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DURFEE CONJECTURE AND COORDINATE FREE CHARACTERIZATION OF HOMOGENEOUS SINGULARITIES

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0. Introduction

This work is a natural continuation of our previous work [14].

The motivation of our work is to solve the Durfee conjecture. Let $f\colon (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. For $\varepsilon > 0$ suitably small and δ yet smaller, the space $V' = f^{-1}(\delta) \cap D_\varepsilon$ (where D_ε denotes the closed disk of radius ε about 0) is a real oriented four-manifold with boundary whose diffeomorphism type depends only on f. It has been proved that V' has the homotopy type of a wedge of two-spheres; the number μ of two-spheres is precisely $\dim \mathbf{C}\{x, y, z\}/(f_x, f_y, f_z)$. Let $\pi\colon (M, A) \to (V, 0)$ be a resolution of $V = \{(x, y, z) : f(x, y, z) = 0\}$ with exceptional set $A = \pi^{-1}(0)$. The geometric genus p_g of the singularity V is the dimension of $H^1(M, \mathscr{O})$. Let $\chi(A)$ be the topological Euler characteristic of A, and K^2 be the self-intersection number of the canonical divisor on M. Laufer's formula (cf. [5]) says that

$$1 + \mu = \chi(A) + K^2 + 12p_g.$$

However the formula does not provide direct comparison between μ and p_g, which are two important numerical measures of the complexity of the singularity. In 1978, Durfee [2] made the following spectacular conjecture which has remained open ever since.

Durfee conjecture. Let σ be the signature of the Milnor fiber V' above. Then

(1) $\sigma \leq 0$,

(2) $6p_g \leq \mu$ with equality only when $\mu = 0$.

In this paper we prove the Durfee conjecture in the weighted homogeneous case. In fact we show that the conjecture itself is not sharp. More precisely, we have the following theorem.

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Theorem A. *Let $(V, 0)$ be a two-dimensional isolated singularity defined by a weighted homogeneous polynomial $f(z_0, z_1, z_2) = 0$. Let μ be the Milnor number, p_g be the geometric genus and ν be the multiplicity of the singularity. Then*

$$\mu - \nu + 1 \geq 6p_g$$

with equality if and only if $(V, 0)$ is defined by the homogeneous polynomial.

In particular $6p_g = \mu$, if and only if $\nu = 1$, if and only if V is smooth at 0, if and only if $\mu = 0$.

Corollary B. *Let $(V, 0)$ be an isolated singularity defined by a weighted homogeneous polynomial. Let σ be the signature of the Milnor fibre, μ be the Milnor number of the singularity, and ν be the multiplicity of the singularity. Then*

$$\sigma \leq -\frac{\mu}{3} - \frac{2}{3}(\nu - 1).$$

The proof of Theorem A makes use of the results of W. V. D. Hodge [3] and Milnor and Orlik [7]. Hodge's result allows us to express p_g in terms of a number of positive integral points in the Newton polyhedron of f. (See §1 for a precise definition.) Thus Theorem A is related to the Main Theorem of [14]. However it does not follow directly from that theorem because the minimal weight of the variables z_i may not be an integer. We need our previous result in [12] that the multiplicity ν is $\inf\{n \in \mathbf{Z}_+ : n \geq \inf(w_0, w_1, w_2)$ where w_i is the weight of $z_i\}$ which was also independently observed by Saeki [9]. The key point there is to prove that if $w_0 \geq w_1 \geq w_2$ and w_2 is not an integer, then $w_2 = [w_2] + \beta$, $0 < \beta < 1$ and β is either w_2/w_0 or w_2/w_1. We then need to get an even sharper estimate in a particular case than those obtained in the Main Theorem of [14] (cf. Theorem 2.4).

It is well known that the Durfee conjecture is not valid for general smoothable singularities. The validity of the Durfee conjecture for hypersurface singularities has the following important implication. It gives a necessary condition for a singularity to be hypersurface.

Give a function f with an isolated singularity at the origin, it is an important question to know wheither f is a weighted homogeneous polynomial or a homogeneous polynomial after a biholomorphic change of variables. The former question was answered by a celebrated paper [10] by Saito in 1973. However the latter question has remained open ever since. In case f is a holomorphic function of three variables, the problem is solved. More precisely we have the following theorem.

Theorem C. *Let $(V, 0)$ be a two-dimensional isolated hypersurface singularity defined by $f(x, y, z) = 0$. Let μ be the Milnor number, p_g be the geometric genus, ν be the multiplicity of singularity and τ = dimension of the semi-universal deformation space of $(V, 0) = \dim \mathbf{C}\{x, y, z\}/(f, f_x, f_y, f_z)$. Then after a biholomorphic change of coordinate f is a homomogeneous polynomial if and only if $\mu - \nu + 1 = 6p_g$ and $\mu = \tau$.*

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1. Preliminaries

Let $f(z_0, z_1, \cdots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $f(z_0, z_1, \cdots, z_n) = \sum_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0} \cdots z_n^{\lambda_n}$. Recall that Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\mathbf{R}^+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbf{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\mathbf{C}^* = \mathbf{C} - \{0\}$.

Let $(V, 0)$ be an isolated hypersurface singularity defined by holomorphic function $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. Let $\pi: M \to V$ be a resolution of the singularity at 0. Define the geometric genus of the singularity $(V, 0)$ to be $p_g = \dim H^{n-1}(M, \mathcal{O})$. Let ω be a holomorphic n-form on $V - \{0\}$. ω is said to be L^2-integrable if $\int_{W-\{0\}} \omega \wedge \overline{\omega} < \infty$ for any sufficiently small relatively compact neighborhood W of 0 in V. Let $L^2(V - \{0\}, \Omega^n)$ be the set of all L^2-integral holomorphic n-forms $V - \{0\}$, which is a linear subspace of $\Gamma(V - \{0\}, \Omega^n)$. Then $p_g = \dim \Gamma(V - \{0\}, \Omega)/L^2(V - \{0\}, \Omega^n)$. (See Laufer [5] for $n = 2$ and Yau [15] for $n > 2$).

We say that a point p of the integer lattice $\mathbf{Z}^{n+1}$ in $\mathbf{R}^{n+1}$ is positive if all the coordinates of p are positive; then we have the following theorem.

Theorem 1.1. *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. Then the geometric genus $p_g = \#\{p \in \mathbf{Z}^{n+1} \cap \Gamma_-(f) : p$ is positive$\}$.*

Notice that in the above formula, positive lattice points on $\Gamma(f)$ are counted. This formula was proved by Hodge [3, §5] for $n = 2$. A corresponding result for all dimensions $n \geq 2$ is due to D. N. Bernstein, A. G. Khovanski, and/or Kouchnirenko. See the remark in [1, p. 19]. However the complete proof of the above theorem was first published by Merle and Teissier [6].

A polynomial $f(z_0, z_1, \cdots, z_n)$ is weighted homogeneous of type $(w_0, w_1, \cdots, w_n)$, where $w_0, w_1, \cdots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z^{i_1} \cdots z_n^{i_n}$ for which $i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1$.

Theorem 1.2 (Milnor and Orlik). *Let $f(z_0, z_1, \cdots, z_n)$ be a weighted homogeneous polynomial of type $(w_0, w_1, \cdots, w_n)$ with isolated singularity at the origin. Then the Milnor number $\mu = (w_0 - 1)(w_1 - 1) \cdots (w_n - 1)$.*

The signature $\sigma(M)$ of an arbitrary real oriented four-manifold M with or without boundary is defined as follows: There is a symmetric bilinear intersection pairing $(\ ,\ )$ on $H_2(M; \mathbf{R})$ defined by setting

$$(x, y) = (x' \cup y')[M]$$

where x' and y' in $H^2(M, \partial M; \mathbf{R})$ are Lefschetz duals to x and y in $H_2(M; \mathbf{R})$, and $[M] \in H_4(M, \partial M; \mathbf{R})$ is the orientation class. The bilinear form may be diagonalized, with diagonal entries $+1$, 0, and -1. The signature $\sigma(M)$ of M is the signature of this bilinear form, namely, the number of positive minus the number of negative diagonal entries.

