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ON THE SPECTRAL GAP FOR COMPACT MANIFOLDS

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1. Introduction

We aim to give lower bounds for the spectral gap of the Laplace operator on a compact Riemannian manifold in terms of a lower bound for the Ricci curvature and an upper bound for the diameter of the manifold.

We apply the maximum principle technique to $|\nabla\varphi|^2 - G(\varphi)$ for appropriate auxiliary functions G. The auxiliary functions are chosen in such a way that the above quantity vanishes identically if φ is replaced by an eigenfunction of an appropriate Neumann boundary problem. For the case of manifolds with nonnegative Ricci curvature it is sufficient to consider radial eigenfunctions for annular regions in constant curvature spaces.

Our approach seems to yield better results than techniques using isoperimetric inequalities (cf. [1]). If additional information about the median value of an eigenfunction is known, a sharper estimate can be obtained which, in particular, improves the result by Zhong and Yang (see [11] and [6], §4]). Our basic examples show that the estimates are in some sense sharp.

2. Statement of the basic gradient estimate

We obtain our basic estimate by comparison with a Neumann problem on a manifold with boundary. The manifold is constructed using Fermi coordinates on a sphere of constant curvature with sufficiently small diameter. This construction is also closely related to the proof of the Lévy-Gromov isoperimetric inequality (see [3], §§XXII.8 and XII.9]). We adopt the notation of Chavel's book.

Let a dimension $n > 1$, a diameter d, and a constant Ricci curvature R be given. We set $\kappa \equiv R/(n-1)$. Suppose that the condition $d \leq \pi/\sqrt{\kappa}$

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is fulfilled if $R > 0$. We set

$$\begin{array}{lll} C_\kappa(t) \equiv \cos(\sqrt{\kappa}t), & S_\kappa(t) \equiv (1/\sqrt{\kappa})\sin(\sqrt{\kappa}t) & \text{for } \kappa > 0, \\ C_\kappa(t) \equiv \cosh(\sqrt{-\kappa}t), & S_\kappa(t) \equiv (1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}t) & \text{for } \kappa < 0, \\ C_0(t) \equiv 1, & S_0(t) \equiv t & \end{array}$$

for every t. Let $\delta \leq 0$ and $d > 0$ be given such that $J_{\kappa,\delta}(t) \equiv (C_\kappa(t) - \delta S_\kappa(t))^{n-1}$ is nonnegative on $[-d/2, d/2]$. We will consider a manifold $\mathbf{M}^n_{R,\delta,d,d_S}$ with boundary containing a hypersurface $\widehat{M}$ which is isometric to an $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}_{d_S}$ of constant curvature, and with diameter d_S which will be specified later. The manifold $\mathbf{M}^n_{R,\delta,d,d_S}$ is up to isometry uniquely determined by the conditions that the exponential map Exp (cf. [3, p. 319]) based on the normal bundle $\mathscr{N}\widehat{M}$ is a diffeomorphism from $\{\zeta \in \mathscr{N}\widehat{M} \mid |\zeta| < d/2\}$ onto the interior of $\mathbf{M}^n_{R,\delta,d,d_S}$, and that the Riemannian metric ds on $\mathbf{M}^n_{R,\delta,d,d_S}$ is given by

$$ds^2(\operatorname{Exp} t\xi) = dt^2 + |(C_\kappa - \delta S_\kappa)(t)\, dp|^2$$

for every vector ξ from a connected component of the unit normal bundle $\mathscr{S}\mathscr{N}\widehat{M} \subset \mathscr{N}\widehat{M}$ and a generic element dp of the tangent spaces of $\mathbf{S}^{n-1}_{d_S}$. We consider a nonconstant solution $\hat{\psi}$ of the Neumann boundary value problem

$$\Delta_{\mathbf{M}^n_{R,\delta,d,d_S}} \hat{\psi} = -\mu\hat{\psi}, \qquad \left.\frac{\partial\hat{\psi}}{\partial n}\right|_{\partial\mathbf{M}^n_{R,\delta,d,d_S}} \equiv 0$$

for the smallest possible eigenvalue μ. For sufficiently small values of d_S the eigenfunction $\hat{\psi}$ can be given by a function $\psi^n_{R,\delta,d}$ on $[-d/2, d/2]$ as follows (notice that $\mu = \mu(n, R, \delta, d)$ and $\psi^n_{R,\delta,d}$ do not depend on d_S):

$$\hat{\psi}(\operatorname{Exp} t\xi) = \psi^n_{R,\delta,d}(t) \quad \text{for every } t \in \left[-\frac{d}{2}, \frac{d}{2}\right].$$

Moreover, $\psi^n_{R,\delta,d}$ is an eigenfunction for the first nontrivial eigenvalue μ of the following Sturm-Liouville equation:

$$\psi'' + (n-1)\frac{(C_\kappa - \delta S_\kappa)'}{C_\kappa - \delta S_\kappa}\psi' + \mu\psi = 0 \quad \text{on } \left[-\frac{d}{2}, \frac{d}{2}\right] \tag{1}$$

with Neumann boundary conditions. The relations among the quantities n, R, δ, d, and μ will be studied in the §§4 and 5. We normalize $\psi^n_{R,\delta,d}$ by $\psi^n_{R,\delta,d}(-d/2) \equiv 1$. Notice that $d \leq \pi/\sqrt{\kappa}$ for $\kappa > 0$.

We remark that in the case of positive Ricci curvature R we could also define the function $\psi^n_{R,\delta,d}$ by means of the first nontrivial "radial" eigenfunction on $\mathbf{M}^n_{R,\delta,d,d_S}$ for $d_S \equiv \pi/\sqrt{\kappa}$. For this choice of d_S the manifold $\mathbf{M}^n_{R,\delta,d,d_S}$ is isomorphic to an annular region in the n-sphere with constant sectional curvature $R/(n-1)$. Similarly, we could consider annular regions in Euclidean space instead of $\mathbf{M}^n_{R,\delta,d,d_S}$ for $R=0$ and $\delta \neq 0$.

