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A NUMERICAL CRITERION FOR VERY AMPLE LINE BUNDLES

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Abstract

Let X be a projective algebraic manifold of dimension n and let L be an ample line bundle over X. We give a numerical criterion ensuring that the adjoint bundle $K_X + L$ is very ample. The sufficient conditions are expressed in terms of lower bounds for the intersection numbers $L^p \cdot Y$ over subvarieties Y of X. In the case of surfaces, our criterion gives universal bounds and is only slightly weaker than I. Reider's criterion. When $\dim X \geq 3$ and $\operatorname{codim} Y \geq 2$, the lower bounds for $L^p \cdot Y$ involve a numerical constant which depends on the geometry of X. By means of an iteration process, it is finally shown that $2K_X + mL$ is very ample for $m \geq 12n^n$. Our approach is mostly analytic and based on a combination of Hörmander's L^2 estimates for the operator $\overline{\partial}$, Lelong number theory and the Aubin-Calabi-Yau theorem.

1. Introduction

Let L be a holomorphic line bundle over a projective algebraic manifold X of dimension n. We denote the canonical line bundle of X by K_X and use an additive notation for the group $\operatorname{Pic}(X) = H^1(X, \mathcal{O}^*)$. The original motivation of this work was to study the following tantalizing conjecture of Fujita [23]: If $L \in \operatorname{Pic}(X)$ is ample, then $K_X + (n+2)L$ is very ample; the constant $n+2$ would then be optimal since $K_X + (n+1)L = \mathcal{O}_X$ is not very ample when $X = \mathbf{P}^n$ and $L = \mathcal{O}(1)$. Although such a sharp result seems at present out of reach, a consequence of our results will be that $2K_X + mL$ is always very ample for L ample and m larger than some universal constant depending only on n.

Questions of this sort play a very important role in the classification theory of projective varieties. In his pioneering work [9], Bombieri proved the existence of pluricanonical embeddings of low degree for surfaces of general type. More recently, for an ample line bundle L over an algebraic surface S, I. Reider [39] obtained a sharp numerical criterion ensuring that the adjoint line bundle $K_X + L$ is very ample; in particular,

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K_S+3L is always spanned, and K_S+4L very ample. Reider's method was further developed by Catanese [11], Sakai [40] and Beltrametti-Francia-Sommese [6], who studied the existence of higher order embeddings via s-jets. Reider's approach is based on the construction of rank two vector bundles associated to some 0-cycles in special position with respect to the linear system $|K_S + L|$ and a use a Bogomolov's inequality for stable vector bundles. Unfortunately, these methods do not apply in dimension ≥ 3 and no similar general result was available. In a somewhat different context, Fujita [22] proved that $K_X+(n+2)L$ is always ample. This result is obtained via Mori's theory of extremal rays [35] and the cone theorem of Kawamata (cf. [27], [28]), but the arguments are purely numerical and give apparently no insight on the very ample property.

Our purpose here is to explain a completely different analytic approach which is applicable in arbitrary dimension. Let us first recall a few usual notations that will be used constantly in the sequel:

$$L_1 \cdots L_p \cdot Y = \int_Y c_1(L_1) \wedge \cdots \wedge c_1(L_p) \tag{1.1}$$

denotes the intersection product of p line bundles $L_1, \cdots, L_p$ over a p-dimensional subvariety $Y \subset X$. In case $L_1 = \cdots = L_p$ we write instead $L^p \cdot Y$ and in case $Y = X$ we omit Y in the notation. Similar notations will be used for divisors. Recall that a line bundle (or a $\mathbb{R}$-divisor) L over X is said to be *numerically effective, nef* for short, if $L \cdot C \geq 0$ for every curve $C \subset X$; in this case L is said to be *big* if $L^n > 0$. More generally, a vector bundle E is said to be nef if the associated line bundle $\mathcal{O}_E(1)$ is nef over $P(E^*)$ = projective space of hyperplanes in E; any vector bundle E such that some symmetric power $S^m E$ is spanned by its global sections is nef. In this context, we shall prove

Main Theorem. *Let X be a projective n-fold and let L be a big nef line bundle over X. Suppose that there is a number $a \geq 0$ such that $TX \otimes \mathcal{O}(aL)$ is nef. Then $K_X + L$ is spanned at each point of a given subset Ξ of X (resp. separates all points in Ξ, resp. generates s-jets at any point of Ξ) provided that $L^n > \sigma_0$ with $\sigma_0 = n^n$ (resp. $\sigma_0 = 2n^n$, resp. $\sigma_0 = (n+s)^n$), and that there exists a sequence $0 = \beta_1 < \cdots < \beta_n \leq 1$ such that any subvariety $Y \subset X$ of codimension $p = 1, 2, \cdots, n-1$ intersecting Ξ satisfies*

$$L^{n-p} \cdot Y > (\beta_{p+1} - \beta_1)^{-1} \cdots (\beta_{p+1} - \beta_p)^{-1} \sum_{0 \leq j \leq p-1} S_j^p(\beta) a^j \sigma_{p-j}$$

with $S_0^p(\beta) = 1$, $S_j^p(\beta)$ = elementary symmetric function of degree j in

$\beta_1, \cdots, \beta_p$ *and*

$$\sigma_p = \left(1 - \left(1 - \frac{\sigma_0}{L^n}\right)^{p/n}\right) L^n, \qquad \sigma_p \in]\sigma_0 p/n, \sigma_0[.$$

The expression "separation of points" used here includes infinitesimal separation, that is, generation of 1-jets at each point (the constant $\sigma_0 = (n+1)^n$ corresponding to $s = 1$ can therefore be replaced by the smaller value $2n^n$). In fact, our proof also gives sufficient conditions for the generation of jets corresponding to arbitrary 0-dimensional subschemes $(\Xi, \mathscr{O}_\Xi)$ of X, simply by changing the value of σ_0; for example, if $(\Xi, \mathscr{O}_\Xi)$ is a local complete intersection, the constant σ_0 can be taken equal to $n^n h^0(\Xi, \mathscr{O}_\Xi)$; unfortunately, this value is in general far from being optimal. Notice that the number a involved in the hypothesis on TX need not be an integer nor even a rational number: the hypothesis then simply means that any real divisor associated to $\mathscr{O}_{TX}(1) + a\pi^\star L$ is nef over $P(T^\star X)$.

As the notation are rather complicated, it is certainly worth examining the particular case of surfaces and 3-folds. If X is a surface, we have $\sigma_0 = 4$ (resp. $\sigma_0 = 8$, resp. $\sigma_0 = (2+s)^2$), and we take $\beta_1 = 0$, $\beta_2 = 1$. This gives only two conditions, namely

$$L^2 > \sigma_0, \qquad L \cdot C > \sigma_1 \tag{1.2}$$

for every curve C intersecting Ξ. In that case, the proof shows that the assumption on the existence of a is unnecessary. These bounds are not very far from those obtained with Reider's method, although they are not exactly as sharp. If X is a 3-fold, we have $\sigma_0 = 27$ (resp. $\sigma_0 = 54$, resp. $\sigma_0 = (3+s)^3$), and we take $\beta_1 = 0 < \beta_2 = \beta < \beta_3 = 1$. Therefore our condition is that there exists $\beta \in]0, 1[$ such that

$$L^3 > \sigma_0, \quad L^2 \cdot S > \beta^{-1}\sigma_1, \quad L \cdot C > (1-\beta)^{-1}(\sigma_2 + \beta a \sigma_1) \tag{1.3}$$

for every curve C or surface S intersecting Ξ.

In general, we measure the "amount of ampleness" of a nef line bundle L on a subset $\Xi \subset X$ by the number

$$\mu_\Xi(L) = \min_{1 \le p \le n} \ \min_{\dim Y = p,\, Y \cap \Xi \ne \varnothing} (L^p \cdot Y)^{1/p}, \tag{1.4}$$

where Y runs over all p-dimensional subvarieties of X intersecting Ξ. The Nakai-Moishezon criterion tells us that L is ample if and only if $\mu_X(L) > 0$. An effective version of this criterion can be easily deduced from the Main Theorem: in fact, a suitable choice of the constants β_p in terms of a, σ_0 and $\mu_\Xi(L)$ yields

Corollary 1. *Let L be a big nef line bundle on X such that $TX \otimes \mathscr{O}(aL)$ is nef for some $a \geq 0$, and let Ξ be an arbitrary subset of X. Then the line bundle $K_X + mL$ spans (resp. separates points, resp. generates s-jets) on Ξ as soon as*

$$m > \frac{1}{\mu_{\Xi}(L)} \max \left\{ B_n\sigma_0,\ (B_n\sigma_0)^{1/(n-k)} \times (a\mu_{\Xi}(L))^{[(k/(n-k)][(1/(n-1)+1/(n-2)+\cdots+1/(k+1)]} \right\}_{1\leq k\leq n-2}$$

with $\sigma_0 = n^n$ (resp. $\sigma_0 = 2n^n$, resp. $\sigma_0 = (n+s)^n$) and with a constant $B_n < 2.005$ depending only on n (Table (11.11) contains the first values of B_n).

When L is ample, the number a always exists and we have $\mu_{\Xi}(L) \geq 1$ for any choice of Ξ. We thus get an explicit lower bound m_0 depending only on n, a such that $K_X + mL$ is spanned or very ample for $m \geq m_0$. Unfortunately, these lower bounds are rather far from Fujita's expected conditions $m \geq n+1$ and $m \geq n+2$ respectively. Observe however that the lower bound for L^n is the Main Theorem is optimal: if $X = \mathbf{P}^n$ and $L = \mathscr{O}(1)$, then $K_X = \mathscr{O}(-n-1)$ so $K_X + nL$ is not spanned, although $(nL)^n = n^n = \sigma_0$. Similarly $K_X + (n+s)L$ does not generate s-jets, although $((n+s)L)^n = (n+s)^n = \sigma_0$. When $X \subset \mathbb{P}^{n+1}$ is the n-dimensional quadric and $L = \mathscr{O}_X(1)$, then $K_X + nL = \mathscr{O}_X$ is not very ample, although $(nL)^n = 2n^n = \sigma_0$.

Another unsatisfactory feature is that our bounds depend on the geometry of X through the number a, while the case of curves or surfaces suggests that they should not. In fact, our proof uses a rather delicate self-intersection inequality for closed positive currents, and this inequality (which is essentially optimal) depends in a crucial way on a bound for the "negative part" of TX. It follows that new ideas of a different nature are certainly necessary to get universal bounds for the very ampleness of $K_X + L$. However, an elementary argument shows that $TX \otimes \mathscr{O}(K_X + nF)$ is always nef when F is very ample (see Lemma 12.1). This observation combined with an iteration of the Main Theorem finally leads to a universal result. Corollary 2 below extends in particular Bombieri's result on pluricanonical embeddings of surfaces of general type to arbitrary dimensions (at least when K_X is supposed to be ample , see 12.10 and 12.11), and can be seen as an effective version of Matsusaka's theorem ([34], [30]).

Corollary 2. *If L is an ample line bundle over X, then $2K_X + mL$ is very ample, resp. generates s-jets, when $(m-1)\mu_X(L) + s > 2C_n\sigma_0$ with* *a constant* $C_n < 3$ *depending only on* n (*see Table* (12.3)). *In particular,* $2K_X + mL$ *is very ample for* $m > 4C_n n^n$ *and generates higher s-jets for* $m > 2C_n\sigma_0$.

Our approach is based on three rather powerful analytic tools. First, we use Hörmander's L^2 estimates for the operator $\overline{\partial}$ with singular plurisubharmonic weights to prove a general abstract existence theorem for sections of $K_X + L$ with prescribed jets at finitely many points; the idea is similar to that of the Hörmander-Bombieri-Skoda theorem, but following an idea of A. Nadel [36], we consider plurisubharmonic functions with logarithmic poles associated to an arbitrary ideal in $\mathcal{O}_{X,x}$ (see Corollary 4.6). We refer to [19] for further results relating ample or nef line bundles to singular hermitian metrics. The second tool is the Aubin-Calabi-Yau theorem. This fundamental result allows us to solve the Monge-Ampère equation $(\omega + \frac{i}{\pi}\partial\overline{\partial}\psi)^n = f$ where $\omega = \frac{i}{2\pi}c(L)$ is the curvature form of L, and the right-hand side f is an arbitrary positive (n, n)-form with $\int_X f = L^n$. We let f converge to a linear combination of Dirac measures and show that the solution ψ produces in the limit a singular weight on L with logarithmic poles. In order to control the poles and singularities, we use in an essential way a convexity inequality due to Hovanski [25] and Teissier [47], [48], which can be seen as a generalized version of the Hodge index theorem for surfaces. Finally we invoke in several occasions the theory of closed positive currents and Lelong numbers (see [3], [32]). In particular, the generalized Lelong numbers introduced in [17] are used as a substitute of the intersection theory of algebraic cycles in our analytic context. The self-intersection inequality 10.7 can be seen as a generalization to currents (and in any dimension) of the classical upper bound $d(d-1)/2$ for the number of multiple points of a plane curve of degree d. It actually gives a bound for the sum of degrees of the irreducible components in the sublevel sets of Lelong numbers of a closed positive $(1, 1)$-current T with integral cohomology class $\{T\}$, in terms of an explicit polynomial in $\{T\}$.

2. Singular hermitian metrics on holomorphic line bundles

Let L be a holomorphic line bundle over a projective algebraic manifold X and $n = \dim X$. If L is equipped with a hermitian metric, we denote by $c(L) = \frac{i}{2\pi}\nabla^2$ the Chern curvature form, which is a closed real $(1, 1)$-form representing the first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$. It is well known that L is ample if and only if L has a smooth hermitian metric such that $c(L)$ is positive definite at every point.

However, we are also interested in singular metrics, because they often give additional information about the existence of sections of high multiples mL. By definition a *singular metric* on L is a metric which is given in any trivialization $\tau: L_{\restriction\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}$ by

$$(2.1) \qquad \|\xi\| = |\tau(\xi)|e^{-\varphi(x)}, \qquad x \in \Omega, \ \xi \in L_x,$$

where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is a weight function. Then the curvature of L is given by the $(1,1)$-current $c(L) = \frac{i}{\pi}\partial\overline{\partial}\varphi$ on Ω. For example, to any divisor $D = \sum \lambda_j D_j$ with coefficients $\lambda_j \in \mathbb{Z}$ is associated the invertible sheaf $\mathscr{O}(D)$ of meromorphic functions f such that $\operatorname{div}(f) + D \geq 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|f\| = |f|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$, then $\tau(f) = f\prod g_j^{\lambda_j}$ defines a trivialization of $\mathscr{O}(D)$ over Ω; thus our singular metric is associated to the weight $\varphi = \sum \lambda_j \log|g_j|$. By the Lelong-Poincaré equation, we find

$$(2.2) \qquad c(\mathscr{O}(D)) = \frac{i}{\pi}\partial\overline{\partial}\varphi = [D],$$

where $[D] = \sum \lambda_j [D_j]$ denotes the current of integration over D.

In the sequel, all singular metrics are supposed to have positive curvature in the sense of currents (cf. [31]); i.e., the weight functions φ are supposed to be plurisubharmonic. Let us recall some results of [18]: consider the real Neron-Severi space $NS_{\mathbb{R}}(X) = (H^2(X, \mathbb{Z} \cap H^{1,1}(X)) \otimes \mathbb{R}$ of algebraic cohomology classes of degree 2, and let $\Gamma_+ \subset NS_{\mathbb{R}}(X)$ (resp. $\Gamma_a \subset \Gamma_+$) be the closed convex cone generated by cohomology classes of effective (resp. ample) divisors D; denote by Γ_+° (resp. Γ_a°) the interior of Γ_+ (resp. Γ_a). Then if ω is a Kähler metric on X and $\varepsilon > 0$, we have the following equivalent properties:

$$(2.3) \qquad c_1(L) \in \Gamma_+ \Leftrightarrow L \text{ has a singular metric with } c(L) \geq 0;$$

$$(2.4) \qquad \begin{aligned} c_1(L) \in \Gamma_+^\circ &\Leftrightarrow \exists\varepsilon, \ L \text{ has a singular metric with } c(L) \geq \varepsilon\omega \\ &\Leftrightarrow \kappa(L) = n; \end{aligned}$$

$$(2.5) \qquad \begin{aligned} c_1(L) \in \Gamma_a &\Leftrightarrow \forall\varepsilon, \ L \text{ has a smooth metric with } c(L) \geq -\varepsilon\omega \\ &\Leftrightarrow L \text{ is nef}; \end{aligned}$$

$$(2.6) \qquad \begin{aligned} c_1(L) \in \Gamma_a^\circ &\Leftrightarrow \exists\varepsilon, \ L \text{ as a smooth metric with } c(L) \geq \varepsilon\omega \\ &\Leftrightarrow L \text{ is ample}. \end{aligned}$$

The notation $\kappa(L)$ stands for the Kodaira dimension of L, that is by definition, the supremum of the generic rank of the rational maps to projective space defined by the nonzero sections in $H^0(X, mL)$ for $m \geq 1$

(if any), and $\kappa(L) = -\infty$; otherwise, alternatively, $\kappa(L)$ is the smallest constant such that $h^0(X, mL) \leq O(m^{\kappa(L)})$. The only thing that will be needed here is the fact that a big nef line bundle satisfies the equivalent properties in (2.4); we shall briefly sketch the proof of this. If L is nef, the Hilbert polynomial of $\chi(X, mL)$ has leading coefficient $L^n/n! \geq 0$, and it is well known that $h^j(X, mL) = O(m^{n-1})$; thus $h^0(X, mL) = (L^n/n!)m^n + O(m^{n-1})$. Hence L is big if and only if $\kappa(L) = n$. Let A be an ample divisor. Then $H^0(X, mL - A)$ is the kernel of $H^0(X, mL) \to H^0(A, mL_{\restriction A})$, and the target has dimension $\leq Cm^{n-1}$. When $\kappa(L) = n$ we get $H^0(X, mL - A) \neq 0$ for m large, so there is an effective divisor E such that $mL \simeq A+E$. Now, $pL+A$ is ample for every $p \geq 0$, so $pL+A$ has a smooth metric with $c(pL+A) \geq \varepsilon_p \omega$, and the isomorphism $(m+p)L \simeq pL + A + E$ gives a metric on L such that

$$c(L) = (m+p)^{-1}(c(pL + A) + [E]) \geq (m+p)^{-1}\varepsilon_p \omega. \tag{2.7}$$

Observe that the singular part $(m+p)^{-1}[E]$ can be chosen as small as desired by taking p large.

3. Basic results on Lelong numbers

These results will be needed in the sequel as an analytic analogue of some standard facts in the intersection theory of algebraic cycles. They are developed in more detail in [13,], [16], [17] (cf. Lelong [31], [32] for an earlier presentation). We first recall a few definitions. Let T be a closed positive current of bidimension (p, p), that is, of bidegree $(n-p, n-p)$, on an open set $\Omega \subset \mathbb{C}^n$. The Lelong number of T at a point $x \in \Omega$ is defined by $\nu(T, x) = \lim_{r \to 0+} \nu(T, x, r)$ where

$$\nu(T, x, r) = \frac{1}{(2\pi r^2)^p} \int_{B(x,r)} T(z) \wedge (i\partial\overline{\partial}|z|^2)^p \tag{3.1}$$

measures the ratio of the mass of T in the ball $B(x, r)$ to the area of the ball of radius r in $\mathbb{C}^p$; this ratio is an increasing function of r (cf. [30]), and the limit $\nu(T, x)$ does not depend on the choice of coordinates. In the case where T is a current of integration $[A]$ over an analytic subvariety, the Lelong number $\nu([A], x)$ coincides with the multiplicity of A at x (Thie's theorem [49]).

