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ON THE LAPLACIAN AND THE GEOMETRY OF HYPERBOLIC 3-MANIFOLDS

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Abstract

Let $N = \mathbf{H}^3/\Gamma$ be an infinite volume hyperbolic 3-manifold which is homeomorphic to the interior of a compact manifold. Let $\lambda_0(N) = \inf\operatorname{spec}(-\Delta)$ where Δ is the Laplacian acting on functions on N. We prove that if N is not geometrically finite, then $\lambda_0(N) = 0$, and if N is geometrically finite we produce an upper bound for $\lambda_0(N)$ in terms of the volume of the convex core. As a consequence we see that $\lambda_0(N) = 0$ if and only if N is not geometrically finite. We also show that if N has a lower bound for its injectivity radius and is not geometrically finite, then its limit set L_Γ has Hausdorff dimension 2.

1. Introduction

In this paper we will study the relationship between the geometry of infinite volume hyperbolic 3-manifolds and the bottom λ_0 of the spectrum of the Laplacian. We will also study the relationship between spectral information and the measure-theoretic properties of the limit set. These relationships have been studied extensively by Patterson (cf. [28], [27]) and Sullivan (cf. [32], [33]), and much of this paper may be regarded as an extension of their work. Recall that a hyperbolic 3-manifold is said to be *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. Our first result is:

Theorem A. *Let N be an infinite volume, topologically tame hyperbolic 3-manifold. Then $\lambda_0(N) = 0$ if N is not geometrically finite. Moreover, there exists a constant K such that if N is geometrically finite, then*

$$\lambda_0(N) \leq K\frac{|\chi(\partial C(N))|}{\operatorname{vol}(C(N))},$$

where $\operatorname{vol}(C(N))$ denotes the volume of N's convex core.

Combining Theorem A with work of Lax and Phillips ([20], [21]) we show that λ_0 detects whether or not a topologically tame hyperbolic 3-manifold is geometrically finite.

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Corollary B. *Let N be an infinite volume topologically tame hyperbolic 3-manifold. N is geometrically finite if and only if $\lambda_0(N) \neq 0$.*

In a forthcoming paper [5], Marc Burger and the author prove that there exists a constant $G > 0$ such that if N is geometrically finite, then

$$\lambda_0(N) \geq \frac{G}{\mathrm{vol}(\mathcal{N}_1(C(N)))^2},$$

where $\mathrm{vol}(\mathcal{N}_1(C(N)))$ denotes the volume of the neighborhood of radius one of the convex core. (This result is analogous to results of Schoen [29] and Dodziuk-Randol [13] for the closed and finite volume cases.) Thus combining this result with Theorem A we see that the volume of the convex core "controls" λ_0. Here is one intuitive explanation for such a relationship. There always exists a positive harmonic function f such that $\Delta f = -\lambda_0 f \cdot \nabla(-\log f)$ is a bounded vector field whose associated flow is volume-increasing and the rate of increase at each point is at least λ_0. Outside of the convex core the geometry is exponentially expanding, so it is easy to construct volume-increasing flows with "large" rates of volume increase. However, the convex core is more congested, and the thicker the convex core is the more difficult it will be to construct flows with a "large" rate of volume increase.

In [32], Sullivan proved that if $N = \mathbf{H}^3/\Gamma$ is a geometrically finite hyperbolic 3-manifold, and D is the Hausdorff dimension of the limit set of Γ, then $\lambda_0(N) = 1$ if $D \leq 1$, while otherwise $\lambda_0(N) = D(2-D)$. Thus combining Theorem A with the above-mentioned result of Burger and Canary makes explicit the intuitive relationship between the thickness (volume) of the convex core and the fuzziness (Hausdorff dimension) of the limit set. (The basis of this second intuitive link is that the convex core is the quotient of the convex hull of the limit set by the group action. It stands to reason that the limit set should be locally complicated if and only if the convex core is thick.) Thus the bottom of the spectrum of the Laplacian, the Hausdorff dimension of the limit set, and the volume of the convex core all serve as measures of "how geometrically finite" N is.

By analogy one would conjecture that the limit set of a topologically tame hyperbolic 3-manifold which is not geometrically finite had Hausdorff dimension 2. We remark that it is shown in [8] that the limit set of a topologically tame hyperbolic 3-manifold either has measure zero or is the entire sphere at infinity. In this paper we prove the following result.

Theorem C. *Let $N = \mathbf{H}^3/\Gamma$ be a topologically tame hyperbolic 3-manifold with a lower bound on its injectivity radius. If N is not geomet-*

rically finite, then the limit set L_Γ *for* Γ*'s action on the sphere at infinity has Hausdorff dimension* 2.

Theorem A and Corollary B hold for analytically tame hyperbolic 3-manifolds, and Theorem C may be extended to analytically tame hyperbolic 3-manifolds with thin parts of uniformly bounded type (see §2 for definitions). The results will be stated and proved in this generality in the text.

In §2 we will review the structure of hyperbolic 3-manifolds. In §3 we will reprove an upper bound on λ_0 due to Buser and also derive a preliminary result about the growth of harmonic functions on analytically tame hyperbolic 3-manifolds. In §4, we will use Buser's upper bound to derive Theorem A and Corollary B and some other consequences of Theorem A concerning the critical exponent of the Poincaré series and the bottom of the essential spectrum. In §5 we will prove Theorem C, by showing that the harmonic function given by Patterson-Sullivan measure has subexponential growth.

2. The structure of hyperbolic 3-manifolds

Let N be a (orientable) hyperbolic 3-manifold with finitely generated fundamental group. N may be represented as the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a group Γ of orientation-preserving isometries of $\mathbf{H}^3$. We recall that the *limit set* L_Γ of Γ is defined to be the smallest closed Γ-invariant subset of the sphere at infinity S^2_∞ for hyperbolic 3-space. We will say that N is *elementary* if Γ is abelian. If N is nonelementary, the *convex core* $C(N)$ of N is defined to be the smallest convex submanifold such that the inclusion map is a homotopy equivalence. Explicitly, $C(N) = CH(L_\Gamma)/\Gamma$, where $CH(L_\Gamma)$ denotes the convex hull (in $\mathbf{H}^3$) of the limit set L_Γ. (See Maskit [24] for basic definitions in the theory of Kleinian groups.) The following structural theorem is central to understanding hyperbolic 3-manifolds and Kleinian groups.

