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GRAPH MANIFOLDS, ENDS OF NEGATIVELY CURVED SPACES AND THE HYPERBOLIC 120-CELL SPACE

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In this paper we show that a rich class of graph manifolds occur as ends of complete Riemannian manifolds with finite volume whose curvature is strictly negative and uniformly bounded from below.

For the purpose of this paper, a graph manifold W is given in the following way (for a more general notion see [16]): Let W_i be a finite collection of building blocks diffeomorphic to $\Sigma_i \times S^1$, where Σ_i is a closed oriented surface with some disjoint open balls removed, and S^1 is the unit circle. The boundary components of W_i are tori $S^1 \times S^1$, where the orientation on the first S^1-factor is induced from the boundary of Σ_i, and the second factor carries the canonical orientation of the S^1-factor of W_i. We obtain W from the W_i by gluing the tori in pairs, interchanging the factors, and preserving all orientations.

Then W can be described by a graph where the vertices correspond to the building blocks W_i, and the number at each vertex indicates the genus of Σ_i. If two building blocks are glued together on a boundary torus, we join the vertices by an edge. Examples are described by the graphs shown in Figure 1.

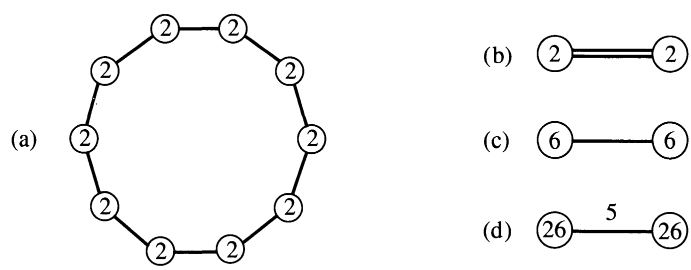

FIGURE 1

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Theorem 0.1. *Let W be a 3-manifold corresponding to a graph in Figure 1 or to a subgraph obtained by deleting some vertices and the corresponding edges. Then there exists a complete Riemannian 4-manifold (M^4, g) with finite volume and sectional curvature K such that $-1 \leq K < 0$ and $M^4 \backslash C$ is a diffeomorphic to $W \times (0, \infty)$ for some compact subset C of M^4.*

This theorem just describes a few examples in a long and rich list of possible end structures of complete manifolds with finite volume and negative curvature K satisfying $-b^2 \leq K < 0$, a phenomenon quite different from the rigid behavior in the case of pinched negative curvature where $-b^2 \leq K \leq -a^2 < 0$. In this more special case all ends are infranilmanifolds [9].

Our theorem implies *in particular* that *the graph manifold* ②—② is the end of a manifold M^4 with negative curvature and finite volume. Note that ②—② is also the end of $\Sigma_{2,1} \times \Sigma_{2,1}$, where $\Sigma_{2,1}$ is a surface of genus 2 with one puncture. By [13] any complete metric of finite volume and nonpositive sectional curvature on $\Sigma_{2,1} \times \Sigma_{2,1}$ is a product metric. Therefore the "compact parts" of M^4 and $\Sigma_{2,1} \times \Sigma_{2,1}$ differ essentially, and the rigidity of $\Sigma_{2,1} \times \Sigma_{2,1}$ does not imply any rigidity of the end of $\Sigma_{2,1} \times \Sigma_{2,1}$.

Our examples are based on the following result.

Theorem 0.2. *Let $(\overline{\Sigma}_i)_{i=1}^N$ be a finite family of compact, totally geodesically embedded, orientable surfaces with simple normal crossings in a compact hyperbolic space $\mathbb{H}^4/\Gamma'$. Then $M^4 := \mathbb{H}^4/\Gamma' \backslash \bigcup_i \overline{\Sigma}_i$ carries a complete, smooth Riemannian metric g with finite volume whose sectional curvature is strictly negative and uniformly bounded from below.*

Here the notion of *simple normal crossings* has been used in the same way as is customary in algebraic geometry: the only singularities allowed for $\bigcup_i \overline{\Sigma}_i$ are simple double points p, where the two sheets intersect perpendicularly and their tangent spaces span $T_p(\mathbb{H}^4/\Gamma')$.

The *end structure* of such a manifold M^4 can be determined by analyzing narrow distance tubes around the set $\bigcup_i \overline{\Sigma}_i$ taken away from $\mathbb{H}^4/\Gamma'$. Clearly the intersection of such a tube with M^4 is diffeomorphic to $W \times (0, \varepsilon)$, where W stands for some graph manifold with building blocks $(\overline{\Sigma}_i \backslash \{\textit{points of intersections}\}) \times S^1$. In §3 it will be shown *explicitly* how to get each graph manifold W listed in Figure 1 as such a boundary of a tubular neighborhood of a family of surfaces in some $\mathbb{H}^4/\Gamma'$. Thereby we deduce Theorem 0.1 from Theorem 0.2. Note that the manifold ②—②

from Figure 1(a) is the simplest (orientable) graph manifold which can be obtained in such a manner.

The obvious restriction for the genus of the building blocks is purely metrical in nature. An arbitrary graph manifold W consisting of building blocks $\Sigma_i \times S^1$ can be obtained by means of the following construction: embed closed oriented surfaces $\overline{\Sigma}_i$ into S^4 with transversal intersections in such a way that each $\overline{\Sigma}_i$ has the same genus as the factor Σ_i of the corresponding building block W_i and such that $\overline{\Sigma}_i$ intersects $\overline{\Sigma}_j$ in at least k_{ij} points, where k_{ij} denotes the multiplicity of the corresponding edge in the graph. Blowing up superfluous intersection points, one gets embeddings $\overline{\Sigma}_i \hookrightarrow Q^4$ with $\#(\overline{\Sigma}_i \cap \overline{\Sigma}_j) = k_{ij}$. Then W is diffeomorphic to the boundary of a small tubular neighborhood of $\bigcup_i \overline{\Sigma}_i$ in Q^4. This suggests the following definition.

Definition. An $(n-1)$-manifold W is called a *generalized graph manifold* iff there is a compact n-manifold Q^n and a finite family $(\overline{V}_i)_{i=1}^N$ of compact, immersed, codimension-2 submanifolds in general position such that W is diffeomorphic to the boundary of a tubular neighborhood of $\bigcup_i \overline{V}_i \subset Q^n$. The least upper bound on the number of sheets through any point $p_0 \in \bigcup_i \overline{V}_i$ will be called the *level* of the generalized graph manifold W.

Here "in general position" means as usual that any k sheets $S_1, \cdots, S_k$ of $\bigcup_i \overline{V}_i$ through any given singular point p_0 are transversal and that $\dim T_{p_0}(\bigcap_{j=1}^k S_j) = n - 2k$. Clearly generalized graph manifolds of level 1 are just products with one S^1-factor. Classical 3-dimensional graph manifolds are generalized graph manifolds of level 2. Notice that the rotations around the various $\overline{V}_i$ give rise to *locally defined S^1-actions* on W. These actions can actually be chosen in such a way that they commute near the singularities of $\bigcup_i \overline{V}_i$. Hence a generalized graph manifold W carries an *F-structure* in the sense of [5]; this F-structure is nonpure as soon as the level of W exceeds 1. It appears that in dimensions $n > 4$ the generalized graph manifolds in the above sense form just the simplest class of manifolds with a nonpure F-structure.

Using this language we extend Theorem 0.1 as follows.

Theorem 0.3. *For every $n \geq 4$ and every $k \leq [\frac{n}{2}]$ there are infinitely many generalized graph manifolds W^{n-1} of level k such that $W \times (0, \infty)$ occurs as an end of a complete Riemannian manifold (M^n, g) with finite volume and sectional curvature K satisfying $-1 \leq K < 0$.*

The basic ingredient in the proof of this theorem is the following result which extends Theorem 0.2 above by means of a general codimension-2 construction in the sense of M. Gromov and W. Thurston (cf. [12] and [1, p. 121f]).

Theorem 0.4. *Let $(\overline{V}_i)_{i=1}^N$ be a finite family of compact, totally geodesically immersed codimension-2 submanifolds in some compact hyperbolic space $\mathbb{H}^n/\Gamma'$. Suppose that the various sheets of $\bigcup_i \overline{V}_i$ intersect pairwise orthogonally in some set of codimension 4. Then $M^n := \mathbb{H}^n/\Gamma \setminus \bigcup_i \overline{V}_i$ carries a complete, smooth Riemannian metric g with finite volume whose sectional curvature is strictly negative and uniformly bounded from below.*

The new metric g on M^n is obtained by *stretching* the hyperbolic metric g_0 in certain directions transversal to the codimension-2 submanifolds $\overline{V}_i$. Explicit formulas for all this will be given in Theorem 1.1 and throughout §1, where all the curvature and volume calculations are done.

It is still open whether the above construction also gives metrics with bounded negative curvature for more *general ambient spaces* than $\mathbb{H}^n$ and $\mathbb{CH}^n$ or not.

Our next result will exhibit a large class of settings where the hypotheses of Theorem 0.4 or even those of the more special Theorem 0.2 are satisfied.

Theorem 0.5. *Let Γ' be a torsionfree, normal subgroup of finite index in some cocompact, discrete group $\Gamma \subset \mathrm{Iso}(\mathbb{H}^n) = O^+(n, 1)$. Suppose in addition that Γ contains k commuting rotations $\rho_1, \cdots, \rho_k$, whose fixed point sets are codimension-2 hyperbolic subspaces intersecting in some $\mathbb{H}^{n-2k} \subset \mathbb{H}^n$. If at most one of the ρ_i's has order 2, then the projection $\mathbb{H}^n \to \mathbb{H}^n/\Gamma'$ maps the subspaces $V_i = \mathrm{Fix}\,\rho_i$ onto compact, connected, totally geodesically embedded submanifolds $\overline{V}_i \subset \mathbb{H}^n/\Gamma'$ of codimension-2, which intersect pairwise orthogonally in codimension-4 subsets such that $\bigcap_{i=1}^k \overline{V}_i \neq \varnothing$.*

Recall that in any arithmetic group Γ the congruence subgroups provide a whole infinite lattice of torsionfree, normal subgroups. So all we need in order to deduce Theorem 0.3 from Theorems 0.4 and 0.5 is a *cocompact arithmetic group* $\Gamma \subset \mathrm{Iso}(\mathbb{H}^n)$ together with appropriate *rotations* $\rho_1, \cdots, \rho_{[n/2]}$.