Our basic result can be stated as follows.

Theorem 1. *Let M be an n-dimensional compact Riemannian manifold with Ricci curvature greater than or equal to R. Let φ be an eigenfunction on M for the smallest positive eigenvalue λ_1. Suppose that we are given a function $\psi^n_{R,\delta,d}$ and a scalar α such that the eigenvalue $\mu = \mu(n,R,\delta,d)$ for $\psi^n_{R,\delta,d}$ coincides with λ_1 and* $\operatorname{range}(\varphi) \subset \operatorname{range}(\alpha\psi^n_{R,\delta,d})$. *Then*

$$|\nabla\varphi(x)|^2 \leq |(\alpha\psi^n_{R,\delta,d})'|^2 \circ (\alpha\psi^n_{R,\delta,d})^{-1} \circ \varphi(x) \quad \textit{for every } x \in M,$$

where $((\alpha\psi)^{-1}$ denotes the inverse function of $\alpha\psi$, and $|(\alpha\psi)'|^2$ denotes the square of the real number $|(\alpha\psi)'|$.)

The proof of Theorem 1 is the objective of §3.

Corollary 1. *Suppose that in the situation of the theorem the stronger assumption* $\operatorname{range}(\varphi) = \operatorname{range}(\alpha\psi^n_{R,\delta,d})$ *holds. Then it follows that* $\operatorname{diam}(M) \geq d$, *where* $\operatorname{diam}(M)$ *denotes the diameter of M.*

Proof. The proof is a consequence of the well-known argument which considers a shortest geodesic joining a maximum and a minimum point of the eigenfunction φ (cf. for instance [6] or [7]).

Remarks. 1. We will see in §6 that it is also possible to give an estimate of the diameter of M without special assumptions on the range of an eigenfunction on M.

2. For any $\tilde{d} > d$ we can choose a sufficiently small d_S such that $\mathbf{M}^n_{R,\delta,d,d_S}$ can be imbedded in a rotational symmetric compact manifold $\widetilde{M}$ with diameter less than $\tilde{d}$ and Ricci curvature greater than or equal to R. For an appropriate choice of $\widetilde{M}$ we can deduce from theorems on the continuous dependence of the solutions of Sturm-Liouville equations from a parameter that the first nontrivial eigenvalue for the Laplacian on $\widetilde{M}$ and the range of the first eigenfunction are arbitrary close to the corresponding quantities for $\mathbf{M}^n_{R,\delta,d,d_S}$. It follows that the bound of Corollary 1 is essentially sharp.

3. The maximum principle technique

Proof of Theorem 1. We can consider $\psi = \psi^n_{R,\delta,d}$ as the solution of an initial value problem for the Sturm-Liouville equation (1) if we are given the values $\psi(0)$ and $\psi'(0)$. The interval $(-d/2, d/2)$ can then be characterized as the maximal interval containing the origin such that $(\psi^n_{R,\delta,d})'$ is different from zero on that interval.

From the standard theorems on the continuous dependence of the solutions on the coefficients of an ordinary differential equation it follows that the endpoints of the maximal interval containing the origin such that on that interval ψ' is different from zero and the maximum and the minimum of ψ are continuous functions of R with n, δ, μ remaining fixed. An appropriate translation with respect to the independent variable yields again a solution of a Neumann boundary value problem of type (1) (with an appropriate $\tilde{\delta}$ in place of δ) on an interval $[-\tilde{d}/2, \tilde{d}/2]$ which is symmetric with respect to the origin.

Therefore we can and will suppose that

$$(1+\varepsilon)\operatorname{range}\varphi \subset \operatorname{range}(\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}}) \tag{2}$$

and

$$(\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}})^{-1}\operatorname{range}\varphi \Subset \left(-\frac{\tilde{d}}{2}, \frac{\tilde{d}}{2}\right)$$

for some $\tilde{R} < R$, $\varepsilon > 0$, and $\tilde{d}$ and $\tilde{\alpha}$ arbitrarily close to d and α, respectively. The proof of the correctness of the assertion under the original assumption can be reduced to the case in (2) by a limit argument. We set

$$G_\beta(u) \equiv (\beta\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}})'^2 \circ (\beta\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}})^{-1}(u)$$

for every $\beta \geq 1$ and every $u \in \operatorname{range}(\beta\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}})$.

By the compactness of M, there is a point $x_0 \in M$ such that the function $|\nabla\varphi|^2 - G_\beta(\varphi)$ achieves its supremum. Suppose that $\sup(|\nabla\varphi|^2 - G_\beta(\varphi)) \geq 0$ for some $\beta \geq 1$. Since this supremum is negative for sufficiently large β, we can choose β in such a way that

$$\sup(|\nabla\varphi|^2 - G_\beta(\varphi)) = (|\nabla\varphi|^2 - G_\beta(\varphi))(x_0) = 0. \tag{3}$$

Recall that by (2)

$$(\beta\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}})^{-1}\operatorname{range}(\varphi) \subset (\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}})^{-1}\operatorname{range}(\varphi) \Subset \left(-\frac{\tilde{d}}{2}, \frac{\tilde{d}}{2}\right).$$

Hence, we have $G_\beta(\lambda) > 0$ for every $\lambda \in \operatorname{range}(\varphi)$. Thus, by (3), we can conclude that $|\nabla\varphi(x_0)| > 0$. The following argument is similar to that used by Li and Yau in [7] although we consider different auxiliary functions (cf. also §4 in [6]). It follows from (3) that

$$\tfrac{1}{2}\nabla(|\nabla\varphi|^2 - G_\beta(\varphi))|_{x_0} = 0 \tag{4}$$

and

$$\tfrac{1}{2}\Delta(|\nabla\varphi|^2 - G_\beta(\varphi))|_{x_0} \le 0. \tag{5}$$

