The line bundle $\Theta_{L,F}$ on $U_C(r, L)$ does not depend on the choice of F and is the generator of the $\operatorname{Pic} U_C(r, L) \cong \mathbf{Z}$.

2. The line bundle Θ_F on $U_C(r, d)$ depends only on the determinant of the vector bundle F. Namely, if F, F' are two choices as above, then

we have the relation

$$\Theta_F = \Theta_{F'} \otimes \det^*(\text{“}\det F \otimes \det F'^{-1}\text{”}),$$

where $\det F \otimes \det F'^{-1}$ is an element of $J^0(C)$ which can be considered naturally as an element of $\operatorname{Pic}^0 J^d(C)$ (see Lemma 2).

We construct now "canonical" choices of line bundles on $U_C(r, d)$ as follows. Let m be an integer such that $m\frac{-d+r(g-1)}{n}$ is an integral linear combination of the numbers $-d+g-1$ and $2g-2$, i.e.,

$$(1) \qquad m\frac{-d+r(g-1)}{n} = \alpha(-d+g-1) + \beta(2g-2).$$

The set of all such m's forms a subgroup of the integers with generator

$$(2) \qquad k_{r,d} = \frac{\text{g.c.d.}(2g-2, -d+g-1)}{\text{g.c.d.}(2g-2, -d+g-1, \frac{-d+r(g-1)}{n})}.$$

Given F (resp. F') with rank and degree as above, we choose a line bundle M (resp. M') of degree $-d+g-1$, such that $M^\alpha = \det F^m \otimes K^{-\beta}$ (resp. $M'^\alpha = \det F'^m \otimes K^{-\beta}$). There are finitely many such choices, namely α^{2g}. The claim is that the line bundle

$$(3) \qquad \Theta_F^m \otimes \det^* \theta_M^{-\alpha}$$

does not depend on the choice of F, M. Indeed, we have that

$$\begin{aligned}\Theta_F^m \otimes \det^* \theta_M^{-\alpha} &= \Theta_{F'}^m \otimes \det^*(\text{“}\det F \otimes \det F'^{-1}\text{”})^m \otimes \det^* \theta_M^{-\alpha} \\ &= \Theta_{F'}^m \otimes \det^*((\text{“}\det F \otimes \det F'^{-1}\text{”})^m \otimes \theta_M^{-\alpha}) \\ &= \Theta_{F'}^m \otimes \det^* \theta_{M'}^{-\alpha},\end{aligned}$$

where the last equality comes from Lemma 3, using that $M^{-\alpha} \otimes \det F^m \otimes \det F'^{-m} = K^\beta \otimes \det F'^{-m} = M'^{-\alpha}$. The line bundles of the above form as in (3) are the canonical choices of line bundles on $U_C(r, d)$. The description of the Picard group of $\mathscr{U}(r, d)$ is given by the following theorem.

Theorem 1. *The restriction of any line bundle on $\mathscr{U}(r, d)$ to the fibers of the map $q\colon \mathscr{U}(r, d) \to \mathscr{M}_g^0$ is such a canonical choice as in* (3). *Even more, for any choice of integers m, α, β satisfying relation* (1), *there exists a line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ which restricts to the above canonical choice $\Theta_F^m \otimes \det^* \theta_M^{-\alpha}$ on the fiber $U_C(r, d)$.*

Remark. As we proved in [6], in the special case of the Jacobians $\mathscr{J}^d \to \mathscr{M}_g^0$, i.e., when $r = 1$, the restriction of a line bundle on the fiber $J^d(C)$ has the form θ_M^α, where $M^\alpha = K^\beta$ for some integers α, β. This corresponds to the above situation when $m = 0$; i.e., the line bundle is trivial on the fibers of the map $\det: \mathscr{U}(r, d) \to \mathscr{J}^d$ and so it is the pullback of a line bundle from $\mathscr{J}^d$.

4. The space of extensions

We first recall some things about symmetric products of curves. The main reference is [2]. For d large enough, the dth symmetric product $C^{(d)}$ of a smooth curve C can be considered as a projectivized vector bundle over the Jacobian variety $J^d(C)$ in the following way: By fixing a point q_0 in C, there exists a normalized Poincaré bundle $\mathscr{P}_{q_0}$ on the product $J^d(C) \times C$. This is characterized by the properties: $\mathscr{P}_{q_0}|_{\{L\}\times C} \cong L$ and $\mathscr{P}_{q_0}|_{J^d(C)\times\{q_0\}} = \mathscr{O}$. To construct $\mathscr{P}_{q_0}$, we define the map

$$\begin{aligned} \phi_{q_0}: J^d(C) \times C &\to J^{g-1}(C), \\ (L, p) &\mapsto L \otimes \mathscr{O}((g-d)q_0 - p). \end{aligned}$$

Then

$$\mathscr{P}_{q_0} \overset{\text{def}}{=} \phi_{q_0}^* \theta \otimes q^* \mathscr{O}((d-g)q_0) \otimes \nu^* \theta^{-1}_{(-d+g-1)q_0}, \tag{4}$$

where ν and q the projections, $\theta = \theta_{\mathscr{O}}$ ($\mathscr{O}$ the trivial line bundle) and $\theta_{(-d+g-1)q_0} \overset{\text{def}}{=} \theta_{\mathscr{O}((-d+g-1)q_0)}$, following the notation of §2. We then have that $C^{(d)} \cong \mathbf{P}(\nu_* \mathscr{P}_{q_0})$, and the fiber of the map $u: C^{(d)} \cong \mathbf{P}(\nu_*\mathscr{P}_{q_0}) \to J^d(C)$ over a point $[L] \in J^d(C)$ is the projective space $\mathbf{P}(H^0(C, L))$. Given a point p in C, the set $\{D \in C^{(d)}$ such that $D - p \geq 0\}$ defines a divisor which we denote by X_p. As it turns out the divisor X_{q_0} is a section of the tautological line bundle $\mathscr{O}_{\mathbf{P}(\nu_*\mathscr{P}_{q_0})}(1)$ (see [2, p. 309]).

