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CONVERGENCE OF CURVATURES IN SECANT APPROXIMATIONS

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1. Introduction

It has long been known that a closed polyhedron P in Euclidean space $\mathbf{E}^n$ admits certain *curvature measures* analogous to classical curvature integrals (cf. [3], [11], [17], [1], [15]). If $P^1, P^2, \ldots$ is a sequence of such polyhedra converging to a smooth submanifold of $M \subset \mathbb{E}^n$, it is natural to ask whether the curvature measures of the P^i converge to the corresponding curvature integrals of M. (In view of well-known examples in area theory (cf. [16, I.1.10]) it is of course necessary to take some care in formulating the hypothesis precisely.) An intrinsic analogue of this equation has been answered positively in [5] by Cheeger, Müller and Schrader, who have also asserted that their method applies equally well to the extrinsic question above. Our aim in the present article is to give a solution to the extrinsic problem that is conceptually much simpler than the solution of [5].

Our approach rests on the observation that the curvature measures (or integrals) of polyhedra P (or smooth submanifolds M) in $\mathbb{E}^n$ may be computed in a universal way from a certain *integral current*, canonically associated to P (or M), living in the tangent sphere bundle $S\mathbb{E}^n \cong \mathbb{E}^n \times S^{n-1}$ (cf. [19], [20], [6]). If M is smooth, then this current is given by integration over the canonically oriented $(n-1)$-manifold $N(M)$ of unit normals to M. We may associate a similar object $N(P)$ to a polyhedron P; although $N(P)$ is no longer a submanifold of $S\mathbb{E}^n$, it is an integral current of dimension $n-1$, called the *normal cycle* to P. To obtain the curvature measures of P, we observe that there are universal differential $(n-1)$-forms $\kappa_0, \cdots, \kappa_{n-1}$ in $S\mathbb{E}^n$ such that the curvature measures of P are given by

$$\Phi_i^P = \pi_\#(N(P) \llcorner \kappa_i), \qquad i = 0, \cdots, n-1,$$

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where $\pi: S\mathbb{E}^n \to \mathbb{E}^n$ is the projection of the bundle. The curvature integrals of M may be computed from $N(M)$ by the same formula.

Recently we have been able to characterize the normal cycle of a compact subset of $\mathbb{E}^n$ in a particularly simple way (Theorem 3.2 of [14]). This characterization is the key to our approach to the convergence theorem. In fact, the main theorem of the present article is a corollary of the following

General convergence theorem. *Let $M \subset \mathbb{E}^n$ be a compact $C^{1,1}$ submanifold with $C^{1,1}$ boundary. Let $P^1, P^2, \cdots \subset \mathbb{E}^n$ be a sequence of polyhedra, all contained within a common compact set in $\mathbb{E}^n$, such that*

(1a) $\mathbf{M}(N(P^j)) \le K < \infty$, $j = 1, 2, \dots$, *and*

(1b) *for a.e.* $(v, t) \in S^{n-1} \times \mathbb{R}$,

$$\lim_{j\to\infty} \chi(P^j \cap H_{v,t}) = \chi(M \cap H_{v,t}),$$

where $H_{v,t}$ is the closed half-space $\{x \in \mathbb{E}^n : x \cdot v \le t\}$.

Then $\lim_{j\to\infty} N(P^j) = N(M)$ in the flat metric topology. In particular, $\lim_{j\to\infty} \Phi_i^{P_j} = \Phi_i^M$ in the sense of weak convergence of measures on $\mathbb{E}^n$, $i = 0, \cdots, n-1$.

Proof. By the compactness theory for integral currents ([10] or [18]), there is a subsequence $P^{j'}$ and an integral current $T \in \mathbf{I}_{n-1}(S\mathbb{E}^n)$ such that $\lim_{j'\to\infty} N(P^{j'}) = T$ in the flat metric topology. It is clear that T is closed, compactly supported and legendrian in the sense of [14]. Using the notation of [14, §3], we have for a.e. $(v, t) \in S^{n-1} \times \mathbb{R}$

$$\lim_{j\to\infty} \mathscr{I}(N(P^{j'}), v, t) = \mathscr{I}(T, v, t)$$

and therefore

$$\begin{aligned}\omega_{n-1}^{-1}\mathscr{I}(T, v, t)(\kappa_0) &= \omega_{n-1}^{-1}\mathscr{I}(N(P^{j'}), v, t)(\kappa_0)\\ &= \lim_{j\to\infty} \chi(P^j \cap H_{v,t})\\ &= \chi(M \cap H_{v,t}).\end{aligned}$$

By Theorem 3.2 [14], it follows that $T = N(M)$. As this outcome is independent of the choice of convergent subsequence, we have $\lim_{j\to\infty} N(P^j) = N(M)$.