More generally, let φ be a continuous plurisubharmonic function with an isolated $-\infty$ pole at x, e.g. a function of the form

$$\varphi(z) = \log \sum_{1 \leq j \leq N} |g_j(z)|^{\gamma_j}, \qquad \gamma_j > 0,$$

where $(g_1, \cdots, g_N)$ is an ideal of germs of holomorphic functions in $\mathcal{O}_x$ with $g^{-1}(0) = \{x\}$. According to [17], the generalized Lelong number $\nu(T, \varphi)$ of T with respect to the weight φ is the limit when t tends to $-\infty$ of

$$(3.2) \qquad \nu(t, \varphi, t) = \int_{\varphi(z)<t} T \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\varphi\right)^p .$$

Because of the logarithmic singularity of φ, the integral is not well defined a priori. In fact, we can use Bedford and Taylor's definition of the Monge-Ampère operator for locally bounded plurisubharmonic functions (see §10) and set

$$(3.3) \qquad \begin{aligned} &\int_{\varphi(z)at} T \wedge (i\partial\overline{\partial}\varphi)^p \\ &= \int_{\varphi(z)<t} T(z) \wedge (i\partial\overline{\partial}\max(\varphi(z), s))^p, \qquad s < t; \end{aligned}$$

observe that the right-hand side is independent of s by Stokes' formula. The relation with ordinary Lelong numbers comes from the equality

$$(3.4) \qquad \nu(T, x, r) = \nu(T, \varphi, \log r), \qquad \varphi(z) = \log|z - x|;$$

in particular $\nu(T, x) = \nu(T, \log|\cdot - x|)$. This equality is in turn a consequence of the following general formula, applied to $\chi(t) = e^{2t}$ and $t = \log r$:

$$(3.5) \qquad \int_{\varphi(z)<t} T \wedge (i\partial\overline{\partial}\chi\circ\varphi)^p = \chi'(t-0)^p \int_{\varphi(z)<t} T \wedge (i\partial\overline{\partial}\varphi)^p,$$

where χ is an arbitrary convex increasing function. To prove the formula, we use a regularization and thus suppose that T, φ and χ are smooth, and that t is a noncritical value of φ. Then Stokes' formula shows that the integrals on the left and on the right of (3.5) are equal respectively to

$$\int_{\varphi(z)=t} T \wedge (i\partial\overline{\partial}\chi\circ\varphi)^{p-1} \wedge i\overline{\partial}(\chi\circ\varphi),$$
$$\int_{\varphi(z)=t} T \wedge (i\partial\overline{\partial}\varphi)^{-1} \wedge i\overline{\partial}\varphi,$$

and the differential form of bidegree $(p-1), p)$ appearing in the integrand of the first integral is equal to $(\chi'\circ\varphi)^p(i\partial\overline{\partial}\varphi)^{p-1} \wedge i\overline{\partial}\varphi$. The expected formula follows.

It is shown in [17] that $\nu(T, \varphi)$ depends only on the asymptotic behavior of φ near the pole x; namely, the Lelong number remains unchanged for a weight ψ such that $\lim_{z\to x}\psi(z)/\varphi(z) = 1$. More generally, if $\limsup_{z\to x}\psi(z)/\varphi(z)=\lambda$, then

$$\nu(T, \psi)\le \lambda^p\nu(T, \varphi). \tag{3.6}$$

Finally, let F be a proper holomorphic map from a neighborhood of x onto a neighborhood of y in $\mathbb{C}^n$, and let ψ be a continuous plurisubharmonic function with an isolated pole at y. The definition of the direct image $F_\star T$ by adjunction of $F^\star$ easily shows that for $t<t_0$ sufficiently small

$$\nu(F_\star T, \psi, t)=\nu(T, \psi\circ F, t), \qquad \nu(F_\star T, \psi)=\nu(T, \psi\circ F). \tag{3.7}$$

For any closed current T of bidimension (p, p) on a complex manifold X and any positive number c, we let $E_c(T)$ be the set of points $z\in X$ where $\nu(T, z)\ge c$. By a theorem of Siu [42], all sublevel sets $E_c(T)$ are closed analytic subsets of X of dimension at most p. Moreover T can be written as a convergent series of closed positive currents

$$T=\sum_{k=1}^{+\infty}\lambda_k[Z_k]+R, \tag{3.8}$$

where $[Z_k]$ is a current of integration over an irreducible analytic set of dimension p, and R is a residual current with the property that $\dim E_c(R)<p$ for every $c>0$. This decomposition is locally and globally unique: the sets Z_k are precisely the p-dimensional components occurring in the sublevel sets $E_c(T)$, and $\lambda_k=\min_{x\in Z_k}\nu(T, x)$ is the generic Lelong number of T along Z_k.

The Lelong number of a plurisubharmonic function w on X can also be defined as

$$\nu(w, x)=\liminf_{z\to x}\frac{w(z)}{\log|z-x|}, \tag{3.9}$$

where $z=(z_1, \cdots, z_n)$ are local coordinates near x, and $|\ |$ denotes an arbitrary norm on $\mathbb{C}^n$. It is well known that $\nu(w, x)$ is equal to the Lelong number $\nu(T, x)$ of the associated positive $(1, 1)$-current $T=\frac{i}{\pi}\partial\bar\partial w$. Accordingly, we set $E_c(w)=E_c(T)$.

4. L^2 estimates and existence of holomorphic sections

We first state the basic existence theorem of Hörmander for solutions of $\bar\partial$ equations, in the form that is most convenient to us.

Proposition 4.1. *Suppose that X is a Stein or compact projective manifold equipped with a Kähler metric ω. Let L be a line bundle with a hermitian metric associated to singular plurisubharmonic weight functions ψ such that $c(L) \geq \varepsilon\omega$ for some $\varepsilon > 0$. For every $q \geq 1$ and every (n, q) form v with values in L such that $\overline{\partial} v = 0$ and $\int_X |v|^2 e^{-2\psi}\, dV_\omega < +\infty$, there is a $(n, q-1)$-form u with values in L such that $\overline{\partial} u = v$ and*

$$\int_X |u|^2 e^{-2\psi}\, dV_\omega \leq \frac{1}{2\pi q\varepsilon} \int_X |v|^2 e^{-2\psi}\, dV_\omega .$$

Here dV_ω stands for the Kähler volume element $\omega^n/n!$, and $|u|^2 e^{-2\psi}$ denotes somewhat abusively the pointwise norm of $u(z)$ at each point $z \in X$, although ψ is only defined on an open set in X. The operator $\overline{\partial}$ is taken in the sense of distribution theory.

Proof. The result is standard when X is Stein and L is the trivial bundle (see [1] and [24]). In general, there exists a hypersurface $H \subset X$ such that $X \setminus H$ is Stein and L is trivial over $X \setminus H$. We then solve the equation $\overline{\partial} u = v$ over $X \setminus H$ and observe that the solution extends to X thanks to the L^2 estimate (cf. [14, Lemma 6.9]). q.e.d.

We will also use the concept of *multiplier ideal sheaf* introduced by A. Nadel [36]. The main idea actually goes back to the fundamental works of Bombieri [8] and H. Skoda [43]. Let φ be a plurisubharmonic function on X; to φ is associated the ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}_X$ of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x. The zero variety $V\mathcal{I}(\varphi)$ is thus the set of points in a neighborhood of which $e^{-2\varphi}$ is nonintegrable. This zero variety is closely related to the Lelong sublevel sets $E_c(\varphi)$. Indeed, if $\nu(\varphi, x) = \gamma$, the convexity properties of plurisubharmonic functions show that

$$\varphi(z) \leq \gamma \log|z - x| + O(1) \quad \text{at } x;$$

hence there exists a constant $C > 0$ such that $e^{-2\varphi(z)} \geq C|z - x|^{-2\gamma}$ in a neighborhood of x. We easily infer that

$$\nu(\varphi, x) \geq n + s \Rightarrow \mathcal{I}(\varphi)_x \subset \mathcal{M}_{X,x}^{s+1}, \tag{4.2}$$

where $\mathcal{M}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. In the opposite direction, it is well known that $\nu(\varphi, x) < 1$ implies the integrability of $e^{-2\varphi}$ in a neighborhood of x (cf. Skoda [43]); that is, $\mathcal{I}(\varphi)_x = \mathcal{O}_{X,x}$. In particular, the zero variety $V\mathcal{I}(\varphi)$ of $I(\varphi)$ satisfies

$$E_n(\varphi) \subset V\mathcal{I}(\varphi) \subset E_1(\varphi). \tag{4.3}$$

Lemma 4.4 [36]. *For any plurisubharmonic function φ on X, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over X.*

Proof. Since the result is local, we may assume that X is the unit ball in $\mathbb{C}^n$. Let E be the set of all holomorphic functions f on X such that $\int_X |f|^2 e^{-2\varphi}\, d\lambda < +\infty$. By the strong noetherian property of coherent sheaves, the set E generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathcal{M}_{X,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let $f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $\overline{\partial} u = \overline{\partial}(\theta f)$ by means of Hörmander's L^2 estimates 4.1, with L equal to the trivial line bundle and with the strictly plurisubharmonic weight

$$\psi(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2 .$$

We get a solution u such that $\int_X |u|^2 e^{-2\varphi} |z-x|^{-2(n+s)}\, d\lambda < \infty$; thus $F = \theta f - u$ is holomorphic, $F \in E$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathcal{M}_{X,x}^{s+1}$. This proves our contention. q.e.d.

Now, suppose that X is a projective n-fold equipped with a Kähler metric ω. Let L be a line bundle over X with a singular metric of curvature $T = c(L) \geq 0$. All sublevel sets $E_c(T)$ are algebraic subsets of X, and if φ is the weight representing the metric of L in an open set $\Omega \subset X$, then $E_c(\varphi) = E_c(T) \cap \Omega$. The ideal sheaf $\mathcal{I}(\varphi)$ is independent of the choice of the trivialization and so it is the restriction to Ω of a global coherent sheaf on X which we shall still call $\mathcal{I}(\varphi)$ by abuse of notation. In this context, we have the following interesting vanishing theorem, which can be seen as a generalization of the Kawamata-Viehweg vanishing theorem [26], [50].

Theorem 4.5 [36]. *Let L be a line bundle over X with $\kappa(L) = n$. Assume that L is equipped with a singular metric of weight φ such that $c(L) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Then $H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geq 1$.*

Proof. Let $\mathcal{F}^q$ be the sheaf of germs of (n, q)-forms u with values in L and with measurable coefficients, such that both $|u|^2 e^{-2\varphi}$ and $|\overline{\partial} u|^2 e^{-2\varphi}$ are locally integrable. The $\overline{\partial}$ operator defines a complex of sheaves $(\mathcal{F}^\bullet, \overline{\partial})$ which is a resolution of the sheaf $\mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi)$: indeed, the kernel of $\overline{\partial}$ in degree 0 consists of all germs of holomorphic n-forms with values in L which satisfy the integrability condition; hence the coefficient function lies in $\mathcal{I}(\varphi)$; the exactness in degree $q \geq 1$ follows from Proposition 4.1 applied on arbitrary small balls. Each sheaf $\mathcal{F}^q$ is a C^∞-module, so $\mathcal{F}^\bullet$ is a fine resolution. Moreover, $H^q(\Gamma(X, \mathcal{F}^\bullet)) = 0$ for $q \geq 1$ by Proposition 4.1 applied globally on X. The theorem follows.

Corollary 4.6. *Let L be a big nef line bundle over X. Assume that L is equipped with a singular metric of weight φ such that $c(L) \geq 0$ and let $x_1, \cdots, x_N$ be isolated points in the zero variety $V\mathcal{I}(\varphi)$. Then for every $\varepsilon > 0$, there is a surjective map*

$$H^0(X, K_X + L) \twoheadrightarrow \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X + L)_{x_j} \otimes (\mathcal{O}_X/\mathcal{I}((1-\varepsilon)\varphi))_{x_j}.$$

Proof. This result can be seen as a generalization of the Hörmander-Bombieri-Skoda theorem [8], [43], [45]; it could be proved directly by using Hörmander's L^2 estimates and cut-off functions. If $c(L) \geq \delta\omega$ for some $\delta > 0$, we apply Theorem 4.5 to obtain the vanishing of the first H^1 group in the long exact sequence of cohomology associated to

$$0 \to \mathcal{I}(\varphi) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(\varphi) \to 0,$$

twisted by $\mathcal{O}(K_X + L)$. The asserted surjectively property follows immediately; as $\mathcal{I}(\varphi) \subset \mathcal{I}((1-\varepsilon)\varphi)$, we see in that case that we can even take $\varepsilon = 0$ and drop the nef assumption on L. If $c(L) \geq 0$ merely, we try to modify the metric so as to obtain a positive lower bound for the curvature. By (2.7) there is a singular metric on L associated to a weight ψ with $\frac{i}{\pi}\partial\overline{\partial}\psi \geq \delta\omega$, $\delta > 0$, and with a singularity of ψ so small that $e^{-2\psi} \in L^1_{\mathrm{loc}}$. Replace the metric on L by the metric associated to the weight $\varphi_\varepsilon = (1-\varepsilon)\varphi + \varepsilon\psi$. Then $e^{-2\varphi_\varepsilon} = (e^{-2\varphi})^{1-\varepsilon}(e^{-2\psi})^\varepsilon$ is integrable on any open set where $e^{-2\varphi}$ is integrable, so $V\mathcal{I}(\varphi_\varepsilon) \subset V\mathcal{I}(\varphi)$ and the points x_j are still isolated in $V\mathcal{I}(\varphi_\varepsilon)$. Moreover, $\mathcal{I}(\varphi_\varepsilon) \subset \mathcal{I}((1-\varepsilon)\varphi)$, for ψ is locally bounded above, and $c(L)_\varepsilon = \frac{i}{\pi}\partial\overline{\partial}\varphi_\varepsilon \geq \varepsilon\delta\omega$. We are thus reduced to the first case.

Example 4.7. Suppose that $\nu(\varphi, x) > n + s$ and that x is an isolated point in $E_1(\varphi)$. Then $\mathcal{I}((1-\varepsilon)\varphi)_x \subset \mathcal{M}_{X,x}^{s+1}$ for ε small enough, and x is isolated in $V\mathcal{I}(\varphi)$ by (4.2), (4.3). We infer that $H^0(X, K_X + L) \twoheadrightarrow J_x^s(K_X + L)$ is surjective onto s-sets of sections at x.

Example 4.8. Suppose that $(z_1, \cdots, z_n)$ are local coordinates centered at x and that

$$\varphi(z) \leq \gamma \log(|z_1| + \cdots + |z_{n-1}| + |z_n|^2) + O(1), \qquad \gamma > n.$$

Then $\mathcal{I}((1-\varepsilon)\varphi) \subset (z_1, \cdots, z_{n-1}, z_n^2)$ for ε small. To check this, observe that for any $\delta > 0$ the Parseval-Bessel formula gives

$$\int_{|x|<\delta} \frac{|\sum a_\alpha z^\alpha|^2 \, d\lambda(z)}{(|z_1| + \cdots |z_{n-1}| + |z_n|^2)^{2n}} = \int_{|z|<\delta} \frac{\sum |a_\alpha|^2 |z^\alpha|^2 \, d\lambda(z)}{(|z_1| + \cdots |z_{n-1}| + |z_n|^2)^{2n}};$$

the integral is divergent unless the coefficients a_0 and $a_{(0,\cdots,0,1)}$ vanish. Indeed, using polar coordinates $z_n = re^{i\theta}$ and setting $z' = (z_1, \ldots, z_{n-1})$, we get

$$\begin{aligned}\int_{|z|<\delta} \frac{|z_n|^2 \, d\lambda(z)}{(|z_1| + \cdots + |z_{n-1}| + |z_n|^2)^{2n}} &\geq 2\pi \int_{|z'|<\delta/2} d\lambda(z') \int_0^{\delta/2} \frac{r^3 \, dr}{(|z'| + r^2)^{2n}} \\ &\geq C \int_{|z'|<\delta/2} \frac{d\lambda(z')}{|z'|^{2n-2}} = +\infty .\end{aligned}$$

Thus, if x is isolated in $E_1(\varphi)$, we are able to prescribe the value of the section at x and its derivative $\partial/\partial z_n$ along the direction $z' = 0$.

Remark 4.9. More generally, it is interesting to consider logarithmic poles of the form

$$\varphi(z) = \gamma \log \left(\sum_{1 \leq j \leq N} |g_j(z)| \right) + O(1),$$

where $\mathscr{J} = (g_1, \cdots, g_N) \subset \mathscr{M}_{X,x}$ is an arbitrary ideal with isolated zero $\{x\}$. However, in this case, we do not know what is the general rule relating the ideal $\mathscr{I}(\gamma \log |g|) =_x$ to the ideal $\mathscr{J}$. Observe that $\mathscr{I}(\gamma \log |g|)_x$ only depends on the integral closure $\overline{\mathscr{J}} = \{$germs f such that $|f| \leq C \sum |g_j|\}$. It is almost obvious by definition that $\mathscr{I}(\varphi)$ itself is always integrally closed. Let $\widetilde{\mathscr{J}} = (\tilde{g}_1, \cdots, \tilde{g}_n) \subset \mathscr{J}$ be the ideal generated by n generic linear combinations $\tilde{g}_k$ of $g_1, \cdots, g_N$. Then $\widetilde{\mathscr{J}}$ and $\mathscr{J}$ have the same integral closure and we have $\sum |\tilde{g}_k| \geq C' \sum |g_j|$ with some $C' > 0$; indeed these ideals have the same multiplicity by a result of Serre [41], and this implies the equality of their integral closures thanks to a result of D. Rees [38]. The ideal $\mathscr{I}(\gamma \log |\tilde{g}|)_x$ associated to $\widetilde{\mathscr{J}}$ thus coincides with $\mathscr{I}(\gamma \log |g|)_x$. We see that there is no loss of generality considering only ideals generated by exactly n generators (as we shall do in §§6, 7). Finally, the proof of the Briançon-Skoda theorem [10] shows that

(4.10) $$\mathscr{I}(\gamma \log |g|)_x = \mathscr{I}(\gamma \log |\tilde{g}|)_x \subset \widetilde{\mathscr{J}} \subset \mathscr{J} \quad \text{when } \gamma > n .$$

In fact, (4.10) is a straightforward consequence of Skoda's division theorem [51], applied to the elements of $\mathscr{I}(\gamma \log|\tilde{g}|)$.

5. Aubin-Calabi-Yau theorem and convexity inequalities

The above results can be applied to construct sections of a given line bundle, provided we are able to produce singular metrics with logarithmic poles. For this, we use in several essential ways the well-known theorem of Aubin-Yau on the Calabi conjecture. What we need is the following existence result about solutions of Monge-Ampère equations.