We denote by x the class in the Neron-Severi group of the divisor X_p. This is independent from the choice of the point p. We also denote by δ the class of the diagonal $\Delta \overset{\text{def}}{=} \{D \in C^{(d)},\ D = D_{d-2} + 2p$ for some $D_{d-2} \in C^{(d-2)},\ p \in C\}$ in $C^{(d)}$. The pullback of the class θ of the theta divisor in $J^d(C)$ by the Abel-Jacobi map $u: C^{(d)} \to J^d(C)$ is given by the MacDonald's formula

$$u^*\theta = (d+g-1)x - \frac{\delta}{2},$$

(see [2, Proposition 5.1 in p. 358] or [6, Lemma 4]). If C is a curve with general moduli, then it is known that the Neron-Severi group of the

$J^d(C)$ is generated by the class of the theta divisor. From this, one concludes that the Neron-Severi group of $C^{(d)}$ is generated by the class of the pullback of θ and the above-defined class x (see [2, p. 359]); using the MacDonald's formula the generators can be chosen to be $\frac{\delta}{2}$ and x. According to this, given a line bundle $\mathscr{L}$ on the universal dth symmetric product $\mathscr{C}_g^{(d)}$, its restriction to a fiber $C^{(d)}$ is algebraically equivalent to an integral combination $ax + b\frac{\delta}{2}$. Since the curve C is not rational, the classes x and $\frac{\delta}{2}$ are linearly independent. In our paper [6] we show that the coefficient a has to satisfy

$$(*) \qquad 2g - 2 | a .$$

In the following we are going to see how the relation $(*)$ imposes conditions to line bundles on $\mathscr{U}(r, d)$. To start with, if D is a stable vector bundle of rank r and degree d, then for d large enough—as we are going to assume from now on—we have an exact sequence (see [9])

$$0 \to \mathscr{O}_C \otimes \mathbf{C}^{r-1} \to E \to L \to 0,$$

where $L = \det E$. The extensions of L by $\mathbf{C}^{r-1}$ are parametrized by the points of $H^1(C, L^{-1} \otimes \mathbf{C}^{r-1})$. Let $\mathbf{P}_L = \mathbf{P}(H^1(C, L^{-1} \otimes \mathbf{C}^{r-1}))$. Take a Poincaré bundle $\mathscr{P}$ on $J^d(C) \times C$ and define

$$\mathbf{P} = \mathbf{P}(R^1\nu_*(\mathscr{P}^{-1} \otimes C^{r-1})) \overset{\text{Serre}}{\cong} \mathbf{P}(\nu_*(\mathscr{P} \otimes q^*K)^\vee \otimes \mathbf{C}^{r-1}),$$

where ν and q are the projections of $J^d(C) \times C$. This is a projectivized vector bundle $v\colon \mathbf{P} \to J^d(C)$. According to [4, Proposition 2, application II], there exist a "universal" vector bundle $\mathbf{E}$ on $\mathbf{P} \times C$ and an exact sequence

$$(5) \qquad 0 \to \mathscr{O}_{\mathbf{P}\times C} \otimes \mathbf{C}^{r-1} \to \mathbf{E} \to p_1^*\mathscr{O}_{\mathbf{P}}(-1) \otimes v^{\#}\mathscr{P} \to 0,$$

(where $v^{\#} = (v \times 1)^*$, and p_1 is the projection $\mathbf{P} \times C \to \mathbf{P}$) such that for every point x in $\mathbf{P}$, with $v(x) = [L] \in J^d(C)$, its restriction to $\{x\} \times C$

$$0 \to \mathscr{O}_C \otimes \mathbf{C}^{r-1} \to \mathbf{E}_x \to x \otimes L \to 0$$

corresponds to the inclusion $x \to H^1(C, L^{-1} \otimes \mathbf{C}^{r-1})$. Let $\mathbf{P}^s$ be the open of $\mathbf{P}$ consisting of points x with $\mathbf{E}_x$ stable vector bundle. We denote by f the forgetful morphism $f\colon \mathbf{P}^s \to U_C(r, d)$ and also by f again the rational map $f\colon \mathbf{P} \to U_C(r, d)$. The complement of $\mathbf{P}^s$ in $\mathbf{P}$ is of codimension ≥ 2 and so, any line bundle on $\mathbf{P}^s$ extends uniquely to a line bundle on $\mathbf{P}$. We denote now by $\mathbf{P}^{un}$ the bundle over $\mathscr{M}_g^0$

whose fiber over $[C] \in \mathcal{M}_g^0$ is the above space $\mathbf{P}$. We have the following diagram:

$$\begin{array}{ccc} \mathbf{P}^{un} & \xrightarrow{\ f\ } & \mathcal{U}(r, d) \\ {\scriptstyle v}\big\downarrow & \swarrow & \big\downarrow \\ \mathcal{J}^d & \longrightarrow & \mathcal{M}_g^0 \end{array}$$

The crucial point here is that the variety $\mathbf{P}^{un}$ is a projective bundle over $\mathcal{J}^d$, with fiber over $[L] \in J^d(C)$ isomorphic to the projective space $\mathbf{P}^{(r-1)(d+g-1)-1}$, *but not in general a projectivized one.* This corresponds to the fact that in general there is no Poincaré bundle on $\mathcal{J}^d \times_{\mathcal{M}_g^0} \mathcal{C}_g$, see application at the end of §5. A way to measuring how far $\mathbf{P}^{un}$ is of being a projectivized vector bundle is to determine the minimum positive number l for which there exists a line bundle on $\mathbf{P}^{un}$ whose restrictions to the fibers of the map f is $\mathcal{O}(l)$, where $\mathcal{O}(1)$ is the hyperplane bundle on $\mathbf{P}^{(r-1)(d+g-1)-1}$. We start with a lemma.