Remark. The reader of [14] will realize that the statement of the theorem above can be broadened considerably.

2. Basic definitions, and statement of theorem

Let $M \subset \mathbb{E}^n$ be a compact k-dimensional submanifold-with-boundary in $\mathbb{E}^n$, of class $C^{1,1}$. Then M has positive reach in the sense of [9]: i.e., there is $\varepsilon > 0$ such that if $\operatorname{dist}(x, M) < \varepsilon$ then there is a *unique* point $\xi(x) \in M$ such that

$$\operatorname{dist}(x, M) = |\xi(x) - x|.$$

It is convenient to work with an extension of M, i.e., an open $C^{1,1}$ submanifold $\widetilde{M} \supset M$ of $\mathbb{E}^n$. Let $\tilde{\xi}$ be the nearest point retraction to $\widetilde{M}$, defined on an open set $U \supset \widetilde{M}$. One computes easily that $\tilde{\xi}$ is differentiable in U, with

$$D\tilde{\xi}(x) = \text{orthogonal projection onto } T_{\tilde{\xi}(x)}\widetilde{M};$$

it follows that $\tilde{\xi}$ is even $C^{1,1}$ in U.

Let $P \subset \mathbb{E}^n$ be a k-dimensional polyhedral submanifold-with-boundary (i.e., a manifold which is a union of affine simplices). We say that P is *inscribed* in M if

(i) all vertices of P lie in M, and

(ii) all vertices of ∂P lie in ∂M.

P is *closely inscribed* in M if additionally

(iii) $P \subset \operatorname{domain} \tilde{\xi}$ and $\tilde{\xi}|P$ is one-to-one, and

(iv) $\partial P \subset \operatorname{domain} \xi_{\partial M}$ and $\xi_{\partial M}|\partial P$ is one-to-one, where $\xi_{\partial M}$ is the projection onto ∂M.

Let $\sigma \in \mathbb{E}^n$ be a k-simplex with vertices $v_0, \cdots, v_k$. The *size* of σ is

$$\eta(\sigma) := \max |v_i - v_j|,$$

and the *fatness* of σ is

$$\Theta(\sigma) := \min\{\mathscr{H}^j(\mu)/\eta(\sigma)^j : \mu \text{ is a } j\text{- dimensional face of } \sigma, \\ j = 0, \cdots, k\}.$$

The *fatness* of a polyhedron P is

$$\Theta(P) := \sup\{\min\{\Theta(\sigma) \colon \sigma \text{ is a } k\text{-simplex of } \Delta\} : \\ \Delta \text{ is a triangulation of } P\}$$

Observe that our definition of the fatness of P generally exceeds that of [5]. This has the effect of weakening the hypothesis in the theorem below. In particular, we do not assume that the approximating polyhedra P^i necessarily become arbitrarily fine; such approximations, satisfying the hypothesis of the theorem, may occur when the limiting submanifold M has large flat regions.

Theorem. *Let $P^1, P^2, \cdots \subset \mathbb{E}^n$ be a sequence of k- dimensional polyhedral submanifolds with boundary, closely inscribed in M. Suppose that $P^j \to M$ and $\partial P^j \to \partial M$ in the Hausdorff metric on subsets of $\mathbb{E}^n$, and that $\Theta(P^j) \geq c$, $j = 1, 2, \cdots$, for some constant $c > 0$. Then $\lim_{j\to\infty} \Phi_i^{P_j} = \Phi_i^M$, $i = 0, \cdots, n-1$, in the sense of weak convergence of measures on $\mathbb{E}^n$.*

To prove this theorem we need only verify conditions (1a) and (1b) of the General Convergence Theorem of §1.

3. Some lemmas

We show first that the hypothesis of the theorem implies Lipschitz convergence of the P^i to M.

Definitions. Let TM be the tangent bundle to M, naturally embedded into $T\mathbb{E}^n$. Given a simplex $\sigma \subset \mathbb{E}^n$, let $\langle\sigma\rangle$ denote the vector subspace of $\mathbb{R}^n$ generated by σ, and let

$$T\sigma := \sigma \times \langle\sigma\rangle \subset \mathbb{E}^n \times \mathbb{R}^n \cong T\mathbb{E}^n.$$

Given a polyhedron $P \subset \mathbb{E}^n$, let

$$TP := \bigcup_\sigma T\sigma,$$

where σ ranges over all the simplices of P.