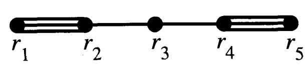
Examples of such groups Γ are obtained for all dimensions n from the *quadratic forms*

$$-a \cdot dx_0^2 + dx_1^2 + \cdots + dx_n^2, \tag{1}$$

where $a = \sqrt{D}$ for any fixed square-free positive integer D, or $a = \frac{1+\sqrt{5}}{2}$, or $a = 2\cos\frac{2\pi}{7}$. The orthogonal group $\Gamma_n(a)$ of each of these quadratic forms, when taken with coefficients in the ring of integers in $\mathbb{Q}(\sqrt{d})$, $\mathbb{Q}(\frac{1+\sqrt{5}}{2})$, or $\mathbb{Q}(\cos\frac{2\pi}{7})$, resp., is known to be a cocompact arithmetic group (cf. [2] and [11]). Moreover, the latter two examples play a basic role when looking for cocompact arithmetic groups which are generated by reflections (cf. [4] and [15]). In order to prove Theorem 0.3 it is sufficient to consider the 90°-rotations ρ_i in the $(2i-1, 2i)$ planes, where $1 \le i \le k \le [n/2]$. These rotations are contained in any group $\Gamma_n(a)$ introduced above. By Theorem 0.5 we get manifolds $M^n = \mathbb{H}^n/\Gamma' \setminus \bigcup_i \overline{V}_i$, whose ends are modelled on level k generalized graph manifolds, for any $k \le [\frac{n}{2}]$. Hence Theorem 0.3 can be deduced from Theorem 0.4 as claimed.

Theorem 0.5 is also useful in the proof of Theorem 0.1, where more precise statements about the topology of the ends in certain 4-dimensional settings have been made. This additional piece of accuracy requires the calculation of a fundamental domain for the cocompact arithmetic group Γ under consideration. For this reason we choose to work with the *Lanner groups*, i.e., those discrete reflection groups $\Gamma \subset \mathrm{Iso}(\mathbb{H}^4)$ which act transitively on a suitable simplicial decomposition of $\mathbb{H}^4$. They are arithmetic and a case-by-case examination (cf. Figure 3 (p. 316) and Proposition 2.1) exhibits the required rotations ρ_1 and ρ_2 in order to apply Theorem 0.5.

A particularly nice example is attached to the reflection group Γ with Coxeter diagram:



Here the 4-manifold is the *hyperbolic* 120-*cell space*, which arises as the quotient of $\mathbb{H}^4$ by a normal subgroup $\Gamma' \triangleleft \Gamma$ of index 14400 [8].

As we shall see in §3, the above construction gives two oriented, totally geodesically embedded surfaces $\overline{\Sigma}_1$, $\overline{\Sigma}_2$ of genus 2, each of them invariant under a subgroup of order 20 in $\mathrm{Iso}(\mathbb{H}^n/\Gamma')$, which is isomorphic to D_{10}. In particular, the $\overline{\Sigma}_i$ allow automorphisms of order 5. A standard application of the Hurwitz theorem shows that this already determines the conformal structure of $\overline{\Sigma}_i$. The surfaces intersect orthogonally at precisely two points. One of these points of intersection can be removed by passing to a suitable 5-fold covering of $\mathbb{H}^4/\Gamma'$. The preimages of the surfaces $\overline{\Sigma}_i$ in this covering consists of five components $\widehat{\Sigma}_i^j$, $1 \le j \le 5$, which intersect as depicted in Figure 2 (next page). This explains Theorem 0.1 not only in the case of Figure 1(b) but also in the case of Figure 1(a) and its subgraphs like ②—②. Figures 1(c) and 1(d) are associated with some further coverings (cf. §3.4), and the proof of Theorem 0.1 is entirely explicit.

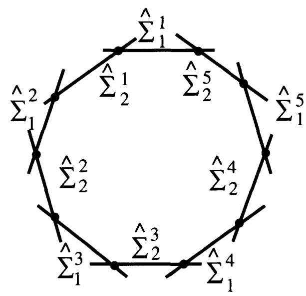


FIGURE 2

Our results fit nicely into the theory of *collapsing Riemannian manifolds* as described in [5]. The hypotheses of Theorem 0.4 imply that M^n carries an F-structure outside of a compact set. By [5, Chapter 5] M^n carries a complete metric of finite volume and bounded curvature. In our more special context we are also able to control the sign of K.

In all our examples, the end or the ends of (M^n, g) come with a foliation by graph manifolds or generalized graph manifolds such that the second fundamental form and thus the intrinsic sectional curvatures of all leaves are uniformly bounded. Their volume and injectivity radius tend to zero. Since, in the terminology of J. Cheeger and M. Gromov, we are collapsing a nonpure F-structure, the *intrinsic diameter* of these graph manifolds must be unbounded (cf. [5, Chapter 3]). In our examples the diameter grows linearly as a function of the distance to some base point. "Adjacent components" of the boundary $M^n(\infty)$ have *Tits distance* equal to $\frac{\pi}{2}$.

This description of $M^n(\infty)$ explains why our approach to Theorem 0.4 differs that much from the standard warping constructions. So it should not be surprising that our curvature calculations are technically much more subtle than K. Fujiwara's [10], who has already handled the case where the $\overline{V}_i$'s are smoothly embedded, totally geodesic submanifolds without intersection.

1. Curvature computations

The goal in this section is to prove Theorem 0.4. We shall construct the complete, strictly negatively curved metric g with finite volume explicitly. Our curvature calculations depend on precise information about the geometry in a neighborhood of the intersecting codimension-2 submanifolds in $\mathbb{H}^n/\Gamma'$. We shall make use of some *exact cancellations*, and so it is doubtful whether one can replace $\mathbb{H}^n/\Gamma'$ by some manifold with pinched negative curvature. However, our calculations can be carried over to the quotients of the complex hyperbolic space $\mathbb{CH}^n$, but this is not the subject of the current paper.

Recall that two subspaces E_1 and E_2 of some Euclidean vector space $\mathbb{R}^n$ *intersect orthogonally* in some l-dimensional subspace iff the intersection $E := E_1 \cap E_2$ has dimension l and the subspaces $E_1 \cap E^{\perp}$ and $E_2 \cap E^{\perp}$ are orthogonal. We shall say that a family $(\overline{V}_i)_{i\in I}$ of embedded, codimension-2 submanifolds $\overline{V}_i \subset \mathbb{H}^n/\Gamma'$ has *normal crossings* if and only if at any point $p \in \overline{V}_i \cap \overline{V}_j$, $i \neq j$, the tangent spaces $T_p\overline{V}_i$ and $T_p\overline{V}_j$ intersect orthogonally in some $(n-4)$-dimensional subspace of $T_p(\mathbb{H}^n/\Gamma')$. This definition is extended to families of immersed, codimension-2 submanifolds $\overline{V}_i$ by putting the same orthogonality condition on all self-intersections.

Theorem 1.1. *Let $(\overline{V}_i)_{i=1}^N$ be a finite family of compact, totally geodesically immersed, codimension-2 submanifolds with normal crossings in a compact, hyperbolic manifold $\mathbb{H}^n/\Gamma'$. Let $\varepsilon_0 > 0$ be so small that the distance function $\operatorname{dist}(\cdot, \bigcup_i \overline{V}_i)$ has no critical points inside the tubes $U_{\varepsilon_0}(\bigcup_i \overline{V}_i)$, and let f be a smooth nonnegative function such that*

$$(2) \qquad \begin{aligned} f(r) &= 0 \quad \textit{for } r \geq \varepsilon_0, \\ f'(r) &\leq 0 \quad \textit{for } r < \varepsilon_0. \end{aligned}$$

Then the metric g on $M^n := \mathbb{H}^n/\Gamma' \bigcup_i \overline{V}_i$, defined by modifying the hyperbolic metric g_0 as

$$(3) \qquad g(X, Y) = g_0(X, Y) + \sum_{\gamma} (f \circ r_\gamma)^2 \, dr_\gamma(X)\, dr_\gamma(Y),$$

has strictly negative sectional curvature K, provided the sum in (3) *is taken over all locally minimizing geodesics γ to the set $\bigcup \overline{V}_i$, and r_γ is the local distance function defined by γ. For more special functions f the metric g has the following additional properties:*

(i) *If $f'(r) \cdot f(r) \geq -a \tanh(r) \cdot (1 + f(r)^2 \cosh^{-2}(r)) \cdot (1 + f(r)^2)$ for some constant $a > 0$, then the curvature of M^n is bounded in terms of a, ε_0, and $k := [\frac{n}{2}]$ by*

$$(4) \qquad -(1+a)(2+a)(1+k\varepsilon_0^2)^2 \leq K < 0.$$

(ii) *If $\int_0^{\varepsilon_0} f(r)\, dr = \infty$, then (M^n, g) is complete.*

(iii) *If $\int_0^{\varepsilon_0} f(r) \cdot \sinh(r)\, dr < \infty$, then $\operatorname{vol}(M^n, g) < \infty$.*

When picking any $\varepsilon < \varepsilon_0$, an appropriate smooth approximation f of the function

$$f_\varepsilon(r) = \begin{cases} \varepsilon/r + r/\varepsilon - 2 & \text{for } r \le \varepsilon, \\ 0 & \text{for } r \ge \varepsilon \end{cases} \tag{5}$$

satisfies all the hypotheses in this theorem, including those in (i)–(iii), provided $a = a(\varepsilon)$ is chosen sufficiently large for each $\varepsilon \in (0, \varepsilon_0)$. We thus get Theorem 0.4 in the introduction as a direct corollary to this more detailed result.

The new metric g on M can be thought of as the hyperbolic metric g_0 stretched into certain directions. The following propositions will prepare for the proof of Theorem 1.1 by establishing general curvature formulas for such metric deformations.