Lemma 5.1 [51]. *Let X be a compact complex n-dimensional manifold with a smooth Kähler metric ω. Then for any smooth volume form $f > 0$ with $\int_X f = \int_X \omega^n$, there exists a Kähler metric $\tilde{\omega}$ in the same Kähler class as ω such that $\tilde{\omega} = f$.*

The method for constructing singular metrics from the Aubin-Calabi-Yau theorem will be explained in detail in §6. Before, we need a useful convexity inequality due to Hovanski [25] and Teissier [47], [48], which is a natural generalization of the usual Hodge index theorem for surfaces. This inequality is reproved along similar lines in [5], where it is applied to the study of projective n-folds of log-general type. For the sake of completeness, we include here a different and slightly simpler proof, based on Yau's Theorem 5.1 instead of the Hodge index theorem. Our proof also has the (relatively minor) advantage of working over arbitrary Kähler manifolds.

Proposition 5.2. *For any dimension n,*

(a) *if $\alpha_1, \cdots, \alpha_n$ are semipositive $(1, 1)$-forms on $\mathbb{C}^n$, then*

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \alpha_n \geq (\alpha_1^n)(\alpha_2^n)^{1/n} \cdots (\alpha_n^n)^{1/n}.$$

(b) *if $u_1, \cdots, u_n$ are semipositive cohomology classes of type $(1, 1)$ on a Kähler manifold X of dimension n, then*

$$u_1 \cdot u_2 \cdots u_n \geq (u_1^n)^{1/n}(u_2^n)^{1/n} \cdots (u_n^n)^{1/n}.$$

By a semipositive cohomology class of type $(1, 1)$, we mean a class in the closed convex cone of $H^{1,1}(X, \mathbb{R})$ generated by Kähler classes. For instance, inequality (b) can be applied to $u_j = c_1(L_j)$ when $L_1, \cdots, L_n$ are nef line bundles over a projective manifold.

Proof. Observe that (a) is a pointwise inequality between (n, n)-forms whereas (b) is an inequality of a global nature for the cup product intersection form. We first show that (a) holds when only two of the forms α_j

are distinct, namely, that

$$\alpha^p \wedge \beta^{n-p} \geq (\alpha^n)^{p/n}(\beta^n)^{(n-p)/n}$$

for all α, $\beta \geq 0$. By a density argument, we may suppose α, $\beta > 0$. Then there is a simultaneous orthogonal basis in which

$$\alpha = i \sum_{1\leq j\leq n} \lambda_j \, dz_j \wedge d\overline{z}_j, \qquad \beta = i \sum_{i\leq j\leq n} dz_j \wedge d\overline{z}_j$$

with $\lambda_j > 0$, and (a) is equivalent to

$$p!(n-p)! \sum_{j_1<\cdots<j_p} \lambda_{j_1}\cdots\lambda_{j_p} \geq n!(\lambda_1\cdots\lambda_n)^{p/n}.$$

As both sides are homogeneous of degree p in (λ_j), we may assume $\lambda_1 \cdots \lambda_n = 1$. Then our inequality follows from the inequality between the arithmetic and geometric means of the numbers $\lambda_{j_1}\cdots\lambda_{j_p}$. Next, we show that statements (a) and (b) are equivalent in any dimension n.

(a) $\Rightarrow$ (b). By density, we may suppose that $u_1, \cdots, u_n$ are Kähler classes. Fix a positive (n, n) form f such that $\int_X f = 1$. Then Lemma 5.1 implies that there is a Kähler metric α_j representing u_j such that $\alpha_j^n = u_j^n f$. Inequality (a) combined with an integration over X yields

$$u_1\cdots u_n = \int_X \alpha_1 \wedge\cdots\wedge \alpha_n \geq (u_1^n)^{1/n}\cdots(u_n^n)^{1/n}\int_X f.$$

(b) $\Rightarrow$ (a). The forms $\alpha_1, \cdots, \alpha_n$ can be considered as constant $(1, 1)$-forms on any complex torus $X = \mathbb{C}^n/\Gamma$. Inequality (b) applied to the associated cohomology classes $u_j \in H^{1,1}(X, \mathbb{R})$ is then equivalent to (a).

Finally we prove (a) by induction on n, assuming the result already proved in dimension $n-1$. We may suppose that α_n is positive definite, say $\alpha_n = i\sum dz_j \wedge d\overline{z}_j$ in a suitable basis. Denote by $u_1, \cdots, u_n$ the associated cohomology classes on the abelian variety $X = \mathbb{C}^n/\mathbb{Z}[i]^n$. Then u_n has integral periods, so some multiple of u_n is the first Chern class of a very ample line bundle $\mathscr{O}(D)$ where D is a smooth irreducible divisor in X. Without loss of generality, we may suppose $u_n = c_1(\mathscr{O}(D))$. Thus

$$u_1\cdots u_{n-1}\cdot u_n = u_{1\restriction D}\cdots u_{n-1\restriction D},$$

and by the induction hypothesis we get

$$u_1\cdots u_n \geq (u_{1\restriction D}^{n-1})^{1/(n-1)}\cdots(u_{n-1\restriction D}^{n-1})^{1/(n-1)}.$$

However $u_{j\restriction D}^{n-1} = u_j^{n-1}\cdot u_n \geq (u_j^n)^{(n-1)/n}(u_n^n)^{1/n}$, since (a) and (b) are equivalent and (a) is already proved in the case of two forms. (b) follows for dimension n, and therefore (a) holds in $\mathbb{C}^n$.

Remark 5.3. In case α_j (resp. u_j) are positive definite, the equality holds in 5.2(a,b) if and only if $\alpha_1, \cdots, \alpha_n$ (resp. $u_1, \cdots, u_n$) are proportional. In our inductive proof, the restriction morphism $H^{1,1}(X, \mathbb{R}) \to H^{1,1}(D, \mathbb{R})$ is injective for $n \geq 3$ by hard Lefschetz theorem; hence it is enough to consider the case of $\alpha^p \wedge \beta^{n-p}$. The equality between arithmetic and geometric means occurs only when all numbers $\lambda_{j_1}, \cdots, \lambda_{j_p}$ are equal, so all λ_j must be equal and $\alpha = \lambda_1 \beta$, as desired. More generally, there is an inequality

$$\begin{aligned} \alpha \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_{n-p} \\ \geq (\alpha_1^p \wedge \beta_1 \wedge \cdots \wedge \beta_{n-p})^{1/p} \cdots (\alpha_p^p \wedge \beta_1 \wedge \cdots \wedge \beta_{n-p})^{1/p} \end{aligned} \tag{5.4}$$

for all $(1, 1)$-forms α_j, $\beta_k \geq 0$. Once again, inequality (5.4) is easier to be proved with cohomology classes rather than forms. By a density argument, we may suppose that all forms β_j are positive definite and have coefficients in $\mathbb{Q}[i]$. Let $u_1, \cdots, u_p$ be the cohomology classes of type $(1, 1)$ associated to $\alpha_1, \cdots, \alpha_p$ on $X = \mathbb{C}^n/\mathbb{Z}[i]^n$. The cohomology class of β_1 is a rational multiple of the first Chern class of a very ample line bundle $\mathcal{O}(Y_1)$, where Y_1 is a smooth irreducible divisor in X, that of $\beta_{2\restriction Y_1}$ is a multiple of such a divisor Y_2 in Y_1, and by induction the cohomology class of $\beta_1 \wedge \cdots \wedge \beta_{n-p}$ is equal to a multiple of the cohomology class of a connected p-dimensional submanifold $Y \subset X$. Then (5.4) is equivalent to the already-known inequality

$$u_{1\restriction Y} \cdots u_{p\restriction Y} \geq (u_{1\restriction Y}^p)^{1/p} \cdots (u_{u\restriction Y}^p)^{1/p}.$$

6. Mass concentration in the Monge-Ampère equation

In this crucial section, we show how the Aubin-Calabi-Yau theorem can be applied to construct singular metrics on ample (or more generally big and nef) line bundles. We first suppose that L is an ample line bundle over a projective n-fold X and that L is equipped with a smooth metric of positive curvature. Then consider the Kähler metric $\omega = \frac{i}{2\pi} c(L)$. Any form $\tilde{\omega}$ in the Kähler class of ω can be written as $\tilde{\omega} = \omega + \frac{i}{\pi} \partial\overline{\partial}\psi$, i.e., is the curvature form of L after multiplication of the original metric by a smooth weight function $e^{-\psi}$. By Lemma 5.1, the Monge-Ampère equation

$$\left(\omega + \frac{i}{\pi} \partial\overline{\partial}\psi\right)^n = f \tag{6.1}$$

can be solved for ψ, whenever f is a smooth (n, n)-form with $f > 0$ and $\int_X f = L^n$. In order to produce logarithmic poles at given points $x_1, \cdots, x_N \in X$, the main idea is to let f converge to a Dirac measure at x_j; then $\widetilde{\omega}$ will be shown to converge to a closed positive $(1, 1)$-current with nonzero Lelong number at x_j.

Let $(z_1, \cdots, z_n)$ be local coordinates centered at x_j, defined on some neighborhood $V_j \simeq \{|z| < R_j\}$. Let $g_j = (g_{j,1}, \cdots, g_{j,n})$ be arbitrary holomorphic functions on V_j such that $g_j^{-1}(0) = \{x_j\}$, and let

$$\log|g_j| = \log\left(\sum_{1\le k\le n} |g_{j,k}|^2\right)^{1/2}. \tag{6.2}$$

Then $\log|g_j|$ has an isolated logarithmic pole at x_j, and $(\frac{i}{\pi}\partial\overline{\partial}\log|g_j|)^n = \rho_j\delta_{x_j}$, where ρ_j is the degree of the covering map $g_j\colon (\mathbb{C}^n, x_j) \to (\mathbb{C}^n, 0)$. Indeed $\partial\overline{\partial}\log|g_j| = g_j^\star\partial\overline{\partial}\log|z|$ has rank $(n-1)$ on $V_j \setminus \{x_j\}$, and formula (3.5) with $\chi(t) = e^{2t}$ gives

$$\begin{aligned}\int_{|g_j(z)|<r}\left(\frac{i}{\pi}\partial\overline{\partial}\log|g_j|\right)^n &= \frac{1}{(2\pi r^2)^n}\int_{|g_j(z)|<r} g_j^\star(i\partial\overline{\partial}|w|^2)^n \\ &= \frac{\rho_j}{(2\pi r^2)^n}\int_{|w|<r}(i\partial\overline{\partial}|w|^2 0^n = \rho_j\end{aligned}$$

for every $r > 0$ small enough. Now, let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a smooth convex increasing function such that $\chi(t) = t$ for $t \ge 0$ and $\chi(t) = -1/2$ for $t \le -1$. We set

$$\alpha_{j,\varepsilon} = \frac{i}{\pi}\partial\overline{\partial}(\chi(\log|g_j|/\varepsilon)). \tag{6.3}$$

Then $\alpha_{j,\varepsilon}$ is a smooth positive $(1, 1)$-form, and $\alpha_{j,\varepsilon} = \frac{i}{\pi}\partial\overline{\partial}\log|g_j|$ over the set of points $z \in V_j$ such that $|g_j(z) > \varepsilon$. It follows that $\alpha_{j,\varepsilon}^n$ has support in the compact set $|g_j(z)| \le \varepsilon$, and Stokes' formula gives

$$\int_{V_j}\alpha_{j,\varepsilon}^n = \int_{V_j}\left(\frac{i}{\pi}\partial\overline{\partial}\log|g_j|\right)^n = \rho_j. \tag{6.4}$$

Hence $\alpha_{j,\varepsilon}^n$ converges weakly to the Dirac measure $\rho_j\delta_{x_j}$ as ε tends to 0. For all positive numbers $\tau_j > 0$ such that $\sigma = \sum\rho_j\tau_j^n < L^n$, Lemma 5.1 gives a solution of the Monge-Ampère equation

$$\omega_\varepsilon^n = \sum_{1\le j\le N}\tau_j^n\alpha_{j,\varepsilon}^n + \left(1 - \frac{\sigma}{L^n}\right)\omega^n \quad \text{with } \omega_\varepsilon = \omega + \frac{i}{\pi}\partial\overline{\partial}\psi_\varepsilon, \tag{6.5}$$

since the right-hand side of the first equation is > 0 and has the correct integral value L^n over X. The solution ψ_ε is merely determined up to a constant. If γ is an arbitrary Kähler metric on X, we can normalize ψ_ε in such a way that $\int_X \psi_\varepsilon \gamma^n = 0$.

Lemma 6.6. *There is a sequence ε_ν converging to zero such that ψ_{ε_ν} has a limit ψ in $L^1(X)$ and such that the sequence of $(1, 1)$-forms ω_{ε_ν} converges weakly towards a closed positive current T of type $(1, 1)$. Moreover, the cohomology class of T is equal to $c_1(L)$ and $T = \omega + \frac{i}{\pi}\partial\overline{\partial}\psi$.*

Proof. The integral $\int_X \omega_\varepsilon \wedge \gamma^{n-1} = L \cdot \{\gamma\}^{n-1}$ remains bounded, so we can find a sequence ε_ν converging to zero such that the subsequence ω_{ε_ν} converges weakly towards a closed positive current T of bidegree $(1, 1)$. The cohomology class of a current is continuous with respect to the weak topology (this can be seen by Poincaré duality). The cohomology class of T is thus equal to $c_1(L)$. The function ψ_ε satisfies the equation $\frac{1}{\pi}\Delta\psi_\varepsilon = \mathrm{tr}_\gamma(\omega_\varepsilon - \omega)$ where Δ is the Laplace operator associated to γ. Our normalization of ψ_ε implies

$$\psi_\varepsilon = \pi G\, \mathrm{tr}_\gamma(\omega_\varepsilon - \omega),$$

where G is the Green operator of Δ. As G is a compact operator from the Banach space of bounded Borel measures into $L^1(X)$, we infer that some subsequence (ψ_{ε_ν}) of our initial subsequence converges to a limit ψ in $L^1(X)$. By the weak continuity of $\partial\overline{\partial}$, we get $T = \lim(\omega + \frac{i}{\pi}\partial\overline{\partial}\psi_{\varepsilon_\nu}) = \omega + \frac{i}{\pi}\partial\overline{\partial}\psi$. q.e.d.

Let $\Omega \subset X$ be an open coordinate patch such that L is trivial on a neighborhood of $\overline{\Omega}$, and let e^{-h} be the weight representing the initial hermitian metric on $L_{\restriction\overline{\omega}}$. Then $\frac{i}{\pi}\partial\overline{\partial}(h + \psi_\varepsilon) = \omega_\varepsilon$, so the function $\varphi_\varepsilon = h + \psi_\varepsilon$ defines a plurisubharmonic weight on $L_{\restriction\Omega}$, as well as its limit $\varphi = h + \psi$. By the continuity of G, we also infer from the proof of Lemma 6.6 that the family (ψ_ε) is bounded in $L^1(X)$. The usual properties of subharmonic functions then show that there is a uniform constant C such that $\varphi_\varepsilon \leq C$ on $\overline{\Omega}$. We use this and equation (6.5) to prove that the limit φ has logarithmic poles at all points $x_j \in \Omega$, thanks to Bedford and Taylor's maximum principle for solutions of Monge-Ampère equations [3]:

Lemma 6.7. *Let u, v be smooth (or continuous) plurisubharmonic functions on $\overline{\Omega}$, where Ω is a bounded open set in $\mathbb{C}^n$. If*

$$u_{\restriction\partial\Omega} \geq v_{\restriction\partial\Omega} \quad \text{and} \quad (i\partial\overline{\partial}u)^n \leq (i\partial\overline{\partial}v)^n \quad \text{on } \Omega,$$

then $u \geq v$ on Ω.

In the application of Lemma 6.7, we suppose that Ω is a neighborhood of x_j and take

$$u = \tau_j(\chi(\log|g_j|/\varepsilon) + \log\varepsilon) + C_1, \qquad v = \varphi_\varepsilon,$$

where C_1 is a large constant. Then for $\varepsilon > 0$ small enough

$$u_{\restriction\partial\Omega} = \tau_j \log|g_j| + C_1, \qquad v_{\restriction\partial\Omega} \le C,$$

$$\left(\frac{i}{\pi}\partial\overline{\partial}v\right)^n = \omega_\varepsilon^n \ge \tau_j^n\alpha_{j,\varepsilon}^n = \left(\frac{i}{\pi}\partial\overline{\partial}u\right)^n \quad \text{on } \Omega.$$

For C_1 sufficiently large, we infer $u \ge v$ on Ω, hence

$$\varphi_\varepsilon \le \tau_j \log(|g_j| + \varepsilon) + C_2 \quad \text{on } \Omega.$$

Corollary 6.8. *The plurisubharmonic weight $\varphi = h + \psi$ on $L_{\restriction\Omega}$ associated to the limit function $\psi = \lim \psi_{\varepsilon_\nu}$ satisfies $\frac{i}{\pi}\partial\overline{\partial}\varphi = T$. Moreover, φ has logarithmic poles at all points $x_j \in \Omega$ and*

$$\varphi(z) \le \tau_j \log|g_j(z)| + O(1) \quad \text{at } x_j.$$

Case of a big nef line bundle. All our arguments were developed under the assumption that L is ample, but if L is only nef and big, we can proceed in the following way. Let A be a fixed ample line bundle with smooth curvature form $\gamma = c(A) > 0$. As $mL + A$ is ample for any $m \ge 1$, by 5.1 there exists a smooth hermitian metric on L depending on m, such that $\omega_m = c(L)_m + \frac{1}{m}c(A) > 0$ and

$$\omega_m^n = \frac{(L + \frac{1}{m}A)^n}{A^n}\gamma^n. \tag{6.9}$$

However, a priori we cannot control the asymptotic behavior of ω_m when m tends to infinity, so we introduce the sequence of not necessarily positive $(1,1)$- forms $\omega'_m = c(L)_1 + \frac{1}{m}c(A) \in \{\omega_m\}$, which is uniformly bounded in $C^\infty(X)$ and converges to $c(L)_1$. Then we solve the Monge-Ampère equation

$$\omega_{m,\varepsilon}^n = \sum_{1\le j\le N} \tau_j^n\alpha_{j,\varepsilon}^n + \left(1 - \frac{\sigma}{(L + \frac{1}{m}A)^n}\right)\omega_m^n \tag{6.10}$$

with $\omega_{m,\varepsilon} = \omega'_m + \frac{i}{\pi}\partial\overline{\partial}\psi_{m,\varepsilon}$ and some smooth function $\psi_{m,\varepsilon}$ such that $\int_X \psi_{m,\varepsilon}\gamma^n = 0$; this is again possible by Yau's Theorem 5.1. The numerical condition needed on σ to solve (6.10) is obviously satisfied for all m if we suppose

$$\sigma = \sum \rho_j\tau_j^n < L^n < \left(L + \frac{1}{m}A\right)^n.$$

The same arguments as before show that there exist a convergent subsequence $\lim_{\nu\to+\infty}\psi_{m_\nu,\varepsilon_\nu} = \psi$ in $L^1(X)$ and a closed positive $(1,1)$-current $T = \lim \omega_{m_\nu,\varepsilon_\nu} = c(L)_1 + \frac{i}{\pi}\partial\overline{\partial} \in c_1(L)$ such that Corollary 6.8 is still valid; in this case, h is taken to be the weight function corresponding to $c(L)_1$. Everything thus works as in the ample case.