Lemma 4. *Let $\mathbf{P}_1^{un}$ be the bundle over $\mathcal{M}_g^0$ whose fiber over the point $[C] \in \mathcal{M}_g^0$ is $\mathbf{P}(\nu_*(\mathcal{P} \otimes q^*K)^\vee)$, where $\mathcal{P}$ and ν, q are as before. Then $\mathbf{P}_1^{un}$ is a projective bundle $v_1 \colon \mathbf{P}_1^{un} \to \mathcal{J}^d$ for which the corresponding number l (definition as above) is the same as that of the projective bundle $\mathbf{P}^{un}$.*

Proof. Over a point $[L] \in J^d(C)$ the fiber of $\mathbf{P}_1^{un}$ is $\mathbf{P}(H^0(C, L \otimes K)^\vee)$ and the fiber of $\mathbf{P}^{un}$ is $\mathbf{P}(H^0(C, L \otimes K)^\vee \otimes \mathbf{C}^{r-1})$. Over a small analytic neighborhood U of $\mathcal{J}^d$ the bundle $\mathbf{P}_1^{un}$ is projectivization of a vector bundle V_U. Let $\phi_1, \cdots, \phi_{d+g-1}$ be a local frame. If $e_1, \cdots, e_{r-1}$ is a frame for the trivial bundle $\mathbf{C}^{r-1}$ over $\mathcal{J}^d$, then over U the bundle $\mathbf{P}^{un}$ is the projectivization of $V_U \otimes \mathbf{C}^{r-1}$ with a local frame $\phi_i \otimes e_j$, $i = 1, \cdots, d+g-1$ and $j = 1, \cdots, r-1$. Consider the diagonal map $V_U \to V_U \otimes \mathbf{C}^{r-1}$ sending $\sum_i a_i\phi_i \mapsto \sum_{ij} a_i\phi_i \otimes e_j$; this induces a morphism $\beta \colon \mathbf{P}_1^{un} \to \mathbf{P}^{un}$. Consider also the map $V_U \otimes \mathbf{C}^{r-1} \to V_U$ sending $\sum_{ij} b_{ij}\phi_i \otimes e_j \mapsto \sum_i (\sum_j b_{ij})\phi_i$; this induces a rational map $\alpha \colon \mathbf{P}^{un} \to \mathbf{P}_1^{un}$. The locus where this is not defined is of codimension $d+g-1$ (≥ 2) in the fibers of $v \colon \mathbf{P}^{un} \to \mathcal{J}^d$. Note that the monodromy on $\mathcal{M}_g^0$ does not cause any problem in the construction of the maps, since the local construction is invariant under the action. Any line bundle on $\mathbf{P}^{un}$ which restricts to $\mathcal{O}(n)$ on the fibers of v pulls back by β to a line bundle on $\mathbf{P}_1^{un}$ with the same property. Similarly, any line bundle on $\mathbf{P}_1^{un}$ with restriction $\mathscr{O}(n)$ pulls back by α and extends *uniquely* to line bundle on $\mathbf{P}^{un}$ with the same property. This proves the lemma.

We have now the following theorem:

Theorem 2. *For the projective bundle $v\colon \mathbf{P}^{un} \to \mathscr{J}^d$ the minimum number l for which there exists a line bundle on $\mathbf{P}^{un}$ whose restriction on the fibers of the map v is $\mathscr{O}(l)$ is given by*

$$l = \text{g.c.d}\,(2g-2\,,\, d+g-1)\,.$$

Proof. By Lemma 4 above, it is enough to prove the result for the bundle $\mathbf{P}_1^{un}$. Consider the maps

$$\mathscr{C}_g^{(d)} \times_{\mathscr{M}_g^0} \mathscr{C}_g \overset{\phi=u\times 1}{\longrightarrow} \mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g \overset{\psi}{\longrightarrow} \mathbf{P}_1^{un}\,,$$

where u is the Abel-Jacobi map, and ψ is the map which sends $(L\,,\,p) \mapsto \{\sigma \in H^0(C\,,\,L\otimes K) \text{ with } \sigma(p) = 0\}$. On $\mathscr{C}_g^{(d)} \times_{\mathscr{M}_g^0} \mathscr{C}_g$ we have a universal bundle $\mathscr{D}_d$; this is the bundle corresponding to the divisor which is the image of the map $\mathscr{C}_g^{(d-1)} \times_{\mathscr{M}_g^0} \mathscr{C}_g \to \mathscr{C}_g^{(d)} \times_{\mathscr{M}_g^0} \mathscr{C}_g$ sending $(D\,,\,p) \mapsto (D + p\,,\,p)$. Note that $\text{class}\,(\mathscr{D}_d|_{C^{(d)}\times\{p\}}) = x$, where x is the class of the divisor X_p defined at the beginning of the section. Let $\mathscr{L}$ be a line bundle on $\mathbf{P}_1^{un}$ which restricts to $\mathscr{O}(s)$ on the fibers of the map $v_1\colon \mathbf{P}_1^{un} \to \mathscr{J}^d$, where s is an integer. Consider the line bundle $\psi^*\mathscr{L}$. If q is the projection $\mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g \to \mathscr{C}_g$, and ω is the relative dualizing sheaf of the family $\mathscr{C}_g \to \mathscr{M}_g^0$, then the line bundle $\mathscr{P} = \psi^*\mathscr{L} \otimes q^*\omega^{-s}$ has the property $\mathscr{P}|_{\{L\}\times C} \cong L^{\otimes s}$. Also, the class in the Neron-Severi group of the $\mathscr{P}|_{J^d(C)\times\{p\}}$ is independent of the choice of $p \in C$, equal say to $n\theta$ where n is an integer independent of C and p; this is because the Neron-Severi group of the Jacobian of a curve with general moduli is generated by the class of the theta divisor, and since algebraic equivalence is an open (topological) condition, n will not vary over the irreducible space $\mathscr{C}_g$. Therefore from the above-mentioned MacDonald's formula it follows that $\text{class}\,(\phi^*\mathscr{P}|_{C^{(d)}\times\{p\}}) = n((d+g-1)x - \frac{\delta}{2})$.

For each $D \in \mathscr{C}_g^{(d)}$ over $[C]$, we have $\phi^*\mathscr{P}|_{\{D\}\times C} \cong \mathscr{D}_d^{\otimes s}|_{\{D\}\times C} \cong \mathscr{O}(D)^{\otimes s}$. By the see-saw principle (see [8]), there exists a line bundle $\mathscr{R}$ on $\mathscr{C}_g^{(d)}$ such that $\phi^*\mathscr{P} \cong \mathscr{D}_d^{\otimes s} \otimes \pi_1^*\mathscr{R}$, where π_1 is the projection on $\mathscr{C}_g^{(d)}$. Therefore, the restriction of $\mathscr{R}$ to a fiber $C^{(d)}$ of the map $\mathscr{C}_g^{(d)} \to \mathscr{M}_g^0$ has class $[n(d+g-1) - s]x - n\frac{\delta}{2}$. From our basic relation $(*)$, we conclude that $2g-2$ has to divide the coefficient of x; i.e.,

$$n(d+g-1) - s = k(2g-2)\,,$$

where k is an integer. This implies that g.c.d $(2g-2, d+g-1)$ has to divide the number s. To conclude the proof of the theorem we have to prove that there exists a line bundle on $\mathbf{P}_1^{un}$ whose restrictions on the fibers of v is $\mathscr{O}(\text{g.c.d.}(2g-2, d+g-1))$. In the following section we construct such a line bundle.