2. Sharp upper estimate of number of integral points in tetrahedron

The following Proposition 2.1, Corollary 2.2 and Theorem 2.3 are proved in our previous paper [14].

Proposition 2.1. *Let N be the number of positive integral solutions of*

$$\frac{x}{r} + \frac{y}{s} \leq 1, \tag{2.1}$$

where $r \geq s > 0$ are real numbers; i.e. $N = \#\{(x, y) \in \mathbf{Z}_+^2 : \frac{x}{r} + \frac{y}{s} \leq 1\}$. Let $s = [s] + \alpha$ with $0 \leq \alpha < 1$, where $[s]$ denotes the largest integer which is less than or equal to s. If $s < 1$, then $N = 0$. If $s > 1$, then

$$N \leq \begin{cases} \frac{r(s-1)}{2} + \frac{r}{8s}, & (2.2) \\ \frac{r(s-1)}{2} + \frac{r-s}{2r} & \text{if } \alpha \geq \frac{s}{r} \text{ and } \frac{s}{r} > \frac{1}{2}, \quad (2.3) \\ \frac{r(s-1)}{2} & \text{if } \alpha < \frac{s}{r}. \quad (2.4) \end{cases}$$

The equality of (2.2) holds only if $s = [s] + \frac{1}{2}$ and $\frac{r}{s} \geq 2$, while the equality of (2.3) holds only if $s = [s] + \frac{r}{s}$ with $\frac{s}{r} < 1$.

Moreover if $r = s =$ integer, then $N = \frac{r(s-1)}{2}$.

Corollary 2.2. *With the notation as in Proposition* 2.1

$$N \leq \begin{cases} \frac{r(s-1)}{2} + \frac{r}{8s}, & (2.5) \\ \frac{r(s-1)}{2} + \frac{r-s}{2r} \quad \text{if } \frac{s}{r} > \frac{1}{2}. & (2.6) \end{cases}$$

The equality of (2.5) *holds only if* $s = [s] + \frac{1}{2}$ *and* $\frac{r}{s} \geq 2$. *The equality of* (2.6) *holds only if* $s = [s] + \frac{r}{s}$ *and* $\frac{s}{r} < 1$.

Theorem 2.3. *Let* $a \geq b \geq c \geq 2$ *be real numbers. Consider*

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1. \tag{2.7}$$

Let P *be the number of positive integral solutions of* (2.7); *i.e.,* $P = \#\{(x, y, z) \in \mathbf{Z}^3_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\}$. *Then*

$$6P \leq (c-1)(ab-a-b) = (a-1)(b-1)(c-1) - c + 1, \tag{2.8}$$

and the quality is attained if and only if $a = b = c = $ *integer*.

Theorem 2.4. *Let* $a \geq b \geq c \geq 2$ *be real numbers. Consider*

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1. \tag{2.9}$$

Let P *be the number of positive integral solutions of* (2.7); *i.e.,*

$$P = \#\{(x, y, z) \in \mathbf{Z}^3_+ : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\}.$$

Suppose c *is not an integer and* $c = [c] + \beta$ *where* β *is either* $\frac{c}{a}$ *or* $\frac{c}{b}$. *Then*

$$6P < (a-1)(b-1)(c-1) - c + \beta. \tag{2.10}$$

Proof. We first remark that if $a < 3$, then $b < 3$, $c < 3$, and $P = 0$. Observe that $a > 2$, and $b > 2$; otherwise $b = c = 2$, which contradicts our hypothesis that c is not an integer. To prove (2.10), we only need to show

$$(a-1)(b-1)(c-1) - c + \beta > 0. \tag{2.11}$$

For the sake of argument, let us assume that $\beta = \frac{c}{a}$. The proof of (2.11) for $\beta = \frac{c}{b}$ is similar. For fixed a, b with $a > 2$ and $b > 2$, we need to prove

$$f_{ab}(c) = (a-1)(b-1)(c-1) - c + \frac{c}{a} > 0.$$

However, $f'_{ab}(c) = (a-1)(b-1) - 1 + \frac{1}{a} \geq \frac{1}{a} > 0$. So it suffices to show $f_{ab}(2) = (a-1)(b-1) - 2 + \frac{2}{a} \geq 0$ for all $a > 2$ and $b > 2$. Let $g_b(a) = f_{ab}(2) = (a-1)(b-1) - 2 + \frac{2}{a}$. We want to show that $g_b(a) \geq 0$ for all $a > 2$ and $b > 2$. Observe that $g'_b(a) = b - 1 - \frac{2}{a^2} > 0$ for $a > 2$

and $b > 2$. Therefore it remains to show that $g_b(2) \geq 0$. However, it is clear that $g_b(2) = b - 2 > 0$. From now on, we shall assume that $a \geq 3$. There are four cases to be considered.

Case I(i). $\frac{a}{c} \geq 2$ and $0 \leq \beta < \frac{c}{b}$.

In this case $\beta = \frac{c}{a}$ and $a > b \geq c$. Following the proof of Theorem 2.3, Case I(i), it suffices to prove

$$\begin{aligned} I &= \frac{ab}{c} + a - 2b - \frac{3a}{2b} - R_1\left(\frac{c}{a}\right) - 2 + \frac{2c}{a} \\ &= \frac{ab}{c} + a - 2b - \frac{3a}{2b} + \frac{2bc}{a^2} + \frac{3b}{a} - 3 + \frac{b}{c} + \frac{3c}{2b} - 2 + \frac{2c}{a} \\ &> 0. \end{aligned} \tag{2.12}$$

For $a \geq 4$, $b \geq 2$, we assert that

$$a - \frac{3a}{2b} - 1 \geq 0 \tag{2.13}$$

and the equality is obtained if and only if $a = 4$ and $b = 2$. This can be seen as follows. Let $h_b(a) = a - \frac{3a}{2b}$. Then $h_b'(a) = 1 - \frac{3}{2b} > 0$. So $h_b(a) > h_b(4) = 4 - \frac{6}{b} - 1 = \frac{3b-6}{b} \geq 0$ for $a > 4$ and our assertion follows.

Hence we have

$$\begin{aligned} I &\geq \frac{ab}{c} + \frac{2bc}{a^2} + \frac{3b}{a} + \frac{b}{c} + \frac{3c}{2b} - \frac{c}{a} - 2b - 4 \\ &= \frac{b}{k_1 k_2} + 2k_1^2 k_2 + 3k_1 + \frac{1}{k_2} + \frac{3}{2}k_2 - k_1 k_2 - 2b - 4 \\ &= \frac{1}{k_1 k_2} I_1 \end{aligned} \tag{2.14}$$

where $k_1 = \frac{b}{a}$, $k_2 = \frac{c}{b}$ and

$$I_1 = b + 2k_1^3 k_2^2 + 3k_1^2 k_2 + k_1 + \frac{3}{2}k_1 k_2^2 - k_1^2 k_2^2 - (2b+4)k_1 k_2.$$

(2.14) is actually a strict inequality. Because if equality in (2.14) is attained, then equality in (2.13) is also attained and hence we have $a = 4$, $b = 2$. It follows that $c = 2$ which contradicts our hypothesis that c is not an integer.

It remains to show that $I_1 \geq 0$ in the region Ω show in Figure 1: $0 < k_1 \leq 1$, $0 < k_2 \leq 1$, $k_1 k_2 \leq \frac{1}{2}$.