Proposition 1.2. *Let (M^n, g_0) be a Riemannian manifold and let $\varphi_j\colon M^n \to \mathbb{R}$, $1 \le j \le k$, be smooth functions. Then the curvature tensor R of the modified metric*

$$g(X, Y) := g_0(X, Y) + \sum_{j=1}^{k} d\varphi_j(X) \cdot d\varphi_j(Y) \tag{6}$$

on M^n is given by

$$\begin{aligned} g(R(X, Y)Z, W) &= g_0(R^0(X, Y)Z, W) \\ &\quad + \sum_{i,j=1}^{k} m^{ij}(D^2_{YZ}\varphi_i \cdot D^2_{XW}\varphi_j - D^2_{XZ}\varphi_i \cdot D^2_{YW}\varphi_j), \end{aligned} \tag{7}$$

where D and R^0 denote the covariant derivative and the curvature tensor of g_0 and $(m^{ij})_{ij}$ is the inverse of the matrix $(\delta_{ij} + g_0(d\varphi_i, d\varphi_j))_{ij}$.

Proof. Formula (7) is obtained directly from the Gauss equations for the embedding of the manifold (M^n, g) as a graph into the product $\overline{M}^{n+k}$ of (M^n, g_0) and a Euclidean factor $\mathbb{R}^k$. This isometric embedding is given by

$$\begin{aligned} F\colon M^n &\to \overline{M}^{n+k} \\ p &\mapsto (p, \Sigma_{j=1}^k \varphi_j(p) \cdot e_j), \end{aligned}$$

where $(e_j)_{j=1}^k$ denotes the standard basis of $\mathbb{R}^k$. For a tangent vector X of the manifold M^n, let $\overline{X} := dF(X)$. By writing $\langle\ ,\ \rangle_{\overline{M}}$ for the product metric on $\overline{M}^{n+k}$, the Gauss equations in their standard form are

$$\begin{aligned} g(R(X, Y)Z, W) &= \langle \overline{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W}\rangle_{\overline{M}} + \langle h(\overline{Y}, \overline{Z}), h(\overline{X}, \overline{W})\rangle_{\overline{M}} \\ &\quad - \langle h(\overline{X}, \overline{Z}), h(\overline{Y}, \overline{W})\rangle_{\overline{M}}, \end{aligned} \tag{8}$$

where $\overline{R}$ is the curvature tensor of $\overline{M}^{n+k}$, and h is the second fundamental form of M^n in $\overline{M}^{n+k}$. Note that

$$(9) \qquad \langle \overline{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W}\rangle_{\overline{M}} = g_0(R^0(X, Y)Z, W).$$

The vectors $\overline{v}_i := (-\operatorname{grad}\varphi_i, e_i)$, $i = 1, \cdots, k$, form a basis of the normal space to M^n. Let $\overline{w}_i$ be the dual basis, i.e., $g_0(\overline{w}_j, \overline{v}_i) = \delta_{ij}$. Then

$$(10) \qquad h(\overline{X}, \overline{Y}) = \sum_{i=1}^{k} \langle h(\overline{X}, \overline{Y}), \overline{v}_i\rangle_{\overline{M}}\, \overline{w}_i.$$

The covariant derivative $\overline{D}$ of $\overline{M}^{n+k}$ is of course the product of D and the flat connection on $\mathbb{R}^k$, hence

$$(11) \quad \langle h(\overline{X}, \overline{Y}), \overline{v}_i\rangle_{\overline{M}} = \langle -\overline{D}_{\overline{X}}\overline{v}_i, \overline{Y}\rangle_{\overline{M}} = g_0(D_X \operatorname{grad}\varphi_i, Y) = D^2_{XY}\varphi_i.$$

Now, combining (8)–(11), we obtain formula (7) with $m^{ij} = g_0(\overline{w}_i, \overline{w}_j)$. The latter matrix is the inverse of

$$(g_0(\overline{v}_i, \overline{v}_j))_{i,j=1}^{k} = (\delta_{ij} + g_0(d\varphi_i, d\varphi_j))_{i,j=1}^{k},$$

hence the proposition. q.e.d.

Let us now return to the curvature calculations required for Theorem 1.1. Outside the tube $U_{\varepsilon_0}(\bigcup_i \overline{V}_i) \subset \mathbb{H}^n/\Gamma'$ the metric g defined in (3) is identical with the hyperbolic metric. The geometry of the tube itself can be studied by considering the analogous situation in $\mathbb{H}^n$: let $V_j \subset \mathbb{H}^n$, $1 \le j \le k := [\frac{n}{2}]$, be mutually orthogonal, codimension-2 hyperbolic subspaces through a common point p_0 such that $\dim(V_i \cap V_j) = n-4$ for $i \ne j$. We set $r_j := \operatorname{dist}(\cdot, V_j)$ and consider the metric

$$(12) \qquad g(X, Y) = g_0(X, Y) + \sum_{j=1}^{k} (f \circ r_j)^2\, dr_j(X)\, dr_j(Y).$$

Clearly, the tube $U_{\varepsilon_0}(\bigcup_i \overline{V}_i) \subset \mathbb{H}^n/\Gamma'$ equipped with the metric g defined in Theorem 1.1 is covered by the open sets which are isometric to the corresponding pieces of the tube $U_{\varepsilon_0}(\bigcup_j V_j) \subset \mathbb{H}^n$ equipped with the metric g given by (12).

We thus have reduced the curvature calculations for Theorem 1.1 to the model situation in $\mathbb{H}^n$ described above. In order to employ Proposition 1.2, we set $\varphi_j := (\int f) \circ r_j$. Notice that

$$(13) \qquad d\varphi_j = (f \circ r_j)\, dr_j.$$

For the sake of brevity we set $f_j := f \circ r_j$, $f'_j := f' \circ r_j$, $th_j := \tanh \circ r_j$, and $v_j := \operatorname{grad} r_j$. The Weingarten map of the hypersurface $\{r_j = \text{const}\}$ will be denoted by $A_j := D \operatorname{grad} r_j = \operatorname{Hess} r_j$, and $\xi_j := \|\tilde{\xi}_j\|^{-1}\tilde{\xi}_j$ stands for the normalized Killing field associated to the rotation about the codimension-2 subspace V_j. Introducing the bilinear forms $p_j^v := g_0(\cdot, v_j) g_0(v_j, \cdot)$ and $p_j^\xi := g_0(\cdot, \xi_j) g_0(\xi_j, \cdot)$, we have

$$g = g_0 + \sum_j f_j^2 \cdot p_j^v, \tag{14}$$

$$\begin{aligned} &g_0(A_j X, Y) \\ &= th_j \cdot (g_0(X, Y) - p_j^v(X, Y)) + \left(\frac{1}{th_j} - th_j\right) \cdot p_j^\xi(X, Y), \end{aligned} \tag{15}$$

and

$$\begin{aligned} D^2_{XY}\varphi_j &= f_j \cdot D^2_{XY} r_j + f'_j \cdot (X r_j)(Y r_j) \\ &= f_j \cdot g_0(A_j X, Y) + f'_j \cdot p_j^v(X, Y) \\ &= f_j th_j \cdot g_0(A_j X, Y) - (-f'_j + th_j f_j) \cdot p_j^v(X, Y) \\ &\quad + f_j \cdot \left(\frac{1}{th_j} - th_j\right) \cdot p_j^\xi(X, Y). \end{aligned} \tag{16}$$

Under the hypothesis of Theorem 1.1 the symmetric bilinear form $D^2\varphi_j$ has signature $(n-1, 1)$, and thus it is by no means a priori clear that the sectional curvature of the new metric g is negative, even without any restriction on the number k of the intersecting subspaces. All this requires more detailed computations.

Proposition 1.3. (i) *Given any two of the mutually orthogonal, codimension-2 hyperbolic subspaces $V_i, V_j \subset \mathbb{H}^n$ introduced above, the angles between the initial vectors v_i and v_j of the minimizing geodesics γ_i and γ_j from some point q to these subspaces are given by*

$$g_0(v_i, v_j) = (1 - th_i^2) \cdot \delta_{ij} + th_i \cdot th_j. \tag{17}$$

(ii) *For the metric g on $\mathbb{H}^n \setminus \bigcup_j V_j$ defined in formula* (12), *the matrix $(m^{ij})_{ij=1}^k$ introduced in Proposition* 1.2 *is given by*

$$\begin{aligned} m^{ij} &= \frac{\delta_{ij}}{1 + f_i^2(1 - th_i^2)} \\ &\quad - \frac{1}{1+\sigma} \cdot \frac{f_i th_i}{1 + f_i^2(1 - th_i^2)} \cdot \frac{f_j th_j}{1 + f_j^2(1 - th_j^2)}, \end{aligned} \tag{18}$$

where

$$\sigma := \sum_{\nu=1}^{k} \frac{f_\nu^2 th_\nu^2}{1 + f_\nu^2(1 - th_\nu^2)}. \tag{19}$$

(iii) *The curvature tensor of the metric* g *from* (12) *can be expressed as*

$$g(R(\cdot, \cdot)\cdot, \cdot) = \frac{-1}{1+\sigma} \cdot \varrho \owedge \varrho - \sum_j c_j \cdot (p_j^v + p_j^\xi) \owedge (p_j^v + p_j^\xi), \tag{20}$$

where the coefficients c_j *and the symmetric bilinear form* ϱ *are given by*

$$c_j := \frac{(-f_j' + f_j th_j) f_j (1/th_j - th_j)}{1 + f_j^2(1 - th_j^2)} \tag{21}$$

and

$$\varrho := g_0 + \sum_j \frac{(-f_j' + f_j th_j) f_j th_j}{1 + f_j^2(1 - th_j^2)} \cdot p_j^v - \sum_j \frac{f_j^2(1 - th_j^2)}{1 + f_j^2(1 - th_j^2)} \cdot p_j^\xi. \tag{22}$$

Note that the wedge product of the symmetric bilinear forms is defined by

$$\begin{aligned} \alpha \owedge \beta(X, Y, Z, W) &:= -\alpha \owedge \beta(X \wedge Y, Z \wedge W) \\ &:= \alpha(Y, Z)\beta(X, W) - \alpha(X, Z)\beta(Y, W). \end{aligned} \tag{23}$$

Formula (20) reveals a nice *block structure* for the curvature operator of the metric g from (12). For instance, a coefficient of the curvature tensor clearly vanishes when it is defined by inserting any of the vectors ξ_j, $1 \le j \le k$, into precisely one slot of $g(R(\cdot, \cdot)\cdot, \cdot)$. This fact can also be seen directly by means of a *parity argument*: for any j there is a reflection which fixes the point q under consideration and maps ξ_j to $-\xi_j$.