Although the sets TM, $TP^i \subset T\mathbb{E}^n$ are not compact, it still makes sense to consider the Hausdorff metric on such sets as the metric induced by the homeomorphism

$$TP \leftrightarrow TP \cap (\mathbb{E}^n \times \overline{B}(0, 1)).$$

We denote the Hausdorff distance between sets A, B by $h(A, B)$, whether in the usual or the generalized sense.

Lemma 1. *Let $M \subset \mathbb{E}^n$ be a compact C^1 submanifold. Suppose that $P^1, P^2, \ldots$ are compact polyhedral k-manifolds inscribed in M with fatness*

$$\Theta(P^i) \geq c > 0, \qquad i = 1, 2, \cdots,$$

and that $\lim_{i\to\infty} P^i = M$ in the Hausdorff metric. Then $\lim_{i\to\infty} TP^i = TM$ in the Hausdorff metric.

The failure of this lemma without the fatness assumption may be regarded as the crucial point of the example of [16] cited in the introduction; cf. the corollary below.

Proof. Let $k = \dim M$. The lemma follows immediately from the following assertion: if $\sigma_1, \sigma_2, \dots$ is a sequence of k-simplices with vertices in M, $p \in M$, and

(i) $\Theta(\sigma_i) \geq c > 0$,

(ii) $\lim_{i\to\infty} \sup_{x\in\sigma_i} \operatorname{dist}(x, M) = 0$,

(iii) $\lim_{i\to\infty} \operatorname{dist}(p, \sigma_i) = 0$,

then $\lim_{i\to\infty}\langle\sigma_i\rangle = T_pM$.

Let $\varepsilon > 0$ be given. Because of the bound on the fatness of the σ_i and the continuity in q of T_qM, there is $\delta > 0$ such that if $\eta(\sigma_i) < \delta$ and $\operatorname{dist}(p, \sigma_i) < \delta$ then

$$h(\langle\sigma_i\rangle, T_pM) \leq h(\langle\sigma_i\rangle, T_{v_0}M) + h(T_{v_0}M, T_pM)$$
$$= O\left(\left\|\frac{(v_1^i - v_0^i)\wedge\cdots\wedge(v_k^i - v_0^i)}{\|(v_1^i - v_o^i)\wedge\cdots\wedge(v_k^i - v_0^i)\|} - \overrightarrow{T_pM}\right\|\right) + O(|p - v_0|) < \varepsilon,$$

where $v_0^i, \cdots, v_k^i$ are the vertices of σ_i and $\overrightarrow{T_pM}$ is a unit k-vector in the direction of T_pM.

On the other hand, $\sigma_1', \sigma_2', \dots$ is a subsequence with $\eta(\sigma_i') \geq \delta$ for all j, then we may find a further subsequence $\sigma_1'', \sigma_2'', \dots$ with vertices $v_0^{i''} \to u_0, \cdots, v_k^{i''} \to u_k$. By the hypothesis, the points $u_0, \cdots, u_k$ are the vertices of a nondegenerate k-simplex $\sigma_\infty \subset M$, with $p \in \sigma_\infty$. Thus $T_pM = \langle\sigma_\infty\rangle = \lim_{i\to\infty}\langle\sigma_i''\rangle$. As this is true independently of the chosen subsequences we find that $T_pM = \lim_{i\to\infty}\langle\sigma_i'\rangle$ in this case.

Corollary. *Under the hypothesis of the theorem, we have*

$$\lim_{i\to\infty} \mathscr{H}^k(P^i) = \mathscr{H}^k(M).$$

Proof. In view of our characterization of the derivative $D\tilde{\xi}$, Lemma 1 implies that the Jacobian of the restriction of $\tilde{\xi}$ to the interior of any k-simplex of P^i is uniformly close to 1 for large i. Using the hypothesis that $\tilde{\xi}|P^i$ is one-to-one, the area formula now gives

$$\lim_{i\to\infty} \mathscr{H}^k(P^i)/\mathscr{H}^k(\tilde{\xi}(P^i)) = 1.$$

On the other hand, the condition that $\partial P^i \to \partial M$ implies that for large i, $\tilde{\xi}(P^i) \subset \widetilde{M}$ differs from M only by a region confined to a narrow band around ∂M; therefore $\lim_{i\to\infty}\mathscr{H}^k(\tilde{\xi}(P^i)) = \mathscr{H}^k(M)$. q.e.d.

For the rest of the paper, the symbol C will denotes a generic positive constant.