At x_0, we rotate the frame so that $\nabla\varphi|_{x_0}$ is in the direction of the first coordinate axis. By (4),

$$\frac{1}{2}\frac{\partial}{\partial x_1}(|\nabla\varphi|^2 - G_\beta(\varphi))|_{x_0} = \varphi_1\left(\varphi_{11} - \frac{1}{2}G'_\beta(\varphi)\right)|_{x_0} = 0,$$

and hence

$$\varphi_{11}(x_0) = \tfrac{1}{2}G'_\beta(\varphi)(x_0).$$

Applying the Bochner-Lichnerowicz formula to (5), we obtain

$$\begin{aligned} 0 \ge\ & |\operatorname{Hess}\varphi|^2 + \langle\nabla\varphi, \nabla\Delta\varphi\rangle + \operatorname{Ric}(\nabla\varphi, \nabla\varphi) \\ & - \tfrac{1}{2}|\nabla\varphi|^2 G''_\beta(\varphi) - \tfrac{1}{2}\Delta\varphi G'_\beta(\varphi)|_{x_0}. \end{aligned}$$

Since

$$|\operatorname{Hess}\varphi|^2 \ge \varphi_{11} + \sum_{i>1}\varphi_{ii}^2 \ge \varphi_{11}^2 + \frac{1}{n-1}(\Delta\varphi - \varphi_{11})^2, \tag{6}$$

it follows that

$$\begin{aligned} 0 \ge\ & \varphi_{11}^2 + \frac{1}{n-1}(\varphi_{11} + \lambda_1\varphi)^2 + (R - \lambda_1)|\nabla\varphi|^2 \\ & - \frac{1}{2}|\nabla\varphi|^2 G''_\beta(\varphi) + \frac{1}{2}\lambda_1\varphi G'_\beta(\varphi)|_{x_0}. \end{aligned}$$

Finally, using $|\nabla\varphi(x_0)|^2 = G_\beta(\varphi(x_0))$ and $\varphi_{11}(x_0) = \frac{1}{2}G'_\beta(\varphi(x_0))$, we arrive at

$$\begin{aligned} 0 \ge\ & -G_\beta(\varphi)\left(\frac{1}{2}G''_\beta(\varphi) + \lambda_1 - R\right) \\ & + \left(\frac{1}{2}G'_\beta(\varphi) + \lambda_1\varphi\right)\left(\frac{1}{2}\frac{n}{n-1}G'_\beta(\varphi) + \frac{1}{n-1}\lambda_1\varphi\right)\bigg|_{x_0}. \end{aligned} \tag{7}$$

Now our key observation is that we obtain an equality instead of an inequality if we replace R by $\tilde{R}$, M by the domain $\mathbf{M}^n_{\tilde{R},\tilde{\delta},\tilde{d},d_S}$, and φ by the corresponding eigenfunction $\hat{\psi}$ with $\hat{\psi}(\operatorname{Exp} t\xi) = \beta\tilde{\alpha}\psi^n_{\tilde{R},\tilde{\delta},\tilde{d}}(t)$:

$$\begin{aligned} 0 =\ & -G_\beta(\hat{\psi})\left(\frac{1}{2}G''_\beta(\hat{\psi}) + \lambda_1 - \tilde{R}\right) \\ & + \left(\frac{1}{2}G'_\beta(\hat{\psi}) + \lambda_1\hat{\psi}\right)\left(\frac{1}{2}\frac{n}{n-1}G'_\beta(\hat{\psi}) + \frac{1}{n-1}\lambda_1\hat{\psi}\right)\bigg|_{x} \end{aligned} \tag{8}$$

for every $x \in \mathbf{M}^n_{\widetilde{R},\tilde{\delta},\tilde{d},d_s}$. Notice in particular that $\operatorname{Ric}(\nabla\hat{\psi}, \nabla\hat{\psi}) \equiv \widetilde{R}|\nabla\hat{\psi}|^2$. Moreover, (6) is actually an equality since the restriction of the Hessian matrix of $\hat{\psi}$ to the orthogonal complement of $\nabla\hat{\psi}$ is a multiple of the identity matrix for each tangent space of $\mathbf{M}^n_{\widetilde{R},\tilde{\delta},\tilde{d},d_s}$. Since $G_\beta(\varphi)(x_0) > 0$, $\widetilde{R} < R$, and $\operatorname{range}(\varphi) \subset \operatorname{range}(\hat{\psi})$, it follows that inequality (7) and equality (8) contradict each other. This completes the proof of the theorem.

Remark. An alternative approach to equations similar to (8) will be given in §5.

4. Relations between diameter and median

We aim to investigate the relations between d and the range of the eigenfunction $\hat{\psi}$ for the first nontrivial eigenvalue μ for the Neumann boundary value problem on $\mathbf{M}^n_{R,\delta,d,d_s}$ stated in §2 if n, R, and μ are fixed.

For $\kappa = 0$ and for $\kappa < 0$, $\delta \leq -\sqrt{-\kappa}$ we can derive Proposition 1 below immediately from Sturm's comparison theorem (cf. the corresponding argument in the proof of Proposition 1). Therefore we can restrict ourselves without loss of generality to the cases $\kappa = +1$; $\delta \leq 0$ and $\kappa = -1$; $-1 < \delta \leq 0$.