There is no such obvious argument for the sign of the sectional curvature of the metric g; nevertheless, its curvature is strictly negative. In Proposition 1.4 below we shall deduce this fact from formula (20).

Proof. (i) The v_j are unit vectors by their very definition, i.e., we want to compute the cosine of the angle between v_i and v_j, whenever $i \neq j$. The two geodesics γ_i and γ_j determine a totally geodesic hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^n$ which intersects the two codimension-2 subspaces V_i and V_j perpendicularly. Thus we are in fact considering a hyperbolic quadrilateral with three 90° angles, and (17) is precisely the well-known formula from planar hyperbolic geometry for the cosine of the fourth angle in such a quadrilateral.

(ii) In view of formula (13) one gets that $g_0(d\varphi_i, d\varphi_j) = f_i f_j \cdot g_0(v_i, v_j)$. Using the expression for $g_0(v_i, v_j)$ from (17), the matrix m introduced in Proposition 1.2 can be described as

$$m = (\mathrm{Id} + D + TT^{\mathrm{tr}})^{-1}, \tag{24}$$

where D denotes the diagonal matrix with entries $f_i^2(1 - th_i^2)$, and T stands for the column vector $(f_i \cdot th_i)_{i=1}^k$. Now the summation formula for the geometric series yields

$$\begin{aligned} m &= (\mathrm{Id} + (\mathrm{Id}+D)^{-1} TT^{\mathrm{tr}})^{-1} \cdot (\mathrm{Id}+D)^{-1} \\ &= \sum_{\mu=0}^{\infty} (-1)^{\mu} \cdot ((\mathrm{Id}+D)^{-1} TT^{\mathrm{tr}})^{\mu} \cdot (\mathrm{Id}+D)^{-1} \\ &= (\mathrm{Id}+D)^{-1} - (\mathrm{Id}+D)^{-1} T \cdot \sum_{\mu=0}^{\infty} (-\sigma)^{\mu} \cdot T^{\mathrm{tr}} (\mathrm{Id}+D)^{-1} \\ &= (\mathrm{Id}+D)^{-1} - \frac{1}{1+\sigma} \cdot (\mathrm{Id}+D)^{-1} TT^{\mathrm{tr}} (\mathrm{Id}+D)^{-1}, \end{aligned} \tag{25}$$

where the scalar σ denotes the inner product $T^{\mathrm{tr}}(\mathrm{Id}+D)^{-1}T$. The claim follows upon evaluating σ in terms of the f_j and th_j.

(iii) By substituting the expression (16) for the Hessian $D^2\varphi_j$ into formula (7), the curvature tensor of the metric g turns out to be

$$\begin{aligned} g(R(\cdot,\cdot)\cdot,\cdot) = \Big(-1 + \sum_{i,j} m^{ij} f_i th_i f_j th_j\Big) \cdot g_0 \owedge g_0 \\ + \sum_i \Big(\sum_j m^{ij} f_j th_j\Big) \cdot \Big[(f_i' - f_i th_i) \cdot (g_0 \owedge p_i^v + p_i^v \owedge g_0) \\ + f_i \Big(\frac{1}{th_j} - th_j\Big) \cdot (g_0 \owedge p_i^{\xi} + p_i^{\xi} \owedge g_0)\Big] \\ + \sum_{i,j} m^{ij} \cdot \Big[-(-f_i' + f_i th_i) \cdot p_i^v + f_i \Big(\frac{1}{th_i} - th_i\Big) \cdot p_i^{\xi}\Big] \\ \owedge \Big[-(-f_j' + f_j th_j) \cdot p_j^v + f_j \Big(\frac{1}{th_j} - th_j\Big) \cdot p_j^{\xi}\Big]. \end{aligned} \tag{26}$$

Now we can use formula (18), which has been established in the previous part of the proposition, to evaluate the various coefficients in the expression (26) as follows:

$$(27)\qquad \begin{aligned}-1+\sum_{i,j} m^{ij} f_i th_i f_j th_j &= -1+\frac{1}{1+\sigma}\cdot\sum_i \frac{f_i^2 th_i^2}{1+f_i^2(1-th_i^2)}\\ &= \frac{-1}{1+\sigma},\end{aligned}$$

$$(28)\qquad (-f_i'+f_i th_i)\cdot\sum_j m^{ij} f_j th_j = \frac{1}{1+\sigma}\cdot\frac{(-f_i'+f_i th_i)f_i th_i}{1+f_i^2(1-th_i^2)},$$

$$(29)\qquad f_i\left(\frac{1}{th_i}-th_i\right)\cdot\sum_j m^{ij} f_j th_j = \frac{1}{1+\sigma}\cdot\frac{f_i^2(1-th_i^2)}{1+f_i^2(1-th_i^2)}.$$

The next two coefficients are only calculated for $i \neq j$:

$$(30)\qquad \begin{aligned}&(-f_i'+f_i th_i)m^{ij}(-f_j'+f_j th_j)\\ &\quad= \frac{-1}{1+\sigma}\cdot\frac{(-f_i'+f_i th_i)f_i th_i}{1+f_i^2(1-th_i^2)}\cdot\frac{(-f_j'+f_j th_j)f_j th_j}{1+f_j^2(1-th_j^2)},\end{aligned}$$

$$(31)\qquad \begin{aligned}&f_i\left(\frac{1}{th_i}-th_i\right)m^{ij}f_j\left(\frac{1}{th_j}-th_j\right)\\ &\quad= \frac{-1}{1+\sigma}\cdot\frac{f_i^2(1-th_i^2)}{1+f_i^2(1-th_i^2)}\cdot\frac{f_j^2(1-th_j^2)}{1+f_j^2(1-th_j^2)}.\end{aligned}$$

The last coefficient, however, will be needed for general pairs (i, j) again:

$$(32)\qquad \begin{aligned}&(-f_i'+f_i th_i)m^{ij}f_j\left(\frac{1}{th_j}-th_j\right)\\ &\quad= c_i\cdot\delta_{ij} - \frac{1}{1+\sigma}\cdot\frac{(-f_i'+f_i th_i)f_i th_i}{1+f_i^2(1-th_i^2)}\cdot\frac{f_j^2(1-th_j^2)}{1+f_j^2(1-th_j^2)}.\end{aligned}$$

Substituting (27)–(32) into (26), collecting terms appropriately, and noticing that all the bilinear forms p_j^v and p_j^ξ have rank 1, which implies that all the products $p_j^v \owedge p_j^v$ and $p_j^\xi \owedge p_j^\xi$ vanish, we get formula (20) and thereby finish the proof of Proposition 1.3.

Proposition 1.4. (i) *If $f(r) \geq 0$ and $f'(r) \leq 0$, then the metric g on $M^n = \mathbb{H}^n \setminus \bigcup_j V_j$ defined by* (12) *has strictly negative sectional curvature.*

(ii) *If, moreover,*

$$f'(r)\cdot f(r) \geq -a\tanh(r)\cdot(1+f(r)^2\cosh^{-2}(r))\cdot(1+f(r)^2)$$

for some constant $a > 0$, then the sectional curvature K of the metric g can be bounded in terms of a, ε_0, and $k = [\frac{n}{2}]$:

$$(33)\qquad -(1+a)(2+a)(1+k\varepsilon_0^2)^2 \leq K < 0.$$

Proof. (i) Recall that the bilinear forms p_j^v and p_j^ξ are represented with respect to the metric g_0 by means of the orthogonal projectors $P_j^v : X \to g_0(X, v_j)v_j$ and $P_j^\xi : X \to g_0(X, \xi_j)\xi_j$. Since the various vectors ξ_j are mutually orthogonal, it is evident that

$$0 < g_0 - \sum_j \frac{f_j^2(1 - th_j^2)}{1 + f_j^2(1 - th_j^2)} \cdot p_j^\xi \leq g_0. \tag{34}$$

The coefficients of p_j^v in (22) are nonnegative by hypothesis, and hence the bilinear form ϱ defined by (22) is positive definite. It follows that

$$\frac{-1}{1+\sigma} \cdot \varrho \owedge \varrho < 0 \quad \text{on } \Lambda^2 T_q M^n. \tag{35}$$

All the remaining terms in (20) have coefficients $-c_j \leq 0$; as $p_j^v + p_j^\xi \geq 0$, all of them are negative semidefinite on $\Lambda^2 T_q M^n$, hence the claim.