Lemma 2. *Under the hypothesis of the theorem, there is a constant C such that for any simplex σ of any approximating polyhedron P^i, the number of simplices of P^i incident to σ is less than C.*

Proof. By the bound on the fatness of P^i, whenever σ is a k-simplex of P^i and $x \in \sigma$, we have

$$\mathscr{H}^k(B(x, r) \cap \sigma) \geq Cr^k,$$

for all small $r > 0$. Therefore, since $D(\tilde{\xi}|\sigma)$ is close to the identity,

$$\mathscr{H}^k(B(\tilde{\xi}(x), r) \cap \tilde{\xi}(\sigma)) \cong \mathscr{H}^k(\tilde{\xi}[B(x, r) \cap \sigma]) \geq Cr^k$$

for all small $r > 0$. Since $\lim_{r\to 0}(\mathscr{H}^k(B(\tilde{\xi}(x), r))/r^k) = \alpha(k) =$ volume of unit k-ball in $\mathbb{E}^k$, and the interiors of the $\tilde{\xi}(\sigma)$ are disjoint, the lemma follows.

Lemma 3. *There are constants C, $\delta_0 > 0$ such that whenever $P \subset \mathbb{E}^n$ is a k-dimensional polyhedral manifold without boundary, and there exists a k-plane $\Pi \subset \mathbb{E}^n$ such that if $\delta := \sup_\sigma h(\langle\sigma\rangle, \Pi) < \delta_0$ (where σ ranges over the k-simplices of P) we have*

$$\operatorname{spt} N(P) \subset P \times \{v \in S^{n-1} : \operatorname{dist}(v, \Pi^\perp) < C\delta\}.$$

Proof. Putting $\Delta_P(x) := \operatorname{dist}(x, P)$, and $P_\varepsilon := \{x \in \mathbb{E}^n : \operatorname{dist}(x, P) \leq \varepsilon\}$, we have by [14] and [12]

$$N(P) = \lim_{\varepsilon \to o} N(P_\varepsilon),$$

$$\operatorname{spt} N(P_\varepsilon) \subset \left\{(x, v) \in \mathbb{E}^n \times S^{n-1}; \Delta_P(x) = \varepsilon,\ v = \sum s_i \frac{x - q_i}{|x - q_i|},\right.$$
$$\left.\text{where, } s_i \geq 0,\ q_i \in P \text{ and } |x - q_i| = \varepsilon\right\}.$$

Thus it is enough to show that if $x \notin P$, $q_i \in P$ for $i = 0, 1$, and $\Delta_P(x) = |x - q_i|$, then

(3a) $$\operatorname{dist}\left(\frac{x - q_i}{\Delta_P(x)}, \Pi^\perp\right) < C\delta, \qquad i = 0, 1,$$

and

(3b) $$|q_0 - q_1| < C\delta\Delta_P(x).$$

To prove (3a) we may assume that Π passes through the point $q = q_i$. It is enough to show that

(3c) $$|\xi_\Pi(x) - q| < \Delta_P(x) O(\delta).$$

If v is any unit vector pointing along Π, then we have for any $t \in \mathbb{R}$

$$\begin{aligned} |x - (q + tv)|^2 &= |(x - q) - tv|^2 \\ &= \Delta_P(x)^2 + t^2 - 2t(x - q) \cdot v . \end{aligned} \tag{3d}$$

Now if δ is sufficiently small then we may express P as the graph of a piecewise-linear function $\varphi \colon \Pi \to \Pi^\perp$; letting $u = v + D_v\varphi(q) \in \Pi \oplus \Pi^\perp = \mathbb{R}^n$, we have $|u - v| < O(\delta)$. The vector u is tangent to P at q in the sense that $q + su \in P$ for small $s > 0$, so in view of the hypothesis that $|x - q| = \Delta_P(x)$ we must have $u \cdot (x - q) \leq 0$. Substituting $u + O(\delta)$ for v in (3d), we get

$$\begin{aligned} |x - (q + tv)|^2 &= \Delta_P(x)^2 + t^2 - 2t(x - q) \cdot u + t\Delta_P(x)O(\delta), \\ &\geq \Delta_P(x)^2 + t^2 + t\Delta_P(x)O(\delta), \end{aligned}$$

and it follows (since $\Delta_P(x) \leq \Delta_\Pi(x)$) that the distance from x to Π is realized within distance $\Delta_P(x)O(\delta)$ of q, which relation is precisely (3c).