We first see that $\partial I_1 / \partial k_2$ does not vanish in Ω. Suppose

$$\frac{\partial I_1}{\partial k_2} = 4k_1^3 k_2 + 3k_1^2 + 3k_1 k_2 - 2k_1^2 k_2 - (2b+4)k_1 = 0$$

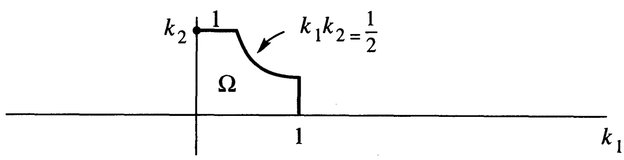

FIGURE 1

in Ω. Then

$$\begin{aligned} k_2 &= \frac{2b+4-3k_1}{3-2k_1+4k_1^2} \\ &= \frac{(2b+1)+(3-3k_1)}{4(k_1-\frac{1}{4})^2+3-\frac{1}{4}} \geq \frac{2b+1}{5} > 1, \end{aligned}$$

since $b > 2$. Hence $\partial I_1/\partial k_2$ does not vanish in Ω. Now

$$(\partial I_1/\partial k_2)(1, \frac{1}{2}) = \frac{3}{2} - 2b < 0.$$

So $\partial I_1/\partial I_2 < 0$ in Ω. It follows that in order to show that $I_1 \geq 0$ in Ω, it suffices to show that $I_1 \geq 0$ on $\{(k_1, 1) : 0 < k_1 \leq \frac{1}{2}\} \cup \{(k_1, k_2) : k_1k_2 = \frac{1}{2}, \frac{1}{2} \leq k_1 \leq 1\}$.

On $\{(k_1, 1) : 0 < k_1 \leq \frac{1}{2}\}$, $I_1 = 2k_1^3 + 2k_1^2 - (2b + \frac{3}{2})k_1 + b$. Its critical points do not lie in the interval $(0, \frac{1}{2}]$. Therefore

$$\begin{aligned} \inf_{0<k_1\leq 1/4} I_1(k_1, 1) &= \min(I_1(0, 1), I_1(\tfrac{1}{2}, 1)) \\ &= \min(b, 0) = 0. \end{aligned}$$

On $\{(k_1, k_2) : k_1k_2 = \frac{1}{2}, \frac{1}{2} \leq k_1 \leq 1\}$, $I_1 = b + \frac{1}{2}k_1 + \frac{3}{2}k_1 + k_1 + \frac{3}{4}k_2 - \frac{1}{4} - b - 2 = 3k_1 + \frac{3}{4}k_2 - \frac{9}{4} = 3k_1 + \frac{3}{8}k_1 - \frac{9}{4} = \frac{1}{8}k_1[24(k_1 - \frac{3}{2})^2 - \frac{3}{8}] \geq 0$ for $\frac{1}{2} \leq k_1 \leq 1$. This finishes the Case I(i).

Case I(ii). $\frac{a}{c} \geq 2$, $\frac{c}{b} \leq \beta < 1$.

In this case $\beta = \frac{c}{b}$. Following the proof of Theorem 2.3 Case I(ii) in [14], it suffices to prove

$$\begin{aligned} &\frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{3a}{2b}\right) + \frac{1}{12}R_2\left(\frac{c}{b}\right) \\ &\qquad < \frac{1}{6}(c-1)(a-1)(b-1) - \frac{c}{6} + \frac{c}{6b}, \end{aligned}$$

which is equivalent to

$$\frac{1}{12}\left[(c-1)\left(a - 2b + \frac{ab}{c} - \frac{3a}{2b}\right) - R_2\left(\frac{c}{b}\right) - 2 + \frac{2c}{b}\right] > 0.$$

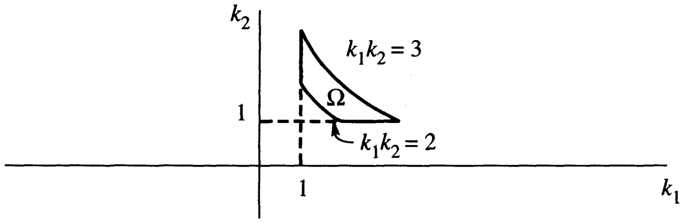

FIGURE 2

Hence we want to prove

$$I = \frac{ab}{c} + a - \frac{3a}{2b} - 2b - R_2\left(\frac{c}{b}\right) - 2 + \frac{2c}{b} > 0.$$

Observe that $R_2(\frac{c}{b}) = -\frac{2ab}{c^2}(\frac{c}{b})^3 + (\frac{3ab}{c^2} + \frac{3a}{c})(\frac{c}{b})^2 - (\frac{3a}{c} + \frac{ab}{c^2} + \frac{3a}{2b})\frac{c}{b} + \frac{3a}{2b} = -\frac{ac}{2b^2} - \frac{a}{c} + \frac{3a}{2b}$. Hence

$$I = \frac{ab}{c} + a - \frac{3a}{2b} - 2b + \frac{ac}{2b^2} + \frac{a}{c} - \frac{3a}{2b} - 2 + \frac{2c}{b} = I_1 + I_2,$$

where $I_1 = \frac{ab}{c} + a - \frac{3a}{2b} - 2b - 2 + \frac{2c}{b}$ and $I_2 = \frac{ac}{2b^2} + \frac{a}{c} - \frac{3a}{2b} = \frac{1}{2b^2c}[2(b - \frac{3}{4}c)^2 - \frac{c^2}{8}] \geq 0$. Let $k_1 = \frac{a}{b}$ and $k_2 = \frac{b}{c}$. Then

$$I_1 = bk_1k_2 + bk_1 - \frac{3k_1}{2} - 2b - 2 + \frac{2}{k_2},$$

$$I_2 = \frac{k_1}{2k_2} + k_1k_2 - \frac{3}{2}k_1.$$

If $k_1k_2 \geq 3$, then

$$\begin{aligned} I \geq I_1 &\geq 3b + \left(b - \frac{3}{2}\right)k_1 - 2b - 2 + \frac{2}{k_2} \\ &= b + \left(b - \frac{3}{2}\right)k_1 - 2 + \frac{2}{k_2} > 0, \end{aligned}$$

since $b \geq 2$. Therefore we only need to prove $I > 0$ on the region shown in Figure 2

$$\Omega : k_1k_2 < 3, \quad k_2k_2 \geq 2, \quad k_1 \geq 2, \quad k_2 \geq 1,$$

$$\begin{aligned} \frac{\partial I}{\partial k_1} &= \frac{\partial I_1}{\partial k_1} + \frac{\partial I_2}{\partial k_1} = k_2b + b - \frac{3}{2} + \frac{1}{2k_2} + k_2 - \frac{3}{2} \\ &\geq 2b - 3 + \frac{1}{2k_2} + k_2 > 0 \end{aligned}$$

in Ω. In order to show that $I > 0$ in Ω, it suffices that $I > 0$ on $\{(1, k_2) : 2 \leq k_2 < 3\} \cup \{(k_1, k_2) : k_1 k_2 = 2, \ 1 \leq k_1 \leq 2\}$.

On $\{(1, k_2) : 2 \leq k_2 < 3\}$,

$$\begin{aligned} I &= \frac{1}{k_2}\left[bk_2^2 + bk_2 - \frac{3}{2}k_2 - (2b+2)k_2 + 2\right] + \frac{1}{2k_2} + k_2 - \frac{3}{2} \\ &= \frac{1}{k_2}\left[bk_2^2 - \left(b + \frac{7}{2}\right)k_2 + 2\right] + \frac{1}{k_2}\left(\frac{1}{2} + k_2^2 - \frac{3}{2}k_2\right) \qquad (2.15) \\ &= \frac{1}{k_2}\left[(b+1)k_2^2 - (b+5)k_2 + \frac{5}{2}\right]. \end{aligned}$$

Observe that the critical point of $(b+1)k_2^2 - (b+5)k_2 + \frac{5}{2}$ is $\frac{b+5}{2(b+1)} = \frac{1}{2} + \frac{2}{b+1} < 2$. Hence

$$\inf_{2 \leq k_2 \leq 3}\left[(b+1)k_2^2 - (b+5)k_2 + \frac{5}{2}\right] \geq (b+1)2^2 - (b+5)2 + \frac{5}{2} = 2b - \frac{7}{2} > 0.$$

It follows from (2.15) that $I > 0$.