7. Choice of the logarithmic singularities

Let us assume (with the notation of §6) that each point x_j is isolated in $E_1(\varphi)$. Then we conclude by (4.3) and Corollary 4.6 that there is a surjective map

$$H^0(X, K_X+L) \twoheadrightarrow \bigoplus_{1\le j\le N} \mathcal{O}(K_X+L)_{x_j}(\mathcal{O}_X/\mathcal{I}((1-\varepsilon)\varphi))_{x_j}. \tag{7.1}$$

However, finding sufficient conditions ensuring that x_j is isolated in $E_1(\varphi) = E_1(T)$ is a harder question. Therefore, we postpone this task to the next section and explain instead how to choose the logarithmic poles $\log|g_j|$ and the constants τ_j to obtain specified ideals and jets of sections at each point x_j.

Suppose that an ideal $\mathcal{J}_i \subset \mathcal{M}_{X,x}$ is given at x_j, in other words, that we are given a 0-dimensional subscheme $(\Xi, \mathcal{O}_\Xi)$ with $\Xi = \{x_1, \cdots, x_N\}$ and $\mathcal{O}_{\Xi,x_j} = \mathcal{O}_{X,x_j}/\mathcal{J}_j$. We want to find sufficient conditions for the surjectivity of the restriction map

$$\begin{aligned} H^0(X, K_X+L) &\to H^0(\Xi, \mathcal{O}_\Xi(K_X+L)) \\ &= \bigoplus_{1\le j\le N} \mathcal{O}(K_X+L)_{x_j}\otimes \mathcal{O}_{X,x_j}\mathcal{J}_j. \end{aligned}$$

By (7.1), we need only find a germ of map $g_j\colon (X, x_j)\to(\mathbb{C}^n, 0)$ and a constant $\tau_{j,0}$ such that $\mathcal{I}(\tau_{j,0}\log|g_j|)\subset\mathcal{J}_j$. For $\tau_j > \tau_{j,0}$ and ε small enough, Corollary 6.8 then implies $\mathcal{I}((1-\varepsilon)\varphi)\subset\mathcal{J}_j$. Thus we have to choose σ slightly larger than $\sigma_0 = \sum \rho_j\tau_{j,0}^n$ where ρ_j is the degree of the covering map g_j; this is possible only if $L^n > \sigma_0$. Let us discuss some specific cases.

Spannedness. To obtain that K_X+L spans at $x\in X$, we consider a single point $x_1 = x$ and take $\mathcal{J}_1 = \mathcal{M}_{X,x}$, $g_1(z) = (z_1, \cdots, z_n)$, $\tau_{1,0} = n$ and $\sigma_0 = \tau_{1,0}^n = n^n$. Then $\mathcal{I}(\tau_{1,0}\log|g_1|)\subset\mathcal{M}_{X,x}$, as desired.

Separation of points. To obtain the separation of two points $x_1\neq x_2$ in X by sections of K_X+L, we make the same choices as above at x_1, x_2 and get $\sigma_0 = \tau_{1,0}^n + \tau_{2,0}^n = 2n^n$. If x_1, x_2 are "infinitely near" in some direction $\xi \in TX$, we choose coordinates $(z_1, \cdots, z_n)$ centered at $x = x_1 = x_2$ so that $\partial/\partial z_n = \xi$, and we set $\mathscr{T}_1 = (z_1, \cdots, z_{n-1}, z_n^2)$. By Example 4.8, we can choose $g_1(z) = (z_1, \cdots, z_{n-1}, z_n^2)$ and $\tau_{1,0} = n$. Then the degree of g_1 is $\rho_1 = 2$ and we find again $\sigma_0 = \rho_1 \tau_{1,0}^n = 2n^n$.

Generation of s-jets. Instead of just considering jets at one point, we wish to look at several points simultaneously which may come into coincidence. Such a concern appears also in the work of Beltrametti-Sommese [6] where an extensive study of the surface case is made. The relevant definition is as follows.

Definition 7.2. We say that L generates s-jets on a given subset $\Xi \subset X$ if $H^0(X, L) \twoheadrightarrow \bigoplus J_{x_j}^{s_j} L$ is onto for any choice of points $x_1, \cdots, x_N \in \Xi$ and integers $s_1, \cdots, s_N$ with $\sum(s_j + 1) = s + 1$. We say that L is s-jet ample if the above property holds for $\Xi = X$.

With this terminology, L is 0-jet ample if and only if L is spanned, and 1-jet ample if and only if L is very ample. In order that $K_X + L$ generates s-jets on Ξ, we take $x_1, \cdots, x_N \in \Xi$ arbitrary, $g_j(z) = (z_1, \cdots, z_n)$ at each x_j and $\tau_{j,0} = n + s_j$. Therefore $\sigma_0 = \max \sum(n + s_j)^n$ over all decompositions $s + 1 = \sum(s_j + 1)$. In fact, if we set $t_j = s_j + 1$, the following lemma gives $\sigma_0 = (n + s)^n$; that is, the maximum is reached when only one point occurs.

Lemma 7.3. *Let $t_1, \cdots, t_N \in [1, +\infty[$. Then*

$$\sum_{1 \le j \le N} (n - 1 + t_j)^n \le \left(n - 1 + \sum_{1 \le j \le N} t_j\right)^n .$$

Proof. The right-hand side of a polynomial with nonnegative coefficients and the coefficient of a monomial t_j^k involving exactly one variable is the same as on the left-hand side (however, the constant term is smaller). Thus the difference is increasing in all variables and we need only consider the case $t_1 = \cdots = t_N = 1$. This case follows from the obvious inequality

$$n^n N = n^n + \binom{n}{1} n^{n-1}(N - 1) \le (n + N - 1)^n .$$

Corollary 7.4. *Let L is a big nef line bundle. A sufficient condition for spannedness (resp. separation of points, s-jet ampleness) of $K_X + L$ on a given set Ξ is $L^n > \sigma_0$ with*

$$\sigma_0 = n^n, \qquad \textit{resp. } \sigma_0 = 2n^n, \qquad \textit{resp. } \sigma_0 = (n + s)^n,$$

provided that the solution ω_ε of (6.5) (resp. the solution $\omega_{m,\varepsilon} = \omega_m' + \frac{i}{\pi}\partial\overline{\partial}\psi_{m,\varepsilon}$ of (6.10)), always has a subsequence converging to a current T for which all points $x \in \Xi \cap E_1(T)$ are isolated in $E_1(T)$.

Case of an arbitrary 0-dimensional subscheme. Let $\mathcal{J}_j = (h_{j,k})_{1\le k\le N}$ be an arbitrary ideal in $\mathcal{O}_{X,x_j}$ with $V\mathcal{J}_j = \{x_j\}$. By Remark 4.9, we can take $g_j = (g_{j,1}, \cdots, g_{j,n})$ to be n generic elements of $\mathcal{J}_j$ and $\tau_{j,0} = n$. Indeed, property (4.10) then shows that $\mathcal{J}(\gamma\log|g_j|) \subset \mathcal{J}_j$ for $\gamma > n$. In this case, we find $\sigma_0 = n^n \sum \rho_j$. Unfortunately, this value is in general very far from a being optimal: for instance, we would get $\sigma_0 = n^n(s+1)^n$ instead of $(n+s)^n$ in the case of s-jets. If $(\Xi, \mathcal{O}_\Xi)$ is a local complete intersection, that is, if each $\mathcal{J}_j$ has $N = n$ generators, we simply take $g_j = (h_{j,1}, \cdots, h_{j,n})$. Thus we obtain $\rho_j = \dim \mathcal{O}_{X,x_j}/\mathcal{J}_j$ and

$$\sigma_0 = n^n h^0(\Xi, \mathcal{O}_\Xi). \tag{7.5}$$

8. Upper bound for the 1-codimensional polar components

The goal of this section is to give a rather simple derivation of numerical conditions ensuring that $\operatorname{codim}(E_1(T), x) \ge 2$ at a given point x. In particular, we will obtain a criterion for very ample line bundles over surfaces. Although these results are only formal consequences of those obtained in the next two sections, we feel preferable to indicate first the basic ideas in a simple case.

Let L again denote an ample line bundle over a projective algebraic manifold X and keep the same notations as in §6. Siu's decomposition formula (3.8) applied to $T = \lim \omega_{\varepsilon_\nu}$ gives

$$T = \sum_{k=1}^{+\infty} \lambda_k [H_k] + R, \tag{8.1}$$

where $[H_k]$ is the current of integration over an irreducible hypersurface H_k and $\operatorname{codim} E_c(R) \ge 2$ for every $c > 0$. As we would like $E_1(T)$ to have isolated points at x_j, a difficulty may come from the singular points of high multiplicities in the hypersurfaces H_k. We thus need to find upper bounds for the coefficients λ_k. The convexity inequality 5.2 can be used for this purpose to obtain a lower bound of the mass of R:

Proposition 8.2. *We have*

$$\sum_{k=1}^{+\infty} \lambda_k L^{n-1} \cdot H_k \le \sigma_1 = (1 - (1 - \sigma/L^n)^{1/n})L^n.$$

Proof. As $\int_X T \wedge \omega^{n-1} = L^n$ and $\int_X [H_k] \wedge \omega^{n-1} = L^{n-1} \cdot H_k$, we need only prove that

$$\int_X R \wedge \omega^{n-1} \geq (1 - \sigma/L^n)^{1/n} L^n . \tag{8.3}$$

Let θ be a smooth function on X such that $0 \leq \theta \leq 1$, $\theta = 1$ in a neighborhood of $\bigcup_{1 \leq k \leq N} H_K$ and $\int_X \theta\omega^n < \varepsilon_0$, where $\varepsilon_0 > 0$ is an arbitrarily small number. This is possible because $\bigcup_{1 \leq k \leq N} H_k$ is a closed set of zero Lebesgue measure in X. Then

$$\begin{aligned}\int_X (1-\theta) T \wedge \omega^{n-1} &\geq \liminf_{\varepsilon \to 0} \int_X (1-\theta)\omega_\varepsilon \wedge \omega^{n-1} \\ &\geq \liminf_{\varepsilon \to 0} \int_X (1-\theta)(\omega_\varepsilon^n)^{1/n}(\omega^n)^{1-1/n} \\ &\geq (1 - \sigma/L^n)^{1/n} \int_X (1-\theta)\omega^n\end{aligned}$$

by the convexity inequality and equation (6.5). By our choice of θ we have $(1-\theta)T \leq \sum_{k>N} \lambda_k[H_k] + R$, so

$$\int_X \left(\sum_{k>N} \lambda_k [H_k] + R \right) \wedge \omega^{n-1} \geq (1 - \sigma/L^n)^{1/n}(L^n - \varepsilon_0) .$$

Since N and ε_0 were arbitrary, we get the expected inequality (8.3). q.e.d.

If L is big and nef, the same result can be obtained by replacing ω with ω_{k_0} and ω_ε with $\omega_{k,\varepsilon}$ in the above inequalities ($\varepsilon \to 0$, $k \to +\infty$) and by letting k_0 tend to $+\infty$ at the end. Now suppose that for any hypersurface H in X passing through a given point x we have

$$L^{n-1} \cdot H > (1 - (1 - \sigma_0/L^n)^{1/n}) L^n . \tag{8.4}$$

We can choose $\sigma = \sigma_0 + \varepsilon$ such that inequality (8.4) is still valid with σ instead of σ_0, and then all hypersurfaces H_k passing through x have coefficients $\lambda_k < 1$ in (8.1). Thus

$$\begin{aligned} E_1(T) \subset E_1(R) \cup \bigcup_k H_{s,\mathrm{sing}} \cup \bigcup_{k \neq l} (H_k \cap H_l) \\ \cup \bigcup_{\lambda_k < 1} (H_k \cap E_{1-\lambda_k}(R)) \cup \bigcup_{\lambda_k \geq 1} H_k ,\end{aligned}$$

because the contribution of $[H_k]$ to the Lelong number of T is equal to 1 at a regular point. As all terms in the union have codimension ≥ 2 except the last ones which do not contain x, condition (8.4) ensures that $\operatorname{codim}(E_1(T), x) \geq 2$. In the case of surfaces, we can therefore apply Corollary 7.4 to obtain

Corollary 8.5. *Let X be smooth algebraic surface, and L a big nef line bundle over X. Then on a given subset $\Xi \subset X$*

$K_X + L$		*is spanned*	*separates points*			*generates s-jets*
when	$L^2 >$	4	8	9	12	$(2+s)^2$
	$\forall C,\ L \cdot C >$	2	6	5	4	$2 + 3s + s^2$

for all curves $C \subset X$ intersecting Ξ. In particular, if L is ample, $K_X + mL$ is always globally spanned for $m \geq 3$ and very ample for $m \geq 5$.

Proof. For s-jets, we have $\sigma_0 = (2+s)^2$, so we find the condition

$$L^2 > (2+s)^2, \qquad L \cdot C > (1 - (1 - (2+s)^2/L^2)^{1/2})L^2.$$

The last constant decreases with L^2 and is thus at most equal to the value obtained when $L^2 = (2+s)^2 + 1$; its integral part is precisely $2 + 3s + s^2$. q.e.d.

The above lower bounds on L^2 are sharp but not those for $L \cdot C$. Reider's method shows in fact that $K_X + mL$ is very ample as soon as $m \geq 4$. In the higher dimensional case, a major difficulty is to ensure that the germs $(E_1(T), x_j)$ do not contain any analytic set of dimension $1, 2, \cdots, n-2$. This cannot be done without considering "self-intersections" of T and prescribing suitable bounds for all intermediate intersection numbers $L^p \cdot Y$.

9. Approximation of closed positive $(1, 1)$-currents by divisors

Let L be a line bundle with $c_1(L) \in \Gamma_+$ and let $T = c(L) \geq 0$ be the curvature current of some singular metric on L. Our goal is to approximate T in the weak topology by divisors which have roughly the same Lelong numbers as T. The existence of weak approximations by divisors has already been proved in [33] for currents defined on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ with $H^2(\Omega, \mathbb{R}) = 0$, and in [15] in the situation considered here. However, the result of [15] is less precise than what we actually need, and moreover the proof contains a small gap; a complete proof will therefore be included here.

Proposition 9.1. *For any $T = c(L) \geq 0$ and any ample line bundle F, there is a sequence of nonzero sections $h_s \in H^0(X, p_sF + q_sL)$ with $p_s, q_s > 0$, $\lim q_s = +\infty$ and $\lim p_s/q_s = 0$, such that the divisors $D_s = (1/q_s)\operatorname{div}(h_s)$ satisfy $T = \lim D_s$ in the weak topology and $\sup_{x \in X} |\nu(D_s, x) - \nu(T, x)| \to 0$ as $s \to +\infty$.*

Remark 9.2. The proof will actually show, with very slight modifications, that Proposition 9.1 also holds when X is a Stein manifold and L is an arbitrary holomorphic line bundle. The last assertion concerning Lelong numbers implies that there is a sequence $\varepsilon_s > 0$ converging to 0 such that $E_c(T) = \bigcap_{s\geq 1} E_{c-\varepsilon_s}(D_s)$. When D is an effective divisor, given locally as the divisor of a holomorphic function h, then $E_c(D)$ is the set of points $x \in X$ such that the derivatives $h^{(\alpha)}(x) = 0$ for all multi-indices α with $|\alpha| < c$. This gives a new proof of Siu's result [42] that $E_c(T)$ is an analytic set, at least in the case of bidegree $(1, 1)$-currents (in fact the case of an arbitrary bidegree is easily reduced to the $(1, 1)$ case by a standard argument due to P. Lelong). Proposition 9.1 is therefore already nontrivial locally.

Proof. We first use Hörmander's L^2 estimates to construct a suitable family of holomorphic sections and combine this with some ideas of [31] in a second step. Select a smooth metric with positive curvature on F, choose $\omega = c(F) > 0$ as a Kähler metric on X and fix some large integer m (how large m must be will be specified later). For all $s \geq 1$ we define

$$w_s(z) = \sup_{1\leq j\leq N} \frac{1}{s} \log \|f_j(z)\|,$$

where $(f_1, \cdots, f_N)$ is an orthonormal basis of the space of sections of $\mathcal{O}(mF + sL)$ with finite global L^2 norm $\int_X \|f\|^2 \, dV_\omega$. Let e_F and e_L be nonvanishing holomorphic sections of F, L on a trivializing open set Ω, and let $e^{-\psi} = \|e_F\|$, $e^{-\varphi} = \|e_l\|$ be the corresponding weights. If f is a section of $\mathcal{O}(mF+sL)$ and if we still denote by f the associated complex-valued function on Ω with respect to the holomorphic frame $e_F^m \otimes e_L^s$, we have $\|f(z)\| = |f(z)|e^{-m\psi(z)-s\varphi(z)}$; here φ is plurisubharmonic, ψ is smooth and strictly plurisubharmonic, and $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, $\omega = \frac{i}{\pi}\partial\overline{\partial}\psi$. In Ω, we can write

$$w_s(z) = \sup_{1\leq j\leq N} \frac{1}{s} \log |f_j(z)| - \varphi(z) - \frac{m}{s}\psi(z).$$

In particular $T_s := \frac{i}{\pi}\partial\overline{\partial}w_s + T + \frac{m}{s}\omega$ is a closed positive current belonging to the cohomology class $c_1(L) + \frac{m}{s}c_1(F)$.