5. Construction of line bundles

We construct here two line bundles on $\mathbf{P}^{un}$ whose restrictions to the fibers of the map v are $\mathscr{O}(d+g-1)$ and $\mathscr{O}(2g-2)$ respectively. For this, we first do the construction on $\mathbf{P}_1^{un}$, and then pull back by the map α on $\mathbf{P}^{un}$ (see proof of Lemma 4 for the definition of α).

The first line bundle is the dual of the relative dualizing sheaf ω_{v_1} of the family $v_1 : \mathbf{P}_1^{un} \to \mathscr{J}^d$. Since the fibers are projective spaces of dimension $d+g-2$, the dual of ω_{v_1} restricts to $\mathscr{O}(d+g-1)$ on the fibers.

The construction of the second line bundle is a little more complicated. We start with a definition.

Definition 1. For a fixed curve C we denote by $\mathscr{L}_{q_0}$ the tautological bundle $\mathscr{O}_{\mathbf{P}_1}(1)$ of the projectivized bundle $\mathbf{P}_1 \overset{\text{def}}{=} \mathbf{P}(\nu_*(\mathscr{P}_{q_0} \otimes q^* K)^\vee)$, where $\mathscr{P}_{q_0}$ is the normalized Poincaré bundle at q_0, and ν, q are the two projections.

Choose a divisor $\sum_{i=1}^{2g-2} p_i \in H^0(C, K)$ and consider the line bundle $\mathscr{L}_K \cong \bigotimes_{i=1}^{2g-2} \mathscr{L}_{p_i}$ on $\mathbf{P}_1$. As we shall see later (see Definition 2 in §7), this line bundle does not depend on the choice of the section in $H^0(C, K)$. We are going now to prove that there exists a line bundle $\mathscr{L}_K^{un}$ on $\mathbf{P}_1^{un}$, which restricts to $\mathscr{L}_K$ on the fibers. To do this we use the following lemma (see [6]).

Lemma 5. *Let $\mathscr{C}_g \xrightarrow{\pi} \mathscr{M}_g^0$ denote the universal curve over $\mathscr{M}_g^0$, and ω_π the relative dualizing sheaf of π. Then, there is a nonempty Zariski open subset $\mathscr{U}$ of $\mathscr{M}_g^0$ such that on $\pi^{-1}(\mathscr{U})$ there is a holomorphic section of ω_π.*

Proof. $\pi_*\omega_\pi$ is an algebraic bundle on $\mathscr{M}_g^0$. Therefore by Serre's theorem, it can be trivialized on a Zariski open of $\mathscr{M}_g^0$. This is the set $\mathscr{U}$ we are asking for. A trivial section of the bundle $\pi_*\omega_\pi$ over $\mathscr{U}$ corresponds to a holomorphic section of ω_π over $\pi^{-1}\mathscr{U}$. q.e.d.

Note first that we can choose $\mathscr{U}$ such that the restriction of the map π to the above holomorphic section gives an unramified covering of $\mathscr{U}$ of degree $2g-2$. We can now cover the Zariski open $\mathscr{U}$ by open analytic subsets $\{U_a\}$ such that over each U_a there are $2g-2$ sections s_i^a of the map π. Locally over each U_a we can construct a collection of $2g-2$ different maps

$$\phi_{i,a}\colon \mathscr{J}_a^d \times_{U_a} \mathscr{C}_{g,a} \to \mathscr{J}_a^{g-1}, \quad (L, p) \mapsto L \otimes \mathscr{O}((g-d)q_0 - p),$$

where the subindex a on the bundles means restriction over U_a. Then we define locally Poincaré bundles

$$\mathscr{P}_{i,a} \overset{\text{def}}{=} \phi_{i,a}^* \theta \otimes q^* \mathscr{O}((d-g)s_i^a) \otimes \nu^* \theta_{(-d+g-1)s_i^a}^{-1},$$

where the maps ν, q are the projections of $\mathscr{J}_a^d \times_{U_a} \mathscr{C}_{g,a}$, and $\theta_{(-d+g-1)s_i^a}$ is the divisor on $\mathscr{J}_a^d$ whose restriction to $J^d(C)$ is the divisor $\theta_{(-d+g-1)s_i^a([C])}$ (by s_i^a we denote either the map or the image, whatever makes sense). Using these locally defined Poincaré bundles, the restriction of $\mathbf{P}_1^{un}$ over the set U_a can be considered as a projectivized bundle over $\mathscr{J}_a^d$ in $2g-2$ different ways. We denote by $\mathscr{L}_{i,a}$ the corresponding tautological bundles. For each U_a let $\mathscr{L}_a^{un}$ denote the tensor product of all these bundles, which is a line bundle over $\mathbf{P}_1^{un}|_{U_a}$ whose construction remains invariant under the action of the monodromy group. Also by construction the $\mathscr{L}_a^{un}$'s coincide on the overlaps of the set U_a's, and so they fit together and give rise to a line bundle on $\mathbf{P}_1^{un}|_{\mathscr{U}}$ and by extension to a line bundle $\mathscr{L}_K^{un}$ on $\mathbf{P}_1^{un}$. Note that although we may have several possible extensions, their restrictions to the fibers over $\mathscr{M}_g^0$ coincide, and are the above-defined line bundles $\mathscr{L}_K$.

To construct now a line bundle on $\mathbf{P}^{un}$ which restricts to $\mathscr{O}(l)$ on the fibers of the map $v\colon \mathbf{P}^{un} \to \mathscr{J}^d$, consider integers a, b such that $a(2g-2) - b(d+g-1) = l$. Then, if $\mathscr{R} \cong \mathscr{L}_K^{un\otimes a} \otimes \omega_{v_1}^{\otimes b}$, the bundle we are asking for is $\mathscr{R}_1 \cong \alpha^* \mathscr{R}$.