We set $\tau_s f(u) \equiv f(u-s)$ for every s, u and every function f. In place of the function $\psi^n_{R,\delta,d}$ we will investigate the translated function $\chi \equiv \tau_{-s}\psi^n_{R,\delta,d}$, where $s \equiv \arctan\delta$ or $s \equiv \operatorname{arctanh}\delta$. The function χ solves the following Sturm-Liouville equation on its interval of definition:

$$\chi'' - \kappa(n-1)\frac{S_\kappa}{C_\kappa}\chi' + \mu\chi = 0 \quad \text{for } \kappa = \pm 1$$

(restricted to $(-\pi/2, \pi/2)$ if $\kappa = 1$). We set $r \equiv -d/2 - s$. Given n, R, r, and μ, we can define d as the difference between r and the next zero $\tilde{r}$ of χ'. Hence, χ is uniquely determined by n, R, r, and μ. We will prefer however to simply write χ instead of $\chi^n_{R,r,\mu}$ whenever this is unlikely to lead to misunderstandings. Let n, R, μ be fixed.

The interlacing property of the zeros of the solutions of a Sturm-Liouville equation yields that $\delta = \tan((r+\tilde{r})/2)$ (or $\tanh((r+\tilde{r})/2)$) is an increasing function of r. Our assumption $\delta \leq 0$ implies that $r + \tilde{r} \leq 0$.

Proposition 1. *Given n, R, and μ, the diameter d is a decreasing function of δ for every δ with $\delta \leq 0$.*

Proof. We set $I \equiv (-\pi/2, \pi/2)$ if $\kappa = +1$ and $I \equiv \mathbf{R}^1$ if $\kappa = -1$. In view of the above remarks we will study d as a function of r. Standard calculations show that

$$f_r(u) \equiv \chi'(u) C_\kappa^{(n-1)/2}(u)$$

satisfies the equation $f_r'' + H \cdot f_r = 0$ with

$$H(u) \equiv \left(\mu - \frac{n-1}{2}\kappa\right) - \left(\frac{(n-1)^2}{4} + \frac{n-1}{2}\right) \frac{S_\kappa^2(u)}{C_\kappa^2(u)}.$$

By definition, $f_r(r) = 0$ and $f_r'(r) \leq 0$. Consider $r_1, r_2 \in I \cap (-\infty, 0]$ with $r_1 < r_2 < -\tilde{r}_2$. Thus, $g \equiv \tau_{r_1 - r_2} f_{r_2}$ satisfies the equation

$$g'' + (\tau_{r_1 - r_2} H) g = 0.$$

We aim to show that g has at least one zero between r_1 and $\tilde{r}_1$. Otherwise, by Picone's formula (see [5, §10.31]; we write f for f_{r_1}), we obtain

$$\frac{d}{du}\left(f f' - f^2 \frac{g'}{g}\right) = (\tau_{r_1 - r_2} H - H) f^2 + \left(f' - f\frac{g'}{g}\right)^2. \tag{9}$$

Once we have established that the integral of $(\tau_{r_1 - r_2} H - H) f^2$ between two consecutive zeros of f is positive, the proof of the proposition can be completed as follows. Integration of (9) between r_1 and the next zero $\tilde{r}_1$ of f would give

$$0 = \int_{r_1}^{\tilde{r}_1} ((\tau_{r_1 - r_2} H - H) f^2)(u)\, du + \int_{r_1}^{\tilde{r}_1} \left(f' - f\frac{g'}{g}\right)^2 (u)\, du.$$

Since the right-hand side of the last equation is positive, we have arrived at a contradiction.

For the proof of $\int_{r_1}^{\tilde{r}_1} ((\tau_{r_1 - r_2} H - H) f^2)(u)\, du > 0$, we can restrict ourselves to the case that $r_2 - r_1$ is infinitesimally small. More precisely, we will prove that $\int_{r_1}^{\tilde{r}_1} (H' f^2)(u)\, du > 0$. In addition, we will suppose that $r_1 < 0 < \tilde{r}_1$ since the assertion is otherwise a consequence of Sturm's comparison theorem (it can easily be checked that H' is positive on $I \cap (-\infty, 0)$). Consider the reflection σf of f at the axis $u = 0$, i.e., $\sigma f(u) \equiv f(-u)$. Since $f(0) = \sigma f(0)$, it follows from the standard results about Sturm-Liouville equations that

$$|f(u)| \geq |\sigma f(u)| \quad \text{for } -\tilde{r}_1 < u < 0,$$
$$|f(u)| \leq |\sigma f(u)| \quad \text{for } 0 < u < \tilde{r}_1.$$

In view of $H'(u) = -H'(-u) > 0$ for $u < 0$, we obtain

$$\int_{r_1}^{\tilde{r}_1} H' f^2(u)\,du > \int_{-\tilde{r}_1}^{\tilde{r}_1} H' f^2(u)\,du = \int_{-\tilde{r}_1}^{0} H'(f^2 - \sigma f^2)(u)\,du \geq 0,$$

which completes the proof. q.e.d.

Now we aim to study the range of $\psi^n_{R,\delta,\cdot}$ for a fixed eigenvalue $\mu > 0$ in dependence on δ. The "asymmetry" of the range of an eigenfunction ψ with respect to 0 can be described by its median value

$$a(\psi) \equiv |(\max\psi + \min\psi)/(\max\psi - \min\psi)|$$

(cf. [6, §4]).

Proposition 2. *Given n, R, and μ, the median a is a decreasing function of δ for every δ with $\delta \leq 0$.*

Proof. It is clear that a depends continuously on δ. We will show that the function $\delta \mapsto a$ is invertible. Suppose that the median a coincides for two different δ_1, δ_2. It follows that $\psi^n_{R,\delta_1,\cdot}$ and $\psi^n_{R,\delta_2,\cdot}$ have the same range. Now, the analysis of §3 shows that

$$\begin{aligned} |(\psi^n_{R,\delta_1,\cdot})'| \circ (\psi^n_{R,\delta_1,\cdot})^{-1} &\leq |(\psi^n_{R,\delta_2,\cdot})'| \circ (\psi^n_{R,\delta_2,\cdot})^{-1} \\ &\leq |(\psi^n_{R,\delta_1,\cdot})'| \circ (\psi^n_{R,\delta_1,\cdot})^{-1}. \end{aligned}$$

By the uniqueness theorems for solutions of ordinary differential equations of first order we obtain that the functions $\psi^n_{R,\delta_1,\cdot}$ and $\psi^n_{R,\delta_2,\cdot}$ coincide up to a translation with respect to the independent variable. This leads to a contradiction.