Step 1. We check that T_s converges to T as s tends to $+\infty$ and that T_s satisfies the inequalities

$$\nu(T, x) - \frac{n}{s} \leq \nu(T_s, x) \leq \nu(T, x)$$

at every point $x \in X$. Note that T_s is defined on Ω by $T_s = \frac{i}{\pi}\partial\overline{\partial}v_{s,\Omega}$

with

$$v_{s,\Omega}(z) = \sup_{1\le j\le N} \frac{1}{s}\log|f_j(z)|, \qquad \int_\Omega |f_j|^2 e^{-2m\psi-2s\varphi}\, dV_\omega \le 1.$$

We suppose here that Ω is a coordinate open set with analytic coordinates $(z_1, \cdots, z_n)$. Take $z \in \Omega' \Subset \Omega$ and $r \le r_0 = \frac{1}{2} d(\Omega', \partial\Omega)$. By the L^2 estimate and the mean value inequality for subharmonic functions, we obtain

$$|f_j(z)|^2 \le \frac{C_1}{r^{2n}} \int_{|\zeta - z|<r} |f_j(\zeta)|^2\, d\lambda(\zeta) \le \frac{C_2}{r^{2n}} \sup_{|\zeta-z|<r} e^{2s\varphi(\zeta)}$$

with constants C_1, C_2 independent of s and r (the smooth function ψ is bounded on any compact subset of Ω). Hence we infer

$$v_{s,\Omega}(z) \le \sup_{|\zeta-z|<r} \varphi(\zeta) + \frac{1}{2s}\log\frac{C_2}{r^{2n}}. \tag{9.3}$$

If we choose for example $r = 1/s$ and use the upper semicontinuity of φ, we infer $\limsup_{s\to+\infty} v_{s,\Omega} \le \varphi$. Moreover, if $\gamma = \nu(\varphi, x) = \nu(T, x)$, then $\varphi(\zeta) \le \gamma \log|\zeta - x| + O(1)$ near x. By taking $r = |z - x|$ in (9.3), we find

$$v_{s,\Omega}(z) \le \sup_{|\zeta-x|<2r} \varphi(\zeta) - \frac{n}{s}\log r + O(1) \le \left(\gamma - \frac{n}{s}\right)\log|z-x| + O(1),$$

$$\nu(T_s, x) = \nu(v_{s,\Omega}, x) \ge \left(\gamma - \frac{n}{s}\right)_+ \ge \nu(T, x) - \frac{n}{s}.$$

In the opposite direction, the inequalities require deeper arguments since we actually have to construct sections in $H^0(X, mF + sL)$. Assume that Ω is chosen isomorphic to a bounded pseudoconvex open set in $\mathbb{C}^n$. By the Ohsawa-Takegoshi L^2 extension theorem [37], for every point $x \in \Omega$, there is a holomorphic function g on Ω such that $g(x) = e^{s\varphi(x)}$ and

$$\int_\Omega |g(z)|^2 e^{-2s\varphi(z)}\, d\lambda(z) \le C_3,$$

where C_3 depends only on n and $\operatorname{diam}(\Omega)$. For $x \in \Omega'$, we set

$$\sigma(z) = \theta(|z-x|/r) g(z) e_F(z)^m \otimes e_L(z)^s, \qquad r = \min(1, 2^{-1} d(\Omega', \partial\Omega)),$$

where $\theta\colon \mathbb{R} \to [0, 1]$ is a cut-off function such that $\theta(t) = 1$ for $t < 1/2$ and $\theta(t) = 0$ for $t \ge 1$. We solve the global equation $\overline{\partial} u = v$ on X with $v = \overline{\partial}\sigma$, after multiplication of the metric of $mF + sL$ with the weight

$$e^{-2n\rho_x(z)}, \qquad \rho_x(z) = \theta(|z-x|/r)\log|z-x| \le 0.$$

The $(0, 1)$-form v can be considered as a $(n, 1)$-form with values in the line bundle $\mathcal{O}(-K_X + mF + sL)$ and the resulting curvature form of this bundle is

$$\mathrm{Ricci}(\omega) + m\omega + sT + n\frac{i}{\pi}\partial\overline{\partial}\rho_x .$$

Here the first two summands are smooth, $i\partial\overline{\partial}\rho_x$ is smooth on $X\setminus\{x\}$ and ≥ 0 on $B(x, r/2)$, and T is a positive current. Hence by choosing m large enough, we can suppose that this curvature form is $\geq \omega$, uniformly for $x \in \Omega'$. By Proposition 4.1, we get a solution u on X such that

$$\int_X \|u\|^2 e^{-2n\rho_x}\, dV_\omega \leq C_4 \int_{r/2<|z-x|<r} |g|^2 e^{-2m\psi-2s\varphi-2n\rho_x}\, dV_\omega \leq C_5 ;$$

to get the estimate, we observe that v has support in the corona $r/2 < |z-x| < r$ and that ρ_x is bounded there. Thanks to the logarithmic pole of ρ_x, we infer that $u(x) = 0$. Moreover

$$\int_\Omega \|\sigma\|^2\, dV_\omega \leq \int_{\Omega'+B(0, r/2)} |g|^2 e^{-2m\psi-2s\varphi}\, dV_\omega \leq C_6 ;$$

hence $f = \sigma - u \in H^0(X, mF + sL)$ satisfies $\int_X \|f\|^2\, dV_\omega \leq C_7$ and

$$\|f(x)\| = \|\sigma(x)\| = \|g(x)\|\, \|e_F(x)\|^m \|e_L(x)\|^s = \|e_F(x)\|^m = e^{-m\psi(x)} .$$

In our orthogonal basis (f_j), we can write $f = \sum \lambda_j f_j$ with $\sum |\lambda_j|^2 \leq C_7$. Therefore

$$e^{-m\psi(x)} = \|f(x)\| \leq \sum |\lambda_j| \sup \|f_j(x)\| \leq \sqrt{C_7 N} e^{s w_s(x)} ,$$
$$w_s(x) \geq \frac{1}{s} \log(C_7 N)^{-1/2} \|f(x)\| \geq -\frac{1}{s}[\log(C_7 N)^{1/2} + m\psi(x)] ,$$

where $N = \dim H^0(X, mF + sL) = O(s^n)$. By adding $\varphi + \frac{m}{s}\psi$, we get $v_{s,\Omega} \geq \varphi - C_8 s^{-1} \log s$. Thus $\lim_{s\to+\infty} v_{s,\Omega} = \varphi$ everywhere, $T_s = \frac{i}{\pi}\partial\overline{\partial} v_{s,\Omega}$ converges weakly to $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$, and

$$\nu(T_s, x) = \nu(v_{s,\Omega}, x) \leq \nu(\varphi, x) = \nu(T, x) .$$

Note that $\nu(v_{s,\Omega}, x) = \frac{1}{s} \min \mathrm{ord}_x(f_j)$ where $\mathrm{ord}_x(f_j)$ is the vanishing order of f_j at x, so our initial lower bound for $\nu(T_s, x)$ combined with the last inequality gives

$$\nu(T, x) - \frac{n}{s} \leq \frac{1}{s} \min \mathrm{ord}_x(f_j) \leq \nu(T, x) . \tag{9.4}$$

Step 2: Construction of the divisors D_s. Select sections $(g_1, \cdots, g_N) \in H^0(X, m_0F)$ with m_0 so large that m_0F is very ample, and set

$$h_{k,s} = f_1^k g_1 + \cdots + f_N^k g_N \in H^0(X, (m_0 + km)F + ksL).$$

For almost every N-tuple $(g_1, \cdots, g_N)$, Lemma 9.5 below, and the weak continuity of $\partial\overline{\partial}$ show that

$$D_{k,s} = \frac{1}{ks}\frac{i}{\pi}\partial\overline{\partial}\log|h_{k,s}| = \frac{1}{ks}\operatorname{div}(h_{k,s})$$

converges weakly to $T_s = \frac{i}{\pi}\partial\overline{\partial}v_{s,\Omega}$ as k tends to $+\infty$, and that

$$\nu(T_s, x) \leq \nu\left(\frac{1}{ks}D_{k,sx}\right) \leq \nu(T, x) + \frac{1}{ks}.$$

This, together with the first step, implies the proposition for some subsequence $D_s = D_{k(s),s}$. We even obtain the more explicit inequality

$$\nu(T, x) - \frac{n}{s} \leq \nu\left(\frac{1}{ks}D_{k,s}, x\right) \leq \nu(T, x) + \frac{1}{ks}.$$

Lemma 9.5. *Let Ω be an open subset in $\mathbb{C}^n$ and let $f_1, \cdots, f_N \in H^0(\Omega, \mathcal{O}_\Omega)$ be nonzero functions. Let $G \subset H^0(\Omega, \mathcal{O}_\Omega)$ be a finite-dimensional subspace whose elements generate all 1-jets at any point of Ω. Finally, set $v = \sup\log|f_j|$ and*

$$h_k = f_1^k g_1 + \cdots + f_N^k g_N, \qquad g_j \in G \setminus \{0\}.$$

Then for all $(g_1, \cdots, g_N)$ in $(G\setminus\{0\})^N$ except a set of measure 0, *the sequence $\frac{1}{k}\log|h_k|$ converges to v in $L^1_{\mathrm{loc}}(\Omega)$ and*

$$\nu(v, x) \leq \nu\left(\frac{1}{k}\log|h_k|\right) \leq \nu(v, x) + \frac{1}{k}, \quad \forall x \in X, \ \forall k \geq 1.$$

Proof. The sequence $\frac{1}{k}\log|h_k|$ is locally uniformly bounded above and we have

$$\lim_{k\to+\infty}\frac{1}{k}\log|h_k(z)| = v(z)$$

at every point z where all absolute values $|f_j(z)|$ are distinct and all $g_j(z)$ are nonzero. This is a set of full measure in Ω because the sets $\{|f_j|^2 = |f_l|^2, \ j \neq l\}$ and $\{g_j = 0\}$ are real analytic and thus of zero measure (without loss of generality, we may assume that Ω is connected and that the f_j's are not pairwise proportional). The well-known uniform integrability properties of plurisubharmonic functions then show that

$\frac{1}{k}\log|h_k|$ converges to v in $L^1_{\mathrm{loc}}(\Omega)$. It is easy to see that $\nu(v, x)$ is the minimum of the vanishing orders $\mathrm{ord}_x(f_j)$; hence

$$\nu(\log|h_k|, x) = \mathrm{ord}_x(h_k) \geq k\nu(v, x).$$

In the opposite direction, consider the set $\mathscr{E}_k$ of all $(N+1)$-tuples

$$(x, g_1, \cdots, g_N) \in \Omega \times G^N$$

for which $\nu(\log|h_k|, x) \geq k\nu(v, x) + 2$. Then $\mathscr{E}_k$ is a constructible set in $\Omega \times G^N$: it has a locally finite stratification by analytic sets, since

$$\mathscr{E}_k = \bigcup_{s\geq 0}\Big(\bigcup_{j, |\alpha|=s}\{x; D^\alpha f_j(x) \neq 0\} \times G^N\Big) \cap \bigcup_{|\beta|\leq ks+1}\{(x, g_l); D^\beta h_k(x) = 0\}.$$

The fiber $\mathscr{E}_k \cap (\{x\} \times G^N)$ over a point $x \in \Omega$ where $\nu(v, x) = \min \mathrm{ord}_x(f_j)$ $= s$ is the vector space of N-tuples $(g_j) \in G^N$ satisfying the equations $D^\beta(\sum f_j^k g_j(x)) = 0$, $|\beta| \leq ks+1$. However, if $\mathrm{ord}_x(f_j)$, the linear map

$$(0, \cdots, 0, g_j, 0, \cdots, 0) \mapsto (D^\beta(f_j^k g_j(x)))_{|\beta|\leq ks+1}$$

has rank $n+1$, because it factorizes into an injective map $J_x^1 g_j \mapsto J_x^{ks+1}(f_j^k g_j)$. It follows that the fiber $\mathscr{E}_k \cap (\{x\} \times G^N)$ has codimension at least $n+1$. Therefore

$$\dim \mathscr{E}_k \leq \dim(\Omega \times G^N) - (n+1) = \dim G^N - 1,$$

and the projection of $\mathscr{E}_k$ on G^N has measure zero by Sard's theorem. By definition of $\mathscr{E}_k$, any choice of $(g_1, \cdots, g_N) \in G^N \setminus \bigcup_{k\leq 1} \mathrm{pr}(\mathscr{E}_k)$ produces functions h_k such that $\nu(\log|h_k|, x) \leq k\nu(v, x) + 1$ on Ω.

10. Self-intersection inequality for closed positive currents

Let L be a nef line bundle over a projective algebraic manifold X and let $T = c(L) \geq 0$ be the curvature current of any singular metric on L. We want to derive a bound for the codimension p components in the sublevel sets $E_c(T)$ in terms of the pth power $\{T\}^p$ of the cohomology class of T. The difficulty is that, in general, T^p does not make sense as a current. However, products of currents can be defined in some special circumstances. Let M be an arbitrary complex manifold and $n = \dim_{\mathbb{C}} X$. Suppose given a closed positive current of bidegree (p, p) on M and a *locally bounded* plurisubharmonic function ψ on M. According to Bedford-Taylor [4], the product $\Theta \wedge i\partial\overline{\partial}\psi$ can then be defined by

$$\Theta \wedge i\partial\overline{\partial}\psi = i\partial\overline{\partial}(\psi\Theta). \tag{10.1}$$

Here Θ is a differential form with measure coefficients, so its product by the locally bounded Borel function ψ is a well-defined current of order 0, and the derivative $\partial\overline{\partial}$ can be taken in the sense of distribution theory. The resulting current $\Theta \wedge i\partial\overline{\partial}\psi$ is again positive, as is easily seen by taking the weak limit with a sequence of smooth approximation of ψ. More generally, if $\psi_1, \cdots, \psi_m$ are locally bounded plurisubharmonic functions, the product $\Theta \wedge i\partial\overline{\partial}\psi_1 \wedge \cdots \wedge i\partial\overline{\partial}\psi_m$ is well defined by induction on m. Various examples (cf. [29]) show that such products cannot be defined in a reasonable way for arbitrary plurisubharmonic functions ψ_j. However, functions with $-\infty$ poles can be admitted if the polar set is sufficiently small.

Proposition 10.2. *Let ψ be a plurisubharmonic function on M such that ψ is locally bounded on $M \setminus A$, where A is an analytic subset of M of codimension $\geq p+1$ at each point. Then $\Theta \wedge i\partial\overline{\partial}\psi$ can be defined in such a way that $\Theta \wedge i\partial\overline{\partial}\psi = \lim_{\nu\to+\infty} \Theta \wedge i\partial\overline{\partial}\psi_\nu$ in the weak topology of currents, for any decreasing sequence $(\psi_\nu)_{\nu\geq 1}$ of plurisubharmonic functions converging to ψ. Moreover, at every point $x \in X$ we have*

$$\nu\left(\Theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi, x\right) \geq \nu(\theta, x)\nu(\psi, x).$$

Proof. When ψ is locally bounded everywhere, we have $\lim \psi_\nu\Theta = \psi\Theta$ by the monotone convergence theorem and we can apply the continuity of $\partial\overline{\partial}$ with respect to the weak topology to conclude that $\Theta \wedge i\partial\overline{\partial}\psi = \lim_{\nu\to+\infty} \Theta \wedge i\partial\overline{\partial}\psi_\nu$.

First assume that A is discrete. Since our results are local, we may suppose that M is a ball $B(0, R) \subset \mathbb{C}^n$ and that $A = \{0\}$. For every $s \leq 0$, the function $\psi^{\geq s} = \max(\psi, s)$ is locally bounded on M, so the product $\Theta \wedge i\partial\overline{\partial}\psi^{\geq s}$ is well defined. For $|s|$ large, the function $\psi^{\geq s}$ differs from ψ only in a small neighborhood of the origin, at which ψ may have a $-\infty$ pole. Let γ be a $(n-p-1, n-p-1)$-form with constant coefficients and set $s(r) = \liminf_{|z|\to r-0} \psi(z)$. By Stokes' formula, we see that

$$\int_{B(0,r)} \Theta \wedge i\partial\overline{\partial}\psi^{\geq s} \wedge \gamma \tag{10.3}$$

does not depend on s when $s < s(r)$, for the difference of two such integrals involves the $\partial\overline{\partial}$ of a current with compact support in $B(0, r)$.

Taking $\gamma = (i\partial\overline{\partial}|z|^2)^{n-p-1}$, we see that the current $\Theta \wedge i\partial\overline{\partial}\psi$ has finite mass on $B(0, r) \setminus \{0\}$ and we can define $\langle \mathbb{1}_{\{0\}}(\Theta \wedge i\partial\overline{\partial}\psi), \gamma\rangle$ to be the limit of the integrals (10.3) as r tends to zero and $s < s(r)$. In this case, the weak convergence statement is easily deduced from the locally bounded case discussed above.

In the case where $\operatorname{codim} A \geq p+1$, we use a slicing technique to reduce the situation to the discrete case. Set $q = n - p - 1$. There are linear coordinates $(z_1, \cdots, z_n)$ centered at any singular point of A, such that 0 is an isolated point of $A \cap (\{0\} \times \mathbb{C}^{p+1})$. Then there are small balls $B' = B(0, r')$ in $\mathbb{C}^q$, $B'' = B(0, r'')$ in $\mathbb{C}^{p+1}$ such that $A \cap (\overline{B}' \times \partial B'') = \varnothing$, and the projection map

$$\pi\colon \mathbb{C}^n \to \mathbb{C}^q, \qquad z = (z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_q)$$

defines a finite proper mapping $A \cap (B' \times B'') \to B'$. These properties are preserved if we slightly change the direction of projection. Take sufficiently many projections π_m associated to coordinate systems $(z_1^m, \cdots, z_n^m)$, $1 \leq m \leq N$, such that the family of (q, q)-forms

$$i\,dz_1^m \wedge d\overline{z}_1^m \wedge \cdots \wedge i\,dz_q^m \wedge d\overline{z}_q^m$$

defines a basis of the space of (q, q)-forms. Expressing any compactly supported smooth (q, q)-form in such a basis, we see that we need only define

$$\begin{aligned}(10.4)\quad &\int_{B'\times B''} \Theta \wedge i\partial\overline{\partial}\psi \wedge f(z', z'')i\,dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge i\,dz_q \wedge d\overline{z}_q \\ &= \int_{B'} \left\{\int_{B''} f(z', \cdot)\Theta(z', \cdot) \wedge i\partial\overline{\partial}\psi(z', \cdot)\right\} i\,dz_1{\wedge}d\overline{z}_1{\wedge}\cdots{\wedge}i\,dz_q{\wedge}d\overline{z}_q,\end{aligned}$$

where f is a test function with compact support in $B' \times B''$, and $\Theta(z', \cdot)$ denotes the slice of Θ on the fiber $\{z'\} \times B''$ of the projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^q$ (see e.g. Federer [20]). Here $\Theta(z', \cdot)$ is defined for almost every $z' \in B'$ and is again a closed positive current of bidegree (p, p) on B''. The right-hand side of (10.4) makes sense since all fibers $(\{z'\} \times B'') \cap A$ are discrete and the double integral is convergent (this will be explained in a moment). The weak convergence statement can be derived from the discrete case by (10.4) and the bounded convergence theorem. Indeed, the boundedness condition is checked as follows: observe that the functions $\psi_1 \geq \cdots \geq \psi_\nu \geq \psi$ are uniformly bounded below on some cylinder

$$K_{\delta,\varepsilon} = \overline{B}((1-\delta)r') \times (\overline{B}(r'') \setminus \overline{B}((1-\varepsilon)r''))$$

disjoint from A, with $\varepsilon \ll \delta \ll 1$ so small that $\operatorname{Supp} f \subset \overline{B}((1-\delta)r') \times \overline{B}((1-\varepsilon)r'')$; for all $z' \in \overline{B}((1-\delta)r')$, the Chern-Levine-Nirenberg inequality [12] shows that

$$\int_{B((1-\varepsilon)r'')} \Theta(z', \cdot) \wedge i\partial\overline{\partial}\psi_\nu^{\geq s}(z', \cdot)$$
$$\leq C_\varepsilon \max_{K_{\delta,\varepsilon}}(\psi_1^+, |s|) \int_{B((1-\varepsilon/2)r'')} \Theta(z', \cdot) \wedge i\partial\overline{\partial}|z''|^2$$

[Proof: introduce a cut-off function $\chi_\varepsilon(z'')$ equal to 1 near $\overline{B}((1-\varepsilon)r'')$ with support in $\overline{B}((1-\varepsilon/2)r'')$, integrate by parts and write $|i\partial\overline{\partial}\chi_\varepsilon| \leq C_\varepsilon i\partial\overline{\partial}|z|^2$]; for s sufficiently large (independent of ν), the left-hand integral does not depend on s and is equal by definition to the corresponding integral involving ψ_ν; the right-hand side, of course, has a bounded integral over $B((1-\delta)r')$ because we integrate Θ against a smooth form. The same argument with ψ instead of ψ_ν shows that the right-hand side of (10.4) is convergent.