On $\{(k_1, k_2) : k_2 k_2 = 2, \ 1 \leq k_2 \leq 2\}$ we have

$$\begin{aligned} I &= 2b + \frac{2b}{k_2} - \frac{3}{k_2} - 2b - 2 + \frac{2}{k_2} + \frac{1}{k_2^2} + 2 - \frac{3}{k_2} \\ &= \frac{2b}{k_2} - \frac{4}{k_2} + \frac{1}{k_2^2} > 0. \end{aligned}$$

This finishes the proof of case I(ii).

Case II(i). $\frac{a}{c} < 2$ and $0 \leq \beta < \frac{c}{b}$.

In this case $\beta = \frac{c}{a}$. It follows that $\beta \geq \frac{c}{a} + \frac{c}{b} - 1$. We consider two subcases.

Subcase (1). $\frac{a}{c} < \frac{3}{2}$.

We are going to count N_2, the number of positive integral points on $z = [c] - 2 = c - \beta - 2 = c - \frac{c}{a} - 2$ level satisfying

$$\frac{x}{\frac{a}{c}(2 + \frac{c}{a})} + \frac{y}{\frac{b}{c}(2 + \frac{c}{a})} \leq 1,$$

i.e.,

$$\frac{x}{a} + \frac{y}{b} + \frac{c - 2 - \frac{c}{a}}{c} \leq 1. \qquad (2.16)$$

It is easy to see that $(x, y) = (1, 1)$, $(1, 2)$, and $(2, 1)$ are positive integral solutions of (2.16). For (x, y) with $4x + y = 4$, the left-hand

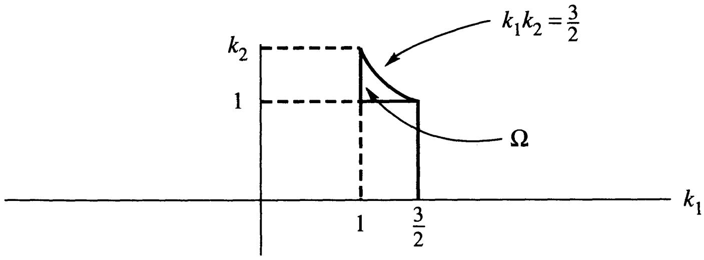

FIGURE 3

side of (2.16) becomes

$$\frac{x-1}{a} + \frac{y}{b} + 1 - \frac{2}{c} \geq \frac{x-1+y}{a} + 1 - \frac{2}{c} = \frac{3}{a} + 1 - \frac{2}{c} > 1,$$

since $\frac{a}{c} < \frac{3}{2}$. Thus we conclude that $N_2 = 3$. Following the proof of Theorem 2.3 in [14], we have

$$\begin{aligned}
P &\leq \sum_{k=1}^{[c]-1} N_k \leq \frac{1}{2}\frac{a}{c}(1+\beta)\left(\frac{b}{c}(1+\beta) - 1\right) + \frac{a-b}{2a} + N_2 \\
&\quad + \sum_{k=3}^{[c]-1}\left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(k+\beta\right) - 1) + \frac{a-b}{2a}\right] \\
&= \sum_{k=1}^{[c]-1}\left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(K+\beta) - 1\right) + \frac{a-b}{2a}\right] + 3 \\
&\quad - \frac{1}{2}\frac{a}{c}(2+\beta)\left[\frac{b}{c}(2+\beta) - 1\right] - \frac{a-b}{2a} < \frac{1}{6}(c-1)(ab-a-b) \\
&\quad + 3 - \frac{1}{2}\frac{a}{c}\left(2+\frac{c}{a}\right)\left[\frac{b}{c}\left(2+\frac{c}{a}\right) - 1\right] - \frac{a-b}{2a} \\
&\leq \frac{1}{6}(c-1)(ab-a-b) + 3 - \left(2\frac{ab}{c^2} + \frac{2b}{c} - \frac{a}{c}\right).
\end{aligned}$$

To prove the desired inequality (2.10), we only need to prove

$$I = \frac{2ab}{c^2} + \frac{2b}{c} - \frac{a}{c} - 3 - \frac{1}{6} + \frac{c}{6a} \geq 0. \tag{2.17}$$

Let $k_1 = \frac{a}{b}$ and $k_2 = \frac{b}{c}$. Then we have $1 \leq k_1 < \frac{3}{2}$, $1 \leq k_2 < \frac{3}{2}$, $1 < k_1k_2 < \frac{3}{2}$ and $I = 2k_1k_2^2 + 2k_2 - k_1k_2 - \frac{19}{6} + \frac{1}{6}k_1k_2$. We are going to show that $I > 0$ on $\Omega = \{(k_1, k_2) : 1 \leq k_1 \leq \frac{3}{2}, 1 \leq k_2 \leq \frac{3}{2}, 1 < k_1k_2 < \frac{3}{2}\} - \{(1, 1)\}$, as shown in Figure 3.

Observe that $\partial I/\partial k_1 = 2k_2^2 - k_2 - \frac{1}{6}k_1^2 k_2 > 0$ on Ω since $k_1 \geq 1$, $k_2 \geq 1$. So the minimal value of I on Ω must be reached on $k_1 = 1$. It remains to prove

$$J(k_2) = I(1, k_2) = 2k_2^2 + 2k_2 - \frac{19}{6} + \frac{1}{6k_2} = 2k_2^2 + k_2 - \frac{19}{6} + \frac{1}{6k_2} > 0$$
$$\text{for } 1 < k_2 \leq \frac{3}{2} \quad \text{and} \quad J(1) \geq 0.$$

Since $dJ/dk_2 = 4k_2 + 1 - \frac{1}{6}k_2^2 > 0$ when $k_2 \geq 1$, the minimal value of J_2 on the interval $[1, \frac{3}{2}]$ is $J(1) = I(1, 1) = 2 + 1 - \frac{19}{6} + \frac{1}{6} = 0$. Our claim that $I > 0$ on Ω is proved.

Subcase (ii). $2 > \frac{a}{c} \geq \frac{3}{2}$ and $b \geq \frac{23}{9}$.