Application. Consider the group

$$A_d = \{n \in Z \text{ such that } \exists \text{a l.b. } \mathscr{P} \text{ on } \mathscr{J}^d \times_{\mathscr{M}_g^0} \mathscr{C}_g \text{ with } \mathscr{P}|_{\{L\}\times C} = L^{\otimes n}\}.$$

Theorem 2 above implies that the generator of the group A_d is the number $l = \text{g.c.d.}(d+g-1, 2g-2)$. Indeed, at first $l \in A_d$. If $\mathscr{R}$ is the above line bundle on $\mathbf{P}_1^{un}$, then as we saw in the proof of the theorem, the line bundle $\mathscr{P} \cong \psi^* \mathscr{R} \otimes q^* \omega_{v_1}^{-l}$ has the property that $\mathscr{P}|_{\{L\}\times C} \cong L^{\otimes l}$. On the other hand, using the map ϕ as in the proof of the theorem we conclude l is the generator of A_d. In particular this implies that there exists a Poincaré bundle on $\mathcal{J}^d \times_{\mathcal{M}_g^0} \mathcal{C}_g$ if and only if $\text{g.c.d}(2g-2, d+g-1) = 1$. The latest has been proven in a different way by Mestrano and Ramanan (see [7]).

6. Imposing conditions

For a fixed curve C, let $f\colon \mathbf{P} \to U_C(r, d)$ be the (rational) map defined in §4. Let Θ_F be the line bundle on $U_C(r, d)$ defined in the same section, where $\operatorname{rk} F = \frac{r}{n}$, $\deg F = \frac{-d+r(g-1)}{n}$, $n = \text{g.c.d}(r, d)$. We recall here from [3] how one calculates the $f^*\Theta_F$; note that since f is not defined in a locus of codim ≥ 2, the pullback of Θ_F is uniquely determined. We have the following diagram:

$$\begin{array}{ccccccc}
C & \xleftarrow{p_2} & \mathbf{P} \times C & \xrightarrow{v\times 1} & J^d(C) \times C & \xrightarrow{q} & C \\
 & & p_1 \downarrow & & \downarrow \nu & & \\
 & & \mathbf{P} & \xrightarrow{v} & J^d(C) & & \\
 & & & f \searrow \quad \nearrow \det & & & \\
 & & & U_C(r, d) & & &
\end{array}$$

Tensoring the exact sequence (5) of §4 by $p_2^* F$ and taking direct images to $\mathbf{P}$ we get the induced long exact sequence

$$\begin{aligned}
0 &\to \mathcal{O}_{\mathbf{P}} \otimes H^0(C, \mathbf{C}^{r-1} \otimes F) \to R^0 p_{1*}(\mathbf{E} \otimes p_2^* F) \\
&\to \mathcal{O}_{\mathbf{P}}(-1) \otimes R^0 p_{1*}(v^\# \mathcal{P} \otimes p_2^* F) \to \mathcal{O}_{\mathbf{P}} \otimes H^1(C, \mathbf{C}^{r-1} \otimes F) \\
&\to R^1 p_{1*}(\mathbf{E} \otimes p_2^* F) \to \mathcal{O}_{\mathbf{P}}(-1) \otimes R^1 p_{1*}(v^\# \mathcal{P} \otimes p_2^* F) \to 0,
\end{aligned}$$

where $v^\# \overset{\text{def}}{=} (v \times 1)^*$. Therefore

$$\det p_{1!}(\mathbf{E} \otimes p_2^* F) = \mathcal{O}_{\mathbf{P}}\left(-(r-1)\frac{d}{n}\right) \otimes \det p_{1!}(v^\# \mathcal{P} \otimes p_2^* F),$$

(note $-(r-1)\frac{d}{n} = \chi(\det \mathbf{E}_x \otimes F)$). In [3] the authors prove that $f^*(-\Theta_F) \cong \det p_{1!}(\mathbf{E} \otimes p_2^* F)$ (see proof of Theorem C). Now since $p_{1!}(v^\# \mathcal{P} \otimes p_2^* F) \cong v^*(\nu_!(\mathcal{P} \otimes q^* F))$ we have

$$f^*(\Theta_F) \cong \mathcal{O}_{\mathbf{P}}\left((r-1)\frac{d}{n}\right) \otimes v^*(\det \nu_!(\mathcal{P} \otimes q^* F))^{-1}. \tag{6}$$

Also if $f_L\colon \mathbf{P}_L \to u(r, L)$ is the restriction map, then

$$f^*(\Theta_{L,F}) \cong \mathcal{O}_{\mathbf{P}_L}\left((r-1)\frac{d}{n}\right),$$

where $\Theta_{L,F}$ is the generator of $\operatorname{Pic} U_C(r, d)$. Combining this with Theorem 2, we impose now conditions for line bundles on $\mathscr{U}(r, d)$. We start with the diagram

$$\begin{array}{ccc} \mathbf{P}^{un} & \xrightarrow{\quad v \quad} & \mathscr{J}^d \\ & {\searrow}^{f} \quad {}^{\det}{\nearrow} & \\ & \mathscr{U}(r, d) & \end{array}$$

Consider a line bundle $\mathscr{L}$ on $\mathscr{U}(r, d)$ which restricts to $\Theta_{L,F}^{\otimes k}$ on the fiber $U_C(r, L)$ of the map det. Then, $f^*\mathscr{L}|_{\mathbf{P}_L} \cong \mathscr{O}_{\mathbf{P}_L}(k(r-1)\frac{d}{n})$, and Theorem 2 implies that

$$\text{g.c.d.}\,(2g-2, d+g-1) | k(r-1)\tfrac{d}{n}.$$

The minimum of such number k is

$$\frac{\text{g.c.d.}\,(2g-2, d+g-1)}{\text{g.c.d.}\,(2g-2, d+g-1, (r-1)\frac{d}{n})}.$$