Theorem 2.1 (Ahlfors' finiteness theorem [1]). *Let* N *be a nonelementary hyperbolic* 3*-manifold with finitely generated fundamental group. The boundary* $\partial C(N)$ *of the convex core* $C(N)$ *is a finite area hyperbolic surface, i.e., there exists a* C^0*-isometric embedding of a finite area hyperbolic surface into* N *with image* $\partial C(N)$.

This statement combines Ahlfors' finiteness theorem [1] with Thurston's observation that the boundary of the convex core is a hyperbolic surface (see Epstein-Marden [17] or Thurston [34]). Thus, in particular, $\partial C(N)$

has area $2\pi|\chi(\partial C(N))|$. If $C(N)$ has finite volume or N is elementary, then N is said to be *geometrically finite.* (If N is elementary we will say that $\mathrm{vol}(C(N)) = 0$.) Recall that N is topologically tame if it is geometrically finite (see, e.g., Marden [23]). We will say that N is *convex cocompact* if $C(N)$ is compact (i.e., if N is geometrically finite and has no cusps).

We will say that a hyperbolic 3-manifold N with finitely generated fundamental group is *analytically tame* if $C(N)$ may be exhausted by a sequence of compact submanifolds $\{C_i\}$ such that $C_i \subset C_j$ if $i < j$, $\cup C_i^0 = C(N)$ (where C_i^0 is the interior of C_i considered as a subset of $C(N)$), and there exist K and L such that the boundary ∂C_i of C_i has area at most K and the neighborhood of radius one of ∂C_i has volume at most L for all i. We only require that our submanifolds C_i have Lipschitz boundary. This regularity assumption is natural, as the boundary of the convex core itself is always a Lipschitz submanifold but is not in general a C^1-submanifold (see Epstein-Marden [17]).

In [8] the following theorem is proved.

Theorem 2.2 [8]. *If N is a topologically tame hyperbolic 3-manifold, then N is analytically tame.*

In the same paper [8] the following generalization of a result of Thurston [34] is established.

Theorem 2.3. *If N is analytically tame hyperbolic 3-manifold, then either $L_\Gamma = S^2_\infty$ or L_Γ has measure zero. Moreover, if $L_\Gamma = S^2_\infty$, then Γ acts ergodically on S^2_∞.*

Work of Bonahon guarantees that there is a large class of hyperbolic 3-manifolds which are topologically tame. (This ordering is historically misleading—Theorem 2.4 was actually used to prove Theorem 2.2; see the remarks at the end of the section for a further discussion.) Let Γ be a discrete subgroup of the group of isometries of hyperbolic 3-space. A finitely generated group Γ of hyperbolic isometries is said to satisfy condition (B) if it is not cyclic, and whenever $\Gamma = G * H$ is a nontrivial free decomposition of Γ there exists a parabolic element γ which is not conjugate to any element of G or H. In particular, condition (B) is satisfied if Γ is freely indecomposable.

Theorem 2.4 (Bonahon [3]). *If $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold and Γ satisfies condition* (B), *then N is topologically tame.*

It will be necessary in the proof of Theorem C to make use of the thick-thin decomposition of a hyperbolic 3-manifold. We recall that the injectivity radius of N at a point x, denoted $\mathrm{inj}(x)$, is defined to be half the length of the shortest (homotopically nontrivial) loop through x. There exists a constant $\mathscr{M}$, called the Margulis constant, such that if $\varepsilon < \mathscr{M}$ and

$$N_{\text{thin}(\varepsilon)} = \{x \in N \mid \text{inj}(x) \leq \varepsilon\},$$

then every component of $N_{\text{thin}(\varepsilon)}$ is either

(a) a torus cusp, i.e., a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z} \oplus \mathbf{Z}$,

(b) a rank one cusp,i.e., a horoball in $\mathbf{H}^3$ modulo a parabolic action of $\mathbf{Z}$, or

(c) a solid torus neighborhood of a geodesic

(see Thurston [34] or Morgan [25]). We also define

$$N_{\text{thick}(\varepsilon)} = \{x \in N \mid \text{inj}(x) \geq \varepsilon\}.$$

We further remark that if ε is chosen to be less than the Margulis constant, that there exists an $L > 0$ (depending only on ε) such that if σ is any geodesic in N, then the distance (in σ) between any two components of $\sigma \cap N_{\text{thin}(\varepsilon)}$ is at least L. (When reading about hyperbolic 3-manifolds it is often easier, on a first reading, to assume that there are no parabolics or even that there is a uniform lower bound on injectivity radius. This caution applies equally well to this paper, especially the proof of Theorem C.)

We furthermore say that N has *thin parts of uniformly bounded type* if there exists J such that if S is any component of $\partial N_{\text{thin}(\varepsilon)} \cap C(N)$, then S has diameter less than J. In particular, if N contains any rank-one cusps, then their intersections with the convex core have finite volume; such rank-one cusps are said to be bounded or doubly cusped in the language of Kleinian groups.

Remarks. (1) Actually Bonahon [3] proved that hyperbolic 3-manifolds satisfying condition (B) are geometrically tame. The main theorem of [8] uses this theorem to prove that hyperbolic 3-manifolds are topologically tame if and only if they are geometrically tame. Analytic tameness is a consequence of geometric tameness. We have developed the structure in this way to avoid introducing simplicial hyperbolic surfaces and the more technical points in the definition of geometric tameness, none of which is necessary for the work in this paper. We urge the reader to consult Bonahon [3], Thurston [34], or Canary [8] for a discussion of geometric tameness.

(2) It is conjectured that all hyperbolic 3-manifolds with finitely generated fundamental groups are topologically tame, and hence both geometrically and analytically tame. However, there are hyperbolic 3-manifolds

which are known to be analytically tame but which are not known to be topologically tame. In particular, Culler and Shalen [12] proved that there is a dense G_δ of analytically tame manifolds in the boundary of the Schottky space of genus 2.

(3) Condition (B) is really a topological condition. Let N_ε^0 be obtained from N by removing the noncompact components of $N_{\text{thin}(\varepsilon)}$. There exists a compact submanifold C of N_ε^0 such that the inclusion map is a homotopy equivalence and C intersects each component of the boundary in either an annulus or a torus (see Feighn-McCullough [18]). Γ satisfies condition (B) if every compressible curve on the boundary ∂C of C intersects the boundary of a noncompact component of $N_{\text{thin}(\varepsilon)}$. (A curve in ∂C is said to be compressible if it is homotopically trivial in C, but not in ∂C.)