In case II(i) of the proof of Theorem 2.3 in [14], we have already shown that

$$P \leq \frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{6(a-b)}{a}\right) + \frac{1}{12}R_3.$$

We want to show

$$\frac{1}{12}(c-1)\left(2ab - \frac{ab}{c} - 3a + \frac{6(a-b)}{a}\right) + \frac{1}{12}R_3$$
$$< \frac{1}{6}(c-1)(a-1)(b-1) - \frac{c}{6} + \frac{c}{6a}.$$

This is equivalent to showing

(2.18)
$$\frac{1}{12}(c-1)\left[\frac{ab}{c} + 3a - \frac{6(a-b)}{a}\right] - \frac{1}{12}R_3\left(\frac{c}{a}\right)$$
$$-\frac{1}{6}(c-1)(a+b) - \frac{1}{6}\left(1 - \frac{c}{a}\right) > 0.$$

Since

$$\begin{aligned}\frac{ab}{c} + 3a - \frac{6(a-b)}{a} - 2(a+b) &= \frac{ab}{c} + a - 2b - \frac{6(a-b)}{a}\\ &\geq 2(a-b) - \frac{6}{a}(a-b)\\ &\geq 0 \quad \text{for } a \geq 3,\end{aligned}$$

in order to prove (2.18), it suffices to prove

$$\text{(2.19)} \quad \frac{1}{12}\left[\frac{ab}{c} + 3a - \frac{6(a-b)}{a}\right] - \frac{1}{12}R_3\left(\frac{c}{a}\right) - \frac{1}{6}(a+b) - \frac{1}{6}\left(1 - \frac{c}{a}\right) > 0.$$

Recall

$$R_3\left(\frac{c}{a}\right) = \frac{4bc}{a^2} + 3 - \frac{3c}{a} - \frac{3b}{a} - \frac{b}{c}.$$

Let

$$I = \frac{ab}{c} + 3a - \frac{6(a-b)}{a} - R_3\left(\frac{c}{a}\right) - 2(a+b) - 2\left(1 - \frac{c}{a}\right)$$
$$= \frac{ab}{c} + a + \frac{9b}{a} + \frac{5c}{a} + \frac{b}{c} - 2b - 11 - \frac{4bc}{a^2}.$$

Let $t_1 = \frac{b}{a}$ and $t_2 = \frac{c}{b}$. Then

$$I(t_1, t_2) = \frac{b}{t_1 t_2} + \frac{b}{t_1} + 9t_1 + 5t_1 t_2 + \frac{1}{t_2} - 2b - 11 - 4t_1^2 t_2.$$

We want to show that $I(t_1, t_2) > 0$ on Ω_1 where

$$\Omega_1 = \{(t_1, t_2) : \tfrac{1}{2} \le t_1 \le 1,\ \tfrac{1}{2} \le t_2 \le 1,\ \tfrac{1}{2} < t_1 t_2 \le \tfrac{3}{2}\}.$$

Observe that on Ω_1

$$\frac{\partial I}{\partial t_2} = -\frac{b}{t_1 t_2^2} + 5t_1 - \frac{1}{t_2^2} - 4t_1^2 = -\frac{1}{t_2^2}\left(\frac{b}{t_1} + 1\right) + 5t_1 - 4t_1^2$$
$$\le -(b+1) + 5\left(\frac{5}{8}\right) - 4\left(\frac{5}{8}\right)^2 = -(b+1) + \frac{25}{16} = -b + \frac{9}{16} < 0,$$

since $b \ge 2$.

Therefore minimal value of I on Ω_1 can only be reached on $\{(t_1, 1) : \frac{1}{2} \le t_1 \le \frac{3}{2}\}$ or $\{(t_1, t_2) : t_2 t_2 = \frac{2}{3},\ \frac{2}{3} \le t_1 \le 1\}$; see Figure 4.

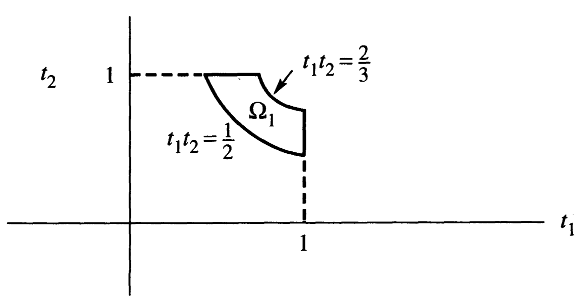

FIGURE 4

On $\{(t_1, 1) : \frac{1}{2} \leq t_1 \leq \frac{2}{3}\}$ we have

$$\begin{aligned} I(t_1, 1) &= \frac{b}{t_1} + \frac{b}{t_1} + 9t_1 + 5t_1 + 1 - 2b - 11 - 4t_1^2 \\ &= \frac{2b}{t_1} - 4t_1^2 + 14t_1 - 2b - 10, \\ \frac{dI(t_1, 1)}{dt_1} &= -\frac{2b}{t_1^2} - 8t_1 + 14, \\ \frac{d^2I(t_1, 1)}{dt_1^2} &= \frac{4b}{t_1^3} - 8 > 4b - 8 \geq 0 \quad \text{for } \frac{1}{2} \leq t_1 \leq \frac{2}{3}. \end{aligned}$$

It follows that for $\frac{1}{2} \leq t_1 \leq \frac{2}{3}$

$$\frac{dI(t_1, 1)}{dt_1} \leq \left.\frac{dI(t_1, 1)}{dt_1}\right|_{t_1=2/3} = -\frac{27b + 52}{6} < 0.$$

Hence $\inf_{1/2 \leq t_1 \leq 2/3} I(t_1, 1) = I(\frac{2}{3}, 1) = 3b - \frac{16}{9} + \frac{28}{3} - 2b - 10 = b - \frac{22}{9} > 0$ by the hypothesis $b \geq \frac{23}{9}$. On $\{(t_1, t_2) : t_1 t_2 = \frac{2}{3}, \frac{2}{3} \leq t_1 \leq 1\}$

$$\begin{aligned} I &= \frac{3}{2}b + \frac{b}{t_1} + 9t_1 + \frac{10}{3} + \frac{3}{2}t_1 - 2b - 11 - \frac{8}{3}t_1 \\ &= \frac{b}{t_1} - \frac{b}{2} + \frac{47}{6}t_1 - \frac{23}{3}. \end{aligned} \tag{2.20}$$

It is easy to check that $\inf(\frac{b}{t_1} + \frac{47}{6}t_1) = \frac{\sqrt{282b}}{3}$. We claim that for $\frac{23}{9} \leq b \leq 92$ we have

$$\sqrt{\frac{282b}{3}} > \frac{b}{2} + \frac{23}{3}, \qquad \text{i.e., } 9b^2 - 852b + 2116 < 0.$$

This can be seen easily by checking that $\frac{23}{9}$ and 92 are lying between the roots of $9b^2 - 852b + 2116 = 0$. We have shown for $\frac{23}{9} \leq b \leq 92$, the right-hand side of (2.20) is strictly larger than zero. On the other hand for $b > 92$, the right-hand side of (2.20)

$$I = \frac{b}{t_1} - \frac{b}{2} + \frac{47}{6}t_1 - \frac{23}{3} > \frac{b}{2} - \frac{23}{3} > 0.$$

This completes of the proof that $I > 0$ on Ω_1.

Subcase (iii). $2 > \frac{a}{c} \geq \frac{3}{2}$ and $b < \frac{23}{6}$.

We first observe that $b < \frac{23}{9}$ implies $c < \frac{23}{9}$ and $\frac{a}{c} < 2$ implies $a < \frac{46}{9}$. On the other hand, $c < \frac{23}{9}$ implies $2 < c < 3$. It follows that we have

$$\begin{aligned} c &= 2 + \frac{c}{a} = 2 + \frac{2}{a} + \frac{c}{a^2} \\ &= 2\left(1 + \frac{1}{a} + \cdots + \frac{1}{a^m} + \cdots\right) = \frac{2}{1 - \frac{1}{a}} \\ &> \frac{1}{1 - \frac{9}{46}} \quad \text{since } a < \frac{46}{9} \\ &= \frac{92}{35}. \end{aligned}$$

But we have already shown $c < \frac{23}{9} < \frac{92}{35}$. The contradiction means that subcase (iii) cannot occur.

Case II(ii). $\frac{a}{c} < 2$ and $1 > \beta \geq \frac{c}{b}$.