Observe that this minimum is the same as the number $k_{r,d}$ in equality (2) of §3. Assume now that the second part of the Theorem 1 is true; i.e., given integers m, α, β satisfying relation (1) of §3, then there exists a line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ which restricts to the canonical choices $\Theta_F^m \otimes \det^* \theta_M^{-\alpha}$ on the fiber $U_C(r, d)$ over the point $[C] \in \mathscr{M}_g^0$. Thus the above discussion leads to the proof of the first part of the Theorem 1. Indeed, let $\mathscr{L}$ be any line bundle on $\mathscr{U}(r, d)$. The fiber of the map det over a point $[L] \in \mathscr{J}^d$ is $U_C(r, L)$, which has Picard group $\operatorname{Pic} U_C(r, L) \cong \mathbf{Z}[\Theta_{F,L}]$ (see §3). Restricting $\mathscr{L}$ to $U_C(r, L)$, by the above discussion we conclude that $\mathscr{L}|_{U_C(r,L)} \cong \Theta_{F,L}^{\otimes k_{r,d}s}$, s an integer. Therefore by taking $m = k_{r,d}s$ we can find integers α, β satisfying relation (1). Let $\mathscr{L}_{m,\alpha}$ be the "corresponding" line bundle on $\mathscr{U}(r, d)$. Then $\mathscr{L}_{m,\alpha}|_{U_C(r,L)} \cong \Theta_{F,L}^{\otimes m}$ and so $\mathscr{L}|_{U_C(r,L)} \cong \mathscr{L}_{m,\alpha}|_{U_C(r,L)}$. By the see-saw principle there exists a line bundle $\mathscr{M}$ on $\mathscr{J}^d$ such that $\mathscr{L} \cong \mathscr{L}_{m,\alpha} \otimes \det^* \mathscr{M}$. Now using the remark following Theorem 1, we conclude the proof of the first part of this theorem.

7. The generator line bundles on $\mathscr{U}(r, d)$

In this section we construct the above mentioned line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ and complete the proof of Theorem 1.

Lemma 6. *Let $\mathscr{P}_{q_0}$ be a normalized Poincaré bundle at the point q_0. If $\nu\colon J^d(C)\times C\to J^d(C)$ is the projection map, then*

$$\det \nu_! \mathscr{P}_{q_0} \cong \theta^{-1}_{(-d+g-1)q_0}.$$

More general, if E is a vector bundle on C of rank r_1 and degree d_1, then

$$\det \nu_!(\mathscr{P}_{q_0}\otimes q^*E) \cong \theta^{-(r_1-1)}_{(-d+g-1)q_0}\otimes \theta^{-1}_{\mathscr{O}((-d-d_1+g-1)q_0)\otimes \det E},$$

where $q\colon J^d(C)\times C\to C$ is the projection map.

Proof. We first claim that $\mathscr{P}_{q_0}|_{J^d(C)\times\{p\}} \cong$ "$\mathscr{O}(q_0-p)$". Indeed, with the notation of the construction of $\mathscr{P}_{q_0}$ (see relation (4)) we have

$$\begin{aligned}\phi^*\theta|_{J^d(C)\times\{p\}}&\\ &\cong \mathscr{O}(\{M\in J^d(C) \text{ such that } h^0(C, M\otimes\mathscr{O}((g-d)q_0-p))\geq 1\})\\ &\cong \theta_{(-d+g)q_0-p},\end{aligned}$$

and so $\mathscr{P}_{q_0}|_{J^d(C)\times\{p\}} \cong \theta_{(-d+g)q_0-p}\otimes\theta_{(-d+g)q_0-p}\otimes\theta^{-1}_{(-d+g-1)q_0}$. Then using Lemma 3, the claim is true.

By the Grothendieck-Riemann-Roch theorem one can show that $\det\nu_!\mathscr{P}_{q_0}$ has class θ^{-1}; see [2, Chapter VIII, §2]. Therefore, there exists a line bundle $L\in J^{-d+g-1}(C)$ with $\det\nu_!\mathscr{P}_{q_0}\cong\theta_L^{-1}$. We want to prove that $L\cong\mathscr{O}((-d+g-1)q_0)$. Fix a generic line bundle D_{d-1} of degree $d-1$ on C, and define the map $\psi_1\colon C\to J^d(C)$ which sends $p\mapsto D_{d-1}\otimes\mathscr{O}(p)$. Now consider the diagram

$$\begin{array}{ccccc} C & \xleftarrow{\pi_2} & C\times C & \xrightarrow{\psi=\psi_1\times 1} & J^d(C)\times C\\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \nu}\\ & & C & \xrightarrow{\psi_1} & J^d(C)\end{array}$$

where π_1, π_2, ν are the projection maps. Note that $\psi^*\mathscr{P}_{q_0}|_{\{p\}\times C}\cong D_{d-1}\otimes\mathscr{O}(p)$ and $\psi^*\mathscr{P}_{q_0}|_{C\times\{p\}}\cong\mathscr{O}(p-q_0)$. The last equality is derived from the above claim and the relation $\psi_1^*\theta_L\cong K\otimes D^{-1}_{d-1}\otimes L^{-1}$. Therefore by the theorem of the cube (see [8]), we have

$$\psi^*\mathscr{P}_{q_0}\cong\pi_1^*\mathscr{O}(-q_0)\otimes\pi_2^*D_{d-1}\otimes\mathscr{O}(\Delta).$$

Consider the exact sequence

$$0\to\mathscr{O}_{C\times C}\to\mathscr{O}_{C\times C}(\Delta)\to\mathscr{O}_\Delta(\Delta)\to 0.$$

Tensoring by $\pi_2^* D_{d-1}$ and taking the induced long exact sequence of the projection π_1, we get $\det \pi_{1!}(\Delta \otimes \pi_2^* D_{d-1}) \cong \det(\mathrm{id}_C)_!(\Delta \otimes D_{d-1}) \cong K^{-1} \otimes D_{d-1}$. Therefore

$$\begin{aligned}\det \pi_{1!}\psi^* \mathscr{P}_{q_0} &\cong \det(\mathscr{O}(q_0)^{-1} \otimes \pi_{1!}(\Delta \otimes \pi_2^* D_{d-1})) \\ &\cong \mathscr{O}((-d+g-1)q_0) \otimes K^{-1} \otimes D_{d-1}.\end{aligned}$$

On the other hand $\psi_1^* \det \nu_! \mathscr{P}_{q_0} \cong \psi_1^* \theta_L^{-1} \cong L \otimes D_{d-1} \otimes K^{-1}$. By the base change property we get that $L \otimes D_{d-1} \otimes K^{-1} \cong \mathscr{O}((-d+g-1)q_0) \otimes K^{-1} \otimes D_{d-1}$ and so $L \cong \mathscr{O}((-d+g-1)q_0)$.