3. Buser's upper bound for λ_0

Let N be a complete Riemannian n-manifold (without boundary). We recall some equivalent definitions of $\lambda_0(N)$ (in this paper the Laplacian $\Delta f = \operatorname{div}(\operatorname{grad} f)$ is a negative definite operator):

$$\begin{aligned}\lambda_0(N) &= \sup\{\lambda \mid \exists f \in C^\infty(N) \text{ s.t. } \Delta f = -\lambda f \text{ and } f > 0\} \\ &= \inf_{f \in C_0^\infty(N)} \left(\frac{\int_N |\nabla f|^2}{\int_N f^2} \right) \\ &= \inf \operatorname{spec}(-\Delta).\end{aligned}$$

We also recall that the Cheeger constant $h(N)$ is defined to be the infimum, over all compact n-submanifolds A of N (with Lipschitz boundary), of $\operatorname{vol}_{n-1}(\partial A)/\operatorname{vol}(A)$. Buser [6] proved that if N has Ricci curvature bounded from below, then $h(N)$ gives an upper bound for $\lambda_0(N)$. (In Cheeger's original paper [9] he proved that the Cheeger constant gives a lower bound on λ_0 with no constraints on the geometry of the manifold, to be precise $\lambda_0(N) \geq h(N)^2/4$.) We give a new proof of this upper bound which also yields a L^2-bound on the growth rate of harmonic functions on analytically tame hyperbolic 3-manifolds.

Theorem 3.1 (Buser [6]). *If the Ricci curvature of a complete Riemannian n-manifold N (without boundary) is bounded below by $-(n-1)\kappa^2$, then*

$$\lambda_0(N) \leq R\kappa h(N),$$

where R depends only on n.

Proof of 3.1. We may assume, by scaling the metric, that $\kappa = 1$. Then Cheng's comparison principle [10] assures us that $\lambda_0(N) \leq (n-1)^2/4$. Let f be a positive eigenfunction of the Laplacian with eigenvalue $-\lambda_0$ (see either Cheng-Yau [11] or Sullivan [38] for a proof that f exists). Now the infinitesimal Harnack inequality of Yau [36] implies that $|\frac{\nabla f}{f}(x)| \leq R$ for some R depending only on n and all $x \in N$.

Now consider $\log f \cdot \nabla \log f = \nabla f / f$ and

$$\Delta \log f = -\lambda_0 - \frac{|\nabla f|^2}{f^2} \leq -\lambda_0.$$

Let A be a compact n-submanifold of N. Then by Stokes' theorem,

$$\int_A \Delta(-\log f) = \int_{\partial A} -\frac{\nabla f}{f} \cdot \hat{n}.$$

But

$$\int_A \Delta(-\log f) \geq \lambda_0 \operatorname{vol}(A)$$

and

$$\int_{\partial A} -\frac{\nabla f}{f} \cdot \hat{n} \leq R \operatorname{vol}_{n-1}(\partial A),$$

so

$$R \frac{\operatorname{vol}_{n-1}(\partial A)}{\operatorname{vol}(A)} \geq \lambda_0,$$

which completes the proof. q.e.d.

When N is analytically tame and h is a positive harmonic function, the same argument applies to prove:

Proposition 3.2. *If N is an analytically tame hyperbolic 3-manifold and h is a positive harmonic function on N, then*

$$\int_{C(N)} \left|\frac{\nabla h}{h}\right|^2 < \infty.$$

Proof of 3.2. Let C_i be a sequence of compact submanifolds exhausting $C(N)$ such that ∂C_i has area less than K. Then

$$\int_{C_i} \Delta(-\log h) = \int_{C_i} \left|\frac{\nabla h}{h}\right|^2 = \int_{\partial C_i} -\frac{\nabla h}{h} \cdot \hat{n} \leq \operatorname{area}(\partial(C_i))R.$$

Therefore,

$$\int_{C(N)} \left|\frac{\nabla h}{h}\right|^2 \leq KR < \infty.$$

Remark. It is a consequence of Theorem 1.2 in Li-Yau [22] that if $u(x, t)$ is any positive solution of the heat equation $(\Delta - \frac{\partial}{\partial t})u(x, t) = 0$ on

$N \times (0, \infty)$, where N is a complete noncompact Riemannian n-manifold without boundary whose Ricci curvature is bounded below by $-(n-1)$, then

$$\frac{|\nabla u|^2}{u^2} - \frac{\alpha u_t}{u} \leq \frac{n\alpha^2(n-1)}{\sqrt{2}(\alpha-1)} + \frac{n\alpha^2}{2t}$$

for all $\alpha > 1$. If f is a positive eigenfunction of the Laplacian with eigenvalue $-\lambda$ on N, then $u(x, t) = e^{-\lambda t} f(x)$ is a positive solution of the heat equation. Applying the above inequality to u and letting $\alpha = 2$ and t go to ∞, we obtain

$$\frac{|\nabla f|^2}{f^2} \leq 2\sqrt{2}n(n-1) - 2\lambda.$$

Therefore in our proof of Buser's theorem we may take R to be $\sqrt{2\sqrt{2}n(n-1)}$. This appears to improve on the constant obtained in Buser's original paper [6].

4. Proofs of Theorem A and Corollary B

Theorem A. *Let N be an infinite volume, analytically tame hyperbolic 3-manifold. Then $\lambda_0(N) = 0$ if N is not geometrically finite. Moreover, there exists $K > 0$ such that if N is geometrically finite, then*

$$\lambda_0(N) \leq K \frac{|\chi(\partial C(N))|}{\operatorname{vol}(C(N))}.$$

Proof of Theorem A. We first suppose that $C(N)$ has infinite volume (i.e., that N is not geometrically finite). Let $\{C_i\}$ be a collection of compact submanifolds exhausting $C(N)$ such that $\operatorname{area}(\partial C_i) \leq K$ for some K. In this case, $\lim_{i\mapsto\infty} \operatorname{vol}(C_i) = \infty$, so $h(N) = 0$. Therefore, applying Buser's Theorem 3.1, we see that $\lambda_0(N) = 0$.