In this case $\beta = \frac{c}{b}$. We are going to count N_1, the number of positive integral points on $z = [c] - 1 = c - \beta - 1 = c - \frac{c}{b} - 1$ level satisfying

$$\frac{c}{\frac{a}{c}(1 + \frac{c}{a})} + \frac{y}{\frac{b}{c}(1 + \frac{c}{a})} \leq 1,$$

i.e.,

$$\frac{x}{a} + \frac{y}{b} + \frac{c - 1 - \frac{c}{b}}{c} \leq 1. \tag{2.21}$$

Setting $(x, y) = (1, 1)$ in (2.21), we have

$$\frac{1}{a} + \frac{1}{b} + 1 - \frac{1}{c} - \frac{1}{b} = 1 - \frac{1}{c} + \frac{1}{a} < 1,$$

since $a > c$. So $(1, 1)$ is a solution of (2.21). For (x, y) with $x + y \geq 3$, the left-hand side of (2.21)

$$\begin{aligned} \frac{x}{a} + \frac{y}{b} + 1 - \frac{1}{c} - \frac{1}{b} &= \frac{x}{a} + \frac{y-1}{b} + 1 - \frac{1}{c} \\ &\geq \frac{x + y - 1}{a} + 1 - \frac{1}{c} \geq \frac{2}{a} + 1 - \frac{1}{c} > 1, \end{aligned}$$

since $\frac{a}{c} < 2$. Therefore we conclude that N_1 is exactly one. Following the proof of Theorem 2.3 in [14], we have

$$\begin{aligned} P &\le \sum_{k=1}^{[c]-1} N_k \le N_1 + \sum_{k=2}^{[c]-1} \left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(k+\beta)-1\right)+\frac{a-b}{2a}\right] \\ &= N_1 - \frac{1}{2}\frac{a}{c}(1+\beta)\left[\frac{b}{c}(1+\beta)-1\right] - \frac{a-b}{2a} \\ &\quad + \sum_{k=1}^{[c]-1} \left[\frac{1}{2}\frac{a}{c}(k+\beta)\left(\frac{b}{c}(k+\beta)-1\right)+\frac{a-b}{2a}\right] \\ &\le 1 - \frac{1}{2}\frac{a}{c}(1+\beta)\left[\frac{b}{c}(1+\beta)-1\right] - \frac{a-b}{2a} + \frac{(c-1)(ab-a-b)}{6}. \end{aligned}$$

To prove the desired inequality (2.10), we only need to prove

$$\frac{1}{2}\frac{a}{c}\left(1+\frac{c}{b}\right)\left[\frac{b}{c}\left(1+\frac{c}{b}\right)-1\right]+\frac{a-b}{2a}-1 > \frac{1}{6}\left(1-\frac{c}{b}\right). \tag{2.22}$$

Left-hand side of (2.22)

$$\begin{aligned} &= \frac{1}{2}\left(\frac{a}{c}+\frac{a}{b}\right)\frac{b}{c}+\frac{a-b}{2a}-1 \\ &= \frac{ab}{2c^2}+\frac{a}{2c}+\frac{1}{2}-\frac{b}{2a}-1 > \frac{ab}{2c^2}-\frac{b}{2a} \quad \left(\text{since } \frac{a}{c}>1\right) \\ &\ge \frac{ab}{2c^2}-\frac{1}{2} \ \left(\text{since } \frac{a}{b}\ge 1\right). \end{aligned}$$

It suffices to prove

$$I = \frac{ab}{2c^2}-\frac{1}{2}-\frac{1}{6}\left(1-\frac{c}{b}\right) = \frac{ab}{2c^2}+\frac{c}{6b}-\frac{2}{3} > 0.$$

Let $k_1 = \frac{a}{b}$ and $k_2 = \frac{b}{c}$. Then $1 \le k_1 < 2$, $1 \le k_2 < 2$, $1 < k_1k_2 < 2$ and

$$I = \frac{1}{2}k_1k_2^2 + \frac{1}{6k_2} - \frac{2}{3}.$$

We want to show that $I > 0$ on

$$\Omega_2\{(k_1, k_2) : 1 \le k_1 < 2,\ 1 \le k_2 < 2,\ 1 < k_1k_2 < 2\};$$

see Figure 5 (next page).

Since $\partial I/\partial k_1 = \frac{1}{2}k_2^2 > 0$ on Ω_2, we only need to check that $I > 0$ on $\{(1, k_2) : 1 < k_2 \le 2\}$ and $I(1, 1) = 0$.

$$I(1, k_2) = \frac{1}{6k_2}(3k_2^3 - 4k_2 + 1).$$

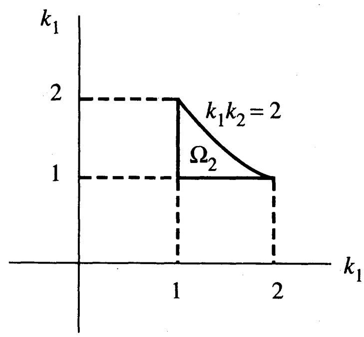

FIGURE 5

So it remains to prove

$$3k_2^3 - k_2 + 1 \begin{cases} > 0 & \text{for } 1 < k_2 < 2, \qquad (2.23) \\ = 0 & \text{for } k_2 = 1. \qquad (2.24) \end{cases}$$

(2.24) is obviously true. Since the derivative of $3k_2^3 - 4k_2 + 1$ is $9k_2^2 - 4$ which is strictly bigger than zero for $1 \leq k_2 < 2$, (2.23) follows from (2.24). This completes the proof.

3. Application to Durfee conjecture for weighted homogeneous hypersurface singularities and coordinate free characterization of homogeneous hypersurface singularities

Let $V = \{z \in \mathbf{C}^3 : f(z) = 0\}$ be the germ of a complex hypersurface with an isolated singular point at the origin. For $\varepsilon > 0$ suitably small and δ yet smaller, the space $V' = f^{-1}(\delta) \cap D_\varepsilon$ (where D_ε denotes the closed disk radius ε about 0) is a real oriented four-manifold with boundary whose diffeomorphism type depends only on V. It has been proved that V' has the homotopy type of a wedge of two-spheres; the Milnor number μ of two-spheres is readily computable. Let σ be the signature of the intersection pairing on the two-dimensional homology of the manifold V'. Let p_g be the geometric genus of the singularity $(V, 0)$; i.e., $p_g = \dim H^1(\widetilde{V}, \mathscr{O}_{\widetilde{V}})$ where $\pi\colon \widetilde{V} \to V$ is a resolution of V. In [2], Durfee conjectured that the signature of the smoothing V' is nonpositive. This conjecture is implied by his other conjecture which says that $6p_g \leq \mu$ with equality only when $\mu = 0$. These conjectures have been open for more than eleven years, although there is an example in [11] of a nonhypersurface singularity (the quotient of $xy^3 + yz^3 + zx^3 = 0$ by a group of order 7) which has a smoothing with positive signature. The purpose of this section is to prove an inequality for $(V, 0)$ with $\mathbf{C}^*$-action, which implies the above conjectures automatically. We also give a coordinate free characterization when $(V, 0)$ is defined by homogeneous polynomial.

A polynomial $f(z_1, z_1, z_2)$ is weighted homogeneous of type (w_0, w_1, w_1), where (w_0, w_1, w_2) are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} z_2^{i_2}$ for which $i_0/w_0 + i_1/w_1 + i_2/w_2 = 1$. (w_0, w_1, w_2) is called the weights of f.

Lemma 3.1. *Let w_0, w_1 and w_2 be the weights of a weighted homogeneous polynomial $f(z_0, z_1, z_2)$ such that $f(z_0, z_1, z_2)$ has an isolated critical point at the origin. Suppose $w_{i_0} \geq w_{i_1} \geq w_{i_2}$ and w_{i_2} is not an integer, where $\{i_0, i_1, i_2\} = \{0, 1, 2\}$. Let $w_{i_2} = [w_{i_2}] + \beta$ with $0 < \beta < 1$. Then β is either w_{i_2}/w_{i_0} or w_{i_2}/w_{i_1}.*

Proof. By a result of Orlik and Wagreich [8] (see also Xu-Yau [12]) $f(z_0, z_1, z_2)$ can be deformed to one of the following seven classes without changing the weights at all.