Corollary 2. *Given n, R and μ, the diameter d is an increasing function of the median a.*

Remark. For the sake of completeness we mention that the behavior of the eigenfunctions $\psi^n_{R,\delta,d}$ can be described in more detail as follows.

First, we consider the case $R = 0$. We obtain the maximal value of d and a if δ is chosen in such a way that $(C_0 - \delta S_0)(t)$ vanishes at the left endpoint of the interval $[-d/2, d/2]$ (recall that $\delta \leq 0$ by convention). The Sonin-Pólya Theorem yields in particular that $|\psi(-d/2)| > |\psi(d/2)|$ (see [2, §X.13, Exercise 4]).

Now, we suppose that $R = +1$. By Lichnerowicz' Theorem (see [3, p. 82]), $\lambda_1 = n/(n-1)$. By Obata's Theorem (see [3, p. 82]), $\lambda_1 = n/(n-1)$ if and only if M is isometric to a sphere with constant sectional curvature $R/(n-1)$. Therefore, we restrict ourselves to the case $\mu > n/(n-1)$.

Again, we obtain the maximal value for d and a if δ is chosen in such a way that $(C_\kappa - \delta S_\kappa)(t)$ with $\kappa = 1/(n-1)$ vanishes for $t = -d/2$.

Moreover, we have $|\psi(-d/2)| > |\psi(d/2)|$. (By assumption, $\psi(-d/2) > 0$. Let t_0 be the unique zero of ψ in $[-d/2, d/2]$. Then $\psi''(t_0) > 0$. Considering the associated Riccati equation, it can be shown that $\psi(2t_0 - t) > \psi(t)$ for every $t \in (t_0, d/2]$.)

Finally, we consider the case $R = -1$. For $\delta = -\sqrt{1/(n-1)}$ we obtain the following translation invariant equation: $\psi'' - \sqrt{n-1}\psi' + \mu\psi = 0$. The first nontrivial eigenvalue for the corresponding Neumann boundary value problem on a finite interval is always bigger than $(n-1)/4$ and tends to $(n-1)/4$ if the length of the interval tends to infinity. By Sturm's Comparison Theorem, the Neumann boundary value problem (1) has no solution if $0 < \mu \le (n-1)/4$ and $\delta < -\sqrt{1/(n-1)}$ (cf. [8]). On the other hand, $\delta = 0$ yields a solution for every $\mu > 0$ (this can be seen by comparison with the particular solution $\int (\cosh u)^{-(n-1)}\, du$ of the equation $\psi''(u) + (n-1)\tanh u\psi'(u) = 0$). By consideration of appropriate initial value problems (cf. the argument before Proposition 3 for $R = +1$), we obtain solutions $\psi^n_{R,\delta,d}$ with arbitrarily large diameter d and median value a tending to 1. For $\mu > (n-1)/4$, we again obtain the maximal value for d and a if $(C_\kappa - \delta S_\kappa)(t)$ with $\kappa = -1/(n-1)$ vanishes for $t = -d/2$.

5. Eigenvalue, median, and diameter for variable dimension

First of all we notice that the right-hand side of (7) is decreasing with respect to the dimension n. For technical reasons we will also consider noninteger values of n although the corresponding differential equations do not admit a geometrical interpretation. For $n \uparrow \infty$ we can conclude from (7) that

$$0 \ge -G(\varphi)(\tfrac{1}{2}G''(\varphi) + \lambda_1 - R) + \tfrac{1}{2}G'(\varphi)(\tfrac{1}{2}G'(\varphi) + \lambda_1\varphi)$$

for every smooth function G which is defined on range φ such that (3) holds. We intend to define G using appropriate solutions ψ of the Sturm-Liouville equation

$$\psi''(u) - (Ru - t)\psi'(u) + \lambda_1\psi(u) = 0 \quad \text{for every } u \in \mathbf{R} \tag{10}$$

for an appropriate real number t. It follows that

$$(Ru - t) = \frac{\psi'' + \lambda_1\psi}{\psi'}$$

on every interval such that ψ' is strictly decreasing, and by differentiation

$$R = \psi'^{-2}(\psi'\psi''' + \lambda_1\psi'^2 - \psi''^2 - \lambda_1\psi\psi''). \tag{11}$$

With $G(\psi) \equiv \psi'^2$ we obtain $\psi' G'(\psi) = 2\psi'\psi''$ and hence $\psi'' = \frac{1}{2}G'(\psi)$. Furthermore, $\psi'\psi''' = \psi'(\frac{1}{2}G'(\psi))' = \frac{1}{2}G(\psi)G''(\psi)$. Thus, we have arrived at the analogue of (8):

$$0 = -G(\psi)(\tfrac{1}{2}G''(\psi) + \lambda_1 - R) + \tfrac{1}{2}G'(\psi)(\tfrac{1}{2}G'(\psi) + \lambda_1 \psi).$$

Remark. A similar calculation as above provides a direct way of obtaining (8) from a Sturm-Liouville equation. Suppose for instance that $R = n - 1$. Then we obtain from $\psi'' + (n-1)(\cot u)\psi' + \lambda\psi = 0$ that

$$(\cot u)^2 = \left(-\frac{1}{n-1}\frac{\psi'' + \lambda\psi}{\psi'}\right)^2$$

and

$$(\cot u)' = \left(-\frac{1}{n-1}\frac{\psi'' + \lambda\psi}{\psi'}\right)'.$$

Summing the last two equalities, we obtain a differential equation which does not explicitly contain the independent variable u. Thus we can show that (8) also holds for noninteger values of n.