If N is geometrically finite, let $C_\varepsilon = C(N) \cap N_{\text{thick}(\varepsilon)}$. Since $C(N) \cap N_{\text{thick}(\varepsilon)}$ is compact, $N_{\text{thin}(\varepsilon)}$ has only finitely many components, so there exists $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ then all the components of $N_{\text{thin}(\varepsilon)}$ are noncompact. Let T be a noncompact component of $N_{\text{thin}(\varepsilon)}$. If T is a torus cusp, then it is isometric to $T^2 \times [c, \infty)$ with the metric

$$ds^2 = \frac{ds_{T^2}^2 + dt^2}{t^2},$$

where $ds_{T^2}^2$ is a Euclidean metric on T^2 and $c > 0$. If T is a rank-one cusp, then $T \cap C(N)$ is isometric to $A \times [c, \infty)$ with metric

$$ds^2 = \frac{ds_A^2 + dt^2}{t^2},$$

where ds_A^2 is an Euclidean metric on the annulus A and $c > 0$. If $\varepsilon \leq \varepsilon_0$, then $\partial C_\varepsilon - \partial C(N) = \partial N_{\mathrm{thin}(\varepsilon)} \cap C(N)$, so

$$\mathrm{area}(\partial C_\varepsilon - \partial C(N)) = \left(\frac{\varepsilon}{\varepsilon_0}\right)^2 \mathrm{area}(\partial C_{\varepsilon_0} - \partial C(N)),$$

which implies

$$\lim_{\varepsilon \mapsto 0} \mathrm{area}(\partial C_\varepsilon) = \mathrm{area}(\partial C(N)) = 2\pi |\chi(\partial C(N))|,$$

while

$$\lim_{\varepsilon \mapsto 0} \mathrm{vol}(C_\varepsilon) = \mathrm{vol}(C(N)).$$

Therefore,

$$h(N) \leq \frac{2\pi |\chi(\partial C(N))|}{\mathrm{vol}(C(N))}$$

and one may again use Theorem 3.1 to complete the argument. q.e.d.

In a series of papers Lax and Phillips ([20], [21]) have studied the spectrum of the Laplacian on finite volume geometrically finite hyperbolic manifolds. In particular, they proved that $\lambda_0 \neq 0$. (One may also see that $\lambda_0 \neq 0$ for geometrically finite hyperbolic 3-manifolds using the techniques of Patterson and Sullivan; see, for example, [32], [33].) We state a portion of their results.

Theorem 4.1 (Lax and Phillips). *Let N be an infinite volume geometrically finite hyperbolic 3-manifold. The intersection of* $\mathrm{spec}(-\Delta)$ *with the interval* $[0, 1)$ *consists entirely of a finite number of point eigenvalues* (*of finite multiplicity*) *all lying in* $(0, 1)$, *and there are no point eigenvalues in* $[1, \infty)$. *Moreover, the spectrum is absolutely continuous and of infinite uniform multiplicity in* $[1, \infty)$.

We combine this with Theorem A to obtain:

Corollary B. *Let N be an infinite volume analytically tame hyperbolic 3-manifold. Then $\lambda_0(N) = 0$ if and only if N is not geometrically finite.*

We recall that the critical exponent of the Poincaré series of a Kleinian group Γ is defined to be

$$\delta = \inf\left\{ s \,\middle|\, \sum_{\gamma \in \Gamma} e^{-sd(0, \gamma(0))} < \infty \right\}.$$

This critical exponent is closely related to λ_0. In fact (see Sullivan [33]), if $\lambda_0 = 1$ then $\delta \leq 1$, otherwise $\delta > 1$ and $\lambda_0 = \delta(2-\delta)$. Therefore Theorem A implies:

Corollary 4.2. *If $N = \mathbf{H}^3/\Gamma$ is analytically tame but not geometrically finite, then the critical exponent of its Poincaré series is 2. Moreover, if N is geometrically finite and $\lambda_0(N) \neq 1$, then*

$$\delta \geq 2 - \frac{K|\chi(\partial C(N))|}{\operatorname{vol} C(N)}.$$

Let n_k denote the number of elements γ of Γ such that $\gamma(0)$ is contained in the ball of (hyperbolic) radius k about 0. Then

$$\delta = \limsup \frac{\log n_k}{k}.$$

If N is further convex cocompact, then there exist constants a and A such that $ae^{k\delta} \leq n_k \leq Ae^{k\delta}$ (see Sullivan [30]). So we can interpret Corollary 4.2 as an asymptotic estimate on the number of lattice points in the ball of radius k in terms of the volume of the convex core.

We recall that the discrete spectrum of N is defined to be the isolated points in $\operatorname{spec}(-\Delta)$ which correspond to eigenvalues of finite multiplicity. The *essential spectrum* of N is the complement of the discrete spectrum in $\operatorname{spec}(-\Delta)$. (See Donnelly [14] for a discussion of the essential spectrum.) One direct consequence of Lax and Phillips' result is that the bottom $\lambda_0^{\mathrm{ess}}(N)$ of the essential spectrum is 1, whenever N is geometrically finite. As a consequence of Theorem A we obtain:

Corollary 4.3. *If N is analytically tame, but not geometrically finite, then $\lambda_0^{\mathrm{ess}}(N) = 0$.*

Proof of 4.3. Theorem A assures us that $0 \in \operatorname{spec}(-\Delta)$. But it is a result of Yau [37] that there are no (nonzero) harmonic functions in $L^2(N)$ when N is a complete infinite volume Riemannian manifold. Therefore 0 is in the essential spectrum. q.e.d.

Another consequence of Theorem A and Lax and Phillips' result is:

Corollary 4.4. *If N is geometrically finite and*

$$\operatorname{vol}(C(N)) > \frac{1}{2\pi K|\chi(\partial C(N))|},$$

then N has nonempty point spectrum.

Remarks. (1) If N is a geometrically finite hyperbolic 3-manifold, then there exists a positive eigenfunction f such that $\Delta f = -\lambda_0 f$ and

$$f(x) = \int_{S^2_\infty} \left(\frac{1-|x|^2}{|x-\xi|^2}\right)^d d\nu,$$

where $1 \leq d < 2$, and ν is a probability measure on the sphere at infinity for the Poincaré ball model for $\mathbf{H}^3$ (see Sullivan [30], Patterson [28], or Nicholls [26]). In particular, $|\nabla f/f| \leq 2$. Therefore, returning to the proof of Buser's theorem, we see that $\lambda_0(N) \leq 2h(N)$, so that the constant K in Theorem A may be taken to be 4π. Notice that the topological term $|\chi(\partial C(N))|$ is necessary in the statement of Theorem A, since when one passes to a finite cover of a geometrically finite hyperbolic 3-manifold λ_0 remains the same.