Class 1. $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} + z_2^{a_2}$.

Clearly this case cannot happen under the hypothesis of the weights of f.

Class 2. $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}$, $a_1 > 1$.

In this case $w_0 = a_0$, $w_1 = a_1$ and $w_2 = a_1 a_2/(a_1 - 1)$. The only possible noninteger weight is $w_2 = a_1 a_2/(a_1 - 1) = a_2 + a_2/(a_1 - 1)$. By our hypothesis, $w_1 > w_2$ since w_1 is an integer and w_2 is not an integer. In particular, we have $a_1 > a_1 a_2/(a_1 - 1)$, i.e., $a_2/(a_1 - 1) < 1$. Hence we deduce that $[w_2] = a_2$ and $\beta = a_2/(a_1 - 1) = w_2/w_1$.

Class 3. $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} z_2 + z_1 z_2^{a_2}$, $a_1 > 1$, $a_2 > 1$.

In this case, we have $w_0 = a_0$, $w_1 = (a_1 a_2 - 1)/(a_2 - 1)$ and $w_2 = (a_1 a_2 - 1)/(a_1 - 1)$. By symmetry, we may assume $w_1 \geq w_2 \notin \mathbf{Z}$.

We claim that $w_1 \neq w_2$. If $w_1 = w_2$, then $a_1 = a_2$ and it follows that $w_1 = w_2 = a_1 + 1 = a_2 + 1 \in \mathbf{Z}$, which contradicts our hypothesis. Hence we conclude that $w_1 > w_2 \notin \mathbf{Z}$,

$$w_2 = \frac{a_1 a_2 - 1}{a_1 - 1} = a_2 + \frac{a_2 - 1}{a_1 - 1}.$$

Since $w_1 > w_2$, we have $a_2 - 1 < a_1 - 1$. Therefore $[w_2] = a_2$ and $\beta = (a_2 - 1)/(a_1 - 1) = w_2/w_1$.

Case 4. $f(z_0, z_1, z_2) = z_0^{a_0} + z_1^{a_1} z_2 + z_0 z_2^{a_2}$, $a_0 > 1$.

In this case, we have $w_0 = a_0$, $w_1 = a_0 a_1/(a_0 - 1)$,

$$w_2 = a_0 a_1 a_2/(a_0 a_1 - a_0 + 1).$$

We need to consider two cases.

(i) $w_1 > w_2 \notin \mathbf{Z}$. As

$$w_2 = a_0a_1a_2/(a_0a_1 - a_0 + 1) = a_2 + a_2(a_0 - 1)/(a_0a_1 - a_0 + 1),$$

we have

$$w_1 > w_2 \Leftrightarrow \frac{a_0a_1}{a_0 - 1} > \frac{a_0a_1a_2}{a_0a_1 - a_0 + 1} \Leftrightarrow \frac{a_2(a_0 - 1)}{a_0a_1 - a_0 + 1} < 1.$$

Thus $[w_2] = a_2$ and $\beta = a_2(a_0 - 1)/(a_0a_1 - a_0 + 1) = w_2/w_1$.

(ii) $w_2 \geq w_1 \notin \mathbf{Z}$.

Since $w_0 \in \mathbf{Z}$ and $w_1 \notin \mathbf{Z}$, we have $w_0 = a_0 > w_1 = a_0a_1/(a_0 - 1)$. This implies $a_1/(a_0 - 1) < 1$. As $w_1 = a_0a_1/(a_0 - 1) = a_1 + a_1/(a_0 - 1)$, we have $[w_1] = a_1$ and $\beta = a_1/(a_0 - 1) = w_1/w_0$.

Class 5. $f(z_0, z_1, z_2) = z_0^{a_0}z_1 + z_1^{a_1}z_2 + z_0z_2^{a_2}$.

In this case we have

$$w_0 = \frac{a_0a_1a_2 + 1}{a_1a_2 - a_2 + 1}, \quad w_1 = \frac{a_0a_1a_2 + 1}{a_0a_2 - a_0 + 1}, \quad w_2 = \frac{a_0a_1a_2 + 1}{a_0a_1 - a_1 + 1}.$$

Without loss of generality, we may assume that w_2 is the minimal weight and $w_2 \notin \mathbf{Z}$. Observe that

$$w_2 = \frac{a_0a_1a_2 + 1}{a_0a_1 - a_1 + 1} = a_2 + \frac{a_1a_2 + a_2 + 1}{a_0a_1 - a_1 + 1}.$$

We claim that $w_0 \neq w_2$. If $w_0 = w_2$, then $w_0/w_2 = 1$ which implies $(a_0a_1 - a_1 + 1)/(a_1a_2 - a_2 + 1) = 1$. Thus $w_2 = a_2 + 1 \in \mathbf{Z}$ which contradicts our hypothesis. So we conclude that $w_0 > w_2$ and hence $(a_1a_2 - a_2 + 1)/(a_0a_1 - a_1 + 1) < 1$. It follows that $[w_2] = a_2$ and $\beta = (a_1a_2 - a_2 + 1)/(a_0a_1 - a_1 + 1) = w_2/w_0$.

Class 6. $f(z_0, z_1, z_2) = z_0^{a_0} + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ with $(a_0 - 1)(a_1b_2 + a_2b_1) = a_0a_1a_2$.

In this case $w_0 = a_0$, $w_1 = a_0a_1/(a_0 - 1)$ and $w_2 = a_0a_2/(a_0 - 1)$. Without loss of generality, we may assume that w_2 is the minimal weight and $w_2 \notin \mathbf{Z}$. Hence we have $w_0 > w_2 = a_0a_2/(a_0 - 1) = a_2 + a_2(a_0 - 1)$. $w_0 = a_0 > w_2 = a_0a_2/(a_0 - 1)$ implies $a_2/(a_0 - 1) < 1$. Hence $[w_2] = a_2$ and $\beta = a_2/(a_0 - 1) = w_2/w_0$.

Class 7. $f(z_0, z_1, z_2) = z_0^{a_0}z_1 + z_0z_1^{a_1} + z_0z_2^{a_2} + z_1^{b_1}z_2^{b_2}$ with $(a_0 - 1)(a_1b_2 + a_2b_1) = a_2(a_0a_1 - 1)$.

In this case,

$$w_0 = \frac{a_0a_1 - 1}{a_1 - 1}, \quad w_1 = \frac{a_0a_1 - 1}{a_0 - 1}, \qquad w_2 = \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)}.$$

There are three cases to consider.

Case (i). w_0 is the minimal weight and $w_0 \notin \mathbf{Z}$. We claim that $w_0 < w_1$. If $w_0 = w_1$, then $a_0 = a_1$ and hence $w_0 = w_1 = a_0 + 1 \in \mathbf{Z}$. This contradicts the hypothesis. Hence we have $w_1 > w_0$, i.e.,

$$\frac{a_0a_1 - 1}{a_0 - 1} > \frac{a_0a_1 - 1}{a_1 - 1} \Leftrightarrow (a_0a_1 - 1)(a_0 - 1) < (a_1 - 1)(a_0a_1 - 1)$$
$$\Leftrightarrow (a_0a_1 - 1)(a_0 - a_1) < 0 \Leftrightarrow a_1 > a_0 \Leftrightarrow \frac{a_0 - 1}{a_1 - 1} < 1.$$

Now $w_0 = (a_0a_1 - 1)/(a_1 - 1) = a_0 + (a_0 - 1)(a_1 - 1)$. Thus $[w_0] = a_0$ and $\beta = (a_0 - 1)/(a_1 - 1) = w_0/w_1$.

Case (ii). w_1 is the minimal weight and $w_1 \notin \mathbf{Z}$.

