COMPACTNESS THEOREMS FOR KÄHLER-EINSTEIN MANIFOLDS OF DIMENSION 3 AND UP

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There has been increasing interest lately in compactness theorems of Riemannian manifolds under various geometric assumptions (see, among others, [3], [10], [1], [7], and [19]). More recently, it has been found that the boundedness condition on the curvature as in [3] and [10] can be replaced by some integral norms of the curvature tensor. One of those often used is the $L^{n/2}$-norm on the curvature tensor, where $n$ is the real dimension of the underlying manifold. For instance, in [1] and [19], the authors show that if $\{(M_i, g_i)\}$ is a sequence of Einstein manifolds of real dimension $2n$ satisfying: (i) $\text{diam}(M_i, g_i) \leq \mu$; (ii) $\int_{M_i} \|Rm(g_i)\|_g^n \, dV_{g_i} \leq \mu$; and (iii) $\text{Vol}(M_i, g_i) \geq \frac{1}{\mu}$, where $\mu$ is a uniform constant, then the subsequence of $\{(M_i, g_i)\}$ converges to an Einstein orbifold with finitely many isolated singular points. Also see [20] for the case of Kähler-Einstein surfaces. The case that the limit is an orbifold does occur in dimension four (cf. [15], [20]). However, in this paper, we show that it cannot occur for Kähler-Einstein manifolds of higher dimension and nonzero scalar curvature. In order to give our main theorem precisely, we need to introduce some notation first. For any fixed constant $\mu > 0$ and positive integer $n > 0$, denote by $K(\mu, n)$ the set of all Kähler-Einstein manifolds $(M, g)$ of complex dimension $n$ satisfying:

\begin{align*}
(0.1) & \quad \text{diam}(M, g) \leq \mu, \\
(0.2) & \quad \int_M |Rm(g)|_g^n \, dV_g \leq \mu, \\
(0.3) & \quad \text{Vol}_g(M) \geq 1/\mu,
\end{align*}

where $Rm(g)$ denotes the curvature tensor of $g$. Let $K_+(\mu, n)$ (resp. $K_-(\mu, n)$) be the subset of all $(M, g)$ in $K(\mu, n)$ with $\text{Ric}(g) = \omega_g$ (resp. $\text{Ric}(g) = -\omega_g$), where $\omega_g$ is the associated Kähler form of $g$. We should point out that the diameters of the manifolds in $K_+(\mu, n)$ are

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bounded from above by a constant depending only on \( n \).

Our first main theorem is stated as follows:

**Theorem 1.** \( K_+ (\mu, n) \) (resp. \( K_- (\mu, n) \) is compact for \( n \geq 3 \).

A related problem is the classification of complete Ricci-flat Kahler manifolds with bounded \( L^n \)-norm of the curvature tensor. The examples of such manifolds can be constructed in the following way (cf. [21], [25]). Let \( \Gamma \subset SU(n) \) be a finite group acting on \( C^n \) with the origin as its unique fixed point. We further assume that \( C^n/\Gamma \) admits a resolution \( M \) such that the push-down of \( dz_1 \wedge ... \wedge dz_n \) on \( C^n \) can be extended nonvanishingly across the exceptional divisor, in other words, the canonical line bundle \( K_M \) is trivial. Note that this assumption is automatically true in the case \( n \leq 3 \). Then \( M \) has a complete Ricci-flat Kahler metric with bounded \( L^n \)-norm of the curvature. In the case \( n = 2 \), it was proved before by Hitchin and P. Kronheimer using a different method ([13], [17]).

**Theorem 2.** Let \( (M, g) \) be a complete Ricci-flat Kahler manifold with the \( L^n \)-norm of its curvature tensor bounded. Then \( M \) is a resolution of \( C^n/\Gamma \) for some \( \Gamma \subset SU(n) \) with \( K_M \) trivial.

The organization of this paper is as follows. In §1, we recall that for any sequence of Kahler-Einstein manifolds in either \( K_+ (\mu, n) \) or \( K_- (\mu, n) \), a subsequence of it converges to a Kahler-Einstein orbifold in the sense of Cheeger-Gromov (cf. Theorem 1.1). We include an outlined proof of it here following the arguments in §3 of [20]. In §2, we prove the continuity of the dimensions of plurianticanonical or pluricanonical divisors under the convergence of Kahler-Einstein manifolds in Cheeger-Gromov's sense. The basic analytic tool is Hörmander's \( L^2 \)-estimate for \( \bar{\partial} \)-operators. We will also discuss some corollaries of this continuity result. In §3, using Kohn's estimate for \( \bar{\partial}_b \)-operators on strongly pseudoconvex CR-manifolds, we study the local structure of the Kahler-Einstein orbifold \( M_\infty \) being the limit of Kahler-Einstein manifolds. In particular, we prove that \( M_\infty \) is in fact a manifold. §4 contains the proof of Theorem 2. In §5, we complete the proof of Theorem 1 based on the discussions in the previous sections.

The key idea of this paper occurred to the author during his attendance in Professor J. Kohn’s class in Princeton University when he was visiting there. The author would like to express his gratitude to both the institute and Professor Kohn.

1. Convergence to Kahler orbifolds

An \( n \)-dimensional complex orbifold \( M \) is a topological space satisfying:

(1) each point \( x \) in \( M \) admits an open neighborhood \( U_x \) homeomorphic
to $D^n/\Gamma_x$, where $D^n$ is the unit disc in $C^n$, and $\Gamma_x \subset U(n)$ is a finite group; and (2) those $U_x$ are patched together by biholomorphic transition functions. Any point $x$ with $\Gamma_x$ trivial is called a regular point of $M$. In particular, $M$ is a manifold near such a regular point. Denote by $M_{\text{reg}}$ the set of all regular points. All other points are singular points of $M$, i.e., $\text{Sing}(M) = M \setminus M_{\text{reg}}$. We will confine ourselves to the special case that $\text{Sing}(M)$ consists of isolated points, although it is not necessary for the following discussions. A Kähler metric is just the one on $M_{\text{reg}}$ such that for each $x$ in $\text{Sing}(M)$, if $\psi_x : D^n \to U_x$ is the local uniformization, then $\psi_x^*g$ can be extended across the origin.

Now suppose $g$ be a Kähler orbifold metric on $M$. In the case $\text{Ric}(g) = \lambda \omega$ on $M$ for some constant $\lambda$, we call $(M, g)$ a Kähler-Einstein orbifold metric.

**Theorem 1.1.** Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein manifolds in either $K_{+}(\mu, n)$ or $K_{-}(\mu, n)$. By taking a subsequence of it, we may assume that $(M_i, g_i)$ converges to Kähler-Einstein orbifold $(M_{\infty}, g_{\infty})$ in Cheeger-Gromov's sense, that is, there are finitely many points $x_{i1}, \ldots, x_{iN}$ in $M_i$, and $x_{\infty 1}, \ldots, x_{\infty N}$ in $M_{\infty}$, where $N$ is a positive integer depending only on $n, \mu$ such that, for any $r > 0$, there are diffeomorphisms $\phi_i$ from $M_i \setminus \bigcup_{\beta=1}^{N} B_r(x_{i\beta}, g_i)$ into $M_{\infty}$ with $K_r = M_{\infty} \setminus \bigcup_{\beta=1}^{N} B_{5r}(x_{\infty \beta}, g_{\infty})$ in the image and satisfying:

1. in the $C^5$-topology, $(\phi_i^{-1})^*g_i$ converges to $g_{\infty}$ uniformly on $K_r$;
2. in the $C^5$-topology, $\phi_{i\ast} \circ J_i \circ (\phi_i^{-1})_{\ast}$ converges to $J_{\infty}$ uniformly on $K_r$, where $J_i, J_{\infty}$ are the almost complex structures of $M_i, M_{\infty}$, respectively.

Theorem 1.1 can be derived from the compactness theorem stated in [1] or [19] (see also [20] for the special case of Kähler-Einstein surface). But for the reader's convenience, we outline a proof of it here. For simplicity, we may assume $(M_i, g_i)$ so in $K_{+}(\mu, n)$ for all $i$. The key analytic tool is Uhlenbeck's Yang-Mills estimate for curvatures of Yang-Mills connections.

**Lemma 1.1.** Let $(M_i, g_i)$ be a Kähler-Einstein manifold given as in Theorem 1.1. Then there are uniform constants $C', C''$, depending only on the upper bound of $n$ and $\mu$, such that for any $f$ in $C^4(M_i, R)$

$$C' \left( \int_{M_i} |f|^{2n/(n-1)} \, dV_{g_i} \right)^{(n-1)/n} - C'' \int_{M_i} |f|^2 \, dV_{g_i}$$

$$\leq \int_{M_i} |\nabla f|^2 \, dV_{g_i},$$

where $\nabla f$ denotes the gradient of $f$. 
Proof. This follows from a combination of results in C. Croke [5] and P. Li [18].

Lemma 1.2. Let $N$ be the integer $[\mu/(C')^n] + 1$, where $C'$ is the Sobolev constant given in (1.1), and $[a]$ denotes the integer part of the real number $a$. Then there is a universal constant $C \geq 0$, such that for any $r \in (0, 1)$ and any Kähler-Einstein manifold $(M_\ell, g_\ell)$ as in Theorem 1.1, there are finitely many points $x_{i1}', \ldots, x_{in}'$ in $M_\ell$ such that for any $x \in M_\ell \backslash \cup_{\beta=1}^N B_\ell(x_{i\beta}', g_\ell)$,

$$\|R(i)\|_{g_\ell}(x) \leq \frac{C}{r^n} \left( \int_{B_{r/4}(x, g_\ell)} \|R(i)\|^2_{g_\ell}(x) \, dV_{g_\ell} \right)^{1/2},$$

where $B_\ell(x_{i\beta}', g_\ell)$ is the geodesic ball with radius $r$ and center at $x_{i\beta}'$, and $\|R(i)\|_{g_\ell}$ is the norm of $R(i)$ with respect to $g_\ell$.