Actually our argument even shows that the *curvature operator* of the new metric g is *negative definite.*

(ii) By hypothesis, we have

$$\frac{(-f_j' + f_j th_j) f_j th_j}{1 + f_j^2(t - th_j^2)} \leq a \cdot th_j^2 + (1 + a) \cdot f_j^2 th_j^2. \tag{36}$$

Combining this estimate with formula (14) and inequality (34), we get the following bounds:

$$\begin{aligned} 0 < \varrho &\leq g_0 + \sum_j \frac{(-f_j' + f_j th_j) f_j th_j}{1 + f_j^2(1 - th_j^2)} \cdot p_j^v \\ &\leq g_0 + \sum_j ((1 + a) \cdot f_j^2 th_j^2 + a \cdot th_j^2) \cdot p_j^v \\ &\leq (1 + a + ak\varepsilon_0^2) \cdot g. \end{aligned} \tag{37}$$

Here we have made use of the bound ε_0 for the support of f and of the upper bound $k = [\frac{n}{2}]$ for the number of orthogonal hyperbolic codimension-2 subspaces V_j in $\mathbb{H}^n$ through one point p_0. As $\sigma \geq 0$, we conclude that

$$\frac{1}{1+\sigma} \cdot \varrho \owedge \varrho \leq (1 + a + ak\varepsilon_0^2)^2 \cdot g \owedge g. \tag{38}$$

In order to estimate the sum of the remaining terms in (20) as well, observe that, by hypothesis, their coefficients satisfy the inequality

$$0 \leq c_j \leq (1 + a)(1 - th_j^2)(1 + f_j^2). \tag{39}$$

All k reflections $\mathrm{Id} - 2P_j^\xi : T_qM^n \to T_qM^n$ are isometric with respect to both metrics g_0 and g. Moreover, they preserve

$$\sum_j c_j(p_j^v + p_j^\xi) \owedge (p_j^v + p_j^\xi).$$

We shall view this expression as a symmetric bilinear form on $\Lambda^2 T_qM^n$. The eigenvectors for all its nonzero eigenvalues are of the form $X \wedge \xi_i$ simply because all its eigenspaces are invariant under the involutions $\mathrm{Id} - 2P_j^\xi$. Using (39), one estimates

$$\tag{40} \begin{aligned} &\sum_j c_j \cdot (p_j^v + p_j^\xi) \owedge (p_j^v + p_j^\xi)(X, \xi_i, \xi_i, X) \\ &\quad = c_i \cdot p_i^v(X, X) \le a(1 + f_i^2) \cdot p_i^v(X, X) \le g(X, X). \end{aligned}$$

Hence

$$\tag{41} \sum_j c_j \cdot (p_j^v + p_j^\xi) \owedge (p_j^v + p_j^\xi) \le (1 + a) \cdot g \owedge g,$$

and the claim follows upon combining (38) and (41). q.e.d.

Using only the completeness of the hyperbolic metric g_0, which is due to our assumptions, the next statement will be evident from inspecting formulas (3) and (12) for the new metric g.

Proposition 1.5. *If $f(r) \ge 0$ and $\int_0^{\varepsilon_0} f(r)\,dr = \infty$, then both the spaces $(\mathbb{H}^n/\Gamma' \setminus \bigcup_i \overline{V}_i, g)$ and $(\mathbb{H}^n \setminus \bigcup_j V_j, g)$ are complete.*

It is a consequence of the arguments given when setting things up for Proposition 1.3 that our next result will actually finish the proof of Theorem 1.1.

Proposition 1.6. (i) *The Lebesgue measures μ_{g_0} and μ_g associated to the hyperbolic metric g_0 and the new metric g respectively satisfy*:

$$\tag{42} \mu_g = \left(1 + \sum_{i=1}^k \sum_{j_1 < \cdots < j_i} s_{j_1 \cdots j_i} \cdot f_{j_1}^2 \cdots f_{j_i}^2\right)^{1/2} \cdot \mu_{g_0},$$

where $s_{j_1 \cdots j_i} := \det(g_0(v_{j_\mu}, v_{j_\nu}))_{\mu,\nu=1}^i$. In particular, $s_{j_1 \cdots j_i} \le 1$, and hence

$$\tag{43} \mu_g \le \prod_{j=1}^k (1 + f_j^2)^{1/2} \cdot \mu_{g_0}.$$

(ii) *If $f(r) \ge 0$ and $\int_0^{\varepsilon_0} f(r) \cdot \sinh(r)\,dr < \infty$, then $\mathrm{vol}_g(W \setminus \bigcup_j V_j) \le \infty$ for any bounded, measurable subset $W \subset (\mathbb{H}^n, g_0)$.*

Proof. (i) By formula (14) we get $\mu_g = \det(\mathrm{Id} + \sum_j f_j^2 \cdot P_j^v)^{1/2} \cdot \mu_{g_0}$. It remains to evaluate the determinant. For short we set $\omega_j := g_0(v_j, \cdot)$ and $\hat{\omega}_j := f_j^2 \omega_j$. Now it is straightforward to calculate that

$$\begin{aligned}
\mu_g &= \det\left(\mathrm{Id} + \sum_{j=1}^k v_j \otimes \hat{\omega}_j\right)^{1/2} \cdot \mu_{g_0} \\
&= \left[1 + \sum_{i=1}^k \sum_{j_1 < \cdots < j_i} \det\left(\sum_{\mu=1}^i v_{j_\mu} \otimes \hat{\omega}_{j_\mu}\right)\right]^{1/2} \cdot \mu_{g_0} \qquad (44)\\
&= \left[1 + \sum_{i=1}^k \sum_{j_1 < \cdots < j_i} f_{j_1}^2 \cdots f_{j_i}^2 \cdot \det\left(\sum_{\mu=1}^i v_{j_\mu} \otimes \omega_{j_\mu}\right)\right]^{1/2} \cdot \mu_{g_0}.
\end{aligned}$$

The claim follows upon identifying each determinant $\det(\sum_{\mu=1}^i v_{j_\mu} \otimes \omega_{j_\mu})$ with the i-dimensional Gram determinant $s_{j_1 \cdots j_i} = \det(g_0(v_{j_\mu}, v_{j_\nu}))_{\mu,\nu=1}^i$, which is always ≤ 1.

(ii) Because of inequality (43), all we need to show is that $\int_W \prod_{j=1}^k (1 + f_j^2)^{1/2} \mathrm{vol}_{g_0}$ converges. When n is odd, let us introduce in addition the distance function $r_0 := \mathrm{dist}_{g_0}(\cdot, V_0)$ to the $2k$-dimensional, totally geodesic, hyperbolic subspace $V_0 := \exp_{p_0}(\bigoplus_{j=1}^k (T_{p_0} V_j)^\perp)$ and its g_0-gradient field $v_0 := \mathrm{grad}\, r_0$. We obtain the following proper maps:

$$(45) \qquad \begin{aligned}
\psi_n &:= (r_j)_{j=1}^k : (\mathbb{H}^n, g_0) \to \mathbb{R}^k \quad &&\text{for } n = 2k, \\
\psi_n &:= (r_j)_{j=0}^k : (\mathbb{H}^n, g_0) \to \mathbb{R}^{k+1} \quad &&\text{for } n = 2k+1.
\end{aligned}$$

The set W is bounded with respect to the hyperbolic metric g_0 and thus is contained in some "polydisc"

$$(46) \qquad \begin{aligned}
Z_{b_1 \cdots b_k} &:= \{p \in \mathbb{H}^n \mid r_j < b_j,\ 1 \leq j \leq k\} \quad &&\text{for } n = 2k, \\
Z_{b_0 \cdots b_k} &:= \{p \in \mathbb{H}^n \mid r_j < b_j,\ 0 \leq j \leq k\} \quad &&\text{for } n = 2k+1.
\end{aligned}$$

The integral over such a larger set can be estimated by using the coarea formula. We show the computation for the case where $n = 2k + 1$; the other case is similar.

For any point $\hat{r} = (\hat{r}_j)_{j=0}^k \in Q := (0, b_0) \times \cdots \times (0, b_k)$ the fiber $\psi^{-1}(\hat{r})$ is a torus $T^k(\hat{r})$. Let $g_{\hat{\tau}}$ denote the restriction of the metric g_0 to this torus. Then

$$(47) \qquad \mathrm{vol}(T^k(\hat{r}), g_{\hat{\tau}}) = (2\pi)^k \cdot \prod_{j=1}^k \sinh \hat{r}_j,$$

and the coarea formula in its standard form becomes:

$$\begin{aligned}(48)\qquad &\int_{Z_{b_0\cdots b_k}\setminus\bigcup_j V_j} h(p)\cdot\|\Lambda^{k+1}(d\psi_{|p})\|\,d\mu_{g_0}(p)\\ &\qquad = \int_Q\left[\int_{T^k(\hat r)} h(p)\,d\mu_{g_{\hat r}}(p)\right]d\mu(\hat r),\end{aligned}$$

where $h\colon Z_{b_0\cdots b_k}\setminus\bigcup_j V_j \to \mathbb{R}$ may be any nonnegative, locally Lebesgue integrable function. By the standard compactness argument there is a constant $C_{b_0\cdots b_k} > 0$ such that $C^{-1}_{b_0\cdots b_k} \le \|\Lambda^{k+1}(d\psi_{|p})\|$ for any $p \in Z_{b_0\cdots b_k}$. Combining this inequality with formulas (43), (47), and (48), we estimate

$$\begin{aligned}&\operatorname{vol}\left(Z_{b_0\cdots b_k}\setminus\bigcup_j V_j,\, g_{\hat r}\right)\\ &\le \int_{Z_{b_0\cdots b_k}\setminus\bigcup_j V_j}\left[\prod_{j=1}^k(1+f_j^2)^{1/2}\right]d\mu_{g_0}(p)\\ (49)\qquad &\le C_{b_0\cdots b_k}\cdot\int_{Z_{b_0\cdots b_k}\setminus\bigcup_j V_j}\left[\prod_{j=1}^k(1+f_j^2)^{1/2}\cdot\|\Lambda^{k+1}(d\psi_{|p})\|\right]d\mu_{g_0}(p)\\ &= (2\pi)^k\cdot C_{b_0\cdots b_k}\cdot\int_Q\left[\prod_{j=1}^k(1+f_j^2)^{1/2}\cdot\prod_{j=1}^k\sinh\hat r_j\right]d\mu(\hat r)\\ &= (2\pi)^k\cdot C_{b_0\cdots b_k}\cdot b_0\cdot\prod_{j=1}^k\left[\int_0^{b_j}\sqrt{1+f(r)^2}\cdot\sinh r\,dr\right].\end{aligned}$$

By hypothesis, each factor of the last product of (49) is bounded. Therefore all the polydiscs and hence also all g_0-bounded, measurable sets W have finite volume with respect to the new metric g as well, despite the fact that $g \ge g_0$ everywhere.

2. Group theoretic arguments

The main purpose of this section is to prove Theorem 0.5 and to explain how it can be applied to all the Lanner groups for the case of dimension 4.

Proof of Theorem 0.5. Since Γ' is supposed to be torsionfree, its action on $\mathbb{H}^n$ is fixed point free, and the quotient space $\mathbb{H}^n/\Gamma'$ is a manifold. Note that $\mathbb{H}^n/\Gamma'$ is compact as required in Theorem 0.4 simply because by assumption $\mathbb{H}^n/\Gamma$ is compact and $[\Gamma : \Gamma'] < \infty$. It is also clear that the projection $\overline{\mathrm{pr}}\colon \mathbb{H}^n \to \mathbb{H}^n/\Gamma'$ maps the codimension-2 subspaces $V_i = \operatorname{Fix}\rho_i \subset \mathbb{H}^n$, $1 \le i \le k$, onto connected, totally geodesically immersed submanifolds $\overline{V}_i$.