For the second identity, if $r_1 = 1$, i.e., $E = \det E$, then a slight modification of the above calculation gives the result. For general rank r_1, we have in the K-group that $[E] = [\mathbf{C}^{r_1 - 1}] \oplus [\det E]$ which proves the lemma.

Lemma 7. *Let E be a vector bundle on a variety X, and let $E' \cong E \otimes L$ where L is a line bundle. Then*

$$\mathscr{O}_{\mathbf{P}E'}(1) \cong \mathscr{O}_{\mathbf{P}E}(1) \otimes \pi^* L^{-1},$$

where π is the canonical map.

Proof. See [4, Chapter II, Lemma 7.9].

Lemma 8. *We have*

$$\mathscr{L}_{q_0} \otimes \mathscr{L}_{p_0}^{-1} \cong v_1^* \text{“}\mathscr{O}(q_0 - p_0)\text{”},$$

where $\mathscr{L}_{q_0}$, $\mathscr{L}_{p_0}$ as in Definition 1 of §5.

Proof. Indeed, by the construction of the normalized Poincaré bundles, we have that $\mathscr{P}_{q_0} \cong \mathscr{P}_{p_0} \otimes \nu^* \text{“}\mathscr{O}(q_0 - p_0)\text{”}$; see claim at the beginning of the proof of Lemma 6. Therefore $\nu_*(\mathscr{P}_{q_0} \otimes q^* K)^\vee \cong \nu_*(\mathscr{P}_{p_0} \otimes q^* K)^\vee \otimes \text{“}\mathscr{O}(p_0 - q_0)\text{”}$, and Lemma 7 concludes the proof.

Definition 2. If M is a line bundle on C, we define $\mathscr{L}_M \overset{\text{def}}{=} \bigotimes_i \mathscr{L}_{p_i} \bigotimes_j \mathscr{L}_{q_j}^{-1}$ where $\sum_i p_i - \sum_j q_j$ is the divisor of a meromorphic section of M. By the above Lemma 8 the definition does not depend on the choice of the divisor. From the same lemma we easily derive the following properties:

(1) $\mathscr{L}_{M_1 \otimes M_2} \cong \mathscr{L}_{M_1} \otimes \mathscr{L}_{M_2}$ where M_1, M_2 are two line bundles.

(2) $v_1^*(\text{“}L_1 \otimes L_2^{-1}\text{”}) \cong \mathscr{L}_{L_1} \otimes \mathscr{L}_{L_2}^{-1}$, where L_1, L_2 are two line bundles of the same degree.

In the following we often use the notation $+$ and $-$ for the “tensor” and the “dual”.

Lemma 9. *If* $L \in J^{-d+g-1}(C)$, *then*

$$v_1^*\theta_L \cong \mathscr{L}_L - \mathscr{L}_K - \omega_{v_1}.$$

Proof. The bundle $v_1 \colon \mathbf{P}_1 = \mathbf{P}(\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee) \to J^d(C)$ has Euler sequence

$$0 \to \mathscr{O} \to v_1^*\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee \otimes \mathscr{O}_{\mathbf{P}_1}(1) \to \Omega_{v_1}^\vee \to 0,$$

where Ω_{v_1} is the sheaf of relative differentials of the map v_1. Therefore

$$\det v_1^*\nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee \cong -\omega_{v_1} + (-d-g+1)\mathscr{L}_{q_0}.$$

Since $\det \nu_*(\mathscr{P}_{q_0} \otimes q^*K)^\vee \cong \theta_{(-d-g+1)\mathscr{O}(q_0)\otimes K}$ (see Lemma 6), we get

$$v_1^*\theta_{(-d-g+1)\mathscr{O}(q_0)\otimes K} \cong -\omega_{v_1} + (-d-g+1)\mathscr{L}_{q_0}.$$

Using Lemma 3 and Lemma 8 yields

$$\begin{aligned} v_1^*\theta_L &\cong v_1^*\theta_{(-d-g+1)\mathscr{O}(q_0)\otimes K} \otimes v_1^*(\text{“}L - ((-d-g+1)\mathscr{O}(q_0) + K)\text{”}) \\ &\cong -\omega_{v_1} + (-d-g+1)\mathscr{L}_{q_0} + \mathscr{L}_L - (-d-g-1)\mathscr{L}_{q_0} - \mathscr{L}_K \\ &\cong -\omega_{v_1} + \mathscr{L}_L - \mathscr{L}_K. \end{aligned}$$

As we saw in the proof of Lemma 4 we have a map $\alpha \colon \mathbf{P} \to \mathbf{P}_1$. On $\mathbf{P}$ we denote again by $\mathscr{L}_L$ the pullback line bundle $\alpha^*\mathscr{L}_L$, and by ω the pull back $\alpha^*\omega_{v_1}$. Note that if we consider $\mathbf{P}$ as a projectivized bundle with the "use" of the Poincaré bundle $\mathscr{P}_{q_0}$, then $\mathscr{O}_{\mathbf{P}}(1) \cong \mathscr{L}_{q_0}$.

Lemma 10. *For the line bundle* Θ_F *on* $U_C(r, d)$,

$$f^*\Theta_F \cong \mathscr{L}_{\det F} - \frac{r}{n}(\mathscr{L}_K + \omega),$$

where $f \colon \mathbf{P} \to U_C(r, d)$ *is the forgetful* (*rational*) *map.*

Proof. By Lemmas 6 and 9 we have

$$\begin{aligned} &\det v^*\nu!(\mathscr{P}_{q_0} \otimes q^*F)^{-1} \\ &\cong v^*(\theta_{(-d+g-1)q_0}^{(r/n)-1} \otimes \theta_{(-d+d/n-(r/n)(g-1)+g-1)\mathscr{O}(q_0)\otimes \det F}) \\ &\cong \left(\frac{r}{n} - 1\right)((-d+g-1)\mathscr{L}_{q_0} - \mathscr{L}_K - \omega) \\ &\quad + \left(-d + \frac{d}{n} - \frac{r}{n}(g-1) + g - 1\right)\mathscr{L}_{q_0} + \mathscr{L}_{\det F} - \mathscr{L}_K - \omega \\ &\cong -(r-1)\frac{d}{n}\mathscr{L}_{q_0} - \frac{r}{n}(\mathscr{L}_K + \omega) + \mathscr{L}_{\det F}. \end{aligned}$$

Now this proves the lemma since

$$f^*\Theta_F \cong (r-1)\tfrac{d}{n}\mathscr{L}_{q_0} \otimes \det v^*\nu!(\mathscr{P} \otimes q^*F)^{-1}$$

(see relation (6)).