Proof. A straightforward computation shows

$$-\Delta_{g_\ell}(\|R(i)\|_{g_\ell}) \leq \|R(i)\|^2_{g_\ell} + C(n)(\|R(i)\|_{g_\ell})^2,$$

where $\Delta_{g_\ell}$ is the laplacian of $g_\ell$, and $C(n)$ is a positive constant depending only on $n$, whose actual value is not important to us. Define

$$E_i = \left\{ x \in M_\ell \left\| \int_{B_{r/4}(x, g_\ell)} \|R(i)\|^2_{g_\ell} \, dV_{g_\ell} \geq \varepsilon \right\}. $$

Then by the well-known covering lemma, $E_i$ can be covered by $N$ geodesic balls of radius $\frac{\varepsilon}{2}$. Take $x_{i1}', \ldots, x_{in}'$ to be the centers of these balls. Then for any $x \in M_\ell \backslash \cup_{\beta=1}^N B_\ell(x_{i\beta}', g_\ell)$,

$$\int_{B_{r/4}(x, g_\ell)} \|R(i)\|^n_{g_\ell} \, dV_{g_\ell} \leq \varepsilon.$$}

Let $\eta: R^1_+ \rightarrow R^1_+ = \{ t \in R^1 \mid t \geq 0 \}$ be a cut-off function satisfying $\eta \equiv 1$ for $t \leq 1$, and $\eta \equiv 0$ for $t \geq 2$ and $|\eta'(t)| \leq 1$.

For any $x \in M_\ell \backslash \cup_{\beta=1}^N B_\ell(x_{i\beta}', g_\ell)$, denote by $\rho_\ell(\cdot)$ the distance function on $M_\ell$ from $x$.

Put $f = \|R(i)\|_{g_\ell}$. Multiplying $\eta^2(8\rho_\ell/r)f$ on both sides of (1.3) and then integrating by parts, one obtains

$$\int_{M_\ell} |\nabla(\eta f)|^2 \, dV_{g_\ell} \leq \int_{M_\ell} \eta^2 f^2 \, dV_{g_\ell} + \int_{M_\ell} |\nabla \eta|^2 f^2 \, dV_{g_\ell} + \int_{M_\ell} \eta^2 f^3 \, dV_{g_\ell}. $$
By Lemma 1.1 and Hölder's inequality,

\begin{align}
& C' \left( \int_{M_1} |\eta f|^{2n/(n-1)} dv_{g_i} \right)^{(n-1)/n} - C'' \int_{M_1} |\eta f|^2 dv_{g_i} \\
\leq & \int_{M_1} \left( \eta^2 + \frac{64|\eta|^2}{r^2} \right) |f|^2 dv_{g_i} \\
& + \left( \int_{M_1} |\eta f|^n dv_{g_i} \right)^{1/n} \left( \int_{B_{r/4}(x, g_i)} |f|^{2n/(n-1)} dv_{g_i} \right)^{(n-1)/n}.
\end{align}

Therefore, for some constant $C \geq 0$ depending only on $n$, we have

\begin{equation}
\left( \int_{B_{r/16}(x, g_i)} |f|^{2n/(n-1)} dv_{g_i} \right)^{(n-1)/n} \leq \frac{C}{r^2 (C' - \sqrt{\varepsilon})} \int_{B_{r/4}(x, g_i)} |f|^2 dv_{g_i}.
\end{equation}

Similarly, by multiplying $\eta^2 f^{(n+1)/(n-1)}$ on both sides of (1.3) and processing as above, we have

\begin{equation}
\left( \int_{B_{r/16}(x, g_i)} |f| \right)^{2(n/(n-1))^2} dv_{g_i} \leq \frac{C}{r^2 \left( \frac{n-1}{2n} C' - \sqrt{\varepsilon} \right)} \int_{B_{r/4}(x, g_i)} |f|^{2n/(n-1)} dv_{g_i}.
\end{equation}

Let $\varepsilon \leq ((n - 1)/4n)^{2k} (C')^2$ and choose $k$ satisfying $(n/(n - 1))^k \geq n$. Continuing the above processes $k$ times, we obtain

\begin{equation}
\left( \int_{B_{r/16}(x, g_i)} |f|^{2(n/(n-1))^k} dv_{g_i} \right)^{(n-1)/n} \leq \frac{C}{r^{n(1 - ((n-1)/n)^k)}} \left( \int_{B_{r/4}(x, g_i)} |f|^2 dv_{g_i} \right)^{1/2}.
\end{equation}

Then (1.2) follows from Moser's iteration as in the proof of Theorem 8.17 in [16]. q.e.d.

We further observe that we may take the set \{ $x_{1/4}^i$, \ldots, $x_{r/4}^i$ \} contained in the union of the balls $B_r(x_{i/\beta}^i, g_i)$. Let \{ $r_j$ \} $j \geq 1$ be a decreasing sequence of positive numbers such that $r_1 \leq 1/4$, $r_j \leq r_{j-1}/4$. If we write $x_{r_j}^i$ as $x_{i/\beta}^r$
and define

(1.11) \[ \Omega^j_i = M_i \setminus \bigcup_{\beta=1}^{N} B_{2r_j}(x^j_{i\beta}, g_i), \]

then

\[ \Omega^j_i \subseteq \Omega^{j+1}_i \left( \frac{r_{j+1}}{8} \right) \quad \text{and} \quad \bigcup_{j \geq 1} \Omega^j_i = M_i \setminus \{x_{i1}, \ldots, x_{iN}\}, \]

where \( x^j_{i\beta} = \lim_{j \to \infty} x^j_{i\beta} \), and for any \( 1 \leq \beta \leq N \),

\[ \Omega^{j+1}_i(e) = \{ x \in \Omega^{j+1}_i \mid \text{dist}_{g_i}(x, \partial \Omega^{j+1}_i) > e \}. \]

The following lemma is essentially a special case of the famous Gromov’s compactness theorem (cf. [10], [12]).

**Lemma 1.3.** Let \( \{(X_i, h_i)\} \) be a sequence of \( n \)-dimensional Kähler-Einstein manifolds (maybe noncompact), and \( \Omega_i \) a sequence of domains in \( X_i \) with boundary \( \partial \Omega_i \). Suppose the following for all \( i \):

(i) The norm \( \|R(h_i)\|_{h_i}(x) \) of the bisectional curvatures \( R(h_i) \) are uniformly bounded for \( x \) in \( \Omega_i \).

(ii) \( \text{InjRad}(x) \geq c_i \) for \( x \in \Omega_i \) and for some constant depending only on \( i \).

(iii) \( 0 \leq C' \leq \text{Vol}_{h_i}(\Omega_i) \leq C'' \) for some uniform constants \( C', C'' \).

Then given any \( e > 0 \), there is a subsequence \( \{\Omega_{i_k}(e), h_{i_k}\}_{k \geq 1} \) of Kähler-Einstein manifolds \( \{\Omega_i(e), h_i\}_{i \geq 1} \), where \( \Omega_i(e) = \{ x \in \Omega_i \mid \text{dist}_{h_i}(x, \partial \Omega_i) > e \} \), and a Kähler-Einstein manifold \( (\Omega_\infty(e), h_\infty) \) such that for the compact subset \( K \subset \Omega_\infty(e) \), there is an \( e' > e \) such that for \( k \) sufficiently large, there are diffeomorphisms \( \phi_k \) of \( \Omega_{i_k}(e') \) into \( \Omega_\infty(e) \) satisfying:

1. \( K \subset \phi_k(\Omega_{i_k}(e')) \) for any \( k \geq 1 \),
2. \( (\phi_k^{-1})^*h_i \) converges uniformly to \( h_\infty \) on \( K \),
3. \( (\phi_k)_* \circ J_i \circ (\phi_k^{-1})_* \) converges uniformly to \( J_\infty \) on \( K \), where \( J_i, J_\infty \) are the almost complex structures of \( \Omega_i, \Omega_\infty(e) \), respectively.

**Proof.** By some standard computations and the assumption that the \((X_i, h_i)\) are Kähler-Einstein manifolds, the bisectional curvature tensor \( R(h_i) \) satisfies a quasi-linear elliptic system. The assumptions (i), (ii), and (iii) imply that the Sobolev inequalities hold on \( \Omega_i(e) \) with uniform Sobolev constants. It follows from some well-known elliptic estimates (cf. [27]) that
where $D^l R(h_i)$ denotes the $l$th covariant derivative of $R(h_i)$ on $\Omega_i$, and the $C(l)$ are uniform constants depending only on $l$. Then by Gromov’s compactness theorem ([10], [12]), there is a subsequence $\{ (\Omega_i(\epsilon), h_i) \}$ and a Riemannian manifold $(\Omega_\infty(\epsilon), h_\infty)$ such that the above (1) and (2) hold. Let $K$ be any compact subset in $\Omega_\infty(\epsilon)$, and $\phi_k$ defined as in the statement of this proposition. For the almost complex structure $J_i$ on $\Omega_i$, it is clear that $(\phi_k)_* \circ J_i \circ (\phi_k^{-1})_*$ is almost complex on $K$. By taking the subsequence of $\{ i_k \}$, we may assume that $(\phi_k)_* \circ J_i \circ (\phi_k^{-1})_*$ converges on $K$. Since $K$ is arbitrary, we obtain an almost complex structure $J_\infty$ on $\Omega_\infty(\epsilon)$. It is easy to check that this $J_\infty$ is integrable, and $h_\infty$ is a Kähler-Einstein metric with respect to this $J_\infty$. 