An additional argument is required in order to guarantee that the $\overline{V}_i$ are actually compact and embedded. The basic ingredient into this is the hypothesis that $\Gamma' \triangleleft \Gamma$ is normal, so that Γ/Γ' acts as a group of isometries on $\mathbb{H}^n/\Gamma'$ by means of covering transformations. In particular, the cosets $\overline{\rho}_i$ of ρ_i act isometrically on $\mathbb{H}^n/\Gamma'$. Hence their fixed point sets $\operatorname{Fix}\overline{\rho}_i$ are compact, totally geodesically embedded submanifolds in $\mathbb{H}^n/\Gamma'$ which may—and in general do—consist of several components. Clearly $\overline{V}_i = \overline{\mathrm{pr}}(\operatorname{Fix}\rho_i)$ is one component of $\operatorname{Fix}\overline{\rho}_i$ for each i.

Note that $\bigcap_{i=1}^k \overline{V}_i$ contains the image $\overline{\mathrm{pr}}(\bigcap_{i=1}^k V_i)$ and hence is not empty. So it remains to show that the $\overline{V}_i$ can only have normal crossings. This is in fact true at any point $\overline{p} \in \bigcap_{i=1}^k \operatorname{Fix}\overline{\rho}_i$, as can be seen from the Jordan decomposition of the commuting rotations $d\overline{\rho}_{i|\overline{p}}$. q.e.d.

It has already been explained in the introduction how to apply this theorem to certain cocompact arithmetic groups and to prove Theorem 0.3 as the final result. The next goal is to handle the *discrete, cocompact reflection groups*, which act on $\mathbb{H}^4$ in such a way that a *simplex* can be chosen as the fundamental domain. There are precisely five such groups (cf. [3], [14], [15]), the so-called *Lanner groups*, which are classified in terms of their Coxeter diagrams listed in Figure 3.

In the cases (a)–(c), the fundamental domain is an orthoscheme rather than just a simplex. Anyway, all five groups in question give rise to nice

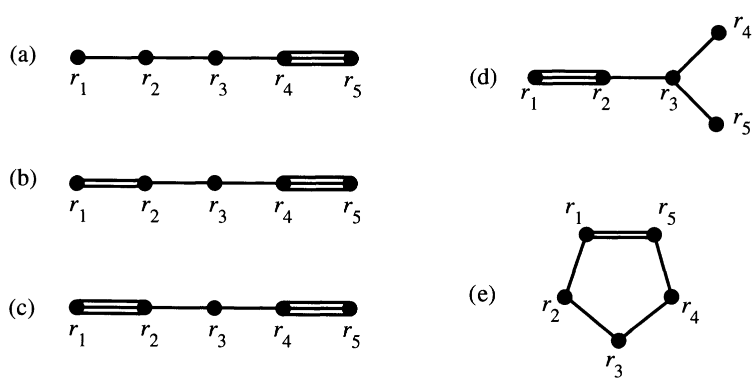


FIGURE 3

tessalations of $\mathbb{H}^4$. Inspecting their Coxeter matrices, all the Lanner groups are moreover recognized to be *arithmetic*, and again we can pick torsionfree, normal subgroups of finite index, making use of the congruence subgroups.

In Proposition 2.1 below we shall exhibit appropriate rotations ρ_1 and ρ_2 for all five Lanner groups, and so in principle each of them is useful in our context. It is example (c) which, when worked out in more detail in the next section, provides the various families of surfaces in compact hyperbolic 4-manifolds necessary to deduce Theorem 0.1 from Theorem 0.2, as announced in the introduction.

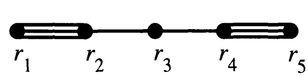
Proposition 2.1. *Each of the five Lanner groups* Γ *on* $\mathbb{H}^4$ *contains two commuting rotations* ρ_1 *and* ρ_2 *such that* $\Sigma_1 = \operatorname{Fix}\rho_1$ *and* $\Sigma_2 = \operatorname{Fix}\rho_2$ *are hyperbolic subspaces of codimension* 2 *in* $\mathbb{H}^4$, $\operatorname{Fix}\rho_1 \cap \operatorname{Fix}\rho_2$ *consists of a single point, and* ρ_1 *has order* > 2.

Proof. (a)–(d): Set $\rho_1 := r_1 r_2$ and $\rho_2 := r_5 r_4$, and the claimed properties follow directly, since in either case the diagram implies that both the hyperplane reflections r_1 and r_2 commute with both the reflections r_4 and r_5.

(e) In this case the rotations ρ_1 and ρ_2 have to be constructed differently. We shall look for them in the subgroup $\Gamma_{1345} = \langle r_1, r_3, r_4, r_5\rangle$; this is the stabilizer of the vertex v_2 of the fundamental simplex, and it is itself a Coxeter group with diagram • = •— • —•. Moreover, it is the automorphism group of the standard 4-dimensional cube in $T_{v_2}\mathbb{H}_4$. Writing this as the cartesian product of two squares, we find a subgroup of rotations in Γ_{1345}, which is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$. Evidently, ρ_1 and ρ_2 can be taken as the generators of this subgroup.

3. Surfaces in the hyperbolic 120-cell space

This hyperbolic 4-manifold has been constructed by M. W. Davis in 1985 [8]. The construction is based on the *Coxeter group* Γ associated to the diagram



This group induces a *tesselation* of $\mathbb{H}^4$ by hyperbolic 120-cells with dihedral angles equal to $\frac{2\pi}{5}$; each of these 120-cells consists of 14400 fundamental orthoschemes. This tesselation has already been known to Coxeter [6], [7]. The point in Davis' paper was to exhibit a *torsionfree* subgroup

$\Gamma' < \Gamma$ which has precisely such a *hyperbolic* 120-*cell* as its fundamental domain. So one gets the hyperbolic 120-cell space as the quotient $\mathbb{H}^4/\Gamma'$.

Important for our purposes is that Γ' comes as a *normal* subgroup in Γ so that Theorem 0.5 can be applied with $\rho_1 = r_1 r_2$ and $\rho_2 = r_5 r_4$. This yields two connected, totally geodesically embedded surfaces $\overline{\Sigma}_1, \overline{\Sigma}_2 \subset \mathbb{H}^4/\Gamma'$ with simple normal crossings.

In this section the main purpose is to determine the *genus*, the *conformal type*, and the *intersection patter* of $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$. In order to obtain Theorem 0.1 for all the graph manifolds listed in Figure 1, we shall also be interested in the corresponding data in certain finite covering spaces of $\mathbb{H}^4/\Gamma'$. For this aim we need a *detailed* description of the hyperbolic 120-cell space. All the required information is collected in §§3.1–3.3, where we recover the hyperbolic 4-manifold in question by means of an *arithmetic* construction, which does not seem to be in the literature so far.

3.1. The standard Coxeter model for Γ. The diagram $\bullet \equiv \bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet \equiv \bullet$ of Γ is defined to be a brief form for the *Coxeter matrix*

$$g = (g(e_i, e_j))_{i,j=1}^{5} = \begin{pmatrix} 2 & -\tau & & & \\ -\tau & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -\tau \\ & & & -\tau & 2 \end{pmatrix}, \tag{50}$$

where $\tau = 2 \cdot \cos\frac{\pi}{5} = \frac{1}{2}(1+\sqrt{5})$, i.e., τ is a root of $\tau^2 - \tau - 1$. This matrix defines a nondegenerate, symmetric bilinear form g on the free $\mathbb{Z}[\tau]$-module E generated by $e_1, \cdots, e_5$. The extension of g to $E_{\mathbb{R}} = E \otimes_{\mathbb{Z}[\tau]} \mathbb{R}$ has signature (4, 1), and so the *reflections*

$$r_i \colon e_j \mapsto e_j - \frac{2 \cdot g(e_i, e_j)}{g(e_i, e_i)} \cdot e_i, \qquad 1 \le i, j \le 5, \tag{51}$$

act on the hyperbolic 4-space $\mathbb{H}^4 = \{[v] \in \mathbb{P}E_{\mathbb{R}} \mid g(v, v) < 0\}$. The Coxeter group $\Gamma := \langle r_1, \cdots, r_5 \rangle$ is clearly a subgroup of $O(E, g) \subset \mathrm{Gl}(5, \mathbb{Z}[\tau])$ and thus arithmetic. Note that $\mathbb{Z}[\tau]$ is the ring of integral elements in $\mathbb{Q}(\sqrt{5})$. Set

(52)

$$\begin{aligned}
v_1 &:= e_1 + 2(1+2\tau)e_2 + (4+7\tau)e_3 + 2(3+5\tau)e_4 + (5+8\tau)e_5, \\
v_2 &:= \tau e_1 + 2e_2 + (2+\tau)e_3 + 2(1+\tau)e_4 + (1+2\tau)e_5, \\
v_3 &:= (1+2\tau)e_1 + 2(1+\tau)e_2 + (2+\tau)e_3 + 2(1+\tau)e_4 + (1+2\tau)e_5, \\
v_4 &:= (1+2\tau)e_1 + 2(1+\tau)e_2 + (2+\tau)e_3 + 2e_4 + \tau e_5, \\
v_5 &:= (5+8\tau)e_1 + 2(3+5\tau)e_2 + (4+7\tau)e_3 + 2(1+2\tau)e_4 + e_5.
\end{aligned}$$

It is straightforward to verify that $g(v_i, e_j) = 0$ for $i \neq j$ and that $g(v_i, v_j) < 0$ for all pairs (i, j). So the points $[v_1], \cdots, [v_5]$ are the vertices of the simplex:

$$P = \{[v] \in \mathbb{P}E_{\mathbb{R}} \mid g(v, e_j) \cdot g(v, v_5) > 0, \ 1 \leq j \leq 5\} \subset 3 \tag{53}$$

Actually, this simplex is an orthoscheme and is the fundamental domain for the action of Γ on $\mathbb{H}^4$.