(2) In [31] Sullivan proves that $\lambda_0(N) = 0$ when N is a "hyperbolic half-cylinder" (see remark (1) at the end of §5 for the definition of a hyperbolic half-cylinder and a discussion of Sullivan's work). Let M be a hyperbolic 3-manifold which fibers over the circle. C. L. Epstein [16] proved that $\lambda_0(N) = \lambda_0^{\mathrm{ess}}(N) = 0$ if N is the cover of M associated to the fiber subgroup. In both cases the manifolds involved are known to be topologically tame.

(3) In [15] Doyle shows that there exists $Y > 0$ such that if Γ is a classical Schottky group, then $\lambda_0(N) \geq Y$, where $N = \mathbf{H}^3/\Gamma$. Therefore, Theorem A implies that $\mathrm{vol}(C(N)) \leq K(2g-2)/Y$, where g is the genus of the classical Schottky group. We can interpret this as a quantitative version of the fact (see Jorgensen-Marden-Maskit [19]) that all algebraic limits of classical Schottky groups are geometrically finite. Recall that Γ is a classical Schottky group of genus g if there exist g mutually disjoint pairs of circles in the sphere at infinity such that Γ is generated by a set of g Möbius transformations each of which takes the interior of a circle to the exterior of its partner circle. In this case, the neighborhood of radius 1 of $C(N)$ is a handlebody of genus g.

(4) If N is convex cocompact and $\lambda_0(N) = 1$, then Γ is either a Schottky group (i.e., the neighborhood of radius 1 of $C(N)$ is a handlebody) or a Fuchsian group (i.e., $C(N)$ is a totally geodesic surface) (see Sullivan [30] or Braam [4]). Presumably, the case with cusps is equally restrictive.

5. The Hausdorff dimension of the limit set

In this section we prove that the limit sets of geometrically infinite, analytically tame hyperbolic 3-manifolds with thin parts of uniformly bounded type have Hausdorff dimension 2. If the limit set is not all of S^2, then such limit sets provide naturally arising examples of sets with measure zero but Hausdorff dimension 2. This phenomenon was first studied by Sullivan [31], who proved this result for "hyperbolic half-cylinders" (see the remarks at the end of the section for a discussion of Sullivan's work).

Theorem C. *If $N = \mathbf{H}^3/\Gamma$ is an analytically tame hyperbolic 3-manifold with thin parts of uniformly bounded type which is not geometrically finite, then its limit set L_Γ has Hausdorff dimension 2.*

Proof of Theorem C. The first step in the Patterson-Sullivan program is the construction of a probability measure μ on S^2_∞ (called Patterson-Sullivan measure) supported on the limit set such that

$$(*) \qquad \mu(\gamma(E)) = \int_E |\gamma'|^\delta \, d\mu,$$

where δ is the critical exponent of the Poincaré series, E is any Borel subset of the sphere, and γ is any element of Γ. (All calculations are done in the Poincaré ball model for hyperbolic 3-space.) We also recall that if ν is any probability measure on S^2_∞ and $d \in [0, 2]$, then we may define a function $\phi_{\nu,d}$ on $\mathbf{H}^3$, where

$$\phi_{\nu,d}(x) = \int_{S^2_\infty} |\alpha_x'|^d \, d\nu,$$

α_x being a hyperbolic isometry taking x to 0. Explicitly,

$$|\alpha_x'(\xi)| = \frac{1 - |x|^2}{|x - \xi|^2}.$$

$\phi_{\nu,d}$ is then a positive eigenfunction of the Laplacian with eigenvalue $d(d-2)$. When μ is the Patterson-Sullivan measure on L_Γ and $d = \delta$, condition $(*)$ guarantees that $\phi_{\mu,\delta}$ is equivariant with respect to Γ. Recall from Corollary 4.2 that in our case $\delta = 2$, so $\phi_{\mu,2}$ descends to a positive harmonic function on N. See Sullivan [30], [32], Patterson [28], or Nicholls [26] for a discussion of Patterson-Sullivan measure.

Our proof depends on the following result of Sullivan (see Theorem 2.15 of [33] or [31]) whose proof we will review. Recall that if ϕ is a function on $\mathbf{H}^3$, we define its exponential growth rate to be

$$e(\phi) = \limsup_{R\to\infty} \left(\frac{\log(\max\{\phi(x) \mid x \in B_R(0)\})}{R} \right).$$

If $e(\phi) \le 0$, then ϕ is said to have *subexponential growth.*

Proposition 5.1 (Sullivan). *Let ν be a probability measure on S^2_∞, and $\phi(x) = \int_{S^2_\infty} |\alpha_x'|^2 \, d\nu$. If ϕ has subexponential growth, then the support of ν has Hausdorff dimension 2.*

Proof of 5.1. If $\xi \in S^2_\infty$, let $\nu(\xi, r)$ denote the ν-mass of a disk $B(\xi, r)$ of (spherical) radius r about ξ. Given $\varepsilon > 0$ choose $T(\varepsilon)$ such that if $d(0, x) \ge T(\varepsilon)$ then $\phi(x) \le e^{\varepsilon d(0,x)}$.

Lemma 5.2. *There exists $C > 0$ such that if $r \le e^{-T(\varepsilon)}$ and $\xi \in S^2_\infty$, then $\nu(\xi, r) \le Cr^{2-\varepsilon}$.*

Proof of 5.2. Let $\sigma(\xi)$ denote the geodesic ray from 0 to ξ, and let $p(\xi, r)$ denote the point along this ray at a distance of $-\log(r)$ away from the origin. Then $p(\xi, r) = ((1-r)/(1+r))\xi$, so

$$|\xi - p(\xi, r)| = \frac{2r}{1+r} \le 2r.$$

If $\hat{\xi} \in B(\xi, r)$, then $|\hat{\xi} - p(\xi, r)| \le 3r$. Moreover

$$1 - |p(\xi, r)|^2 = \frac{4r}{(1+r)^2}.$$

If $\hat{\xi} \in B(\xi, r)$ and α_p is a hyperbolic isometry taking $p(\xi, r)$ to 0, then

$$|\alpha'_p(\hat{\xi})| \ge \frac{4r}{9r^2(1+r)^2} \ge \frac{1}{9r}$$

since $r \le 1$. Therefore,

$$\phi(p(\xi, r)) \ge \int_{B(\xi, r)} |\alpha'_p|\, d\nu \ge \frac{1}{81r^2}\nu(\xi, r),$$

which implies

$$\nu(\xi, r) \le 81r^2\phi(p(\xi, r)) \le 81r^2 e^{-\varepsilon \log r} = 81r^{2-\varepsilon}. \quad \text{q.e.d.}$$

If $\{B(\xi, r_i)\}$ is a covering of the support of ν by a countable collection of balls of radius $r_i \le e^{-T(\varepsilon)}$ centered at ξ_i, then

$$1 \le \sum \nu(\xi, r_i) \le \sum Cr_i^{2-\varepsilon},$$

which shows that $\operatorname{supp}(\nu)$ has positive $(2-\varepsilon)$-dimensional Hausdorff measure. In particular, the Hausdorff dimension of $\operatorname{supp}(\nu)$ is at least $2-\varepsilon$. But since this holds for all $\varepsilon > 0$, $\operatorname{supp}(\nu)$ has Hausdorff dimension 2, and hence Proposition 5.1 is proved.