Since $\text{diam}(M_i, g_i) \leq \mu$ and $\text{Vol}(M_i, g_i) \geq \frac{1}{\mu}$ for all $i$, by an estimate on the injectivity radius in [4], one can prove that assumptions (i)-(iii) in Lemma 1.3 are fulfilled by $(\Omega_i, g_i)$, $i, j \geq 1$. Therefore, we have a sequence of open Kähler-Einstein manifolds $(\Omega^i, g^i)$. Furthermore, one can identify $\Omega^i$ naturally with a subdomain in $\Omega^{i+1}_\infty$ such that the restriction of $g^{i+1}$ to $\Omega^i$ coincides with $g^i$. Therefore the $\{ (\Omega_i^j, g^i) \}$ can be glued together to be a Kähler-Einstein manifold $(M^i, g^i)$. By Fatou’s lemma,

$$\int_{M^i_\infty} \| \text{Rm}(g^i) \|^n \text{d}V_{g^i} \leq \mu.$$ 

Also, it follows from the Volume Comparison Theorem [2] that $M^i_\infty$ has only finitely many connected components.

Let $\rho_i$ be the distance function on $M_i \times M_i$ induced by $g_i$, and let $\rho_{\infty}$ be the limit of $\rho_i$. Obviously, $\rho_{\infty}$ is Lipschitz on $M_{\infty} = M'_{\infty}$. According to [10], one may attach finitely many points $x_{i,1}, \ldots, x_{i,N}$ to $M'_{\infty}$ such that $M_{\infty} = M'_{\infty} \cup \{ x_{i,1}, \ldots, x_{i,N} \}$ becomes a compact length space with length function $\rho_{\infty}$ extending that $\rho_{\infty}$ on $M'_{\infty} \times M'_{\infty}$. We need to give a Kähler orbifold structure on $M_{\infty}$.

Lemma 1.4. There is a decreasing positive function $\varepsilon(r)$, satisfying $\lim_{r \to \infty} \varepsilon(r) = 0$ such that for any point $x$ in $M_{\infty}$, we have

$$\| \text{Rm}(g_{\infty}) \|(x) \leq \frac{\varepsilon(r(x))}{r^2(x)},$$

where $r(x) = \min \{ \rho_{\infty}(x_{i,j}, x) | 1 < j \leq N \}$. 

\(1.12\)  \[\| D^l R(h_i) \|_{h_i} (x) \leq C(l), \quad l = 1, 2, \ldots, \infty, \]
This is simply a corollary of Lemma 1.2. Using the trick of blowing up and the curvature estimate in Lemma 1.4, one can endow $M_\infty$ with a topological orbifold structure at $x_\infty^\beta$ (1 $\leq$ $\beta$ $\leq$ $N$). Precisely, for each $\beta$, there is an open neighborhood $U_\beta$ of $x_\infty^\beta$ such that each connected component $U_{\beta j}$ (1 $\leq$ $j$ $\leq l_\beta$) of $U_\beta \cap M_\infty'$ is covered by a smooth manifold $\tilde{U}_{\beta j}$ diffeomorphic to the punctured ball $D_r^*$ in $C^n$. The covering group $\Gamma_{\beta j}$ is isomorphic to a finite group in $U(n)$. Moreover, let $\phi_{\beta j}^*$ be the diffeomorphism from $D^*_r$ onto $\tilde{U}_{\beta j}$ and let $\pi_{\beta j}: \tilde{U}_{\beta j} \to U_{\beta j}$ be the covering map. Then $\phi_{\beta j}^* \circ \pi_{\beta j}^* g_\infty$ extends to be a $C^0$-metric on $D^n_r$, where $D^n_r = \{x \in C^n, |x| < r\}$, $D^*_r = D^n_r \setminus \{0\}$. We refer readers to §3 in [20] for the details of its proof.

In order to obtain a Kähler orbifold structure on $M_\infty'$, we have to prove that the curvature tensor $Rm(g_\infty)$ is in fact bounded. From Lemma 1.4 follow the topological orbifold structure of $M_\infty$ and the analogy of Uhlenbeck's removable singularity theorem [27]. In §4 of [20], this boundedness of $Rm(g_\infty)$ is proved for surfaces, i.e., for $n = 2$. However, the whole argument can be generalized to higher dimensions without substantial change. Next, as the author did in Lemma 4.4 and 4.5 of [20], one can construct a diffeomorphism $\psi$ from $D^*_r$ into itself such that $\psi^* \circ \phi_{\beta j}^* \circ \pi_{\beta j}^* g_\infty$ extends smoothly across the origin, where $\phi_{\infty j}$ and $\pi_{\beta j}$ are the same as in last paragraph. Therefore, $(M_\infty, g_\infty)$ is a Kähler-Einstein orbifold with $\text{Ric}(g_\infty) = \omega_{g_\infty}$.

Note that $M_\infty$ is in fact connected (cf. [20]). However, we do not need this fact in the following arguments, and the sketched proof of Theorem 1.1 is finished.

2. Convergence of pluricanonical or plurianticanonical divisors

Let $\{(M_i, g_i)\}_{i \geq 1}$ be a sequence of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$. By Theorem 1.1, we may assume that $(M_i, g_i)$ converges to a Kähler-Einstein orbifold $(M_\infty, g_\infty)$ in the sense of Cheeger-Gromov. In this section we will apply the $L^2$-estimate for $\bar{\partial}$ operators to show the convergence of $H^0(M_i, K_i^{-m})$ to $H^0(M_\infty, K_\infty^{-m})$ for any integer $m$ as $(M_i, g_i)$ approaches $(M_\infty, g_\infty)$. Recall that $M_\infty$ is a Kähler orbifold with only isolated quotient singularities.

A line bundle $L$ on $M_\infty$ is a line bundle on the regular part $M_\infty'$ such that for each local uniformization $\pi_x: \tilde{U}_x \to M_\infty$ of a singular
point $x$, the pullback $\pi^*L$ on $\tilde{U}_x \setminus \pi^{-1}(x)$ can be extended to the whole $\tilde{U}_x$. The natural line bundles on $M_\infty$ are pluricanonical and plurianticanonical ones $K_{M_\infty}^m$ $(m \in \mathbb{Z})$. A global section of $K_{M_\infty}^m$ is an element in $H^0(M_\infty, K_{M_\infty}^m)$, which can be extended across the singular set in the above sense. Then $H^0(M_\infty, K_{M_\infty}^m)$ is just the linear space of all the global sections of $K_{M_\infty}^m$. Note that the metric $g_\infty$ induces natural hermitian orbifold metrics on $K_{M_\infty}^m$.

**Lemma 2.1.** Let $\{(M_i, g_i)\}$ be the sequence of Kähler-Einstein manifolds given at the beginning of this section and let $S^i$ be a global holomorphic section in $H^0(M_i, K_{M_i}^{-m})$ with $\int_{M_i} \|S^i\|_{g_i}^2 \, dV_{g_i} = 1$, where $m$ is a fixed positive integer. Then there is a subsequence $\{i_k\}$ of $\{i\}$ such that the sections $S^{i_k}$ converge to a global holomorphic section $S^\infty$ in $H^0(M_\infty, K_{M_\infty}^{-m})$. In particular, if $\{S^i_\beta\}_{0 \leq \beta \leq N_m}$ is an orthogonal basis of $H^0(M_i, K_{M_i}^{-m})$ with respect to the induced inner product by $g_i$, then by taking a subsequence, we may assume that $\{S^i_\beta\}_{0 \leq \beta \leq N_m}$ converges to an orthonormal basis of a subspace in $H^0(M_\infty, K_{M_\infty}^{-m})$, where $N_m + 1 = \dim_C H^0(M_i, K_{M_i}^{-m})$.

**Remark.** Before we prove this lemma, we should justify the meaning of the convergence of $\{S^i\}$ in the above lemma since these sections are no longer on the same Kähler manifold. Recall that for any compact subset $K \subset M_\infty \setminus \text{Sing}(M_\infty)$, there are diffeomorphisms $\phi_i$ from compact subsets $K_i \subset M_i$ onto $K$ such that $(\phi_i^{-1})^* g_i$ and $\phi_i \circ J_i \circ (\phi_i^{-1})^*$ converge to $g_\infty$ and $J_\infty$ on $K$, respectively. Now with $\phi_i$ as above, we can push the sections $S^i$ down to the sections $\phi_i^*(S^i)$ of $\bigotimes^m (\Lambda^m (TM_\infty \oplus T\overline{M_\infty}))$ on $K$. The convergence in Lemma 2.1 means that for any compact subset $K$ of $M_\infty \setminus \text{Sing}(M_\infty)$ and $\phi_i$ as above, the sections $\phi_i^*(S^i)$ converge to a section $S^\infty$ of $K_{M_\infty}^{-m}$ on $K$ in the $C^\infty$-topology. Note that the limit $S^\infty$ is automatically holomorphic.