Equality (10) has a particular simple form if $R = 0$. As a consequence we obtain the following result which improves an estimate given by Zhong and Yang (see [11] or [6], §4]).

Corollary 3. *Let M be a compact Riemannian manifold with nonnegative Ricci curvature. Let φ be an eigenfunction on M for the smallest positive eigenvalue λ_1. Then the following holds:*

$$(\operatorname{diam} M)^2 \lambda_1 \geq \pi^2 + \{\ln(\max\varphi / -\min\varphi)\}^2.$$

Proof. Let z be the complex number $z \equiv \ln(\max\varphi/-\min\varphi)/d + \pi i/d$. The function $\psi : u \mapsto \operatorname{Re}\exp(zu)$ is a solution of (10) for $\lambda_1 \equiv |z|^2$ and an appropriate t. The difference between the values of the independent variable at two consecutive extrema of ψ is equal to d, and the ratio of the values of ψ at those points is equal to $\exp(\pm\ln(\max\varphi/-\min\varphi))$. Hence the assertion follows by a similar argument as Theorem 1 and Corollary 1.

Remark. Zhong and Yang proved that

$$(\operatorname{diam} M)^2 \lambda_1 \geq \pi^2 + \frac{6}{\pi}\left(\frac{\pi}{2} - 1\right)^4 \left(\frac{\max\varphi + \min\varphi}{\max\varphi - \min\varphi}\right)^2.$$

We notice that (in contrast to our result) the expression on the right-hand side of this inequality remains bounded if $\max\varphi/-\min\varphi \to +\infty$ (or $+0$).

Remark. Using the above methods and the upper bounds for eigenvalues obtained by Cheng (see [4]), it can be deduced that the median of an eigenfunction on an n-dimensional compact Riemannian manifold with Ricci curvature bounded below by a constant R can be estimated above by a constant $a(n, R, \varepsilon) < 1$ if λ_1 is bounded below by $((n-1)(-R)/4)(1+\varepsilon)$ for a positive ε. For the sake of simplicity we restrict ourselves to the case of a manifold with nonnegative Ricci curvature. The assertion immediately follows from the above corollary and Cheng's estimate $(\operatorname{diam} M)^2\lambda_1 \le 2n(n+4)$.

However, it seems to be impossible to obtain sharp estimates for the value of $a(n, R, \varepsilon)$ by a simple combination of Cheng's results with our results since Cheng considered functions which approximate the eigenfunctions in some symmetric situations where the value of the median is 0. In particular, we cannot prove in this manner that the maximum value of the median is attained for a manifold of the form $\mathbf{M}^n_{R,\delta,d,d_S}$, where δ is chosen such that $(C_\kappa(t) - \delta S_\kappa(t)^{n-1})$ vanishes at an endpoint of the interval $[-d/2, d/2]$ (cf. the remark at the end of the previous section). This makes the following considerations necessary.

We aim to show that (1) has for some d and δ solutions with median value arbitrarily close to 1 if we choose a sufficiently large n with all the other quantities remaining fixed.

If $R = 0$, we can deduce the above statement from the proof of the above corollary if we take into account that we can approximate an arbitrarily real number t uniformly on an interval $[-d/2, d/2]$ of length d by functions of the form $u \mapsto n/(u-s)$ for appropriate values of s and n large.

Now, we consider the case $R = +1$. By a limit argument, we can restrict ourselves to the proof of the assertion for the differential equation (10) with $t = 0$ in place of (1).

First, we notice that we can find a finite interval (depending only on λ_1) such that every solution of (10) has at most one local extremum in each of the two connected components of the complement of the above interval. This follows if we differentiate (10) and apply Sturm's Comparison Theorem to the resulting equation for ψ' and an appropriate equation with constant coefficients. (10) has the following odd solution:

$$\psi(u) = u + \sum_{\nu=1}^{\infty} \frac{(1-\lambda_1)\cdot(3-\lambda_1)\cdots(2\nu-1-\lambda_1)}{(2\nu+1)!} u^{2\nu+1}.$$

We can deduce from Lichnerowicz' Theorem that $\lambda_1 > R$ for every finite n. The modulus of every solution of (10) tends to ∞ if $u \to \pm\infty$ (cf. [10, §16.5]). A simple calculation shows that the derivative of the above odd solution ψ has a local maximum at 0 if $\lambda_1 > R = +1$. Hence, the above solution has at least two local extrema. By consideration of the solution of the initial value problem $\psi(u) = 1$, $\psi'(u) = 0$ for every u, we obtain that there exist solutions of (10) such that the ratio of two particular consecutive extremal values of the solution is arbitrarily large if $R = +1$ and the sum of the corresponding values of the independent variable is positive.

Finally, the assertion in the case $R = -1$ follows from the remark at the end of §4.

We can now state the following (rather technical) result which is needed for the proof of Theorem 2 below.

Proposition 3. *Suppose that* $n, R, \lambda_1 > 0$, *and an* a *with* $0 \le a < 1$ *are given such that*

$$\left|\frac{\max\psi + \min\psi}{\max\psi - \min\psi}\right| < a$$

for every solution $\psi = \psi^n_{R,\delta,d}$ *of the Neumann problem* (1) *with eigenvalue* $\mu = \lambda_1$. *Then there exist real numbers* $\tilde{n}$ *with* $\tilde{n} > n$ *and* $\tilde{d}$ *such that the smallest eigenvalue for the Neumann boundary problem for*

$$\omega'' + (\tilde{n} - 1)\frac{S'_\kappa}{S_\kappa}\omega' + \mu\omega = 0 \quad \textit{with } \kappa = \frac{R}{\tilde{n} - 1} \textit{ on } [0, \tilde{d}] \tag{12}$$

is equal to λ_1 *and such that for the corresponding eigenfunction* $\omega = \omega^{\tilde{n}}_{R,\tilde{d}}$ *the following holds*:

$$\left|\frac{\max\omega + \min\omega}{\max\omega - \min\omega}\right| = a.$$

Moreover, $\tilde{d} > d$ *for every solution* $\psi^n_{R,\delta,d}$ *of* (1) *with* $\mu = \lambda_1$.