It only remains to prove the final statement concerning Lelong numbers. Assume that $M = B(0, r)$ and $x = 0$. By definition

$$\nu\left(\Theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi, x\right) = \lim_{r\to 0}\int_{|x|\leq r} \Theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\log|z|\right)^{n-p-1}.$$

Set $\gamma = \nu(\psi, x)$ and

$$\psi_\nu(z) = \max(\psi(z), (\gamma-\varepsilon)\log|z| - \nu)$$

with $0 < \varepsilon < \gamma$ (if $\gamma = 0$, there is nothing to prove). Then ψ_ν decreases to ψ and

$$\int_{|z|\leq r} \Theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\log|z|\right)^{n-p-1}$$
$$\geq \limsup_{\nu\to+\infty}\int_{|z|\leq r} \Theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi_\nu \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\log|z|\right)^{n-p-1}$$

by the weak convergence of $\Theta \wedge i\partial\overline{\partial}\psi_\nu$; here $(\frac{i}{\pi}\partial\overline{\partial}\log|z|)^{n-p-1}$ is not smooth on $\overline{B}(0, r)$, but the integrals remain unchanged if we replace $\log|z|$ by $\chi(\log|z|/r)$ with a smooth convex function χ such that $\chi(t) = t$ for $t \geq -1$ and $\chi(t) = 0$ for $t \leq -2$. Now, we have $\psi(z) \leq \gamma\log|z| + C$ near 0, so $\psi_\nu(z)$ coincides with $(\gamma-\varepsilon)\log|z| - \nu$ on a small ball $B(0, r_\nu) \subset B(0, r)$ and we infer

$$\int_{|z|\leq r} \Theta \wedge \frac{i}{\pi}\partial\overline{\partial}\psi_\nu \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\log|z|\right)^{n-p-1}$$
$$\geq (\gamma-\varepsilon)\int_{|z|\leq r_\nu} \Theta \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\log|z|\right)^{n-p} \geq (\gamma-\varepsilon)\nu(\Theta, x).$$

As $r \in]0, R[$ and $\varepsilon \in]0, \gamma[$ where arbitrary, the desired inequality follows.

Corollary 10.5. *For $1 \leq j \leq p$, let $T_j = \frac{i}{\pi}\partial\overline{\partial}\psi_j$ be closed positive $(1, 1)$-currents on a complex manifold M. Suppose that there are analytic sets $A_2 \supset \cdots \supset A_p$ in M with $\operatorname{codim} A_j \geq j$ at every point such that each ψ_j, $j \geq 2$, is locally bounded on $M \setminus A_j$. Let $\{A_{p,k}\}_{k\geq 1}$ be the irreducible components of A_p of codimension p exactly and let $\nu_{j,k} = \min_{x\in A_{p,k}} \nu(T_j, x)$ be the generic Lelong number of T_j (or ψ_j) along $A_{p,k}$. Then $T_1 \wedge \cdots \wedge T_p$ is well defined and*

$$T_1 \wedge \cdots \wedge T_p \geq \sum_{k=1}^{+\infty} \nu_{1,k} \cdots \nu_{p,k}[A_{p,k}].$$

Proof. By induction on p, Proposition 10.2 shows that $T_1 \wedge \cdots \wedge T_p$ is well defined. Moreover, we get

$$\nu(T_1 \wedge \cdots \wedge T_p, x) \geq \nu(T_1, x) \cdots \nu(T_p, x) \geq \nu_{1,k} \cdots \nu_{p,k}$$

at every point $x \in A_{p,k}$. The desired inequality is a consequence of Siu's decomposition theorem (3.8). q.e.d.

Now, let X be a projective n-fold and let T be a closed positive $(1, 1)$-current on X. By the Lebesgue decomposition theorem, we can write $T = T_{\mathrm{abc}} + T_{\mathrm{sing}}$ where T_{abc} has absolutely continuous coefficients with respect to the Lebesgue measure and the coefficients of T_{sing} are singular measures. In general, T_{abc} and T_{sing} are positive but nonclosed. We fix an arbitrary set $\Xi \subset X$ and for $p = 1, 2, \cdots, n, n+1$ we set

$$(10.6) \qquad b_p = b_p(T, \Xi) = \inf\{c > 0 ;\ \operatorname{codim}(E_c(T), x) \geq p,\ \forall x \in \Xi\},$$

with the convention that a germ has codimension $> n$ if and only if it is empty. Then $0 = b_1 \leq b_2 \leq \cdots \leq b_n \leq b_{n+1}$ with $b_{n+1} = \max_{x\in\Xi} \nu(T, x) < +\infty$, and for $c \in]b_p, b_{p+1}]$, $E_c(T)$ has codimension $\geq p$ at every point of Ξ and has at least one component of codimension p exactly which intersects Ξ. We call $b_1, b_2, \ldots$ the *jumping values* of the Lelong numbers of T over Ξ. Our goal is to prove the following fundamental inequality for the Lelong sublevel sets $E_c(T)$, when T is the curvature current $c(L)$ of a line bundle (this restriction is unnecessary but the general case is more involved; see [20] for a general proof).

Theorem 10.7. *Suppose that there is a semipositive line bundle G over X and a constant $a \geq 0$ such that $\mathscr{O}_{TX}(1)+a\pi^* G$ is nef; set $u = ac(G) \geq 0$ with any smooth semipositive metric on G. Let $T = c(L) \geq 0$ be the curvature current of a nef line bundle L, let $\Xi \subset X$ be an arbitrary subset and*

$b_p = b_p(T, \Xi)$. *Denote by* $\{Z_{p,k}\}_{k\geq 1}$ *the irreducible components of codimension* p *in* $\bigcup_{c>b_p} E_c(T)$ *which intersect* Ξ *and let* $\nu_{p,k} \in\ ]b_p, b_{p+1}]$ *be the generic Lelong number of* T *along* $Z_{p,k}$. *Then the De Rham cohomology class* $(\{T\}+b_1\{u\})\cdots(\{T\}+b_p\{u\})$ *can be represented by a closed positive current* Θ_p *of bidegree* (p, p) *such that*

$$\Theta_p \geq \sum_{k\geq 1}(\nu_{p,k} - b_1)\cdots(\nu_{p,k} - b_p)[Z_{p,k}] + (T_{\mathrm{abc}} + b_1 u) \wedge \cdots \wedge (T_{\mathrm{abc}} + b_p u).$$

The same is true for $\Xi = X$ *if we only suppose* $c_1(L) \in \Gamma_+$ *instead of* L *nef.*

Here $\bigwedge(T_{\mathrm{abc}} + b_j u)$ is computed pointwise as a (p, p)-form. It follows in particular from our inequality that T^p_{abc} has locally integrable coefficients for all p. Let ω be a Kähler metric on X. If we take the wedge product of the fundamental inequality 10.7 by ω^{n-p}, integrate over X and neglect T_{abc} in the right-hand side, we get

Corollary 10.8. *With the notation of Theorem* 10.7, *the degrees with respect to* ω *of the p-codimensional components* $Z_{p,k}$ *of* $\bigcup_{c>b_p} E_c(T)$ *intersecting* Ξ *satisfy*

$$\sum_{k=1}^{+\infty}(\nu_{p,k} - b_1)\cdots(\nu_{p,k} - b_p)\int_X [Z_{p,k}] \wedge \omega^{n-p}$$
$$\leq (\{T\} + b_1\{u\})\cdots(\{T\} + b_p\{u\}) \cdot \{\omega\}^{n-p}.$$

In particular, if D is a nef divisor and if $L = \mathcal{O}(D)$ is equipped with the singular metric such that $T = c(L) = [D]$, we get a bound for the degrees of the p-codimensional singular strata of D in terms of a polynomial of degree p in the cohomology class $\{D\}$. The case $X = \mathbb{P}^n$ is of course especially simple: Since $T\mathbb{P}^n$ is ample, we can take $u = 0$, and then the bound is simply $\{D\}^p \cdot \{\omega\}^{n-p}$; the same is true more generally as soon as TX is nef. It is natural to try to find an interpretation of the (p, p)-form involving T_{abc} in the general inequality. Unfortunately this (p, p)-form is not closed and so it does not correspond to an intrinsic cohomology class that would have a simple counterpart in algebraic geometry. Nevertheless, the additional Monge-Ampère mass provided by T_{abc} is absolutely crucial for the purely algebraic application which we shall make in the next section. Our intuition is that the additional (p, p)-form must be understood as an *excess of self-intersection,* measuring asymptotically the amount of freedom a divisor in the linear system $H^0(X, sL)$ can be kept while being moved through the fixed singular strata prescribed by sT, when s tends to infinity.

Proof of Theorem 10.7. By the first step of Theorem 9.1 and by (9.4), there is a positive line bundle F with the following property: for every $s \geq 0$ there exist sections $f_i \in H^0(X, F + sL)$, $1 \leq i \leq N(s)$, with

$$\nu(T, x) - \frac{n}{s} \leq \frac{1}{s} \min \operatorname{ord}_x(f_i) \leq \nu(T, x), \qquad \forall x \in X.$$

The main idea is to decrease the Lelong numbers by replacing each section f_i by some of its high order derivatives, or rather by some jet section. In this way, the polar components with low generic Lelong number disappear, and we can decrease the dimension so as to be able to take intersections of currents (thanks to Proposition 10.2 or Corollary 10.5). Of course, introducing jet sections also introduces symmetric powers of the cotangent bundle; this is the reason why the curvature of TX plays an important role in the inequality.

First step: Killing Lelong numbers in the singular metric of L. Consider the m-jet section $J^m f_i$ with values in the vector bundle $E_m = J^m\mathcal{O}(F + sL)$ of m-jets. First suppose that a is rational. There are exact sequences

$$0 \to S^m T^\star X \otimes \mathcal{O}(F + sL) \to E_m \to E_{m-1} \to 0,$$

and $S^m TX \otimes \mathcal{O}(maG)$ is nef by our assumptions. By induction on m we easily infer that

$$E_m^\star \otimes \mathcal{O}(2F + sL + maG)$$

is ample (in an exact sequence of vector bundles with ample extreme terms, so is the middle term). Hence there is a symmetric power of order q with $qa \in \mathbb{N}$ such that

$$S^q E_m^\star \otimes \mathcal{O}(2qF + qsL + qmaG)$$

is generated by holomorphic sections g_j. We use the pairing of $S^q E_m$ and $S^q E_m^\star$ to get sections

$$S^q(J^m f_i) \cdot g_j \in H^0(X, \mathcal{O}(2qF + qsL + qmaG)).$$

By means of these sections, for each pair (s, m) we define a new singular metric $\|\ \|_{s,m}$ on L such that

$$\|\xi\|_{s,m} = \frac{\|\xi\|}{\sum_{i,j} \|S^q(J^m f_i) \cdot g_j\|^{1/(qs)}}, \qquad \xi \in L,$$

where $\|\ \|$ denotes the original singular metric on L as well as the induced metric on $\mathcal{O}(2qF + qsL + qmaG)$; here the metric of F (resp. G) is smooth and has positive (resp. semipositive) curvature. Denote by φ the weight of the original metric on L, by $\varphi_{s,m}$ the new one, and by ψ_F, ψ_G the weights of F, G on some trivializing open set $\Omega \subset X$. Then

$$(10.9) \qquad \varphi_{s,m} = \frac{1}{qs} \log \sum_{i,j} |S^q(J^m f_i) \cdot g_j| - \frac{2}{s}\psi_F - \frac{m}{s} a\psi_G,$$

because $e^{-\varphi}$ appears in the numerator an $\exp(-2q\psi_F - qs\varphi - qma\psi_G)^{1/(qs)}$ in the denominator of $\|\xi\|_{s,m}$. As ψ_F, ψ_G are smooth and the g_j's do not vanish simultaneously, we get

$$\nu(\varphi_{s,m}, x) = \frac{1}{s} \min_i \operatorname{ord}_x(J^m f_i) = \frac{1}{s}(\min_i \operatorname{ord}_x(f_i) - m)_+ .$$

Hence we have the inequality

$$(10.10) \qquad \left(\nu(T, x) - \frac{m+n}{s}\right)_+ \leq \nu(\varphi_{s,m}, x) \leq \left(\nu(T, x) - \frac{m}{s}\right)_+ ;$$

that is, we have been able to construct a new curvature current $\frac{i}{\pi}\partial\overline{\partial}\varphi_{s,m}$ on L in which all the Lelong numbers that were $\leq m/s$ have been killed. Unfortunately the curvature is no longer ≥ 0, but by (10.9) we have

$$(10.11) \qquad \frac{i}{\pi}\partial\overline{\partial}\varphi_{s,m} \geq -\frac{2}{s}c(F) - \frac{m}{s}ac(G) = -\frac{2}{s}\omega - \frac{m}{s}u,$$

where $\omega = c(F) > 0$. Only the term $\frac{2}{s}\omega$ can be made arbitrarily small. Now, for each s, select an integer m such that $b_p < m/s \leq b_p + \frac{1}{s}$. By (10.9) and (10.10), we see that $\varphi_{s,m}$ is locally bounded on $X \setminus E_{m/s}(T)$, and the definition of b_p implies that $E_{m/s}(T)$ has codimension $\geq p$ in a neighborhood of Ξ.

Second step: Construction of the pth intersection current Θ_p. By induction on p, we suppose that Θ_{p-1} has already been constructed ($\Theta_1 = T$ satisfies the requirements for $p = 1$). By Proposition 10.2, the wedge product $\Theta_{p-1} \wedge \frac{i}{\pi}\partial\overline{\partial}\varphi_{s,m}$ is well defined in a neighborhood of Ξ. However, this is not satisfactory when $\Xi \neq X$, because we need a current defined everywhere on X. This is the reason why we have to assume L nef when $\Xi \neq X$. Under this assumption, there is for each s a smooth metric on L, associated to some weight ρ_s on the trivializing open set Ω, such that $\frac{i}{\pi}\partial\overline{\partial}\rho_s \geq -\frac{1}{s}\omega$. We introduce the weight

$$\psi_{s,m,A,B} = \sup(\varphi, \varphi_{s,m} - A, \rho_s - B),$$

where A, $B > 0$ are large constants. This weight corresponds to the singular metric on L given by

$$\|\xi\|_{s,m,A,B} = \inf(\|\xi\|, e^A\|\xi\|_{s,m}, e^B\|\xi\|_{\rho_s}).$$

Clearly $\psi_{s,m,A,B}$ converges to φ as A, B tend to $+\infty$, and $\psi_{s,m,A,B}$ is locally bounded; therefore the curvature current $T_{s,m,A,B} = \frac{i}{\pi}\partial\overline{\partial}\psi_{s,m,A,B}$ converges weakly to $T = \frac{i}{\pi}\partial\overline{\partial}\varphi$ as A, B tend to $+\infty$. Moreover, the assumed lower bound on $\frac{i}{\pi}\partial\overline{\partial}\rho_s$ combined with (10.11) implies

$$T_{s,m,A,B} \geq -\frac{2}{s}\omega - \frac{m}{s}u;$$

this is easily seen by adding $\frac{2}{s}\psi_F + \frac{m}{s}a\psi_G$ to each term in the supremum formula defining $\varphi_{s,m,A,B}$. Now, the positive (p, p)-current

$$\Theta_{p,s,m,A,B} = \Theta_{p-1} \wedge \left(T_{s,m,A,B} + \frac{2}{s}\omega + \frac{m}{s}u\right)$$

is well defined over X since $\psi_{s,m,A,B}$ is locally bounded. Its cohomology class is independent of A, B and converges to $\{c_1(L)\} \cdot (c_1(L) + b_p u)$ when s tends to $+\infty$ (by the choice of m made at the end of the first step, we have $\lim m/s = b_p$). Hence the family $(\theta_{p,s,m,A,B})$ is weakly compact. First extract a weak limit $\Theta_{p,s,m,A}$ by taking some subsequence $B_\nu \to +\infty$. By Proposition 10.2 we see that in a neighborhood of Ξ

$$\Theta_{p,s,m,A} = \lim_{B\to+\infty} \Theta_{p,s,m,A,B} = \Theta_{p-1} \wedge \left(T_{s,m,A} + \frac{2}{s}\omega + \frac{m}{s}u\right),$$

where

$$T_{s,m,A} = \frac{i}{\pi}\partial\overline{\partial}\psi_{s,m,A}, \qquad \psi_{s,m,A} = \sup(\varphi, \varphi_{s,m} - A).$$

Indeed the codimension of the set of poles of $\psi_{s,m,A}$ is at least p in a neighborhood of Ξ. Now, by (10.10), we have

$$\nu(\psi_{s,m,A}, x) \geq \min(\nu(\varphi, x), \nu(\phi_{s,m}, x)) \geq \left(\nu(T, x) - \frac{m+n}{s}\right)_+ .$$

Proposition 10.2 shows that

$$\nu(\Theta_{p,s,m,A}, x) \geq \nu(\Theta_{p-1}, x)\left(\nu(T, x) - \frac{m+n}{s}\right)_+ \qquad \text{near } \Xi.$$

By induction on p, we conclude that the generic Lelong number of $\Theta_{p,s,m,A}$ along $Z_{p,k}$ is at least equal to

$$(\nu_{p,k} - b_1)\cdots(\nu_{p,k} - b_{p-1})\left(\nu(T, X) - \frac{m+n}{s}\right)_+ .$$

In fact, $Z_{p,k}$ meets Ξ at some point x, and therefore the inequality holds at least on a neighborhood of x in $Z_{p,k}$. Siu's decomposition formula (3.8) yields

$$\Theta_{p,s,m,A} \geq (\nu_{p,k} - b_1)\cdots(\nu_{p,k} - b_{p-1})\left(\nu(T, x) - \frac{m+n}{s}\right)_+ [Z_{p,k}].$$

Now, extract a weak limit $\Theta_{p,s,m}$ for some subsequence $A_\nu \to -\infty$ and then a weak limit Θ_p for some subsequence $m_\nu/s_\nu \to b_p$ with $s_\nu \to +\infty$. We obtain a current Θ_p such that $\{\Theta_p\} = \{\Theta_{p-1}\} \cdot (c_1(L) + b_p\{u\})$ and

$$\Theta_p \geq (\nu_{p,k} - b_1) \cdots (\nu_{p,k} - b_p)[Z_{p,k}].$$

It only remains to show by induction on p that

$$\Theta_{p,\mathrm{abc}} \geq (T_{\mathrm{abc}} + b_1 u) \wedge \cdots \wedge (T_{\mathrm{abc}} + b_p u).$$

As the coefficients of $[Z_{p,k}]$ are singular with respect to the Lebesgue measure, Θ_p will actually be larger than the sum. By construction, there exists a subsequence $(s_\nu, m_\nu, A_\nu, B_\nu)$ such that

$$\Theta_p = \lim \Theta_{p-1} \wedge \left(\frac{i}{\pi}\partial\overline{\partial}\psi_{s_\nu, m_\nu, A_\nu, B_\nu} + \frac{2}{s_\nu}\omega + \frac{m_\nu}{s_\nu}\right),$$
$$\psi_{s_\nu, m_\nu, A_\nu, B_\nu} = \sup(\varphi, \varphi_{s_\nu, m_\nu} - A_\nu, \rho_{s_\nu} - B_\nu).$$

The desired lower bound follows from Lemma 10.12 below. At the beginning of the proof, a was supposed to be rational, but this extra assumption can be removed as above by extracting a weak limit $\Theta_{p,a_\nu} \to \Theta_p$ with a sequence of rational numbers decreasing to $a \in \mathbb{R}_+$. If $\Xi = X$, everything works even if we omit the term $\rho_k - B$ in the definition of $\psi_{k,m,A,b}$: we can start directly with $\psi_{k,m,A}$ because its polar set has codimension $\geq p$ on the whole space X. Hence the nef assumption on L is not necessary.