**Proof of Lemma 2.1.** Let $\Delta_i$ be the laplacian of the metric $g_i$. Then by a direct computation, we have

$$\Delta_i(\|S^i\|_{g_i}^2)(x) = \|D_i S_i^i\|_{g_i}^2(x) - nm \|S^i\|_{g_i}^2(x),$$

where $D_i$ is the covariant derivative with respect to $g_i$. Since $\int_{M_i} \|S^i\|_{g_i}^2 \, dV_{g_i} = 1$, by Lemma 1.1 and applying Moser's iteration to (2.1), there is a constant $C(n, m)$ depending only on $m$ such that

$$\sup_{M_i} \|S^i\|_{g_i}^2(x) \leq C(n, m).$$
Let $K$ be a compact subset in $M_{\infty} \backslash \text{Sing}(M_{\infty})$, and $\phi_i$ the diffeomorphism from $K_i$ onto $K$ as in the above remark. To prove the lemma, it suffices to show

\[(*) : \text{for any integer } l > 0, \text{ the } l\text{th covariant derivatives of } \phi_i(S^i) \text{ with respect to } g_{\infty} \text{ are bounded in } K \text{ by a constant } C'_l \text{ depending only on } l \text{ and } K.\]

There is an $r > 0$, depending only on $K$, such that for any point $x$ in $K$, the geodesic ball $B_r(x, g_i)$ is uniformly biholomorphic to an open subset in $C^n$. On each $B_r(x, g_i)$, the section $S_i$ is represented by a holomorphic function $f_{i,x}$. By (2.1), the function $f_{i,x}$ is uniformly bounded. Therefore, by the well-known Cauchy integral formula, one can easily prove that at $x$ the $l$th covariant derivative of $S^i$ is uniformly bounded by a constant depending only on $l, K$. $(*)$ follows since $(\phi_i^{-1})^*g_i$ uniformly converges to $g_{\infty}$ in $K$. Hence the lemma is proved.

The following proposition can be easily proved by modifying the proof of [14, p. 92, Theorem 4.4.1] with the use of the Bochner-Kodaira Laplacian formula (see, e.g., [16]).

**Proposition 2.1.** Suppose that $(X, g)$ is a complete Kähler orbifold of complex dimension $n$, $L$ a line bundle on $X$ with the hermitian orbifold metric $h$, and $\varphi$ a function on $X$ which can be approximated by a decreasing sequence of smooth functions $\{\varphi_i\}_{1 \leq i \leq +\infty}$. If, for any tangent vector $\nu$ of type $(1, 0)$ at any point of $X$ and for each $l$,

\[
(2.3) \quad \left\langle \frac{\partial \bar{\nu}}{\partial} \varphi_i + \frac{2\pi}{\sqrt{-1}}(\text{Ric}(h) + \text{Ric}(g)), \nu \wedge \bar{\nu} \right\rangle_g \geq C\|\nu\|_g^2,
\]

where $C$ is a constant independent of $l$, and $\langle , \rangle_g$ is the inner product induced by $g$, then for any $C^\infty$ $L$-valued $(0, 1)$-form $w$ on $X$ with $\bar{\partial}w = 0$ and $\int_X \|w\|^2 e^{-\varphi} dV_g$ finite, there exists a $C^\infty$ $L$-valued function $u$ on $X$ such that $\bar{\partial}u = w$ and

\[
(2.4) \quad \int_X \|u\|^2 e^{-\varphi} dV_g \leq \frac{1}{C} \int_X \|w\|^2 e^{-\varphi} dV_g,
\]

where $\| \cdot \|$ is the norm induced by $h$ and $g$.

**Lemma 2.2.** Any section $S$ in $H^0(M_{\infty}, K_{M_{\infty}}^{-m})$ is the limit of some sequence $\{S^i\}$ with $S^i$ in $H^0(M_i, K_{M_i}^{-m})$. In particular, this implies that the dimension of $H^0(M_{\infty}, K_{M_{\infty}}^{-m})$ is the same as that of $H^0(M_i, K_{M_i}^{-m})$, that is, plurianticanonical dimensions are invariant under the degeneration of Kähler-Einstein manifolds in either $K_+(\mu, n)$ or $K_-(\mu, n)$. 


Proof. We may assume that \( \int_{M_i} \|S\|_{g_i}^2 \, dV_{g_i} = 1 \). Let \( \{r_i\} \) be a sequence of positive numbers with \( \lim_{i \to \infty} r_i = 0 \) such that for each \( i \), there is a diffeomorphism \( \phi_i \) from \( M_i \setminus \bigcup_{\beta=1}^{N} B_{r_i} \) into \( M_\infty \setminus \text{Sing}(M_\infty) \) as given in Theorem 1.1, where \( N \) is defined in Lemma 1.2, and \( x_{i\beta} \) are defined in (1.3). Then \( \phi_i \) satisfies the following facts:

1. \( \lim_{i \to \infty} (\text{Im}(\phi_i)) = \text{just } M_\infty \setminus \text{Sing}(M_\infty) \),
2. \( (\phi_i^{-1})_* g_i \) uniformly converges to \( g_\infty \) on any compact subset of \( M_\infty \setminus \text{Sing}(M_\infty) \) in the \( C^\infty \)-topology,
3. \( \phi_i \circ J_i \circ (\phi_i^{-1})_* \) converges to \( J_\infty \), where \( J_i, J_\infty \) are the almost complex structures on \( M_i, M_\infty \), respectively.

Define a cut-off function \( \eta: \mathbb{R} \to \mathbb{R}^+ \) satisfying \( \eta(t) = 0 \) for \( t \leq 1 \), and \( \eta(t) = 1 \) for \( t \geq 2 \) and \( |\eta'| \leq 1 \). Also let \( \pi_i \) be the natural projection from the bundle \( \bigotimes^m(\Lambda^n(TM_i) \oplus \overline{TM_i}) \) onto \( K_i^{-m} = \bigotimes^m(\Lambda^n TM_i) \).

For each \( i \), we have a smooth section \( v_i = \eta(\rho_i(x)/2r_i) \cdot \pi_i((\phi_i^{-1})_* S) \) of \( K_i^{-m} \) on \( M_i \), where \( \rho_i(x) \) is a Lipschitz function defined by \( \rho_i(x) = \min_{1 \leq \beta \leq N} \{ \text{dist}(x, x_{i\beta}) \} \). Then by facts (2) and (3) above, there is a decreasing function \( \varepsilon_3(r) \) on \( r \) with \( \lim_{r \to 0} \varepsilon_3(r) = 0 \) such that

\[
\sup \left\{ \| \overline{\partial}_i \pi_i((\phi_i^{-1})_* S) \|_{g_i} \right\} \leq \varepsilon_3(r) \tag{2.5}
\]

\[
\left| \int_{M_i} \|v_i\|_{g_i}^2 \, dV_{g_i} - 1 \right| \leq \varepsilon_3(r) \tag{2.6}
\]

where \( \overline{\partial}_i \) is the corresponding \( \overline{\partial} \)-operator on \( M_i \).

By (2.5), we have

\[
\int_{M_i} \| \overline{\partial}_i v_i \|_{g_i}^2 \, dV_{g_i} \leq \varepsilon_3(r) \text{ Vol}_{g_i}(M_i)
\]

\[
+ \sum_{\beta=1}^{N} \int_{B_{4r_i}(x_{i\beta}, g_i)} \left\| \overline{\partial}_i \left( \eta \left( \frac{\rho_i}{2r_i} \right) \right) \cdot \pi_i((\phi_i^{-1})_* S) \right\|_{g_i}^2 \, dV_{g_i}
\]

\[
\leq \varepsilon_3(r) \text{ Vol}_{g_i}(M_i) \sum_{\beta=1}^{N} \frac{1}{4r_i^2} \text{ Vol}(B_{4r_i}(x_{i\beta}, g_i))
\]

\[
\times \sup \left\{ \| (\phi_i^{-1})_* S \|_{g_i}^2 \right\} \text{ for } x \in M_i \setminus \bigcup_{\beta=1}^{N} B_{2r_i} \text{ and } g_i \right\}.
\]
As in the proof of Lemma 2.1, one may bound $\sup_{M_\infty} (\|S\|_{g_{\infty}}^2 (x))$ by the constant $C(n, m)$ in (2.2). Thus by (2.7), the Volume Comparison Theorem, and the convergence of $(\phi_i^{-1})^* g_i$ in fact (2) above, there is a constant $C$ independent of $i$ such that

$$\int_{M_i} \|\partial_i v_i\|_{g_i}^2 (x) dV_{g_i} \leq C(r_i^{2n-2} + \varepsilon_3 (r_i)).$$

Now applying Proposition 2.1, i.e., the $L^2$-estimate of $\bar{\partial}$-operators, we have a $C^\infty$-smooth $K_{M_i}^{-m}$-valued function $u_i$ such that

$$\bar{\partial} u_i = \partial v_i,$$

$$\int_{M_i} \|u_i\|_{g_i}^2 (x) dV_{g_i} \leq \frac{1}{m+1} \int_{M_i} \|\bar{\partial}_i v_i\|_{g_i}^2 (x) dV_{g_i},$$

$$\leq \frac{C}{m+1} (r_i^{2n-2} + \varepsilon_3 (r_i)).$$

By (2.9), for each $i$, the norm function $\|u_i\|_{g_i}^2$ satisfies the elliptic equation

$$\Delta_i (\|u_i\|_{g_i}^2 (x)) = \|D_i u_i\|_{g_i}^2 (x) - nm \|u_i\|_{g_i}^2 (x) + 2 \text{Re} (h_i^m (u_i, \bar{\partial}_i v_i))(x),$$

where $\bar{\partial}_i^*$ is the adjoint operator of $\bar{\partial}_i$ on a $K_{M_i}^{-m}$-valued function with respect to $g_i$. As in (2.5), we also have

$$\sup \{\|\partial_i \bar{\partial}_i v_i\|_{g_i}^2 (x) | x \in M_i \setminus B_{4r_i} (x_{i\beta}, g_i) \} \to 0 \text{ as } i \to \infty.$$

Using (2.9), (2.10), (2.11), and (2.12), we see that $u_i$ converges uniformly to zero in the sense of the remark after Lemma 2.1 as $i$ goes to infinity. Put

$$S^i (x) = \frac{(u_i (x) - u_i (x))}{(\int_{M_i} \|v_i - u_i\|_{g_i}^2 (x) dV_{g_i})^{1/2}}.$$

Then $\{S^i\}$ is the required sequence.