Evidently, the subgroup $\Gamma_{1234} = \langle r_1, r_2, r_3, r_4 \rangle$ is the *stabilizer* in Γ of the vertex $[v_5]$ of P. Γ_{1234} is a *spherical Coxeter group*; its diagram is • ≡ •—•—• , and so the star of $[v_5]$, i.e., the set $\overline{\Gamma_{1234} \cdot P}$, is recognized as a hyperbolic 120-cell. It has dihedral angles $\frac{2\pi}{5}$, as we have deleted a triple line in the diagram of Γ when passing to Γ_{1234}. $\mathbb{H}^4$ is tesselated into such hyperbolic 120-cells. The dual tesselation of $\mathbb{H}^4$ also consists of hyperbolic 120-cells; this time their centers are the points $\Gamma \cdot \{[v_1]\}$. The reason for this self-duality is the *symmetry* of the diagram • ≡ •— • —• ≡ • of Γ, which can be expressed in a more concrete way in terms of the involution $\vartheta \in O(E, g) \backslash \Gamma$ given by

$$\vartheta : e_j \mapsto e_{6-j}, \qquad 1 \leq j \leq 5. \tag{54}$$

Later on, we shall also make use of the elements ε_{123} and ε_{345} given by

$$\varepsilon_{123} = \begin{pmatrix} -1 & & & 1+2\tau & \\ & -1 & & 2(1+\tau) & \\ & & -1 & 2+\tau & \\ & & & 1 & \\ & & & 0 & 1 \end{pmatrix}, \tag{55}$$

$$\varepsilon_{345} = \begin{pmatrix} 1 & 0 & & & \\ & 1 & & & \\ & 2+\tau & -1 & & \\ & 2(1+\tau) & & -1 & \\ & 1+2\tau & & & -1 \end{pmatrix}. \tag{56}$$

They act as $-\mathrm{Id}$ on $\mathrm{span}\{e_1, e_2, e_3\}$ and as $+\mathrm{Id}$ on its orthogonal complement, respectively as $-\mathrm{Id}$ on $\mathrm{span}\{e_3, e_4, e_5\}$ and as $+\mathrm{Id}$ on the orthogonal complement. Both elements ε_{123} and ε_{345} lie in Γ and not only in $O(E, g)$. More precisely, ε_{123} generates the center of $\Gamma_{123} := \langle r_1, r_2, r_3\rangle$; the explanation is that Γ_{123} (its Coxeter graph is $\bullet \equiv \bullet\!\!-\!\!\!-\!\!\bullet$) is the symmetry group of the icosahedron, whose center is known to be generated by the antipodal map. In fact, $\Gamma_{123}/Center \cong A_5$. Similarly, ε_{345} generates the center of $\Gamma_{345} := \langle r_3, r_4, r_5\rangle$.

There is also an antipodal map $\varepsilon_{1234} \in \Gamma_{1234}$. On $\mathbb{H}^4$ this element acts as the reflection at the point $[v_5]$; it is given by the matrix

$$\varepsilon_{1234} = \begin{pmatrix} -1 & & & & 2(5+8\tau) \\ & -1 & & & 4(3+5\tau) \\ & & -1 & & 2(4+7\tau) \\ & & & -1 & 4(1+2\tau) \\ & & & & 1 \end{pmatrix}. \tag{57}$$

3.2. Natural quotient maps. Mapping τ to $\frac{1}{2}(1+\sqrt{5})$, the Gaussian integers $\mathbb{Z}[\tau]$ can be considered as a subring of $\mathbb{Z}_5[\sqrt{5}]$, where $\mathbb{Z}_5$ stands for the 5-adic integers. We are going to employ the valuation $\nu_5\colon \mathbb{Z}_5[\sqrt{5}] \to \mathbb{R}_+$, $z \mapsto 5^{-\operatorname{ord}_5(z)}$ and the corresponding ring homomorphism $\Phi\colon \mathbb{Z}_5[\sqrt{5}] \to \mathbb{F}_5$ onto the residue field. Note that $\Phi(\tau) = 3$. The following is an analogue to Minkowski's theorem:

Proposition 3.1. *The kernels of the induced homomorphisms*

$$\Phi_n\colon \mathrm{Gl}(n, \mathbb{Z}_5[\sqrt{5}]) \to \mathrm{Gl}(n, \mathbb{F}_5), \qquad n \in \mathbb{N}, \tag{58}$$

are torsionfree.

Proof. Suppose, conversely, that there exists some torsion element $x \in \ker\Phi_n$. Without loss of generality, we may assume that $x \neq 1$, but $x^p = 1$ for some prime number p. Clearly x can be written in the form

$$x = 1 + 5^{\xi}\cdot a \quad \text{with } \xi \in \tfrac{1}{2}\cdot\mathbb{N},\ a \in \mathrm{gl}(n, \mathbb{Z}_5[\sqrt{5}]). \tag{59}$$

Increasing ξ if necessary, we may also assume that

$$\nu_5(a) := \max\{\nu_5(a_{ij}) \mid 1 \le i, j \le n\} = 1. \tag{60}$$

Case $p \neq 5$: Calculating modulo $5^{2\xi}$, we get $1 = x^p \equiv 1 + p\cdot 5^{\xi}\cdot a$, a formula which contradicts condition (60).

Case $p = 5$: Here, taking into account that $5^{5\xi} \equiv 0 \mod 5^{2\xi+1}$ by (59), the full binomial formula yields a similar contradiction:

$$\begin{aligned} 1 = x^5 &= 1 + 5^{\xi+1}\cdot a + 5^{2\xi+1}\cdot 2a^2 + 5^{3\xi+1}\cdot 2a^3 + 5^{4\xi+1}\cdot a^4 + 5^{5\xi}\cdot a^5 \\ &\equiv 1 + 5^{\xi+1}\cdot a \mod 5^{2\xi+1}. \quad \text{q.e.d.} \end{aligned}$$

Let us now restrict Φ_5 to the image of the inclusion $O(E, g) \hookrightarrow O(E_{\mathbb{Z}_5[\sqrt{5}]}, g) \subset \mathrm{Gl}(E_{\mathbb{Z}_5[\sqrt{5}]})$. This defines a homomorphism

$$\varphi\colon O(E, g) \hookrightarrow \mathrm{Gl}(E_{\mathbb{F}_5}), \tag{61}$$

whose image is of course a subgroup of $O(E_{\mathbb{F}_5}, \bar{\bar{g}}) \cap (\{\pm 1\} \times \mathrm{Sl}(E_{\mathbb{F}_5}))$, where $\bar{\bar{g}}$ is the induced bilinear form on $E_{\mathbb{F}_5}$.

Corollary 3.2. (i) $\ker(\varphi\colon O(E, g) \to \mathrm{Gl}(E_{\mathbb{F}_5}))$ *and thus also* $\Gamma'' := \Gamma \cap \ker\varphi$ *are torsionfree, normal subgroups of finite index in* $O(E, g)$ *and* Γ, *respectively.*

(ii) φ *maps the stabilizers* $\Gamma_{1\cdots\hat{j}\cdots 5}$ *of the vertices* $[v_j]$ *injectively onto their images* $\bar{\bar{\Gamma}}_{1\cdots\hat{j}\cdots 5} \subset O(E_{\mathbb{F}_5}, \bar{\bar{g}})$.

The point behind the second claim is that the stabilizers in question are pure torsion groups, and so this claim also follows directly from Proposition 3.1. The induced bilinear form $\bar{\bar{g}}$ on the quotient $E_{\mathbb{F}_5}$, however, is degenerate. More precisely, we have (cf. formulas (62) and (63) below)

Lemma 3.3. $\mathrm{rad}(\bar{\bar{g}}) = \mathbb{F}_5 \cdot \bar{\bar{v}}_5$, *where* $\bar{\bar{v}}_5 = \varphi(v_5)$ *denotes the image of the vertex* v_5 *under the* $\mathrm{mod}(\sqrt{5})$*-reduction homomorphism.*

So it is natural to consider the quotient map $\psi\colon E_{\mathbb{F}_5} \to E_{\mathbb{F}_5}/\mathrm{rad}(\bar{\bar{g}})$. For the sake of brevity, we set $\bar{E} := E_{\mathbb{F}_5}/\mathrm{rad}(\bar{\bar{g}})$; the induced bilinear form on the quotient space $\bar{E}$ will be denoted by $\bar{g}$. In the same spirit, we set $\bar{\bar{\Gamma}} := \varphi(\Gamma)$ and $\bar{\Gamma} := \psi(\bar{\bar{\Gamma}}) = \psi \circ \varphi(\Gamma)$; more generally, a double overbar will always indicate objects related to $E_{\mathbb{F}_5}$ and a single overbar will indicate corresponding objects for $\bar{E}$.