To prove (3b), let $\sigma(t)$ be the straight line path of unit speed joining $\xi_\Pi(q_0)$ to $\xi_\Pi(q_1)$. Realizing P as the graph of $\varphi \colon \Pi \to \Pi^\perp$ as above, put $\gamma(t) := (\sigma(t), \varphi(\sigma(t)))$ to be the corresponding path in P connecting q_0 to q_1. By the hypothesis we have

$$|\gamma'(t) - \sigma'(t)| < C\delta$$

whenever $\gamma'(t)$ exists. It follows that

$$|(q_0 - q_1) - (\xi_\Pi(q_0) - \xi_\Pi(q_1))| < C\delta \,\mathrm{length}(\sigma)$$

and, putting $\overline{\sigma}$ to be the segment connecting q_0 and q_1,

$$h(\overline{\sigma}, \gamma) \leq h(\overline{\sigma}, \sigma) + h(\sigma, \gamma) < C\delta \,\mathrm{length}(\overline{\sigma}).$$

Since

$$\begin{aligned} \mathrm{dist}(x, \overline{\sigma}) &= \left(\Delta_P(x)^2 - \left(\frac{\mathrm{length}(\overline{\sigma})}{2}\right)^2\right)^{1/2} \\ &< \Delta_P(x) - \frac{1}{8\Delta_P(x)}(\mathrm{length}(\overline{\sigma}))^2 \end{aligned}$$

(see Figure 1, next page), we have

$$\begin{aligned} \Delta_P(x) &= \mathrm{dist}(x, \gamma) \leq \mathrm{dist}(x, \overline{\sigma}) + C\delta \,\mathrm{length}(\overline{\sigma}) \\ &< \Delta_P(x) - \frac{1}{8\Delta_P(x)} \mathrm{length}(\overline{\sigma})^2 + C\delta \,\mathrm{length}(\overline{\sigma}) \end{aligned}$$

or

$$|q_0 - q_1| = \mathrm{length}(\overline{\sigma}) < C\delta\Delta_P(x),$$

as claimed.

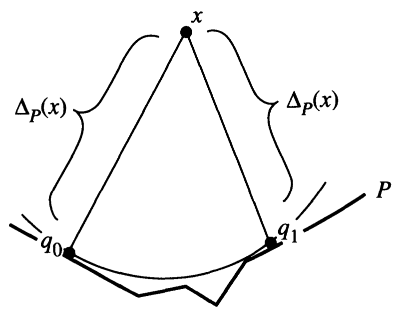

FIGURE 1

The next two lemmas give a mild generalization of the Poincaré-Hopf theorem. Let M be a $C^{1,1}$ manifold without boundary, and let $U \subset M$ be a Lipschitz domain. Given a Lipschitz function $\alpha\colon M \to \mathbb{R}$, let $D^*\alpha$ be its generalized differential in the sense of Clarke [7, Chapter 2]. Let μ be a Lipschitz vector field defined on a neighborhood of $\overline{U}$. Then μ is said to *point out* of U at $x \in \operatorname{bdry} U$, provided there exist a neighborhood V of x in M and a Lipschitz function $\alpha\colon V \to \mathbb{R}$ such that $U \cap V = \alpha^{-1}(-\infty, 0)$ and such that for every $\xi \in D^*\alpha(x)$ we have $\langle \mu, \xi \rangle > 0$. We say that μ *points out* of U if μ points out of U at every $x \in \operatorname{bdry} U$.

Lemma 4. *Suppose the Lipschitz vector field μ points out of the Lipschitz domain U and has finitely many zeros in U. Then*

$$\chi(U) = \sum_{\mu(z)=0} \operatorname{index}(\mu, z).$$

Proof. Let $\varphi(t, x)$ be the flow of the vector field $-\mu$. Then φ maps $[0, \infty) \times U$ into U. For if $p \in \operatorname{bdry} U$, and $\alpha\colon V \to \mathbb{R}$ is as above then for $x \in V$

$$\left.\frac{d}{dt}\right|_{t=0} \alpha(\varphi(t, x)) \leq \sup\{\langle \xi, -\mu(x)\rangle : \xi \in D^*\alpha(x)\} < 0 \qquad \text{(by [7, pp. 25, 27])}$$

for x sufficiently close to p, since $D^*\alpha$ is upper semicontinuous. Now apply the Lefschetz fixed point theorem as in e.g. [8, VII. 6.6].