**Lemma 2.3.** Let $\{(M_i, g_i)\}$ and $(M_\infty, g_\infty)$ be given as in Theorem 1.1. For each integer $m > 0$, we have orthonormal bases $\{S^i_{m\beta}\}_{0 \leq \beta \leq N^m}$ (resp. $\{S^\infty_{m\beta}\}$) of $H^0 (M_i, K_{M_i}^{-m})$ (resp. $H^0 (M_\infty, K_{M_\infty}^{-m})$). Then

$$\lim_{i \to \infty} \left( \inf_{M_i} \left\{ \sum_{\beta=0}^{N^m} \|S^i_{m\beta}\|_{g_i}^2 (x) \right\} \right) \geq \inf_{M_\infty} \left\{ \sum_{\beta=0}^{N^m} \|S^\infty_{m\beta}\|_{g_\infty}^2 (x) \right\}.$$
Proof. By direct computations, we have
\[(2.15) \quad \Delta_i(\|D_i S_{\beta \gamma \mu}^i\|_{g_i}^2)(x) = \|D_i D_i S_{\beta \gamma \mu}^i\|_{g_i}^2(x) - ((n+1)m-2)\|D_i S_{\beta \gamma \mu}^i\|_{g_i}^2(x),\]
where $\Delta_i$ (resp. $D_i$) is the laplacian (resp. covariant derivative) with respect to $g_i$. Then by (2.1), Lemma 1.1, and a standard Moser's iteration, there is a constant $C'(n, m)$ depending only on $n, m$ such that
\[(2.16) \quad \sup\{\|D_i S_{\beta \gamma \mu}^i\|_{g_i}^2(x)|0 \leq \beta \leq N_m, \, x \in M_i\} \leq C'(n, m).
\]
Combining this with (2.2), we conclude that the first derivatives of $\sum_{\beta=0}^{N_m} \|S_{\beta}^i\|_{g_i}^2(x)$ are uniformly bounded independent of $i$. Then (2.14) follows from this and Lemmas 2.1 and 2.2.

**Theorem 2.1.** There exist a universal integer $m_0 > 0$ and a universal constant $C > 0$ such that for any Kähler-Einstein surface $(M', g')$ in either $K_+(\mu, n)$ or $K_-(\mu, n)$, we have
\[(2.17) \quad \inf_{M'} \left\{ \sum_{\beta=0}^{N_m} \|S_{\beta}^i\|_{g_i}^2 \right\} \geq C > 0,
\]
where $N_{m+1}$ is the complex dimension of $H^0(M', K^{-m_0}_{M'})$, and $\{S_{\beta}^i\}_{0 \leq \beta \leq N}$ is an orthonormal basis of $H^0(M', K^{-m_0}_{M'})$ with respect to the inner product induced by $g'$.

Proof. It suffices to prove that for any sequence of a Kähler-Einstein surface $\{(M_i, g_i)\}$ converging to a Kähler-Einstein orbifold $(M_\infty, g_\infty)$ in the sense of Theorem 1.1, there exist $m_0 > 0$ and $C > 0$ such that (2.17) holds for these $(M_i, g_i)$. By Lemma 2.3, it is sufficient to find a large $m$ such that
\[(2.18) \quad \inf_{M_\infty} \left\{ \sum_{\gamma=0}^{N_m} \|S_{\gamma \beta}^{\infty}\|_{g_\infty}^2(x) | x \in M_\infty \right\} > 0,
\]
where $\{S_{\gamma \beta}^{\infty}\}$ and $N_m$ are given as in Lemma 2.3. This is equivalent to the fact that for any point $x$ in $M_\infty$, there is a holomorphic global section $S$ in $H^0(M_\infty, K^{-m_0}_{M_\infty})$ such that $S(x) \neq 0$. The latter can be achieved by the application of an $L^2$-estimate (Proposition 2.1) as follows. Let $x_{\infty 1}, \cdots, x_{\infty N}$ be the singular points of $M_\infty$. There is a small positive number $r$ independent of $\beta$ such that for any $x_{\infty \beta}$ in $M_\infty$, the closure of each connected component in $B_r(x_{\infty \beta}, g_\infty) \setminus \{x_{\infty \beta}\}$ is locally uniformized by a neighborhood $\tilde{U}_{\beta j}$ $(1 \leq j \leq l_\beta)$ of the origin $o$ in $C^n$ with finite
uniformization group $\Gamma_\beta$. Let $\pi_{\beta j}: \tilde{U}_{\beta j} \rightarrow B_r(x_{\infty \beta}, g_\infty)$ be the natural projection with $\pi_{\beta j}(o) = x_{\infty \beta}$ and $q = \prod_{1 \leq \beta \leq N}(\prod_{1 \leq j \leq R_\beta} q_{\beta j})$, where $q_{\beta j}$ is the order of the finite group $\Gamma_\beta$. Let $m = pq$. We will choose $p$ later. We may take $r$ to be sufficiently small such that the function $\rho_\beta = \text{dist}(\cdot, x_{\infty \beta})$ is smooth on $B_r(x_{\infty \beta}, g_\infty) \setminus \{x_{\infty \beta}\}$ for any $\beta$. Now fix an $x_{\infty \beta}$ and $\tilde{U}_{\beta j}$.

Let $(z_1, \ldots, z_n)$ be a coordinate system on $\tilde{U}_{\beta j}$, and define a $q$-anticanonical section $v$ by

$$v(y) = \sum_{\sigma \in \Gamma_{\beta j}} \sigma^* \left( \left( \frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_n} \right)^{q} \right)(y), \quad y \in \tilde{U}_{\beta j}.$$  

By the definition of $q$, we have $v(o) \neq 0$. Let $\eta: R^1 \rightarrow R_+^1$ be a cut-off function such that $\eta(t) = 1$ for $t \leq 1$, and $\eta(t) = 0$ for $t \geq 2$ and $|\eta'(t)| \leq 1$. Then $w = \eta(4\rho_\beta/r^2)(\pi_{\beta j})_*(v^p)$ is a $C^\infty$-global section of the line bundle $K_{M_\infty}^{-m}$. Choose a large $p$ depending only on $r$ such that for tangent vector $v$ of type $(1, 0)$,

$$(2.19) \left\langle \partial \bar{\partial} \left( 4n\eta \left( \frac{4\rho_\beta}{r^2} \right) \log \left( \frac{\rho_\beta}{r^2} \right) \right) + \frac{2\pi i}{\sqrt{-1}} \omega_{g_\infty}, v \wedge \bar{v} \right\rangle \geq \|v\|_{g_\infty}^2.$$  

Applying Proposition 2.1, we obtain a $C^\infty$ smooth $K_{M_\infty}^{-m}$-valued function $u$ satisfying $\partial u = \bar{\partial} w$ and

$$\int_{M_\infty} \|u\|_{g_\infty}^2 e^{-4n\eta \log(\rho_\beta/r^2)} dV_{g_\infty} \leq \int \|\bar{\partial} w\|_{g_\infty}^2 e^{-4n\eta \log(\rho_\beta/r^2)} dV_{g_\infty} < +\infty.$$  

It follows that the pullback $\pi_{\beta j}^* u$ of $u$ vanishes up to order 2 at the origin in $\tilde{U}_{\beta j} \subset C^n$. Put

$$(2.20) S_{\beta j} = \frac{w - u}{(\int_{M_\infty} \|w - u\|_{g_\infty}^2 dV_{g_\infty})^{1/2}};$$  

then $S_{\beta j} \in H^0(M_\infty, K_{M_\infty}^{-m})$ and $\inf_{\tilde{U}_{\beta j}} \{\pi_{\beta j}^* \|S_{\beta j}\|_{g_\infty}(x)\} > 0$. By the same arguments as in the proof of Lemma 2.3, one can bound the first derivatives of these $S_{\beta j}$ by a uniform constant. So if $r$ is taken sufficiently
small, we have
\begin{align*}
\inf \left\{ \sum_{y=0}^{N_m} \| S_{\beta_j}^\infty \|_{g_{\infty}}^2 (x) \right\} & = \inf \left\{ \| S_{\beta_j} \|_{g_{\infty}}^2 (x) \right\} \\
& \geq \inf \left\{ \| S_{\beta_j} \|_{g_{\infty}}^2 (x) \right\} > 0.
\end{align*}

For any point \( x \) in \( M_{\infty} \setminus \bigcup_{r=1}^{N} B_r(x_{\infty}^{\beta}, g_{\infty}) \), define \( \rho_x = \text{dist} (\cdot, x) \). As above, by applying Proposition 2.1 to \( K_{M_{\infty}}^{-m} \)-valued \( \overline{\partial} \)-equation with the weight function \( 4n \eta (4\rho_x^2/r^2) \log (\rho_x^2/r^2) \), one can easily construct a holomorphic section \( S_x \) in \( H^0(M_{\infty}, K_{M_{\infty}}^{-m}) \) such that \( S_x(x) \neq 0 \). Thus the inequality (2.18) is proved, and so is Theorem 2.1.

**Corollary 2.1.** The Kähler-Einstein orbifold \( (M_{\infty}, g_{\infty}) \) is irreducible.

Since we do not need this result, we omit its proof here and refer readers to Proposition 5.2 in [20].