Proof. There is nothing to prove if $R < 0$ and $\mu \le -(n-1)R/4$ (cf. the remark at the end of §4).

In view of the above considerations the assertion is established once we have shown that the diameter $\tilde{d}$ and the median of the eigenfunction $\omega^{\tilde{n}}_{R,\tilde{d}}$ are increasing with respect to $\tilde{n}$ for R and μ fixed. The proof is similar to the proofs of Propositions 1 and 2.

For $R \le 0$ it follows from Sturm's Comparison Theorem that $\tilde{d}$ is a decreasing function of $\tilde{n}$. Therefore we restrict ourselves to the case $R = +1$. The function

$$f(u) \equiv \omega'(u)\sin^{(\tilde{n}-1)/2}\left(u\sqrt{\frac{R}{\tilde{n}-1}}\right)$$

satisfies the equation

$$f'' + Hf = 0 \quad \text{with } H \equiv \left(\mu - \frac{1}{2}R\right) - R\left(\frac{\tilde{n}-1}{4} + \frac{1}{2}\right)\cot^2\left(\sqrt{\frac{R}{\tilde{n}-1}}u\right).$$

As in the proof of Proposition 1, we can derive the assertion from the Picone's formula. We only have to show that

$$\int_0^{\tilde{d}} \left(\frac{\partial}{\partial \tilde{n}} H\right) f^2(u)\, du < 0.$$

This follows in a similar way as in the proof of Proposition 1 since

$$\frac{\partial}{\partial \tilde{n}} H \leq -R\left(\frac{\tilde{n}-1}{4} + \frac{1}{2}\right) \frac{\sqrt{R}}{(\tilde{n}-1)^{3/2}} \frac{\cos(u\sqrt{R/(\tilde{n}-1)})}{\sin^3(u\sqrt{R/(\tilde{n}-1)})}.$$

Finally, suppose that $\omega^n_{R,d}$ and $\omega^{\tilde{n}}_{R,\tilde{d}}$ have the same range for n and $\tilde{n}$ with $n < \tilde{n}$. Since the right-hand side of (7) is decreasing with respect to n, we can apply the argument of Theorem 1 and Corollary 1 with $G_\beta \equiv (\beta\tilde{\alpha}\omega^{\tilde{n}}_{R,\tilde{d}})'^2 \circ (\beta\tilde{\alpha}\omega^{\tilde{n}}_{R,\tilde{d}})^{-1}$ to conclude that $d \geq \tilde{d}$. Thus, we have arrived at a contradiction.

6. Eigenvalue estimates by comparison with auxiliary problems

Our main result is the following theorem.

Theorem 2. *Let M be an n-dimensional compact Riemannian manifold with Ricci curvature greater than or equal to R, and φ be an eigenfunction on M for the smallest positive eigenvalue λ_1. Then*

$$\operatorname{diam}(M) \geq v - u,$$

for every solution ψ of the Sturm-Liouville equation

$$\psi'' + (n-1)\frac{(C_\kappa - \delta S_\kappa)'}{C_\kappa - \delta S_\kappa}\psi' + \lambda_1\psi = 0$$

on an interval $[u, v]$ for a real parameter δ with $C_\kappa - \delta S_\kappa \neq 0$ on (u, v) (recall that $\kappa \equiv R/(n-1)$), and ψ strictly decreasing on $[u, v]$ such that

$$\left|\frac{\max\psi + \min\psi}{\max\psi - \min\psi}\right| \leq \left|\frac{\max\varphi + \min\varphi}{\max\varphi - \min\varphi}\right|.$$

Proof. The result is an immediate consequence of the proof of Theorem 1 (take into account that the right-hand side of (7) is a decreasing function of n), and of the proofs of Corollary 1, Corollary 2, and Proposition 3.

Remark. If we choose $\delta = 0$ and consider an odd solution ψ of the above equation with $\psi'(0) < 0$ on the maximal interval $[-v, v]$ with ψ

decreasing, then we obtain a lower bound for the first nontrivial eigenvalue on M without special assumptions on the range of φ.

7. Examples

The following simple example shows that the maximum principle technique yields in general sharper estimates than techniques using isoperimetric inequalities.

Example 1. Let M be a two-dimensional compact manifold with nonnegative Ricci curvature. It was shown in [1] that the isoperimetric function $h\colon [0, 1] \to \mathbf{R}$ with

$$h(\beta) \equiv \inf\{\operatorname{vol}(\partial\Omega)/\operatorname{vol}(M) | \Omega \subset M, \operatorname{vol}(\Omega)/\operatorname{vol}(M) = \beta\}$$

is bounded below by $2\operatorname{Is}(\beta)/\operatorname{diam}(M)$, where $\operatorname{Is}(\beta) = \sqrt{\beta(1-\beta)}$ denotes the isoperimetric function of the 2-sphere $\mathbf{S}^2$ with constant curvature $+1$. By consideration of a family of truncated cones K_β, $0 < \beta < \frac{1}{2}$, in $\mathbf{R}^3$ with

$$K_\beta \equiv \{|(x_2, x_3)| = \varepsilon x_1 \text{ for } x_1 \in [\sqrt{\beta/(1-\beta)}, \sqrt{(1-\beta)/\beta}]\}$$

for sufficiently small ε, it can easily be seen that the above estimate for the isoperimetric function is sharp, since $\Omega \equiv M \cap \{x_1 \leq 1\}$ yields

$$\frac{\operatorname{vol}(\Omega)}{\operatorname{vol}(\Omega) + \operatorname{vol}(\Omega \backslash M)} = \frac{1}{1 + (\sqrt{(1-\beta)/\beta})^2} = \beta,$$

and $\operatorname{vol}(\partial\Omega)\operatorname{diam}(M)/\operatorname{vol}(M) \approx 2\sqrt{\beta(1-\beta)}$ if ε is small. Suppose that $\operatorname{diam}(M) = 2$. It follows that $\lambda_1(M) \geq \lambda_1(\mathbf{S}^2) = 2$ (see [1]).