The proof of Theorem C is then completed by the following proposition.

Proposition 5.3. *Let N be an analytically tame hyperbolic 3-manifold with thin parts of uniformly bounded type which is not geometrically finite, and μ its associated Patterson-Sullivan measure. Then $\phi_{\mu,2}$ has subexponential growth.*

Proof of 5.3. Throughout this proof we will fix a value of $\varepsilon > 0$ which is less than the Margulis constant. For convenience we will assume that $0 \in C(N) \cap N_{\text{thick}(\varepsilon)}$, and ϕ will serve as shorthand for $\phi_{\mu,2}$.

We will need the following easy consequence of elliptic theory:

Lemma 5.4. *Given any $\delta > 0$ and $\varepsilon > 0$, there exists $A > 0$ with the following property. If M is any complete hyperbolic 3-manifold, $x \in M_{\text{thick}(\varepsilon)}$, and h is any positive harmonic function on M such that $|\frac{\nabla h}{h}(x)| \geq \delta$, then*

$$\int_{B_{\varepsilon/2}(x)} \left|\frac{\nabla h}{h}\right|^2 \geq A.$$

Proof of 5.4. Since $x \in M_{\text{thick}(\varepsilon)}$ we may assume that $M = \mathbf{H}^3$ and $x = 0$. Suppose that the lemma is false. Then there exists a sequence u_n of positive harmonic functions on $\mathbf{H}^3$ such that $u_n(0) = 1$,

$$\left|\frac{\nabla u_n}{u_n}(0)\right| \geq \delta, \quad \text{and} \quad \int_{B_{\varepsilon/2}(0)} \left|\frac{\nabla u_n}{u_n}\right|^2 \leq \frac{1}{n}.$$

The Harnack inequality [36] assures us that $|\frac{\nabla u_n}{u_n}(x)| \leq R$ for some $R > 0$, all n, and all $x \in \mathbf{H}^3$, so that $u_n(x) \leq e^{Rd(0,x)}$ for all $x \in \mathbf{H}^3$. Therefore, by elliptic theory (see Aubin [2] for example), there exists a subsequence $\{u_j\}$ which converges in the C^1-topology to a positive harmonic function u. But this would imply that

$$\left|\frac{\nabla u}{u}(0)\right| \geq \delta \quad \text{and} \quad \int_{B_{\varepsilon/2}(0)} \left|\frac{\nabla u}{u}\right|^2 = 0,$$

which contradicts the fact that u is C^∞, and completes the proof of Lemma 5.4. q.e.d.

We now recall, from Proposition 3.2, that

$$\int_{C(N)} \left|\frac{\nabla \phi}{\phi}\right|^2 < \infty.$$

Thus $|\frac{\nabla\phi}{\phi}(x)|$ goes to 0 uniformly on $C(N) \cap N_{\text{thick}(\varepsilon)}$, i.e., given $\delta > 0$, there exists a compact submanifold Y_δ of $C(N) \cap N_{\text{thick}(\varepsilon)}$ such that $|\frac{\nabla\phi}{\phi}(x)| \leq \delta$ on $(C(N) \cap N_{\text{thick}(\varepsilon)}) - Y_\delta$. Let M_δ denote $\max\{\phi(x) \mid x \in Y_\delta\}$. Let $L > 0$ be such that if σ is any geodesic in N, then the distance (measured in σ) between components of $\sigma \cap N_{\text{thin}(\varepsilon)}$ is at least L. Let $J > 0$ be a uniform bound on the diameter of each component of $\partial N_{\text{thin}(\varepsilon)} \cap C(N)$.

Lemma 5.5. *If $x \in C(N) \cap N_{\text{thick}(\varepsilon)}$, then*

$$\phi(x) \leq (M_\delta e^{\delta J}) e^{C_1 \delta d(0,x)},$$

where $C_1 = 1 + J/L$.

Proof of 5.5. Let $x \in C(N) \cap N_{\text{thick}(\varepsilon)}$, and let σ be a path joining 0 to x and lying entirely in $C(N) \cap N_{\text{thick}(\varepsilon)}$. We may integrate ϕ, over the portion of σ which does not lie in Y_δ, to obtain

$$\phi(x) \leq M_\delta e^{\delta l(\sigma)},$$

where $l(\sigma)$ denotes the length of σ. Now let σ' be the shortest geodesic joining 0 to x; notice that σ' lies entirely in $C(N)$ and $l(\sigma') = d(0, x)$. Now σ' intersects at most $1 + d(0, x)/L$ components of $N_{\text{thin}(\varepsilon)}$. We may replace each component of $\sigma' \cap N_{\text{thin}(\varepsilon)}$ by a path lying entirely in $\partial N_{\text{thin}(\varepsilon)} \cap C(N)$ of length of most J to form a new path $\hat{\sigma}$ joining 0 to x, lying entirely in $C(N) \cap N_{\text{thick}(\varepsilon)}$, and having length at most

$$d(0, x) + \left(1 + \frac{d(0, x)}{L}\right) J = C_1 d(0, x) + J.$$

(Notice that the new path $\hat{\sigma}$ need not be homotopic to the original path.) Therefore,

$$\phi(x) \leq M_\delta e^{\delta l(\hat{\sigma})} \leq M_\delta e^{C_1 \delta d(0, x) + \delta J},$$

proving Lemma 5.5. q.e.d.

Let $R\colon N \to C(N)$ denote the nearest point retraction, i.e., $R(x)$ is the nearest point of $C(N)$ to x (see Canary-Epstein-Green [7] or Epstein-Marden [17] for a discussion).