### 3. Application of Kohn's estimates of CR-manifolds

Let \( \{(M_i, g_i)\} \) be the sequence of Kähler-Einstein manifolds in either \( K_+(\mu, n) \) or \( K_-(\mu, n) \) as in §1. By Theorem 1.1 and Corollary 2.1, these \((M_i, g_i)\) converge to a Kähler-Einstein orbifold \((M_{\infty}, g_{\infty})\). Precisely, there are points \( x_i, \cdots, x_{iN} \) in \( M_i \) and \( x_{\infty 1}, \cdots, x_{\infty N} \) in \( M_{\infty} \) satisfying: for \( r > 0 \), there are diffeomorphisms \( \phi_i^* g_i \) and \( \phi_i^* \circ J_i \circ (\phi_i^{-1})^* \) converging to \( g_{\infty} \) and \( J_{\infty} \), respectively, in \( C^3 \)-norms. The purpose of this section is to study the holomorphic structure of \( B_r(x_{i\beta}, g_i) \) for sufficiently small \( r \) and large \( i \). The main analytic tool is Kohn's estimate for \( \square_b \)-operators.

Let \( \rho_{\infty} (\cdot, \cdot) \) be the distance function on \( M_{\infty} \times M_{\infty} \). For simplicity, we may assume that \( N = 1 \) and write \( x_i \) for \( x_{i1} \), and \( x_{\infty} \) for \( x_{\infty 1} \). For each sufficiently small \( r > 0 \), the level surface \( \partial B_r(x_{\infty}, g_{\infty}) \) of \( \rho_{\infty} (\cdot, x_{\infty}) \) is smooth. The Levi form on \( \partial B_r(x_{\infty}, g_{\infty}) \) is the natural hermitian form on the \((n-1)\)-dimensional space \( T^{(1,0)} M_{\infty} \cap (T_R H_{\infty} \otimes C) \) given by

\[ (L_1, L_2) = 2(\partial \overline{\partial} \rho_{\infty} (\cdot, x_{\infty}), L_1 \wedge \overline{L_2}), \]

where \( H_{\infty r} \) denotes the level surface \( \partial B_r(x_{\infty}, g_{\infty}) \).

It is easy to see that this form is positive definite for \( r \) small. In fact, \( \rho_{\infty} (x_{\infty}, \cdot) \) is convex near \( x_{\infty} \). Therefore, each \( H_{\infty r} \) is a strongly pseudoconvex CR-manifold. Similarly, if we define \( H_{ir} \) to be the level surface
then the $H_{ir}$ are also smooth strongly pseudoconvex CR-manifolds.

Define the following for $r > 0$:

$$
\tilde{g}_{\infty} = \frac{1}{r^2} g_{\infty}, \quad \tilde{g}_{ir} = \frac{1}{r^2} g_i,
$$

$$(L_1, L_2)_{\infty} = \frac{2}{r^2} (\partial \overline{\partial} \rho_{\infty}, L_1 \cap \overline{L_2}) \quad \forall L_1, L_2 \in T^{(1,0)} M_{\infty} \cap (T_R H_{\infty} \otimes C),$$

$$(L_1, L_2)_{ir} = \frac{2}{r^2} (\partial \overline{\partial} (\rho_{\infty} \cdot \phi^{-1}), L_1 \wedge \overline{L_2}) \quad \forall L_1, L_2 \in T^{(1,0)} M_i \cap (T_R H_{ir} \otimes C).$$

Lemma 3.1. As $r$ goes to zero, $(H_{\infty}, \tilde{g}_{\infty}, (\cdot, \cdot)_{\infty})$ converges to $(S^{2n-1}/\Gamma, ds^2, (\cdot, \cdot)_s)$, where $\Gamma \subset U(n)$ is a finite group, $ds^2$ is the metric with constant curvature $+1$, and $(\cdot, \cdot)_s$ is induced by the standard Levi-form on the unit sphere.

Proof. It follows trivially from the boundedness of the curvature tensor $Rm(g_{\infty})$.

Lemma 3.2. There is a subsequence $\{i_j\}$ such that there are diffeomorphisms $\psi_j$ from $S^{2n-1}$ onto $H_{ir}$, where $r_j = 1/j$, satisfying:

1. $\|\psi_j^* \tilde{g}_{ij} - ds^2\|_{C^1(S^{2n-1})} \leq \varepsilon(j)$, and
2. $\|\psi_j^* (\cdot, \cdot)_{ij} - (\cdot, \cdot)_s\|_{C^0(S^{2n-1})} \leq \varepsilon(j)$,

where $\varepsilon(j) \rightarrow 0$ as $j \rightarrow \infty$.

In other words, $(H_{ir}, \tilde{g}_{ij}, (\cdot, \cdot)_{ij})$ converges to $(S^{2n-1}, ds^2, (\cdot, \cdot)_s)$ as $j$ tends to infinity.

Proof. Because of the convergence of $(M_i, g_i)$ to $(M_{\infty}, g_{\infty})$, for each $j$ there is a diffeomorphism $\phi_j$ from $M_{\infty} \setminus B_{r_j/10}(x_\infty, g_{\infty})$ into $M_i$ for some $i_j$ satisfying:

1. $M_j \setminus B_{1/2r_j}(x_j, g_{ij}) \subset \text{Im}(\phi_j)$,
2. $\|\phi_j^* g_{ij} - g_{\infty}\|_{C^0(M_{\infty})} \leq 1/j$, and
3. $\|\phi_j^* J_{ij} - J_{\infty}\|_{C^0(M_{\infty})} \leq \frac{1}{j}$, where $J_{ij}$ and $J_{\infty}$ are almost complex structures on $M_i$ and $M_{\infty}$, respectively.

By Lemma 3.1, there are diffeomorphisms $\theta_j$ from $S^{2n-1}$ onto $H_{\infty}$, such that

1. $\|\theta_j^* \tilde{g}_{\infty} - ds^2\|_{C^0(S^{2n-1})} \leq \varepsilon'(j)$, and
2. $\|\theta_j^* (\cdot, \cdot)_{\infty} - (\cdot, \cdot)_s\|_{C^0(S^{2n-1})} \leq \varepsilon'(j)$,
where \( \epsilon'(j) \to 0 \) as \( j \to \infty \). Now our \( \psi_j \) are just the compositions of \( \phi_j \) with \( \theta_j \). q.e.d.

Given a complex manifold \( X \) with strongly pseudoconvex boundary \( Y \), we define \( \mathcal{B}^{p,q}(Y) \) to be the space of smooth sections of the vector bundle \( \Omega^{p,q}(X) \cap \Lambda^p \Lambda^q(T^*_Y \otimes C) \) on \( Y \). The \( \overline{\partial}_b \)-operator of \( X \) induces the \( \overline{\partial}_b \)-operator from \( \mathcal{B}^{p,q}(Y) \) into \( \mathcal{B}^{p,q+1}(Y) \), explicitly, \( \overline{\partial}_b \phi \) is the projection of \( \overline{\partial} \phi \) onto \( \mathcal{B}^{p,q+1}(Y) \). Let \( \overline{\partial}_b^* \) be the adjoint operator of \( \overline{\partial}_b \) on \( Y \) with respect to the induced metric on \( Y \) from \( X \) and the Levi form.

Since \( \overline{\partial}^2 = 0 \), it follows that \( \overline{\partial}_b^2 = 0 \), so we have the boundary complex

\[
0 \to \mathcal{B}^{p,0} \xrightarrow{\overline{\partial}_b} \mathcal{B}^{p,1} \to \cdots \xrightarrow{\overline{\partial}_b} \mathcal{B}^{p,p-1} \to 0.
\]

Then the cohomology of the above boundary complex is called the Kohn-Rossi cohomology and is denoted by \( H^{p,q}(\mathcal{B}) \). We recall the following proposition.

**Proposition 3.1.** Let \( X, Y \) be as above. Then for \( 1 \leq q \leq n-2 \), the cohomology \( H^{p,q}(\mathcal{B}) \) is finite dimensional, and the range of \( \overline{\partial}_b: \mathcal{B}^{p,q-1} \to \mathcal{B}^{p,q} \) is closed in the \( C^\infty \)-topology.

Let \( \tilde{H}_j \) be the universal covering of \( H_{j-i} \); then they are diffeomorphic to \( S^{2n-1} \). In fact, \( \psi_j \) induces these diffeomorphisms from \( S^{2n-1} \) onto \( \tilde{H}_j \), still denoted by \( \psi_j \).

**Lemma 3.3.** Let \( n \geq 3 \). There is a uniform constant \( C > 0 \) such that for \( j \) sufficiently large,

\[
C \| u \|_2^2 \leq \| \overline{\partial}_b u \|_2^2 + \| \overline{\partial}_b^* u \|_2^2
\]

for any \( u \) in \( \mathcal{B}^{0,1}(\tilde{H}_j) \), where \( \| \cdot \|_2 \) denotes the \( L^2 \)-norm induced by the metric \( g_{i,j} \) and Levi form \( (\cdot, \cdot)_{i,j} \).

**Proof.** Let \( \lambda_j \) be the smallest eigenvalue of the operator of \( \Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b \) on \( \mathcal{B}^{0,1}(\tilde{H}_j) \). Then (3.1) is equivalent to \( \lambda_j \geq c > 0 \).

Suppose that the lemma is false. Then we may assume that \( \lambda_j \to 0 \) as \( j \to \infty \). By Proposition 3.1, the eigenspace of \( \lambda_j \) is of finite dimension. Pick up an eigenfunction \( u_j \) for \( \lambda_j \) with \( \| u_j \|_2 = 1 \). Then

\[
\lambda_j \| u_j \|_2^2 = \| \overline{\partial}_b u_j \|_2^2 + \| \overline{\partial}_b^* u_j \|_2^2.
\]

Since \( (\tilde{H}_j, g_{i,j}, (\cdot, \cdot)_{i,j}) \) converges to \( (S^{2n-1}, ds^2, (\cdot, \cdot)_s) \) in the \( C^5 \)-topology, by Kohn’s estimate for \( \Box_b \), these \( u_j \) converge to \( u_\infty \) in \( \mathcal{B}^{0,1}(S^{2n-1}) \) satisfying

\[
\| u_\infty \|_2 = 1 \quad \text{and} \quad \Box_b u_\infty = 0.
\]
In particular, \( u_\infty \) gives a nontrivial cohomological class in \( H^{0,1}(\mathcal{B}(S^{2n-1})) \). However, it follows from Theorem A in [26] that \( H^{0,1}(\mathcal{B}(S^{2n-1})) = 0 \) for \( n \geq 3 \), a contradiction. Therefore, (3.1) holds.