Lemma 10.12. *Let $\Omega \subset \mathbb{C}^n$ be an open subset and let φ be an arbitrary plurisubharmonic function on Ω. Set $\varphi_\nu = \max(\varphi, \psi_\nu)$ where ψ_ν is a decreasing sequence of plurisubharmonic functions converging to $-\infty$, each ψ_ν being locally bounded in Ω (or perhaps only in the complement of an analytic subset of codimension $\geq p$). Let Θ be a closed positive current of bidegree $(p-1, p-11)$. If $\Theta \wedge i\partial\overline{\partial}\nu_\nu)$ converges to a weak limit Θ', then*

$$\Theta'_{\mathrm{abc}} \geq \Theta_{\mathrm{abc}} \wedge (i\partial\overline{\partial}\varphi)_{\mathrm{abc}}.$$

Proof. Let (ρ_ε) (resp. $(\tilde{\rho}_\varepsilon)$) be a family of regularizing kernels on $\mathbb{C}^n$ (resp. on $\mathbb{R}^2$), and let $\max_\varepsilon(x, y) = (\max \star \tilde{\rho}_\varepsilon)(x, y)$ be a regularized max function. For $\varepsilon > 0$ small enough, the function

$$\varphi_{\nu,\varepsilon} = \max_\varepsilon(\varphi \star \rho_\varepsilon, \psi_\nu \star \rho_\varepsilon)$$

is plurisubharmonic and well defined on any preassigned open set $\Omega' \Subset \Omega$. As $\varphi_{\nu,\varepsilon}$ decreases to φ_ν when ε decreases to 0, Proposition 10.2 shows that

$$\lim_{\varepsilon \to 0} \Theta \wedge i\partial\overline{\partial}\varphi_{\nu,\varepsilon} = \Theta \wedge i\partial\overline{\partial}\varphi_\nu$$

in the weak topology. Let (β_j) be a sequence of test forms which is dense in the space of test forms of bidegree $(n-p, n-p)$ and contains strongly positive forms with arbitrary large compact support in Ω. Select $\varepsilon_\nu > 0$ so small that

$$\langle \Theta \wedge i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu} - \Theta \wedge i\partial\overline{\partial}\varphi_\nu, \beta_j\rangle \leq \frac{1}{\nu} \quad \text{for } j \leq \nu.$$

Then the sequence $\Theta \wedge i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu}$ is locally uniformly bounded in mass and converges weakly to the same limit Θ' as $\Theta \wedge i\partial\overline{\partial}\varphi_\nu$. Moreover, at every point $x \in \Omega$ such that $\varphi(x) > -\infty$, we have $\varphi_{\nu,\varepsilon_\nu}(x) \geq \varphi(x) \geq \psi_\nu \star \rho_{\varepsilon_\nu}(x) + 1$ for ν large, because $\lim_{\nu\to-\infty} \psi_\nu = -\infty$ locally uniformly. Hence $\varphi_{\nu,\varepsilon_\nu} = \varphi \star \rho_{\varepsilon_\nu}$ on a neighborhood of x (which may depend on ν) and $i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu}(x) = (i\partial\overline{\partial}\varphi) \star \rho_{\varepsilon_\nu}(x)$ for $\nu \geq \nu(x)$. By the Lebesgue density theorem, if μ is a measure of absolutely continuous part μ_{abc}, the sequence $\mu \star \rho_{\varepsilon_\nu}(x)$ converges to $\mu_{\mathrm{abc}}(x)$ at almost every point. Therefore $\lim i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu}(x) = (i\partial\overline{\partial}\varphi)_{\mathrm{abc}}(x)$ almost everywhere. For any strongly positive test form $\alpha = i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \overline{\alpha}_{n-p}$ of bidegree $(n-p, n-p)$ on Ω, we get

$$\begin{aligned}\int_\Omega \Theta' \wedge \alpha &= \lim_{\nu\to+\infty} \int_\Omega \Theta \wedge i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu} \wedge \alpha \\ &\geq \liminf_{\nu\to+\infty} \int_\Omega \Theta_{\mathrm{abc}} \wedge i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu} \wedge \alpha \geq \int_\Omega \Theta_{\mathrm{abc}} \wedge i\partial\overline{\partial}\varphi_{\mathrm{abc}} \wedge \alpha.\end{aligned}$$

Indeed, the first inequality holds because $i\partial\overline{\partial}\varphi_{\nu,\varepsilon_\nu}$ is smooth, and the last one results from Fatou's lemma. This implies $\Theta'_{\mathrm{abc}} \geq \Theta_{\mathrm{abc}} \wedge (i\partial\overline{\partial}\varphi)_{\mathrm{abc}}$ and Lemma 10.12 follows.

11. Proof of the criterion in arbitrary dimension

We return here to the point where we arrived at the end of §6, and apply our self-intersection inequality 10.7 to complete the proof of the Main Theorem. First suppose, with the notation of §6, that L is an ample line bundle over X. The idea is to apply inequality 10.7 to the $(1, 1)$-current $T = \lim \omega_{\varepsilon_\nu}$ produced by equation (6.5), and to integrate the inequality with respect to the Kähler form $\omega = c(L)$. Before doing this, we need to estimate the excess of intersection in terms of T^n_{abc}.

Proposition 11.1. *The absolutely continuous part T_{abc} of T satisfies*

$$T^n_{\mathrm{abc}} \geq \left(1 - \frac{\sigma}{L^n}\right) \omega^n \quad \textit{a.e. on } X.$$

Proof. The result is local, so we can work in an open set Ω which is relatively compact in a coordinate patch of X. Let ρ_δ be a family of smoothing kernels. By a well-known lemma (see e.g. [3, Proposition 5.1]), the operator $A \mapsto (\det A)^{1/n}$ is concave on the cone of nonnegative hermitian $n \times n$ matrices; hence we get

$$[(\omega_\varepsilon \star \rho_\delta(x))^n]^{1/n} \geq (\omega_\varepsilon^n)^{1/n} \star \rho_\delta(x) \geq \left(1 - \frac{\sigma}{L^n}\right)^{1/n} (\omega^n)^{1/n} \star \rho_\delta(x),$$

thanks to equation (6.5). As ε_ν tends to 0, $\omega_{\varepsilon_\nu} \star \rho_\delta$ converges to $T \star \rho_\delta$ in the strong topology of $C^\infty(\Omega)$, and thus

$$((T \star \rho_\delta)^n)^{1/n} \geq \left(1 - \frac{\sigma}{L^n}\right)^{1/n} (\omega^n)^{1/n} \star \rho_\delta \quad \text{on } \Omega.$$

Now, take the limit as δ goes to 0. By the Lebesgue density theorem $T \star \rho_\delta(x)$ converges a.e. to $T_{\mathrm{abc}}(x)$ on Ω, so we are done. q.e.d.

According to the notation used in §10, we consider an arbitrary subset $\Xi \subset X$ and introduce the jumping values

$$b_p = \inf\{c > 0 ; \operatorname{codim}(E_c(T), x) \geq p, \ \forall x \in \mathbf{X}\}.$$

By Proposition 11.1 and inequality 5.2(a), we have

$$T_{\mathrm{abc}}^j \wedge \omega^{n-j} \geq \left(1 - \frac{\sigma}{L^n}\right)^{j/n} \omega^n. \tag{11.2}$$

Now, suppose that $\mathscr{O}_{TX}(1) + a\pi^\star L$ is nef for some constant $a \geq 0$. We can then apply Theorem 10.7 with $u = a\omega$ and

$$\{\Theta_p\} = (1 + b_1 a) \cdots (1 + b_p a)\{\omega^p\}.$$

By taking the wedge product of Θ_p with ω^{n-p}, we get

$$\begin{aligned}
&(1 + b_1 a) \cdots (1 + b_p a) \int_X \omega^n \\
&\quad \geq \sum_{k \geq 1} (\nu_{p,k} - b_1) \cdots (\nu_{p,k} - b_p) \int_X [Z_{p,k}] \wedge \omega^{n-p} \\
&\qquad + \int_X (T_{\mathrm{abc}} + b_1 a\omega) \wedge \cdots \wedge (T_{\mathrm{abc}} + b_p a\omega) \wedge \omega^{n-p}.
\end{aligned}$$

Combining this inequality with (11.2) for T_{abc}^{p-j} yields

$$\begin{aligned}
(1 + b_1 a) \cdots (1 + b_p a) L^n &\geq \sum_{k \geq 1} (\nu_{p,k} - b_1) \cdots (\nu_{p,k} - b_p) L^{n-p} \cdot Z_{p,k} \\
&\quad + \sum_{0 \leq j \leq p} S_j^p(b) a^j \left(1 - \frac{\sigma}{L^n}\right)^{(p-j)/n} L^n,
\end{aligned}$$

where $S_j^p(b)$, $1 \le j \le p$, denotes the elementary symmetric function of degree j in $b_1, \cdots, b_p$ and $S_0^p(b) = 1$. As $\prod(1+b_j a) = \sum S_j^p(b) a^j$, we get

$$\begin{aligned} &\sum_{k\geq 1}(\nu_{p,k} - b_1)\cdots(\nu_{p,k} - b_p)L^{n-p}\cdot Z_{p,k} \\ (11.3)\qquad &\leq \sum_{0\leq j\leq p} S_j^p(b)a^j\left(1 - \left(1 - \frac{\sigma}{L^n}\right)^{(p-j/n)}\right)^n . \end{aligned}$$

If L is only supposed to be big and nef, we follow essentially the same arguments and replace ω in all our inequalities by $\omega_m = c(L)_m + \frac{1}{m}c(A)$ with A ample (see §6). Note that all (n, n)-forms ω_m^n were defined to be proportional to $\gamma^n = c(A)^n$, so inequality 11.1 becomes in the limit

$$T_{\mathrm{abc}}^n \geq \left(1 - \frac{\sigma}{L^n}\right)\frac{L^n}{A^n}\gamma^n = \left(1 - \frac{\sigma}{L^n}\right)\frac{L^n}{(L+\frac{1}{m}A)^n}\omega_m^n .$$

The intersection inequality (11.3) is the expected generalization of Proposition 8.2 in arbitrary codimension. In this inequality, $\varphi_{p,k}$ is the generic Lelong number of T along $Z_{p,k}$, and $Z_{p,k}$ runs over all p-dimensional components Y of $\bigcup_{c>b_p} E_c(T)$ intersecting Ξ; by definition of b_j we have $\max_k \nu_{p,k} = b_{p+1}$. Hence we obtain

Theorem 11.4. *Let L be a big nef line bundle such that $TX \otimes \mathscr{O}(aL)$ is nef, and let $T \in c_1(L)$ be the positive curvature current obtained by concentrating the Monge-Ampère mass L^n into a finite sum of Dirac measures with total mass σ, plus some smooth positive density spread over X (equations (6.5) and (6.10)). Then the jumping values b_p of the Lelong number of T over an arbitrary subset $\Xi \subset X$ satisfy the inductive inequalities*

$$(11.5)\quad (b_{p+1} - b_1)\cdots(b_{p+1} - b_p) \leq \frac{1}{\min_Y L^{n-p}\cdot Y}\sum_{0\leq j\leq p-1} S_j^p(b)a^j\sigma_{p-j},$$

where $\sigma_j = (1 - (1 - \sigma/L^n)^{j/n})L^n$, and Y runs over all p-codimensional subvarieties of X intersecting Ξ.

Observe that σ_j is increasing in j; in particular $\sigma_j < \sigma_n = \sigma$ for $j \le n-1$. Moreover, the convexity of the exponential function shows that $t \mapsto \frac{1}{t}(1 - (1 - \sigma/L^n)^t)L^n$ is decreasing, and thus $\sigma_j > \sigma_p j/p$ for $j < p$; in particular $\sigma_j > \sigma j/n$ for $j \le n-1$. We are now in a positive to prove the following general result, which contains the Main Theorem as a special case:

Theorem 11.6. *On a projective n-fold X, let $g_j: (X, x_j) \to (\mathbb{C}^n, 0)$ be germs of finite holomorphic maps with covering degree ρ_j. Let*

$\mathcal{J}_1 \subset \mathcal{O}_{X,x_1}, \cdots, \mathcal{J}_N \subset \mathcal{O}_{X,x_N}$ *be the associated ideals* $\mathcal{I}(\tau_{j,0}\log|g_j|)_{x_j}$ *and let*

$$\sigma_0 = \sum \rho_j \tau_{j,0}^n, \qquad \sigma_p = (1-(1-\sigma_0/L^n)^{p/n})L^n, \quad 1 \le p \le n-1,$$

where L *is a big nef line bundle such that* $L^n > \sigma_0$. *Suppose that* $\mathcal{O}_T X(1) + a\pi^\star L$ *is nef over* $P(T^\star X)$ *and that there is a sequence* $0 = \beta_1 < \cdots < \beta_n \le 1$ *with*

$$L^{n-p} \cdot Y > (\beta_{p+1}-\beta_1)^{-1} \cdots (\beta_{p+1}-\beta_p)^{-1} \sum_{0 \le j \le p-1} S_j^p(\beta) a^j \sigma_{p-j}$$

for every subvariety $Y \subset X$ *of codim* $p = 1, 2, \cdots, n-1$ *passing through one of the points* x_j. *Then there is a surjective map*

$$H^0(X, K_X + L) \twoheadrightarrow \bigoplus_{1 \le j \le N} \mathcal{O}(K_X+L)_{x_j} \otimes (\mathcal{O}_{X,x_j}/\mathcal{J}_j).$$

Proof. Select $\tau_j > \tau_{j,0}$ so that $L^{n-p} \cdot Y$ still satisfies the above lower bound with the corresponding value $\sigma > \sigma_0$. Then apply Theorem 11.4 with $\Xi = \{x_1, \cdots, x_N\}$. Inequality (11.5) shows inductively that $b_p < \beta_p$ for $p \ge 2$, so $b_n < 1$ and we get $\operatorname{codim}(E_1(T), x_j) = n$ at each point x_j. Thanks to (4.3), Corollaries 4.6 and 6.8 imply the desired surjectivity property.

Proof of Corollary 1. This is only a matter of straightforward calculations, but adjusting the constants β_p to get optimal exponents of σ_0, a and $\mu_\Xi(L)$ in the lower bound of m requires some care. By the convexity argument already explained, we have $\sigma_{p-j} \le (p-j)\sigma_1 \le p(1-1/p)^j\sigma_1$. As $\beta_1 = 0$, we find

$$\sum_{0 \le j \le p-1} S_j^p(\beta) a^j \sigma_{p_j} \le p\sigma_1 \sum_{0 \le j \le p-1} S_j^p(\beta) \left(\frac{p-1}{p} a\right)^j$$
$$= p\sigma_1 \left(1 + \beta_2 \frac{p-1}{p} a\right) \cdots \left(1 + \beta_p \frac{p-1}{p} a\right).$$

When we replace L by mL, the constant a is replaced by a/m, and by definition of $\mu = \mu_\Xi(L)$ we have $(mL)^k \cdot Y \ge (m\mu)^k$. Hence Theorem 11.6 yields the sufficient condition

$$(m\mu)^{n-p} \ge \prod_{j \le p} (\beta_{p+1}-\beta_j)^{-1} p\sigma_1 \left(1 + \beta_2 \frac{p-1}{p}\frac{\sigma}{m}\right) \cdots \left(1 + \beta_p \frac{p-1}{p}\frac{a}{m}\right)$$

with $0 = \beta_1 < \cdots < \beta_n < 1$. When $p = 1$ we get $(m\mu)^{n-1} \ge \beta_2^{-1}\sigma_1$, and when $p = n-1$ the inequality implies $m\mu > (n-1)\sigma_1 > (1-1/n)\sigma_0$.

We suppose in fact $m\mu \geq \lambda\sigma_0$ where $\lambda > 1 - 1/n$ is a constant which will be adjusted later to an optimal value; in particular $m\mu \geq \lambda n^n$. We will choose β_j/β_{j+1} so small that $\Pi_{j\leq p}(\beta_{p+1} - \beta_j) \geq U_p^{-1}\beta_{p+1}^p$ with a constant U_p slightly larger than 1. We are thus led to define β_p inductively by the formula

$$(11.8)\quad \beta_{p+1}^p = U_p \frac{p\sigma_1}{(m\mu)^{n-p}}\left(1 + \beta_2 \frac{p-1}{p}\frac{a}{m}\right)\cdots\left(1 + \beta_p \frac{p-1}{p}\frac{a}{m}\right), \qquad 1 \leq p \leq n-1,$$

and m has to be taken so large that $\beta_n < 1$; suppose that this is the case. The first step is to determine admissible constants $1 < U_1 < \cdots < U_{n-1}$. For $j \leq p$, (11.8) implies

$$\frac{1}{j-1}\left(\frac{\beta_j}{m\mu}\right)^{j-1} \leq \frac{1}{p-1}\left(\frac{\beta_p}{m\mu}\right)^{n-p-1} \Rightarrow \beta_j \leq \left(\frac{j-1}{n-1}\right)^{1/(j-1)}(m\mu)^{-\left(\frac{n-j}{j-1}\right)}$$

by taking $p = n$ in the first inequality. In general, for $j \leq p$ we get

$$\begin{aligned}\left(\frac{\beta_j}{\beta_p}\right)^{p-1} &\leq \frac{j-1}{p-1}\left(\frac{\beta_j}{m\mu}\right)^{p-j} \\ &\leq \frac{j-1}{p-1}\left(\frac{j-1}{n-1}\right)^{\frac{p-j}{j-1}}(m\mu)^{-(p-j)\left(1+\frac{n-j}{j-1}\right)},\end{aligned}$$

$$(11.9)\qquad \frac{\beta_j}{\beta_p} \leq \frac{(j-1)^{\frac{1}{j-1}}}{(p-1)^{\frac{1}{p-1}}(n-1)^{\frac{1}{j-1}-\frac{1}{p-1}}}(m\mu)^{-(n-1)\left(\frac{1}{j-1}-\frac{1}{p-1}\right)} \leq \frac{\gamma_j}{\gamma_p}$$

with $\gamma_j = \left(\frac{j-1}{n-1}\right)^{1/(j-1)}(\lambda n^n)^{-(n-1)(1/(j-1)-1/(n-1))}$. The sequence (γ_j) is strictly increasing and satisfies $\gamma_j/\gamma_{j+1} < 1/n$. Thus we can take

$$U_p = \prod_{2\leq j\leq p}(1 - \gamma_j/\gamma_{p+1})^{-1}.$$

Let k be the largest integer in $\{1, \cdots, n-1\}$ such that $\beta_k a/m \leq 1$ and let $t = \max(\beta_k a/m, \gamma_k/\gamma_{k+1}) \in [\gamma_k/\gamma_{k+1}, 1]$. Inequality (11.9) implies

$$\left(1 + \beta_j \frac{p-1}{p}\frac{a}{m}\right) \leq \begin{cases} \left(1 + \dfrac{p-1}{p}\dfrac{\gamma_j}{\gamma_k}t\right) & \text{for } 2 \leq j \leq k, \\ \left(\dfrac{p-1}{p} + \dfrac{\gamma_k}{\gamma_j}t^{-1}\right)\beta_j\frac{a}{m} & \text{for } k+1 \leq j \leq p. \end{cases}$$

The product of all factors $(\cdots t^{\pm 1})$ over $j = 2, \cdots, p$ is a polynomial with positive coefficients in $\mathbb{R}[t, t^{-1}]$, hence is a convex function of t.