Lemma 5.6. *If $R\colon N \to C(N)$ is the nearest point retraction and $0 \in C(N)$, then $d(0, R(x)) \leq d(0, x)$ and $\phi(x) \leq \phi(R(x))$ for all $x \in N$.*

Proof of 5.6. We may assume that $x \in N - C(N)$. Let P be the totally geodesic hyperplane which passes through $R(x)$ and is perpendicular to the geodesic segment $\overline{xR(x)})$ through x and $R(x)$. Notice that $C(N)$ lies entirely on one side of P (see Epstein-Marden [17]). The geodesic segment $\overline{0R(x)}$ lies entirely within $C(N)$ and thus makes an obtuse angle with the geodesic segment $\overline{xR(x)}$. Therefore by considering the geodesic triangle with vertices 0, x, and $R(x)$ we see that $d(0, R(x)) < d(0, x)$.

We will now see that $|\alpha_x'(\xi)| \leq |\alpha_{R(x)}'(\xi)|$ for all $\xi \in L_\Gamma$, which clearly implies that $\phi(R(x)) \geq \phi(x)$. If $\xi \in L_\Gamma$, then the geodesic half ray $\overline{R(x)\xi}$ lies entirely in $C(N)$ and is perpendicular to the horoball

$$H = \left\{ y \in \mathbf{H}^3 \,\middle|\, \frac{1 - |y|^2}{|y - \xi|^2} = \frac{1 - |R(x)|^2}{|R(x) - \xi|^2} \right\}$$

based at ξ and passing through $R(x)$. Therefore, since $\overline{R(x)\xi}$ makes an obtuse angle with $\overline{xR(x)}$, we have

$$|\alpha'_x(\xi)| = \frac{1-|x|^2}{|x-\xi|^2} \le \frac{1-|R(x)|^2}{|R(x)-\xi|^2} = |\alpha'_{R(x)}(\xi)|,$$

which completes the proof of Lemma 5.6. q.e.d.

We now deal with compact components of $N_{\text{thin}(\varepsilon)}$.

Lemma 5.7. *If T is any compact component of $N_{\text{thin}(\varepsilon)}$ and $x \in T$, then*

$$\phi(x) \le (M_\delta e^{\delta(C_1 J + J)}) e^{C_1 \delta d(0,x)}.$$

Proof of 5.7. Let S be the boundary of T. The maximum principle (cf. Aubin [2]) implies that the maximum of ϕ over T occurs at a point $\hat{x}$ on S. Lemma 5.6 shows that $\hat{x} \in S \cap C(N)$ (since if $y \in T$, then $R(y) \in T$). Consider $\hat{\sigma}$, the shortest geodesic joining 0 to x. Notice that $\hat{\sigma}$ lies entirely within $C(N)$, and let y be the first point of intersection of $\hat{\sigma}$ with T. Since y is within J of $\hat{x}$, we have $d(0,\hat{x}) \le d(0,x) + J$. We may then apply Lemma 5.5 to see that

$$\phi(x) \le \phi(\hat{x}) \le (M_\delta e^{\delta J}) e^{C_1 \delta d(0,x) + C_1 J \delta} = (M_\delta e^{\delta(C_1 J + J)}) e^{C_1 \delta d(0,x)},$$

proving Lemma 5.7. q.e.d.

We now need only deal with noncompact portions of $N_{\text{thin}(\varepsilon)}$. In §2 of [32] Sullivan establishes that the eigenfunction corresponding to Patterson-Sullivan measure behaves roughly like $e^{(2-\delta)d(0,x)}$ on torus cusps and like $e^{(1-\delta)d(0,x)}$ on rank-one cusps of bounded type (see also Patterson [27]). To both be precise and avoid introducing the construction of the Patterson-Sullivan measure we will use an explicit version of Sullivan's result, which is obtained in the proof of Theorem 3.5.9 in Nicholls [26].

If M is any complete hyperbolic 3-manifold and T is any noncompact component of $M_{\text{thin}(\varepsilon)}$ having boundary S, then there exists a map $F_T \colon T \to S$ which takes a point $x \in T$ to the nearest point on S. (If Γ_∞ is a group of parabolic elements preserving ∞ in the upper half-space model for $\mathbf{H}^3$ and T is isometric to $\{(z,t) \mid t \ge 1\}/\Gamma_\infty$, then $F(z,t) = (z,1)$.)

Lemma 5.8. *Let M be a complete hyperbolic 3-manifold, ν its associated Patterson-Sullivan measure, and T a bounded cusp of rank k. Then given any point y in the boundary S of T and any $\alpha > 0$ there exists D such that if $p \in F_T^{-1}(y)$ then*

$$\phi_{\nu,\delta}(p) \le D e^{(k+\alpha-\delta)d(p,y)},$$

where δ is the critical exponent of the Poincaré series.

We then improve this slightly to obtain:

Lemma 5.9. *If T is any noncompact component of N, then given any $\alpha > 0$ there exists $B(T, \alpha)$ such that if $x \in T \cap C(N)$, then*

$$\phi(x) \le B(T, \alpha) e^{\alpha d(0,x)}.$$

Proof of 5.9. Pick $y \in S \cap C(N)$, and let D be such that if $p \in F_T^{-1}(y)$, then $\phi(p) \le De^{\alpha d(y,p)}$. If $x \in T \cap C(N)$, then there exists a point $\hat{x} \in F_T^{-1}(y)$ such that $d(\hat{x}, x) \le J$. Now notice that $d(\hat{x}, y) \le d(0, x) + 2J$, therefore $\phi(\hat{x}) \le De^{\alpha(d(0,x)+2J)}$. But, since $|\frac{\nabla\phi}{\phi}(y)| \le R$ for all $y \in N$ and $d(\hat{x}, x) \le J$,

$$\phi(x) \le e^{RJ}\phi(\hat{x}) \le De^{RJ+2\alpha J}e^{\alpha d(0,x)},$$

from which we obtain the assertion in Lemma 5.9. q.e.d.

(Notice that this is the only point at which we have used the construction of Patterson-Sullivan measure; if N has no cusps, then the proof applies when μ is any measure supported on the limit set satisfying condition $(*)$.)