**Lemma 3.4.** There exist embeddings \( \iota_j : \tilde{H}_j \to C^n \) such that the \( \iota_j(\tilde{H}_j) \) converge to \( S^{2n-1} \) as submanifolds in \( C^n \) in the \( C^3 \)-topology.

**Proof.** Let \( z_1, \cdots, z_n \) be the standard coordinates in \( \mathbb{R}^n \). The restrictions of these to \( S^{2n-1} \) are CR-functions denoted by the same letters for simplicity. Define

\[
z_{jj} = z_i \circ \psi_j^{-1}, \quad 1 \leq i \leq n, \ j \gg 0.
\]

Then \( \sup \{ \| \bar{\partial}_b(z_i \circ \psi_j) \|_{C^r(\tilde{H}_j)} \} \leq C \varepsilon(j), \) where \( C \) is a uniform constant, and \( \varepsilon(j) \to 0 \) as \( j \to \infty \).

By Lemma 3.3, there are \( v_{ij} \) solving

\[
\Box_b v_{ij} = \bar{\partial}_b(z_i \circ \psi_j^{-1}) \quad \text{on} \quad \tilde{H}_j
\]

with

\[
\| \bar{\partial}_b v_{ij} \|_2^2 + \| \bar{\partial}_b^* v_{ij} \|_2^2 + \| v_{ij} \|_2^2 \leq C_1 \| \bar{\partial}_b(z_i \circ \psi_j^{-1}) \|_2^2 \leq C_1 C \varepsilon(j).
\]

Define \( z_{ij} = z_i \circ \psi_j^{-1} - \bar{\partial}_b^* v_{ij} \); then \( \partial_b z_{ij} = 0 \).

Using Kohn’s estimate for the \( \bar{\partial}_b \)-operator, we have

\[
\sup_{1 \leq i \leq n} \{ \| \bar{\partial}_b v_{ij} \|_{C^r(\tilde{H}_j)} \} \leq C_3 \sup_{1 \leq i \leq n} \| \bar{\partial}_b(z_i \circ \psi_j^{-1}) \|_{C^r(\tilde{H}_j)} \leq C \varepsilon(j).
\]

The required maps \( \iota_j \) assign \( x \) in \( \tilde{H}_j \) to \( (z_{ij}(x), \cdots, z_{nj}(x)) \) in \( C^n \). Since \( \partial_b z_{ij} = 0 \) and \( (\tilde{H}_j, g_{ij}, (\cdot, \cdot)_{ij}) \) converge to \( (S^{2n-1}, ds^2, (\cdot, \cdot)_s) \) through \( \psi_j \), these \( \iota_j \) are CR-embeddings of \( \tilde{H}_j \) such that the images approach \( S^{2n-1} \). Hence the lemma is proved. q.e.d.

Choose a large \( m \) such that the basis \( \{ s_0, \cdots, s_{\dim N_m} \} \) of \( H^0(M_\infty, K_{M_\infty}^{-m}) \) gives a Kodaira’s embedding of \( M_\infty \) into \( CP^{N_m} \), where \( N_m = \dim C H^0(M_\infty, K_{M_\infty}^{-m}) \). Moreover, we may arrange these \( S_\beta \) such that \( S_0(\infty) \neq 0 \), and \( S_\beta(\infty) = 0 \) for \( \beta \geq 1 \). By Theorem 2.3 in the previous section, there are bases \( \{ s_{ij}^\beta \} \) of \( H^0(M_j, K_{M_j}^{-m}) \) converging to \( \{ S_\beta \} \).

In particular, for \( j \) sufficiently large, these bases \( \{ s_{ij}^\beta \} \) give embeddings of \( M_j \) into \( CP^{N_m} \). Fix a small \( r > 0 \); then for \( j \) large we have local embeddings

\[
\tau_j : B_r(x_j) \to C^{N_m}, \quad \tau_\infty : B_r(x_\infty) \to C^{N_m}.
\]
Denote by $w_1, \ldots, w_{N_m}$ the coordinate functions. Let $\pi_j: \tilde{H}_j \to H_{ij}$ be the covering maps. Then the compositions $w_\beta \circ \pi_j$ ($1 \leq \beta \leq N_m$) are CR-functions on $\tilde{H}_j$. Now by the previous lemma, $\tilde{H}_j$ bound strongly pseudo-convex domains $B_j$ in $\mathbb{C}^n$. Moreover, these $B_j$ converge to the unit ball in $\mathbb{C}^n$ as $j$ approaches infinity.

**Lemma 3.5.** Each $w_\beta \circ \pi_j$ can be extended to be a holomorphic function $h_\beta$.

*Proof.* Since $B_j$ is a domain in $\mathbb{C}^n$, there is a nonconstant holomorphic function on $B_j$. This lemma then follows from Theorem 5.3.2 in [6]. q.e.d.

Define

$$\tilde{\tau}_j = (h_{ij}, \ldots, h_{N_m}) : \to \mathbb{C}^{N_m}.$$ 

Then $\tilde{\tau}_j$ coincides with $\tau_j \circ \pi_j$ on $\tilde{H}_j$, so by the analytic unique continuation, the image $\tilde{\tau}_j(B_j)$ coincides with part of $\tau_j(B_j(x_{ij}))$. It follows that there are holomorphic maps $\tau_j^{-1} \circ \tilde{\tau}_j$ from $B_j$ onto the domain in $B_r(x_{ij})$ enclosed by $H_{ij}$, in particular, $\tau_j^{-1} \circ \tilde{\tau}_j$ immersions near $\tilde{H}_j$ and finite maps on $B_j$. For simplicity, denote $\tau_j^{-1} \circ \tilde{\tau}_j$ by $\pi_j$.

**Lemma 3.6.** Let $\Gamma$ be the fundamental group of $H_{ij}$. Then $\Gamma$ acts on $\tilde{H}_j$ as a CR-isomorphism group, and can be extended to be automorphisms of $B_j$. In particular, $\Gamma \subset U(n)$.

*Proof.* It is clear that each $\sigma \in \Gamma$ preserves the CR-structure of $\tilde{H}_j$ as a deck transformation. Therefore, the CR-functions $z_1 \circ \sigma, \ldots, z_n \circ \sigma$ can be extended to be holomorphic ones in $B_j$ (cf. proof of Lemma 3.5), that is, $\sigma$ extends to be a holomorphic map from $B_j$ into itself. The extension must be an automorphism since $\sigma$ has degree one near $\tilde{H}_j$. q.e.d.

As a finite group in $U(n)$, $\Gamma$ has at least a fixed point in $B_j$ if it is nontrivial. This implies that $\mathcal{B}_j/\Gamma$ is singular, contradicting to the fact that $B_r(x_{ij})$ is smooth for each $j$. Therefore, $\Gamma = \{\text{id}\}$, and $M^\infty$ is in fact smooth.

Summarizing the above, we have

**Theorem 3.1.** Let $\{(M_i, g_i)\}$ be a sequence of Kähler-Einstein manifolds in $K_+(\mu, n)$ (resp. $K_-(\mu, n)$). Then either $(M_i, g_i)$ converges to a Kähler-Einstein manifold in the $C^5$-topology, or there is a smooth Kähler-Einstein manifold $(M^\infty, g^\infty)$ in $K_+(\mu, n)$ (resp. $K_-(\mu, n)$) such that a subsequence of $\{(M_i, g_i)\}$, say $\{(M_i', g_i')\}$ itself, converges to $(M^\infty, g^\infty)$ outside finitely many points in the $C^5$-topology.
4. Proof of Theorem 2

In this section, we classify all complete Ricci-flat Kähler manifolds \((X, g)\) with euclidean volume growth and \(\int_X |Rm(g)|^n dV_g < \infty\), where \(n = \dim_X X\). Let us fix one of them, say \((X, g)\).

**Lemma 4.1.** There is a decreasing positive function \(\varepsilon(r)\) with \(\lim_{r \to \infty} \varepsilon(r) = 0\) such that

\[
\|Rm(g)\|_g (x) \leq \frac{\varepsilon(r)}{r(x)^2},
\]

where \(r(x)\) is the distance function from some fixed points.