In order to apply this we will need

Lemma 5. *Let U, U' be Lipschitz domains in M, and μ a Lipschitz vector field pointing out of U at every point of* bdry $U \cap U'$, *and out of U' at every point of* bdry U' *Then μ points out of $U \cap U'$.*

Proof. We need only verify the criteria of the definition above for points $x \in \operatorname{bdry} U \cap \operatorname{bdry} U'$. Let α, $\alpha'\colon V \to \mathbb{R}$ be functions as in that definition for U and U' respectively, $x \in V$. Then $\alpha \vee \alpha' := \max\{\alpha, \alpha'\}$ has $(\alpha\vee\alpha')^{-1}(-\infty, 0) = U' \cap U \cap V$, and $D^*(\alpha\vee\alpha')(x) \subset \{t\xi + (1-t)\eta : \xi \in D^*\alpha(x),\ \eta \in D^*\alpha'(x),\ 0 \le t \le 1\}$ by [7, 2.3.12], from which the lemma follows.

4. Proof of the theorem

We establish first the mass bounds (1a). We first bound the mass $\mathbf{M}(N(P^i) \llcorner \pi^{-1}(\operatorname{int} P^i))$ of the part of $N(P^i)$ lying over the interior of P^i. Since M is $C^{1,1}$ (i.e., has bounded principal curvatures), we may find constants $A < \infty$ and $\delta_0 > 0$ such that if σ is any k-simplex with vertices in M and

$$\Theta(\sigma) \ge c, \qquad \eta(\sigma) < \delta_0,$$

then

$$\operatorname{dist}(\langle\sigma\rangle, T_pM) < A\eta(\sigma), \tag{4a}$$

where $p \in M$ may be taken to be any vertex of σ. By the bound on the fatness of the P^i there is $\delta_1 > 0$ such that if τ is a j-simplex of P^i with $\eta(\tau) < \delta_1$, then any k-simplex σ of P^i incident to τ has size $\eta(\sigma) < \delta_0$. Let E^i_j be the set of all interior j-simplices (i.e., simplices not included in ∂P_i) of P^i of size $< \delta_1$, and let F^i_j be the set of all other interior j-simplices.

Now by the additivity property

$$N(X \cup Y) = N(X) + N(Y) - N(X \cap Y)$$

(cf. [14], [6] or [20]) and the simple nature of the normal cycle of a simplex, we find that

(4b) If τ is any simplex of P^i, then
$$\operatorname{spt}(N(P^i)\llcorner\pi^{-1}(\operatorname{int}\tau)) \subset \tau \times (\tau^\perp \cap S^{n-1}) \subset \mathbb{E}^n \times S^{n-1},$$

and

(4c) the multiplicities of the integral currents $N(P^i)$ are uniformly bounded by a constant C,

as follows at once from Lemma 2. Thus if $\tau \in E^i_j$ then

(4d)
$$\begin{aligned}
&\mathbf{M}(N(P^i)\llcorner\pi^{-1}(\operatorname{int}\tau)) \\
&\quad \le C\mathscr{H}^{n-1}(\operatorname{spt} N(P^i) \cap \pi^{-1}(\operatorname{int}\tau)) \\
&\quad \le C\mathscr{H}^j(\tau)\mathscr{H}^{n-j-1}(\{v \in S^{n-1} : \operatorname{dist}(v, T_pM^\perp) < C\eta(\tau),\ v \in \tau^\perp\})
\end{aligned}$$

by Lemma 3, where p is a vertex of τ. The last factor is maximized when the $(n-j)$-plane $\tau^{\perp}$ contains the $(n-k)$-plane $(T_pM)^{\perp}$, in which case it is $O(\eta(\tau)^{k-j-1})$; so we conclude that for $\tau \in E_j^i$ we have

$$\text{(4e)} \qquad \begin{aligned} \mathbf{M}(N(P^i)\llcorner\pi^{-1}(\operatorname{int}\tau)) &= O(\mathscr{H}^j(\tau)\eta(\tau)^{k-j}) \\ &= O(\eta(\tau)^k) \end{aligned}$$

by the fatness bound. On the other hand

$$\text{(4f)} \qquad \begin{aligned} M\left(N(P^i)\llcorner\bigcup_{\tau\in F_j^i}\tau^{-1}(\operatorname{int}\tau)\right) &\le C\sum_{\tau\in F_j^i}\mathbf{M}(N(\tau)\llcorner\pi^{-1}(\operatorname{int}\tau)) \\ &= C\sum_{\tau\in F_j^i}\mathscr{H}^j(\tau)\mathscr{H}^{n-j-1}(S^{n-j-1}) \\ &= O(\mathscr{H}^j(\cup F_j^i)), \end{aligned}$$

which remains bounded as $i \to \infty$ by the corollary to Lemma 1. Thus we need only bound the sum of the expressions (4e), i.e., $\sum_{\tau\in E_j^i}\eta(\tau)^k$. But by the fatness bound, we have $\eta(\tau)^k = O(\sum_{\sigma^k>\tau}\mathscr{H}^k(\sigma))$, where the sum runs over all k-simplices σ with $\tau \subset \partial\sigma$, and by Lemma 2 we have $\sum_{\tau\in E}\sum_{\sigma^k>\tau}\mathscr{H}^k(\sigma) = O(\mathscr{H}^k(P^i))$, so the desired uniform bound follows from the corollary to Lemma 2.