Let $B(C_1\delta)$ denote the maximum of $B(T, C_1\delta)$ taken over the (finitely many) noncompact components of $N_{\mathrm{thin}(\varepsilon)}$. Recall from Lemma 5.6 that $\phi(x) \le \phi(R(x))$, and from Lemma 5.2 that $d(0, x) \le d(0, R(x))$. Thus, by combining Lemmas 5.5, 5.7, and 5.9 we see that

$$\phi(x) \le \phi(R(x)) \le (M_\delta e^{\delta(C_1J+J)} + B(C_1\delta))e^{C_1\delta d(0,x)}$$

for all points $x \in N$. Therefore, $e(\phi) \le C_1\delta$, but since this is true for all $\delta > 0$, $e(\phi) \le 0$. This completes the proof of Proposition 5.3 and hence of Theorem C.

Remarks. (1) Let N be a hyperbolic 3-manifold homeomorphic to $S \times \mathbf{R}$ whose convex core is homeomorphic to $S \times [0, \infty)$ and which has a uniform lower bound on its injectivity radius. N is said to be a "hyperbolic half-cylinder" if there exists an embedded surface $\widehat{S}$, homotopic to $S \times \{0\}$, such that if $\rho(x)$ denotes the distance from x to $\widehat{S}$, there exists K such that given any n there exists $d \in [n, n+1]$ such that the portion of $\rho^{-1}(d)$ contained in the convex core has diameter less than K. With these assumptions, Sullivan proves that ϕ has linear growth on the convex core, and that Patterson-Sullivan measure is ergodic, hence unique.

(2) Examples of topologically tame hyperbolic 3-manifolds which are not geometrically finite but do have a lower bound on their injectivity radius may be given by using the techniques of Thurston [35] or Jorgenson. The space $QF(S)$ of geometrically finite hyperbolic structures without cusps on $S \times R$ is parametrized by $\mathscr{T}(S) \times \mathscr{T}(S)$, where $\mathscr{T}(S)$ denotes the space of marked hyperbolic structures on the closed surface

S of genus $g \geq 2$. If (σ, τ) is any point in $QF(S)$, and ϕ is any pseudo-Anosov homeomorphism of S, then the sequence of hyperbolic manifolds $(\sigma, \phi^n(\tau))$ converges, both geometrically and algebraically (at least up to subsequence), to a topologically tame hyperbolic 3-manifold with a lower bound on its injectivity radius. These examples have limit sets of measure zero. (Recall that a homeomorphism $\phi: S \to S$ is said to be pseudo-Anosov if it is not homotopic to a finite order homeomorphism and no finite collection of disjoint simple closed curves on S is preserved up to isotopy by ϕ.) This construction provides a $(6g-6)$-dimensional space of hyperbolic 3-manifolds with a lower bound on there injectivity radius, however one still expects such examples to be rare in the boundary of $QF(S)$.

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References

[1] L. V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. **86** (1964) 413–29.
[2] T. Aubin, *Nonlinear analysis of manifolds. Monge-Ampère equations*, Springer, New York, 1982.
[3] F. Bonahon, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) **124** (1986) 71–158.
[4] P. Braam, *A Kaluza-Klein approach to hyperbolic three-manifolds*, Enseignement Math. **34** (1988) 275–311.
[5] M. Burger & R. Canary, in preparation.
[6] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École. Norm. Sup. **15** (1982) 213–230.
[7] R. D. Canary, D. B. A. Epstein & P. Green, *Notes on notes of Thurston*, Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, Cambridge, 1987, 3–92.
[8] R. D. Canary, *Ends of hyperbolic* 3-*manifolds*, preprint.
[9] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis, Princeton University Press, Princeton, NJ, 1970, 195–199.
[10] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975) 289–297.
[11] S. Y. Cheng & S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975) 333–354.
[12] M. Culler & P. Shalen, *Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic* 3-*manifolds*, J. Amer. Math. Soc. **5** (1992) 231–288.
[13] J. Dodziuk & B. Randol, *Lower bounds for* λ_1 *on a finite-volume hyperbolic manifold*, J. Differential Geometry **24** (1986) 133–139.

[14] H. Donnelly, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981) 1-14.

[15] P. Doyle, *On the bass note of a Schottky group*, Acta. Math. **160** (1988) 249–284.

[16] C. L. Epstein, *The spectral theory of geometrically periodic hyperbolic* 3-*manifolds*, Mem. Amer. Math. Soc. No. 335 (1985).

[17] D. B. A. Epstein & A. Marden, *Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces*, Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, Cambridge, 1987, 113–253.

[18] M. Feighn & D. McCullough, *Finiteness conditions for* 3-*manifolds with boundary*, Amer. J. Math. **109** (1987) 1155–69.

[19] T. Jorgensen, A. Marden & B. Maskit, *The boundary of classical Schottky space*, Duke Math. J. **46** (1979) 441-6.

[20] P. Lax & R. S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*, J. Funct. Anal. **46** (1982) 280–350.

[21] ——, *Translation representations for automorphic solutions of the wave equation in non-Euclidean spaces.* I, II, III, Comm. Pure Appl. Math. **37** (1984) 303–324, 779–813; **38** (1985) 197–208.

[22] P. Li & S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986) 153–201.

[23] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. (2) **99** (1974) 383–462.

[24] B. Maskit, *Kleinian groups*, Springer, Berlin, 1988.

[25] J. W. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, The Smith Conjecture (J. Morgan and H. Bass, eds.), Academic Press, New York, 1984, 37–125.

[26] P. J. Nicholls, *The ergodic theory of discrete groups*, Cambridge University Press, Cambridge, 1989.

[27] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976) 241–273.

[28] ——, *Lectures on limit sets of Kleinian groups*, Analytical and Geometrical Aspects of Hyperbolic Spaces, Cambridge University Press, Cambridge, 1987, 281–323.

[29] R. Schoen, *A lower bound for the first eigenvalue of a negatively curved manifold*, J. Differential Geometry **17** (1982) 233–238.

[30] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Haute Études Sci. Publ. Math. **50** (1979) 419–450.

[31] ——, *Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension* 2, Geometry Symposium (Utrecht 1980), Lecture Notes in Math., Vol. 894, Springer, Berlin, 1981, 127–144.

[32] ——, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta. Math. **153** (1984) 259–277.

[33] ——, *Aspects of positivity in Riemannian geometry*, J. Differential Geometry **25** (1987) 327–351.

[34] W. Thurston, *The geometry and topology of* 3-*manifolds*, lecture notes.

[35] ——, *Hyperbolic structures on* 3-*manifolds*, II: *Surface groups and* 3-*manifolds which fiber over the circle*, preprint.

[36] S. T. Yau, *Harmonic function on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975) 201–228.

[37] ——, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976) 659–670.

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