**Proof.** Choose \(\varepsilon(r)\) to be a decreasing positive function such that \(\lim_{r \to \infty} \varepsilon(r) = 0\) and

\[
\int_{B_r(x, g)} \|Rm(g)\|_g^n dV_g \leq \varepsilon(r) \quad \text{for } x \in \partial B_{2r}(x_0).
\]

Now for each fixed \(x\) in \(\partial B_{2r}(x_0)\), define a new metric \(g_x = g/r^2\); then \(g_x\) has vanishing Ricci curvature, and

\[
\int_{B_r(x, g_x)} \|Rm(g_x)\|_{g_x}^n dV_{g_x} \leq \varepsilon(r).
\]

On the other hand, since \((X, g)\) has the euclidean volume growth, there is a constant \(C'\), independent of \(r\), such that

\[
\text{Vol}_g(B_{2r}(X, g)) \geq C' r^{2n},
\]

so by the Volume Comparison Theorem [2],

\[
\text{Vol}_g(B_1(X, g)) \geq \frac{1}{4^{2n}} \text{Vol}_g(B_{4r}(X, g)) \geq \frac{1}{4^{2n}} \text{Vol}_g(B_{2r}(X_0, g)) \geq \frac{C'}{4^{2n}} r^{2n}.
\]

It follows that \(\text{Vol}_{g_x}(B_1(x, g_x)) = \text{Vol}_g(B_r(x, g_x))/r^2\) is not less than a uniform positive constant \(C'/4^{2n}\). So we can apply Lemma 1.2 to \((B_1(x, g_x), g_x)\) and obtain

\[
\|Rm(g_x)\|_{g_x} \leq C\varepsilon(r),
\]

where \(C\) is a constant independent of \(x\). Take \(\varepsilon(r) = C\varepsilon(r)\). Then (4.1) is nothing else but (4.2), and the lemma is proved. \(\text{q.e.d.}\)
Consider a sequence of complete Ricci-flat Kähler manifolds $(X_i, g_i) = (X, g/\tilde{t}^2)$. By Lemma 4.1, $\|Rm(g_i)\|_{g_i}$ are bounded by $\varepsilon(i)/\delta$ outside $B_\delta(x_0, g_i)$ for any $\delta > 0$. Therefore, we can proceed as in §1 to show that $(X_i, g_i)$, by taking subsequences, converges to a complete Kähler orbifold $(X_\infty, g_\infty)$. In fact, the proof in this case is much similar, and $(X_\infty, g_\infty)$ is flat because of Lemma 4.1. Therefore, $X_\infty = C^n/\Gamma$ with unique singular point $o$ in $U(n)$. In particular, there are smooth diffeomorphisms $\psi_i$ from $X_i \setminus B_{1/2}(x_0, g)$ into $X_\infty \setminus B_{1/4}(0, g_\infty)$ such that $\|(\psi_i^{-1})^* g_i - g_\infty\|_{C^i(X_\infty, g_\infty)} = o(1)$ as $i$ goes to infinity. Put $\Sigma_i = \psi_i^{-1}(\partial B_1(0, g_\infty))$, and let $\tilde{\Sigma}_i$ be its universal covering. Then the $\tilde{\Sigma}_i$ are strongly pseudoconvex CR-manifolds and converge to $S^{2n-1}$ in $C^n$. Thus by Lemma 3.4, for $i$ sufficiently large, these $\tilde{\Sigma}_i$ can be holomorphically embedded into $C^n$ and bound domains $B^n_i$ there. Moreover, $\Gamma$ acts on $B^n_i$ by holomorphic transformations.

On the other hand, if we denote by $\rho^2$ the square of the euclidean distance function from $o$ in $C^n/\Gamma$, then the $\psi^* \rho^2$ are convex functions near $\Gamma$. So by Grauert's theorem [8], for each large $i$, there is a holomorphic map $\upsilon_i: E_i \to C^{N_i}$ which is actually an embedding near $\Gamma_i = \partial E_i$, where $E_i$ is the bounded domain enclosed by $\Sigma_i$.

**Lemma 4.2.** For each fixed $i$, if $w_1, \ldots, w_{N_i}$ are coordinate functions of $C^{N_i}$, then the CR-functions $w_j \circ \pi_i: \tilde{\Sigma}_i \to C^{N_i}$ can be extended to be holomorphic ones in $B^n_i$, where $\pi_i: \tilde{\Sigma}_i \to \Sigma_i$ are natural projections.

We omit its proof (cf. Lemma 3.5).

It follows that there are holomorphic maps $\phi_i: B^n_i/\Gamma \to \upsilon_i(E_i)$, which are embeddings in the neighborhoods of $\Sigma_i$.

**Lemma 4.3.** For each $i$, there is a holomorphic map $p_i: E_i \to B^n_i/\Gamma$ such that $\upsilon_i = \phi \circ p_i$.

**Proof.** It is easy to see that $\phi_i^{-1}(x)$ contains exactly one point in $B^n_i/\Gamma$ for $x$ in $\upsilon_i(E_i)$. Let $D_1, \ldots, D_{l_i} \in E_i$ be analytic subvarieties such that $v_i^{-1} \circ \upsilon_i(D_{i_j})$ contains more than one point. Then the $\upsilon_i(D_{ij})$ are isolated points. Define $p_i = \phi_i^{-1} \circ \upsilon_i$ outside these $D_{i_1}, \ldots, D_{i_l}$; then $p_i$ is a holomorphic map from $E_i \setminus \bigcup_{\beta=1}^{l_i} D_{i_\beta}$ into $B^n_i$. Since $B^n_i$ is bounded, the map $p_i$ can be extended across $D_{i_\beta}$. In particular, this implies that $l_i = 1$, i.e., there is only one connected component, and $\upsilon_i(E_i)$ has only one singular point, so $\upsilon_i(E_i) \cong B^n_i/\Gamma$. q.e.d.

It follows that $X$ is the resolution of $C^n/\Gamma$. Hence Theorem 2 is proved.
5. Proof of Theorem 1

In this section, we will finish the proof of Theorem 1.

Let \( \{ (M_i, g_i) \} \) be a sequence of Kähler-Einstein manifolds either in \( K_+(\mu, n) \) or \( K_-(\mu, n) \). By Theorem 3.1, \( (M_i, g_i) \) converges to a smooth Kähler-Einstein manifold \( (M^\infty, g^\infty) \) outside finitely many points. Precisely, there are \( x_{i1}, \cdots, x_{iN} \) satisfying: for each \( r > 0 \), there are diffeomorphisms \( \phi_{ir} \) from \( M^\infty \setminus \bigcup_{\beta=1}^{N} B_r(x_{i\beta}, g^\infty) \) into \( M_i \) containing \( M_i \setminus \bigcup_{\beta=1}^{N} B_{2r}(x_{i\beta}, g_i) \) such that \( \phi_{ir}^\ast g_i \) converges to \( g^\infty \) in the \( C^5 \)-topology. Each \( B_r(x_{i\beta}, g^\infty) \) with small \( r \) is a smooth ball in \( C^n \). So \( B_r(x_{i\beta}, g_i) \) are smooth balls in \( C^n \), too.

We need to show that the \( Rm(g_i) \) are uniformly bounded in \( \bigcup_{\beta=1}^{N} B_{2r}(x_{i\beta}, g_i) \). Suppose it is not true. Then by taking the subsequence, we may assume that \( \mu_i = \|Rm(g_i)\|_{g_i} \to +\infty \) for some \( y_i \) in \( B_{r_i}(x_{i1}, g_i) \), where \( \lim_{i \to \infty} r_i = 0 \). Define new metrics on \( M_i \) by

\[
h_i = \mu_i^2 g_i.
\]

Then the pointed manifolds \( (B_r(x_{i1}, g_i), h_i, y_i) \) converge to a complete Ricci-flat Kähler manifold \( (X, h) \) with \( \int_X \|Rm(h)\|_h^2 dV_h < \infty \), where \( r \) is a fixed small positive number.

**Lemma 5.1.** \( X \) is a Stein manifold.

**Proof.** Let \( (M_i, g_i) \) be in \( K_+(\mu, n) \) for all \( i \). The proof of the other case is identical.

Fix an \( m > 0 \) such that the basis of \( H^0(M^\infty, K_M^{-m}) \) gives an embedding of \( M^\infty \) into some projective space. In particular, there is a positive constant \( C \) satisfying

\[
\min_{M^\infty} \left\{ \sum_{\beta=0}^{N} \|S^\infty_\beta\|^2_{g^\infty}(x) \right\} \geq 2C > 0,
\]

where \( N = \dim_C H^0(M^\infty, K_M^{-m}) \), and \( \{S^\infty_\beta\} \) is an orthonormal basis of \( H^0(M^\infty, K_M^{-m}) \) with respect to \( g^\infty \).

By Theorem 2.1, for \( i \) sufficiently large,

\[
(5.1) \quad \min_{M_i} \left\{ \sum_{\beta=0}^{N} \|S^i_\beta\|^2_{g_i}(x) \right\} \geq C > 0,
\]

where the \( \{S^i_\beta\} \) are orthonormal bases of \( H^0(M_i, K_M^{-m}) \) with respect to
Let $\widetilde{S}_i$ be the section of $K_{M_i}^{-m}$ satisfying:

1. $\int_{M_i} |\widetilde{S}_i|^2 dV_{g_i} = 1$,
2. $|\widetilde{S}_i|_{g_i}(y_i) = \sup \{||S||_{g_i}(y_i)| \int_{M_i} ||S||^2 dV_{g_i} = 1\}$.

Then for $i$ sufficiently large and $r$ sufficiently small,

$$\min_{B_r(x_{i1}, g_i)} (||\widetilde{S}_i||) \geq C > 0.$$ 

Define $u_i(x) = -\log(||\widetilde{S}_i||_{g_i}(x)/||\widetilde{S}_i||_{g_i}(y_i)))$. Then the $u_i$ are uniformly bounded smooth functions in $B_r(x_{i1}, g_i)$ satisfying:

$$\omega_{g_i} = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \bar{\partial} u_i \text{ in } B_r(x_{i1}, g_i),$$

$$u_i(y_i) = \min_{B_r(x_{i1}, g_i)} u_i = 0.$$ 

Therefore, $\omega_h = \sqrt{-1} \partial \bar{\partial} (\mu^2 u_i)/(2\pi)$, and $\mu^2 u_i$ converge to a smooth function $u$ in $X$ such that $\omega_h = \sqrt{-1} \partial \bar{\partial} u/2\pi$. This implies that $X$ is Stein, and hence the lemma is proved. q.e.d.

By Theorem 2, $X$ is a smooth resolution of some $C^n/\Gamma$. Therefore, $X$ has to be $C^n/\Gamma$, and $\Gamma$ is trivial since $X$ is Stein.

Thus $(X, h)$ must be flat, contradicting that $\max_x \|Rm(h)\|_h = 1$. This finishes the proof of Theorem 1.

References