To bound the mass of $N(P^i)$ lying over ∂P^i, we may argue as above with ∂P^i and ∂M replacing P^i and M, respectively. The only point that needs to be checked is the bound on the support of $N(P^i)\llcorner\pi^{-1}(P^i)$, i.e., that for large i and small j-simplices τ of ∂P^i we have

$$\text{(4g)} \qquad \begin{aligned} &\operatorname{spt}(N(P^i)\llcorner\pi^{-1}(\operatorname{int}\tau)) \\ &\qquad \subset \tau\times\{v\in S^{n-1}: \operatorname{dist}(v, T_p(\partial M)^{\perp}) < C\eta(\tau)\} \end{aligned}$$

(cf. (4d)) for appropriate points p. To see this we need to modify Lemma 3 as follows:

Lemma 3′. *There are constants C, $\delta_0 > 0$ such that whenever $P \subset \mathbb{E}^n$ is a k-dimensional polyhedral manifold with boundary, and there exist a k-plane Π and a $(k-1)$-plane $\pi \subset \Pi$ such that*

$$\text{(4h)} \qquad \delta_1 := \sup_{\sigma^k\in P}\operatorname{dist}(\langle\sigma^k\rangle, \Pi) < \delta_0$$

and

$$\text{(4j)} \qquad \delta_2 := \sup_{\tau^{k-1}\in\partial P}\operatorname{dist}(\langle\tau^{k-1}\rangle, \pi) < \delta_0,$$

then we have, putting $\delta := \max\{\delta_1, \delta_2\}$,

$$\begin{aligned}\operatorname{spt} N(P) \subset P \times \{v \in S^{n-1} : \operatorname{dist}(v, \Pi^\perp) < C\delta\} \\ \cup\, \partial P \times \{w \in S^{n-1} : \operatorname{dist}(w, \pi^\perp) < C\delta\}.\end{aligned}$$

Proof. We may follow the proof of (4h) in Lemma 3 to show that

$$\operatorname{dist}\left(\frac{x-q}{\Delta_P(x)}, \mu^\perp\right) < C\delta$$

for $x \notin P$ and $|x-q| = \Delta_P(x)$, where $\mu = \Pi$ or π depending on whether q belongs to $\operatorname{int} P$ or ∂P. It remains to prove (4j) in the case where $q_0 \in \operatorname{int} P$ and $q_1 \in \partial P$ (it is not necessary to consider the case where q_0 and q_1 belong to distinct components of ∂P, since we are only interested in the case of points x lying very close to P). For this we again express P as the graph of a function $\varphi: U \to \Pi^\perp$, where U is a domain with polyhedral boundary $\xi_\Pi(\partial P)$. Let $\sigma(t)$ be a unit speed path in U joining $\xi_\Pi(q_0)$ to $\xi_\Pi(q_1)$. This may no longer be taken to be a straight line, but condition (4j) implies that we may choose σ so that, for all t,

$$\left|\sigma'(t) - \frac{\xi_\Pi(q_0) - \xi_\Pi(q_1)}{|\xi_\pi(q_0) - \xi_\Pi(q_1)|}\right| < C\delta.$$

The proof then proceeds like that of Lemma 3. This completes the proof of the mass bounds (1a).

We now prove (1b) whenever t is a regular value of the restriction of M of the height function $x \mapsto x \cdot v$. Fixing $v \in S^{n-1}$ for the rest of the section, we denote this height function by h. Given a set $X \subset \mathbb{E}^n$ and subset $J \subset \mathbb{R}$ we put also $X_J := X \cap h^{-1}(J)$; we abbreviate $X_{\{t\}}$ to X_t for $t \in \mathbb{R}$. We may also express M as

$$M = \widetilde{M} \cap \varphi^{-1}(-\infty, 0],$$

where $\varphi: \mathbb{E}^n \to \mathbb{R}$ is a $C^{1,1}$ function and 0 is a regular value of $\varphi|\widetilde{M}$. The condition that t be a regular value of $h|M$ includes the condition that $\widetilde{\Delta} h$ and $\widetilde{\Delta}\varphi$ are antipodal nowhere in the neighborhood of $\varphi^{-1}(0) \cap \widetilde{M}_t = (\partial M)_t$, where $\widetilde{\Delta}$ denotes the gradient in the manifold $\widetilde{M}$. It follows that the Lipschitz vector-field $\mu := \widetilde{\nabla} h/|\widetilde{\nabla} h| + \widetilde{\nabla}\varphi/|\widetilde{\nabla}\varphi|$, defined in the neighborhood of $\partial M \cap h^{-1}(t)$, satisfies