From Lemmas 9 and 10 one concludes easily

Theorem 3. *The pullback by the map f of the canonical choices of line bundles on $U_C(r, d)$ is*

$$f^*(\Theta_F^m \otimes \det{}^* \theta_M^{-\alpha}) \cong \left(\alpha + \beta - \frac{mr}{n}\right)\mathscr{L}_K + \left(\alpha - \frac{mr}{n}\right)\omega;$$

see relations (1), (3) *for the notation.*

Proof. Recall that $M^\alpha \cong \det F^m \otimes K^{-\beta}$, so that $\alpha\mathscr{L}_M \cong m\mathscr{L}_{\det F} - \beta\mathscr{L}_K$. Thus we have

$$\begin{aligned} f^*(\Theta_F^m \otimes \det{}^* \theta_M^{-\alpha}) &\cong m\mathscr{L}_{\det F} - \frac{mr}{n}\mathscr{L}_K - \frac{mr}{n}\omega - \alpha\mathscr{L}_M + \alpha\mathscr{L}_K + \alpha\omega \\ &\cong \left(\alpha + \beta - \frac{mr}{n}\right)\mathscr{L}_K + \left(\alpha - \frac{mr}{n}\right)\omega. \quad \text{q.e.d.} \end{aligned}$$

We now prove the existence of the line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$. Let $\mathbf{P}_s^{un}$ denote the subset of $\mathbf{P}^{un}$ corresponding to stable points; the complement is of codimension ≥ 2 in $\mathbf{P}^{un}$. In §5, we saw that there exist on $\mathbf{P}_s^{un}$ globally defining line bundles which restrict to $\mathscr{L}_K$ and ω on the fiber over the point $[C] \in \mathscr{M}_g^0$. Therefore there is a line bundle $\mathscr{F}$ on $\mathbf{P}_s^{un}$ which restricts to $(\alpha+\beta-\frac{mr}{n})\mathscr{L}_K + (\alpha - \frac{mr}{n})\omega$ on the fiber over $[C] \in \mathscr{M}_g^0$. The restriction of this bundle to the fibers of the map $f\colon \mathbf{P}_s^{un} \to \mathscr{U}(r, d)$ is trivial (pullback of a line bundle from $U_C(r, d)$). We give now a see-saw principle argument which implies that the above-defined canonical choices of line bundles on the fibers of the map $q\colon \mathscr{U}(r, d) \to \mathscr{M}_g^0$ are actually restrictions of globally defined line bundles on $\mathscr{U}(r, d)$. We are going to use a resolution of the map $f\colon \mathbf{P}_s^{un} \to \mathscr{U}(r, d)$ constructed in [3]. Following that paper, one can construct over $\mathscr{U}(r, d)$ a bundle $\mathbf{T}$ whose fiber over a point $[E] \in \mathscr{U}(r, d)$ is a bundle over the Grassmannian $\mathbf{Gr}(r-1, H^0(C, E))$ with fiber over $[H] \in \mathbf{Gr}(r-1, H^0(C, E))$ to be $\mathbf{P}(\mathrm{Hom}(\mathbf{C}^{r-1}, H))$; see [3, p. 88]. As it turns out the space $\mathbf{P}_s^{un}$ is included in the space $\mathbf{T}$, and the map $f\colon \mathbf{P}_s^{un} \to \mathscr{U}(r, d)$ is extended to the canonical map of the bundle $f_1\colon \mathbf{T} \to \mathscr{U}(r, d)$. The complement of $\mathbf{P}_s^{un}$ in $\mathbf{T}$ is fiberwise a union of two irreducible divisors. Now having the line bundle $\mathscr{F}$ on $\mathbf{P}^{un}$ which is trivial on the fibers of the map f, one can find an extension $\mathscr{F}_1$ of $\mathscr{F}$ to $\mathbf{T}$, which remains trivial on the fibers of the map f_1: for this, just take any extension of $\mathscr{F}$ and then "correct it" by an appropriate combination of the line bundles defined by the above complement divisors. For the map f_1 we can now apply see-saw principle and so there exists a line bundle $\mathscr{L}_{m,\alpha}$ on $\mathscr{U}(r, d)$ such that $\mathscr{F}_1 \cong f_1^* \mathscr{L}_{m,\alpha}$. Using the fact that the pullback map f^* is one-to-one (see [3]), we get that the restrictions of $\mathscr{L}_{m,\alpha}$ to the fibers of the map $q: \mathscr{U}(r, d) \to \mathscr{M}_g^0$ are the above canonical choices, and this concludes the proof of Theorem 1.

Remark 1. In the case of the Jacobian variety $\mathscr{J}^d$, a canonical choice of a line bundle on the fiber $J^d(C)$ has the form θ_L^α, where $L^\alpha \cong K^\beta$. Working with the symmetric product $C^{(d)} \cong \mathbf{P}(\nu_* \mathscr{P})$—assume that d is large enough—we can prove the analogue of the Lemma 10 and Theorem 3 in this case. The corresponding formulas are

(1) $u^* \theta_L \cong \omega_u - \mathscr{L}_L$,
(2) $u^* \theta_L^\alpha \cong \alpha \omega_u - \beta \mathscr{L}_K$,

where $u: C^{(d)} \to J^d(C)$ is the Abel-Jacobi map, and $\mathscr{L}_L$ is defined in a similar way as above. In the same way as before we can see now that there exists a line bundle $\mathscr{L}_\alpha$ which restricts to the above canonical choices on the fibers. This gives a proof of this fact different from that we gave in [6].

Remark 2. The following is also true. If we have a canonical way of choosing a line bundle on the general fiber of the family $q: \mathscr{U}(r, d) \to \mathscr{M}_g^0$, these choices fit together and give rise to a line bundle on $\mathscr{U}(r, d)$.

Acknowledgment

It was a pleasure for me to spend many hours during this year with Tony Pantev in studying together vector bundles and having numerous conversations about this problem. I am grateful to him for his friendship and help.

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