On the other hand, Theorem 2, Corollary 3, or the estimate given by Zhong and Yang (see [6, §4]) gives the sharp lower bound $\lambda_1(M) \geq \pi^2/4$ (> 2). The reason why the technique using isoperimetric inequalities does not yield the optimal result seems to be that the function $h(\beta)$ approaches its minimum with respect to M for different manifolds if β varies.

Arguments of the following type can provide better bounds for the first eigenvalue if the mass of the manifold is mainly concentrated "close" to one end of a diameter.

Remark. Let M be an n-dimensional compact manifold with diameter d_0 and Ricci curvature greater than or equal to R. Assume that we are given a subset M_0 of M with $\operatorname{diam}(M \backslash M_0) \leq d_1$ for some $d_1 < d_0$ and a positive number λ_0.

Suppose that for every $x \in M_0$ and every solution $\psi^n_{R,\delta,d}$ of the Neumann problem (1) for $d \le d_0$, $\mu \le \lambda_0$, and an arbitrary δ (we admit also positive δ, the only restriction is that $C_\kappa - \delta S_\kappa \ne 0$ on $(-d/2, d/2)$) the following holds:

$$\int_M \psi^n_{R,\delta,d}\left(\min\left\{\frac{d}{2} - d(x, z); -\frac{d}{2}\right\}\right) dz < 0;$$

$d(x, z)$ stands for the Riemannian distance of x and z.

Then it follows that either the first nontrivial eigenvalue λ_1 on M is larger than λ_0 or that an eigenfunction φ for λ_1 exists such that the distance between the points where φ attains its maximum and minimum value is less than d_1.

Proof. Suppose that $\lambda_1 \le \lambda_0$ and that an eigenfunction φ for λ_1 exists with $\min\varphi = \varphi(x_0)$ for some $x_0 \in M_0$. A similar argument as in the proof of Theorem 2 shows that there exist $d \le d_0$, δ, and a solution $\psi^n_{R,\delta,d}$ of the Neumann problem (1) with $\mathrm{range}(\psi^n_{R,\delta,d}) = \mathrm{range}(\varphi)$. By Theorem 1, we obtain

$$\varphi(z) \le \psi^n_{R,\delta,d}\left(\frac{d}{2} - d(x, z)\right)$$

for every z with $d(x, z) \le d$. Hence,

$$\int_M \varphi(z)\, dz \le \int_M \psi^n_{R,\delta,d}\left(\min\left\{\frac{d}{2} - d(x, z); -\frac{d}{2}\right\}\right) dz < 0,$$

which gives a contradiction.

We notice that there always exists an eigenfunction for λ_1 with median zero if the dimension of the corresponding eigenspace is larger than 1.

The following example shows that our results are not always sharp even in the case of manifolds with constant curvature.

Example 2. Let $\mathbf{L}_3(l\colon 1, 1) \equiv \mathbf{S}^3/A_l$ be a three-dimensional lens space, where A_l is the cyclic subgroup of $\mathrm{U}(n)$ generated by $(z_1, z_2) \mapsto (e^{2\pi i/l} z_1, e^{2\pi i/l} z_2)$ for $(z_1, z_2) \in \mathbf{C}^2 \cong \mathbf{R}^4$, l an integer with $l \ge 2$, and $\mathbf{S}^3$ the unit sphere in $\mathbf{R}^4$ (cf. [9]). $\mathbf{L}_3(2\colon 1, 1)$ is the projective space $\mathbf{P}^3$. The diameter of $\mathbf{L}_3(l\colon 1, 1)$ is equal to $\pi/2$ and the first nontrivial eigenvalue is equal to 8 for every even l.

Now suppose that l is an even integer with $l \ge 4$. Then the eigenspace for the eigenvalue 8 has dimension 3, and every eigenfunction can be written in the form $\varphi_{\alpha,\beta}(z_1, z_2) = \alpha(|z_1|^2 - |z_2|^2) + \mathrm{Re}(\beta z_1 \bar{z}_2)$ for a real parameter α and a complex parameter β. Hence, $\varphi_{\alpha,\beta}(z_1, z_2) = -\varphi_{\alpha,\beta}(e^{i\omega t}\bar{z}_2, -e^{i\omega t}\bar{z}_1)$ for every (z_1, z_2) and every real t. In particular, the median value of $\varphi_{\alpha,\beta}$ is zero. Moreover, $\varphi_{\alpha,\beta}$ attains its maximum and minimum value at points with distance equal to the diameter of the lens space.

However, the estimate for the diameter of $\mathbf{L}_3(l\colon 1, 1)$ given by Theorem 2 cannot be sharp since the eigenfunction $(z_1, z_2) \mapsto (\operatorname{Re} z_1)^2 - \frac{1}{4}$ on the projective space has a median value larger than zero (recall that the diameter and the first nontrivial eigenvalue coincide for the projective space $\mathbf{P}^3$ and the lens space $\mathbf{L}_3(l\colon 1, 1))$.

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