(4k) $$\langle d\varphi, \mu\rangle > 0,$$

(4m) $$\langle dh, \mu\rangle > 0,$$

and we may extend μ to a Lipschitz vector field, again denoted by μ and defined on all of $\widetilde{M}$, such that (4k) holds on $(\partial M)_t$ and (4m) holds on M_t. In other words, the vector field μ points out of $M_{(-\infty, t]}$.

After performing a suitable homotopy on M_t fixing a neighborhood of ∂M_t, we may alter μ so that it has a unique zero at a point $p \in M_{(-\infty, t)}$. By the generalize Poincaré-Hopf theorem (Lemma 4), we have

$$\chi(M_{(-\infty, t]}) = \operatorname{index}(\mu, p).$$

Since $\tilde{\xi}|P^i$ is one-to-one for each i, in order to verify (1b) we may check that $\chi(M_{(-\infty, t]}) = \chi(\tilde{\xi}(P^i_{(-\infty, t]}))$ for all large i. The hypothesis of our theorem implies that, for large i, the point p is the unique zero of μ lying in the domain $U_i = \tilde{\xi}(P^i_{(-\infty, t]})$. Thus if we apply Lemma 4 it is enough to show that μ points out of U_i. Applying Lemma 5, this is accomplished by showing that there are domains U_i' and U_i'' such that μ points out of both U_i' and U_i'', and $U_i = U_i' \cap U_i''$.

We take $U_i' := \xi(P^i)$. By Lemma 1 and the fact that $D\tilde{\xi}(x)$ is projection onto $T_{\tilde{\xi}(x)}\widetilde{M}$, we see that $\tilde{\xi}(P^i)$ is a Lipschitz domain in $\widetilde{M}$, with boundary close to ∂M is the Lipschitz sense. From this it is easy to see that μ points out of U_i' at all points x at the boundary with $(h \circ (\tilde{\xi}|P^i)^{-1}(x) - t)_+$ sufficiently small.

To construct U_i'', we consider $Q^i := \partial\tilde{\xi}(P^i_{(-\infty, t]}) = \tilde{\xi}(\partial P^i)$. This is an open Lipschitz hypersurface in $\widetilde{M}$, and for large i it is Lipschitz-close to the Lipschitz hypersurface $\widetilde{M}_t$. Thus it admits a Lipschitz extension $\overline{Q^i}$ which is again Lipschitz close to $\widetilde{M}_t$; for in $C^{1,1}$ local coordinates this latter hypersurface may be expressed as the graph of a constant function, and the former as the graph of a Lipschitz function g with small Lipschitz constant. We may assume that $\widetilde{M}_{(-\infty, t]}$ corresponds to the set of points lying below the graph of the function. This second function g is only defined up to points corresponding to $\tilde{\xi}(\partial P^i)$; taking a Lipschitz extension $\overline{g}$, we take U_i'' to be the set of points lying below the graph of $\overline{g}$. The Lipschitz constant of $\overline{g}$ may be controlled by that of g ([8, 2.10.43], and it follows that μ points out of U_i'' for large enough i.

5. Concluding remarks

It should be noted that the authors of [5] obtained a slightly stronger result than simple convergence of curvature measures, namely an estimate of the rate of convergence in terms of the magnitudes of the curvature tensor of M and of its covariant derivative. As the method of the present article relies on a compactness theorem, we have obtained no such estimate; on the other hand, the manifolds M subject to our theorem will generally have a curvature tensor which is discontinuous, so the estimate of [5] cannot even be formulated in these cases.

Another interesting convergence theorem was obtained by Brehm and Kühnel [2]. These authors considered the approximation of a compact polyhedral surface without boundary in $\mathbb{E}^3$ by smooth surfaces. Using intricate geometric constructions they showed that one may find such an approximation in which the *total absolute* curvature measures converge. This sort of result is beyond the range of our method.

We would like finally to mention that the possibility of an argument as in the present paper was suggested to us by M. Zähle, cf. [21].

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