Therefore, the product is at most equal to the maximum of its values for $t=1$ or $t=\gamma_k/\gamma_{k+1}$:

$$\max\left\{\prod_{2\le j\le k}\left(1+\frac{p-1}{p}\frac{\gamma_j}{\gamma_k}\right)\prod_{k+1\le j\le p}\left(\frac{p-1}{p}+\frac{\gamma_k}{\gamma_j}\right),\right.$$
$$\left.\prod_{2\le j\le k}\left(1+\frac{p-1}{p}\frac{\gamma_j}{\gamma_{k+1}}\right)\prod_{k+1\le j\le p}\left(\frac{p-1}{p}+\frac{\gamma_{k+1}}{\gamma_j}\right)\right\}.$$

We have $(p-1)/p+\gamma_k/\gamma_j\le 1-1/p+1/n\le 1$ for $j>k$; moreover the sequence γ_j/γ_{j+1} is increasing. If we introduce the increasing sequence

$$V_p=\prod_{2\le j\le p}\left(1+\frac{p-1}{p}\frac{\gamma_j}{\gamma_p}\right),$$

it is then easy to check that the above maximum is bounded by V_p for $p\ge k$. Therefore (11.8) gives

$$\beta_{p+1}^p\le U_pV_p\frac{p\sigma_1}{(m\mu)^{n-p}}\beta_{k+1}\cdots\beta_p\left(\frac{a}{m}\right)^{p-k}\qquad\text{for } p\ge k.$$

As $U_pV_pp\le W_n=U_{n-1}V_{n-1}(n-1)$, by induction these inequalities yield

$$\beta_{p+1}\le\left(\frac{W_n\sigma_1}{(m\mu)^{n-k}}\right)^{1/k}(a\mu)^{1/p+1/(p+1)+\cdots+1/(k+1)}\qquad\text{for } p\ge k,$$

where the exponent of $a\mu$ is understood to be 0 if $p=k$. Finally, we have $(mL)^n\ge(m\mu)^n\ge(\lambda\sigma_0)^n$ by definition of μ and by our initial hypothesis $m\mu\ge\lambda\sigma_0$; hence $\sigma_0/(mL)^n\le\lambda^{-n}\sigma_0^{1-n}\le\lambda^{-n}n^{-n(n-1)}$. It follows again from a convexity argument that

$$\sigma_1/\sigma_0\le(1-(1-\lambda^{-n}n^{-n(n-1)})^{1/n})\lambda^n n^{n(n-1)}=T_n.$$

Hence $W_n\sigma_1\le B_n\sigma_0$ with $B_n=W_nT_n=(n-1)U_{n-1}V_{n-1}T_n$. Therefore, sufficient conditions in order that $\beta_n<1$ are

(11.10) $\qquad m\mu>B_n\sigma_0\quad$ for $k=n-1$,

(11.10$_k$) $\qquad (m\mu)^{n-k}>B_n\sigma_0(a\mu)^{k[1/(n-1)+1/(n-2)+\cdots+1/(k+1)]}$

for $k\in\{1,\cdots,n-2\}$. These conditions are equivalent to the inequality stated in Corollary 1. Observe that our constant B_n depends on λ. The initial hypothesis $m\mu\ge\lambda\sigma_0$ will be automatically satisfied if we adjust λ so that $B_n(\lambda)=\lambda$; this is always possible because $B_n(\lambda)$ is decreasing in λ and $B_n(\lambda)>1-1/n$. With this choice, a numerical calculation shows that $B_n < 2.005$ for all n and $\lim_{n\to+\infty} B_n = 2$. For small values of n, we find (with rounding by above):

n	2	3	4	5	6	7	8	9	10	11	12
B_n	0.625	1.019	1.309	1.485	1.603	1.687	1.748	1.794	1.830	1.858	1.879

TABLE (11.11)

and $B_n < 2$ for $n \geq 46$.

12. Universal bounds for very ample line bundles

Let X be an ample line bundle over a projective n-fold X. In order to find universal conditions for $K_X + L$ to be very ample, our main theorem would require a universal value a depending only on $n = \dim_{\mathbb{C}} X$ such that $TX \otimes \mathscr{O}(aL)$ is always nef. However, this is clearly impossible as the example of curves already show: if X is a curve of genus g and L has degree 1, then $TX \otimes \mathscr{O}(aL)$ is nef if and only if $a \geq 2g - 2$. In general; it is an interesting unsolved question to know whether such a value a can be found explicitly in terms of geometric invariants of X (Chern classes, ...). Here, these difficulties can be avoided by means of the following simple lemma.

Lemma 12.1. *Let F be a very ample line bundle over X. Then the vector bundle $TX \otimes \mathscr{O}(K_X + nF)$ is nef and generated by global sections.*

Proof. By the very ample assumption, the 1-jet bundle J^1F is generated by its sections. Consider the exact sequence

$$0 \to T^\star X \otimes F \to J^1 F \to F \to 0$$

where $\operatorname{rank}(J^1F) = n+1$ and $\det(J^1F) = K_X+(n+1)F$. The nth exterior power $\bigwedge^n(J^1F)$ is also generated by sections and there is a surjective morphism

$$\begin{aligned}\bigwedge^n(J^1F) &= (J^1F)^\star \otimes \det(J^1F)\\ &\to (TX \otimes F^\star) \otimes \det(J^1F) = TX \otimes \mathscr{O}(K_X + nF).\end{aligned}$$

Hence $TX \otimes \mathscr{O}(K_X + nF)$ is generated by sections and, in particular, it is nef. q.e.d.

The next idea consists in the following iteration trick: Lemma 12.1 suggests that a universal lower bound for the nefness of $TX \otimes \mathscr{O}(aL')$ can be achieved with $L' = K_X + L$ if L is sufficiently ample. Then it follows from the Main Theorem that $K_X + L' = 2K_X + L$ is very ample under suitable numerical conditions. Lemma 12.1 applied with $F = 2K_X + L$ shows that $TX \otimes \mathscr{O}((2n+1)K_X + nL)$ is nef, and thus $TX \otimes \mathscr{O}((2n+1)L'')$ is nef with $L'' = K_X + \frac{1}{2}L \leq L'$. Hence we see that the Main Theorem can be iterated. The special value $a = 2n+1$ will play an important role.

Lemma 12.2. *Let L' be an ample line bundle over X. Suppose that $TX \otimes \mathscr{O}((2n+1)L')$ is nef. Then $K_X + L'$ is very ample (resp. generates s-jets) as long as $\mu_X(L') > C_n\sigma_0$ with the corresponding value of σ_0 (resp. a constant $C_n < 3$ depending only on n).*

Proof. If $\mu = \mu_X(L')$, $a = 2n+1$ and $\sigma_0 = 2n^n$ (resp. $\sigma_0 = (n+s)^n$), then the first arguments in the proof of Corollary 1 give the sufficient condition

$$\mu^{n-p} > \prod_{2 \leq j \leq p} (\beta_{p+1} - \beta_j)^{-1} p\sigma_1 \left(1 + \frac{p-1}{p}\beta_2(2n+1)\right) \times \cdots \times \left(1 + \frac{p-1}{p}\beta_p(2n+1)\right)$$

with $0 = \beta_1 < \cdots < \beta_n = 1$. We suppose $\mu \geq \lambda\sigma_0$ (in particular $\mu \geq 2\lambda n^n$) and choose

$$\beta_p = (\alpha^{n-p+1}(2n^n)^{n-p})^{1/(p-1)}, \qquad 2 \leq p \leq n-1$$

with suitable constants λ, α to be determined later. In analogy with the proof of Corollary 1, we introduce the constants

$$U'_p = \prod_{2 \leq j \leq p} \left(1 - \frac{\beta_j}{\beta_{p+1}}\right)^{-1}, \; V'_p = \prod_{2 \leq j \leq p} \left(1 + \frac{p-1}{p}\beta_j(2n+1)\right)$$
$$T'_n = (1 - (1 - \lambda^{-n}(2n^n)^{-(n-1)})^{1/n})\lambda^n(2n^n)^{n-1}.$$

We have $\sigma_1 \leq T'_n\sigma_0$ and our conditions become

$$\mu^{n-p} > U'_p\beta^p_{p+1}pT'_n\sigma_0V'_p = \begin{cases} pU'_pV'_pT'_n\alpha^{n-p}(2n^n)^{n-p-1}\sigma_0 & \text{for } p \leq n-2, \\ (n-1)U'_{n-1}V'_{n-1}T'_n\sigma_0 & \text{for } p = n-1. \end{cases}$$

As $\sigma_0 \geq 2n^n$, a sufficient condition is

$$\mu > \max\{(n-1)U'_{n-1}V'_{n-1}T'_n, \alpha(pU'_pV'_pT'_n)^{1/(n-1)}\}_{1 \leq p \leq n-2}\sigma_0.$$

We adjust λ and α so that

$$(n-1)U'_{n-1}V'_{n-1}T'_n = \alpha \max_{1 \leq p \leq n-2}(pU'_pV'_pT'_n)^{1/(n-p)} = \lambda,$$

and we take this common value to be our constant C_n. A numerical calculation gives $C_n < 3$ for all n and $\lim_{n\to+\infty} C_n = 3$. The first values are given by

n	2	3	4	5	6	7	8	9	10	11	12
C_n	0.563	0.737	0.995	1.201	1.370	1.510	1.629	1.730	1.817	1.893	1.959

TABLE (12.3)

Hence Lemma 12.2 is proved.

Lemma 12.4. *Let F be a line bundle which generates s-jets at every point. Then $L^p \cdot Y \geq s^p$ for every p-dimensional subvariety $Y \subset X$.*

Proof. Fix an arbitrary point $x \in Y$. Then consider the singular metric on F given by

$$\|\xi\|^2 = \frac{|\xi|^2}{\sum |u_j(z)|^2},$$

where $(u_1, \cdots, u_N)$ is a basis of $H^0(X, F \otimes \mathscr{M}_x^s)$. By our assumption, these sections have an isolated common zero of order s at x. Hence F possesses a singular metric such that the weight $\varphi = \frac{1}{2}\log\sum|u_j|^2$ is plurisubharmonic and has an isolated logarithmic pole of Lelong number s at x. By the comparison inequality (3.6) with $\psi(z) = \log|z - x|$, we get

$$L^p \cdot Y \geq \int_{B(x,\varepsilon)} [Y] \wedge \left(\frac{i}{\pi}\partial\bar{\partial}\varphi\right)^p \geq s^p \nu([Y], \psi) = s^p \nu(Y, x) \geq s^p.$$

Proof of Corollary 2. As L is ample, there exists an integer q (possibly very large) such that

$$(12.5) \qquad \begin{cases} K_X + qL & \text{is ample,} \\ TX \otimes \mathscr{O}((2n+1)(K_X + qL)) & \text{is nef,} \\ \mu_X(K_X + qL) > C_n \sigma_0. \end{cases}$$

By Lemma 12.2 applied to $L' = K_X + qL$, we find that $F = K_X + L' = 2K_X + qL$ is very ample and generates s-jets. In particular $K_X + \frac{q}{2}L$ is an ample $\mathbb{Q}$-divisor, and for any p-dimensional subvariety $Y \subset X$ we have

$$\begin{aligned}(K_X + (q-1)L)^p \cdot Y &= \left(\frac{1}{2}F + (q/2 - 1)L\right)^p \cdot Y \\ &= \sum_{0 \leq k \leq p} \binom{p}{k} 2^{k-p} (q/2 - 1)^k F^{p-k} \cdot L^k \cdot Y.\end{aligned}$$

By the convexity inequality 5.2 (b) and Lemma 12.4 we get

$$F^{p-k} \cdot L^k \cdot Y \geq (F^p \cdot Y)^{1-k/p} (L^p \cdot Y)^{k/p} \geq s^{p-k} (\mu_X(L))^k .$$

Hence $(K_X + (q-1)L)^p \cdot Y \geq ((q/2 - 1)\mu_X(L) + s/2)^p$ and

$$\mu_X(K_X + (q-1)L) \geq \frac{1}{2}((q-2)\mu_X(L) + s) .$$

Moreover, Lemma 12.1 applied to F shows that

$$TX \otimes \mathscr{O}(K_X + nF) = TX \otimes \mathscr{O}((2n+1)K_X + nqL)$$

is nef. As $nq/(2n+1) \leq q/2 \leq q-1$ for $q \geq 2$, we find that all properties (12.5) except perhaps the last one remain valid with $q-1$ in place of q:

$$(12.6) \qquad \begin{cases} K_X + (q-1)L & \text{is ample}, \\ TX \otimes \mathscr{O}((2n+1)(K_X + (q-1)L)) & \text{is nef}, \\ \mu_X(K_X + (q-1)L) \geq \frac{1}{2}((q-2)\mu_X(L) + s) . \end{cases}$$

By induction we conclude that (12.6) is still true for the smallest integer $q - 1 = m$ such that

$$(q-2)\mu_X(L) + s = (m-1)\mu_X(L) + s > 2C_n\sigma_0 .$$

For this value of m Lemma 12.2 implies that $2K_X + mL$ is very ample, resp. generates s-jets.

Remark 12.7. If G is a nef line bundle, the proof of Corollary 2 can be applied without modification to show that $2K_X + mL + G$ is very ample, resp. generates s-jets, for $(m-1)\mu_X(L) + s > 2C_n\sigma_0$: indeed, adding G can only increase the numbers $\mu_X(K_X + qL + G)$ occurring in the induction.

Remark 12.8. The condition $(m-1)\mu_X(L) + s > C_n\sigma_0$ is never satisfied for $m = 1$. However, it is still possible to obtain a sufficient condition in order that $2K_X + L$ generates s-jets. Indeed, the last step of the iteration shows that $2K_X + 2L$ generates s'-jets and that $\mu_X(K_X + L) \geq s'/2$ if $\mu_X(L) + s' > 2C_n(n+s')^n$. Choose $s' > 2C_n(n+s)^n$. Then $\mu_X(K_X + L) > C_n(n+s)^n$ and we can perform another iteration to conclude that $2K_X + L$ generates s-jets. Of course, the corresponding lower bound for $\mu_X(L)$ is extremely large, of the order of magnitude of $(2C_n)^{n+1}(n+s)^{n^2}$.

Remark 12.9. A numerical computation of $4C_n n^n$ in Corollary 2 gives the following bounds for $2K_X + mL$ to be very ample when L is ample:

n	2	3	4	5	6
$m \geq$	10	80	1019	15010	255537

We now list a few immediate consequences of our results, in connection with some classical questions of algebraic geometry.

Corollary 12.10. *Let X be a projective n-fold of general type with K_X ample. Then mK_X is very ample for $m \geq 12n^n$.*

Corollary 12.11. *Let X be a Fano n-fold, that is, a n-fold such that $-K_X$ is ample. Then $-mK_X$ is very ample for $m \geq 12n^n$.*

Corollaries 12.10 and 12.11 follow easily from Corollary 2 applied to $L = \pm K_X$: then we obtain that $2K_X + mL$ is very ample for $m > C_n n^n$, and a numerical check shows that $4C_n n^n + 2 < 12n^n$ for all n. Hence we get pluricanonical embeddings $\Phi\colon X \to \mathbb{P}^N$ such that $\Phi^\star\mathcal{O}(1) = \pm mK_X$ with $m = 12n^n$. The image $Y = \Phi(X)$ has degree

$$\deg(Y) = \int_Y c_1(\mathcal{O}(1))^n = \int_X c_1(\pm mK_X)^n = m^n |K_X^n|.$$

It can be easily reproved from this that there are only finitely many deformation types of Fano n-folds, as well as of n-folds of general type with K_X ample, corresponding to a given discriminant $|K_X^n|$; such results were already known by the fundamental finiteness theorem. In the Fano case, it is conjectured that there is a universal bound $(-K_X)^n \leq A_n$: if such a universal bound could be proved, it would become possible to obtain an explicit upper bound for the number of deformation types of Fano n-folds in any dimension n.

Finally, let L be an ample line bundle over an arbitrary projective n-fold X. It follows from Mori's theory that $K_X + (n+1)L$ is always nef (see [22]). If $K_X + tL$ is nef for some integer $t \geq 0$, Fujita conjectures that $m(K_X + tL)$ is spanned for every positive integer $m > n + 1 - t$. Although such a sharp result seems very hard to prove, our results allow us to prove that some explicit multiple of $K_X + (t+\varepsilon)L$ is very ample for every $\varepsilon > 0$ rational.

Corollary 12.12. *If L is an ample line bundle such that $K_X + tL$ is nef for some integer $t \geq 0$, the line bundle $m(K_X + (t+\varepsilon)L)$ is very ample for every $\varepsilon > 0$ and every integer $m > 0$ such that $m\varepsilon \in \mathbb{N}$ and $m\varepsilon > 8C_n n^n - 2t - 1$.*

Proof. First suppose that $m = 2p$ is even. Then either $p\varepsilon$ or $p\varepsilon - 1/2$ is an integer. Apply Corollary 2 to the ample line bundles

$$L' = (p-1)(K_X + tL) + (p\varepsilon + t)L,$$

and

$$L' = (p-1)(K_X + tL) + (p\varepsilon + t - 1/2)L$$

separately. In the first case, we find $\mu_X(L') \geq (p\varepsilon + t)\mu_X(L) \geq p\varepsilon + t$, and hence $2K_X + 2L' = 2p(K_X + (t+\varepsilon)L)$ is very ample when $\mu_X(L') + 1 \geq p\varepsilon + t + 1 > 4C_n n^n$; in the second case, we get the condition $p\varepsilon + t + 1/2 > 4C_n n^n$ and we apply Remark 12.7 to conclude that $2K_X + 2L' + L$ is very ample. When $m = 2p+1$ is odd, we argue in the same way with

$$L' = (p-1)(K_X + tL) + ((2p+1)\varepsilon/2 + t)L,$$

and

$$L' = (p-1)(K_X + tL) + ((2p+1)\varepsilon/2 + t - 1/2)L$$

separately, and conclude that $2K_X + 2L' + (K_X + tL)$ or $2K_X + 2L' + (K_X + (t+1)L)$ is very ample when $(2p+1)\varepsilon/2 + t + 1/2 > 4C_n n^n$.

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