Similar argument as in case (i) will lead to $w_0 > w_1$, $[w_1] = a_1$, $\beta = (a_1 - 1)/(a_0 - 1) = w_1/w_0$.

Case (iii). w_2 is the minimal weight and $w_2 \notin \mathbf{Z}$.

Without loss of generality, we may assume that $w_0 > w_2$ and $w_1 > w_2$, otherwise we would be in case (i) or case (ii) again.

$$w_0 > w_2 \Rightarrow \frac{a_0a_1 - 1}{a_1 - 1} > \frac{a_2(a_0a_1 - 1)}{a_1(a_0 - 1)} \Rightarrow 1 > \frac{a_2(a_1 - 1)}{a_1(a_0 - 1)}.$$

Now $w_2 = a_2(a_0a_1 - 1)/a_1(a_0 - 1) = a_2 + +a_2(a_1 - 1)/a_1(a_0 - 1)$. Hence $[w_2] = a_2$ and $\beta = a_2(a_1 - 1)/a_1(a_0 - 1) = w_2/w_0$.

Theorem 3.2. *Let $(V, 0)$ be a two-dimensional isolated singularity defined by a weighted homogeneous polynomial $f(z_0, z_1, z_2) = 0$. Let μ be the Milnor number, p_g be the geometric genus, and ν be the multiplicity of the singularity. Then*

$$\mu - \nu + 1 \geq 6p_g \tag{3.1}$$

with equality if and only if $(V, 0)$ is defined by homogeneous polynomial.

Proof. Let w_0, w_1, w_2 be the weights of x, y and z respectively so that $f(x, y, z)$ is a weighted homogeneous polynomial. By the theorem of Saito [10], we may assume without loss of generality that $w_0 \geq w_1 \geq w_2 \geq 2$. In view of Theorem 1.1, p_g is precisely the number of positive integral solutions of

$$\frac{x}{w_0} + \frac{y}{w_1} + \frac{z}{w_2} \leq 1;$$

i.e., $p_g = \#\{(x, y, z) \in \mathbf{Z}_+^3 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\}$. On the other hand, a result of Milnor and Orlik says that $\mu = (w_0 - 1)(w_1 - 1)(w_2 - 1)$. Therefore Theorem 2.3 implies that

$$6p_g \leq \mu - w_2 + 1 \tag{3.2}$$

with equality if and only if $w_0 = w_1 = w_2 =$ integer. Recall that $\nu = \inf\{n \in \mathbf{Z}_+ : n \geq \inf(w_0, w_1 w_2)\}$ (cf. [12]). If w_2 is an integer, then $\nu = w_2$ and (3.1) follows directly from (3.2). If w_2 is not an integer, then write $w_2 = [w_2] + \beta$ with $0 < \beta < 1$. By Lemma 3.1, β is either w_2/w_0 or w_2/w_1. From Theorems 1.1 and 2.4 it follows that

$$6p_g < (w_0 - 1)(w_1 - 1)(w_2 - 1) - w_2 + \beta = \mu - [w_2]. \tag{3.3}$$

Since $\nu = [w_2] + 1$, in view of (3.3), we have

$$6p_g < \mu - \nu + 1.$$

Now we have proved (3.1) and also that the equality in (3.1) holds if $(V, 0)$ is defined by homogeneous polynomial.

It remains to prove that if $(V, 0)$ is defined by homogeneous polynomial of degree ν, then the equality in (3.1) holds, i.e., $\mu - \nu + 1 = 6p_g$. By a result of [12], $(V, 0)$ has the same topological type as $(W, 0)$ where $W = \{(z_0, z_1, z_2) \in \mathbf{C}^3 : z_0^\nu + z_1^\nu + z_2^\nu = 0\}$. On the other hand by a result of [16], $(V, 0)$ and $(W, 0)$ have the same p_g, which is equal to $\frac{1}{6}\nu(\nu - 1)(\nu - 2)$. Therefore

$$\begin{aligned} \mu - \nu + 1 &= (\nu + 1)^3 - \nu + 1 = (\nu - 1)[(\nu - 1)^2 - 1] \\ &= \nu(\nu - 1(\nu - 2) = 6p_g. \end{aligned}$$

Theorem 3.3. *Let $(V, 0)$ be a two-dimensional isolated hypersurface singularity defined by $f(x, y, z) = 0$. Let μ be the Milnor number, p_g be the geometric genus, ν be the multiplicity of singularity, and $\tau = \dim \mathbf{C}\{z, y, z\}/(f, f_x, f_y, f_z)$. Then after a biholomorphic change of coordinate f is a homogeneous polynomial if and only if $\mu - \nu + 1 = 6p_g$ and $\mu = \tau$.*

Proof. Recall by a theorem of [10], after a biholomorphic change of coordinate f is a weighted homogeneous polynomial if and only if $\mu = \tau$. The result follows immediately from Theorem 3.2.

Corollary 3.4 (Durfee conjecture for weighted homogeneous singularity). *Let $(V, 0)$ be an isolated singularity defined by weighted homogeneous polynomial $f(x, y, z) = 0$. Let p_g be the geometric genus, and μ be the Milnor number of the singuarility. Then $6p_g \leq \mu$ with equality if and only if $\mu = 0$, if and only if V is smooth.*

Proof. By Theorem 3.2, we have $6p_g \leq \mu - \nu + 1 \leq \mu$. If $6p_g = \mu$, then $\nu \leq 1$, This implies $\nu = 1$ and V is smooth. In particular $\mu = 0$. Conversely if $\mu = 0$, then V is smooth and hence $p_g = 0$.

Corollary 3.5. *Let $(V, 0)$ be an isolated singularity defined by a weighted homogeneous polynomial. Let σ be the signature of the Milnor fibre, μ be the Milnor number of singularity, and ν be the multiplicity of the singularity. Then*

$$\sigma \leq -\frac{\mu}{3} - \frac{2}{3}(\nu - 1).$$

Proof. Observe that

$$\mu = \sigma_+ + \sigma_0 + \sigma_- \tag{3.4}$$

and

$$\sigma = \sigma_+ - \sigma_-, \tag{3.5}$$

where σ_+, σ_0 and σ_- are respectively the number of positive, zero and negative eigenvalues of the intersection form of the Milnor fiber. By Proposition 3.1 of [2] we have

$$2p_g = \sigma_+ + \sigma_0. \tag{3.6}$$

(3.4) and (3.6) imply $\sigma_- = \mu - 2p_g$ and $\sigma_+ = 2p_g - \sigma_0$. Putting these into (3.5) gives

$$\begin{aligned} \sigma = 4p_g - \sigma_0 - \mu &\leq 4p_g - \mu \\ &\leq \frac{4}{6}(\mu - \nu + 1) - \mu = -\frac{\mu}{3} - \frac{2}{3}(\nu - 1). \end{aligned}$$

Corollary 3.6. *Let $(V, 0)$ be an isolated singuarility defined by a weighted homogeneous polynomial. Let $\pi: (M, A) \to (V, 0)$ be a resolution of singularity where $A = \pi^{-1}(0)$ is the exceptional set. Let $\chi(A)$ denote the topological Euler characteristic of the exceptional set A, and let K^2 denote the self-intersection of the canonical divisor on M. Then*

$$-1 - \mu + 2\nu \leq K^2 + \chi(A) \leq 1 + \mu, \tag{3.7}$$

where μ and ν are Milnor numbers and multiplicity of the singularity respectively.

Proof. The right-hand inequality of (3.7) follows trivially from the following Laufer's formula:

$$1 + \mu = 12p_g + \chi(A) + K^2 . \tag{3.8}$$

On the other hand, by Theorem 3.2 we have

$$1 + \mu \le 2(\mu - \nu + 1) + K^2 + \chi(A),$$

which implies

$$K^2 + \chi(A) \ge -1 - \mu + 2\nu.$$

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