For most subsequent calculations concerning the homomorphisms φ and $\pi := \psi \circ \varphi$, it will be convenient to work with respect to the basis

$$\begin{aligned}
\bar{\bar{b}}_1 &:= 2\cdot\bar{\bar{e}}_1 + 4\cdot\bar{\bar{e}}_2 + 0 + \bar{\bar{e}}_4 + 4\cdot\bar{\bar{e}}_5, \\
\bar{\bar{b}}_2 &:= 4\cdot\bar{\bar{e}}_1 + \bar{\bar{e}}_2 + 0 + 4\cdot\bar{\bar{e}}_4 + 2\cdot\bar{\bar{e}}_5, \\
\bar{\bar{b}}_3 &:= 3\cdot\bar{\bar{e}}_1 + 2\cdot\bar{\bar{e}}_2 + 0 + 2\cdot\bar{\bar{e}}_4 + 3\cdot\bar{\bar{e}}_5, \\
\bar{\bar{b}}_4 &:= 0 + 0 + \bar{\bar{e}}_3 + 0 + 0, \\
\bar{\bar{b}}_5 &:= 4\cdot\bar{\bar{e}}_1 + \bar{\bar{e}}_2 + 0 + 4\cdot\bar{\bar{e}}_4 + \bar{\bar{e}}_5.
\end{aligned} \tag{62}$$

With respect to this basis, the bilinear form $\bar{\bar{g}}$, the reflections $\bar{\bar{r}}_1, \cdots, \bar{\bar{r}}_5$, and the endomorphisms $\bar{\bar{\vartheta}}$, $\bar{\bar{\varepsilon}}_{123}$, $\bar{\bar{\varepsilon}}_{345}$, and $\bar{\bar{\varepsilon}}_{1234}$ are represented by the matrices:

$$
\begin{aligned}
&\overline{\overline{g}}^{B} = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 0 & 1 & \\ & & 1 & 2 & \\ & & & & 0 \end{pmatrix}, \qquad
&&\overline{\overline{r}}_1^{B} = \begin{pmatrix} 4 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 3 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
&\overline{\overline{r}}_2^{B} = \begin{pmatrix} 4 & & & 1 & \\ & 1 & & 0 & \\ 2 & & 1 & 4 & \\ & & & 1 & \\ 2 & 0 & 0 & 4 & 1 \end{pmatrix}, \qquad
&&\overline{\overline{r}}_3^{B} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & 4 & 4 & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
(63) \quad &\overline{\overline{r}}_4^{B} = \begin{pmatrix} 1 & & & 0 & \\ & 4 & & 1 & \\ & 2 & 1 & 4 & \\ & & & 1 & \\ 0 & 3 & 0 & 1 & 1 \end{pmatrix}, \qquad
&&\overline{\overline{r}}_5^{B} = \begin{pmatrix} 1 & & & & \\ & 4 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix}, \\
&\overline{\overline{\vartheta}}^{B} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \qquad
&&\overline{\overline{\varepsilon}}_{123}^{B} = \begin{pmatrix} 4 & & & & \\ & 1 & & & \\ & & 4 & & \\ & & & 4 & \\ 3 & 0 & 1 & 0 & 1 \end{pmatrix}, \\
&\overline{\overline{\varepsilon}}_{345}^{B} = \begin{pmatrix} 1 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 4 & \\ 0 & 2 & 4 & 0 & 1 \end{pmatrix}, \qquad
&&\overline{\overline{\varepsilon}}_{1234}^{B} = \begin{pmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 4 & \\ 3 & 4 & 1 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

In particular, the proof of Lemma 3.3 is a mere inspection of the matrix $\overline{\overline{g}}^{B}$. It is also worthwhile noticing that $\overline{\overline{b}}_5 = \overline{\overline{v}}_5$, and hence, with respect to this basis, the homomorphism ψ is given by passing to the upper left 4×4 submatrix. Later in §3.4, where we calculate the properties of the surfaces $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ with respect to coverings, the basis $\overline{\overline{b}}_1, \cdots, \overline{\overline{b}}_5$ will turn out to have some further nice properties.

Let us now proceed with

Proposition 3.4. (i) *The homomorphism $\pi = \psi \circ \varphi : O(E, g) \to O(\overline{E}, \overline{g})$ maps the symmetry group Γ_{1234} of the* 120*-cell isomorphically onto its image $\overline{\Gamma}_{1234}$.*

(ii) *In fact $\overline{\Gamma}_{1234} = \overline{\Gamma}$, and this is the index-*2 *subgroup of $O(\overline{E}, \overline{g})$ generated by the reflections at the vectors $\overline{v} \in \overline{E}$ with $\overline{g}(\overline{v}, \overline{v}) = 2$ (quadratic* *nonresidues*). *Note that* $O(\overline{E}, \overline{g}) \cong O_+(4, \mathbb{F}_5)$, *where "+" indicates the standard quadratic form on* $\mathbb{F}_5^4$.

(iii) *The homomorphism* $\zeta := (\pi_{|\Gamma_{1234}})^{-1} \circ \pi_{|\Gamma}: \Gamma \to \Gamma_{1234}$ *is a left inverse to the embedding* $\Gamma_{1234} \hookrightarrow \Gamma$, *and as such it is characterized by the identity*

$$\zeta(r_5) = \varepsilon_{123} \cdot \varepsilon_{1234}, \tag{64}$$

where ε_{123} *and* ε_{1234} *are the antipodal maps of the dodecahedron and the* 120-*cell, respectively* (*cf. formulas* (55) *and* (57)).

(iv) ζ *is unique up to twisting with an* $\langle\varepsilon_{1234}\rangle$-*valued character. More concretely, the only other left inverse to the embedding* $\Gamma_{1234} \hookrightarrow \Gamma$ *is the homomorphism* $\check{\zeta}: \Gamma \to \Gamma_{1234}$ *given by* $\check{\zeta}(r_5) = \varepsilon_{123}$.

Proof. (i) Clearly Γ_{1234} is itself a discrete reflection group; it is characterized by the Coxeter diagram • ≡ •— • —•, and its standard model is a group acting on the $\mathbb{Z}[\tau]$-submodule $E^{(4)} = \operatorname{span}\{e_1, \cdots, e_4\} \subset E$, which is of course equipped with the metric $g^{(4)} = g_{|E^{(4)} \times E^{(4)}}$. It is the group $\Gamma^{(4)} \subset O(E^{(4)}, g^{(4)})$ generated by the reflections

$$r_i^{(4)}: e_j \mapsto e_j - \frac{2 \cdot g(e_i, e_j)}{g(e_i, e_i)} \cdot e_i, \qquad 1 \le i, j \le 4.$$

The identification with Γ_{1234} is done by means of the isomorphism

$$\begin{aligned} \iota: \Gamma^{(4)} &\to \Gamma_{1234} \subset \Gamma \\ r_i^{(4)} &\mapsto r_i, \qquad 1 \le i \le 4. \end{aligned} \tag{65}$$

Notice that the composition $E^{(4)} \hookrightarrow E \xrightarrow{\varphi} E_{\mathbb{F}_5} \xrightarrow{\psi} \overline{E}$ maps e_j to $\overline{e}_j = \overline{\overline{e}}_j + \operatorname{rad}(\overline{\overline{g}})$, $1 \le j \le 4$, and therefore it coincides with the $\operatorname{mod}(\sqrt{5})$-reduction homomorphism $\varphi^{(4)}: E^{(4)} \to E^{(4)}_{\mathbb{F}_5}$, up to the canonical identification of $E^{(4)}_{\mathbb{F}_5}$ and $\overline{E}$ by means of $E^{(4)}_{\mathbb{F}_5} \hookrightarrow E_{\mathbb{F}_5} \xrightarrow{\psi} \overline{E}$, of course. So there is a commutative diagram:

$$\begin{array}{lcl} \Gamma^{(4)} \subset O(E^{(4)}, g^{(4)}) & \xrightarrow{\varphi^{(4)}} & O(E^{(4)}_{\mathbb{F}_5}, \overline{\overline{g}}^{(4)}) \cong O(\overline{E}, \overline{g}) \\ \iota\downarrow & & \hfill \cup \\ \Gamma_{1234} \subset \Gamma \xrightarrow{\varphi} \overline{\overline{\Gamma}} & \xrightarrow{\psi} & \hfill \overline{\Gamma} \end{array} \tag{66}$$

As a matter of fact $g^{(4)}$ is positive definite, and $\Gamma^{(4)}$ and $O(E^{(4)}, g^{(4)})$ are finite groups. Using Proposition 3.1 in the same way as in Corollary 3.2(ii), we therefore see that $\varphi^{(4)}: O(E^{(4)}, g^{(4)}) \to O(\overline{E}, \overline{g})$ is injective, and by the above diagram $(\psi \circ \varphi)_{|\Gamma_{1234}}$ must be injective as well.

(ii) Looking at the matrices (63), we see that $\det \overline{g}^B = 1$, i.e., $\det \overline{g} \in (\mathbb{F}_5^*)^2$, and therefore $(\overline{E}, \overline{g})$ is isomorphic to $\mathbb{F}_5^4$ equipped with the standard quadratic form. This identifies the orthogonal group in question with $O_+(4; \mathbb{F}_5)$, a group of order $2 \times 4 \times 30 \times 120 = 28800$. Note that $|\Gamma_{1234}| = 14400$ and that $\overline{\Gamma}$ is generated by the reflections $\overline{r}_1, \cdots, \overline{r}_5$ at the vectors $\overline{e}_1, \cdots, \overline{e}_5 \in \overline{E}$ with $\overline{g}(\overline{e}_i, \overline{e}_i) = 2$ for $1 \leq i \leq 5$. Now, from the general theory of Chevalley groups it can be read off that $[O(\overline{E}, \overline{g}) : \overline{\Gamma}] = 2$, hence $\overline{\Gamma}_{1234} = \overline{\Gamma} \subset O(\overline{E}, \overline{g})$.

In this particular case, however, the results also follows from an easy explicit calculation based on the character of $\chi: O(\overline{E}, \overline{g}) \to O(\overline{E}, \overline{g})/\overline{\Gamma}_{1234} \cong \mathbb{Z}/2\mathbb{Z}$; evidently $\chi(\overline{r}_i) = 1$ for $1 \leq i \leq 4$, and moreover

$$\chi(\overline{r}_5) = \chi(\overline{r}_4) \cdot \chi(\overline{r}_5) \cdot \chi((\overline{r}_4\overline{r}_5)^5) = (\chi(\overline{r}_4\overline{r}_5))^6 = 1. \tag{67}$$

(iii) In view of (i) and (ii), $(\pi_{|\Gamma_{1234}})^{-1} \circ \pi_{|\Gamma}$ is a well-defined homomorphism and, in fact, is a left inverse to the embedding $\Gamma_{1234} \hookrightarrow \Gamma$. As such, it is determined by the image of r_5. Formula (64) is obtained by comparing the upper left 4×4 blocks in the matrices given in (63).

(iv) By (i) it is equivalent to classify all homomorphisms $\overline{\zeta}: \Gamma \to \overline{\Gamma}_{1234}$ such that $\overline{\zeta}_{|\Gamma_{1234}} = \pi_{|\Gamma_{1234}}$. Such a homomorphism is evidently determined by its image $\overline{\zeta}(r_5)$. This element must be an involution which commutes with $\overline{r}_1, \overline{r}_2, \overline{r}_3$; in particular, it must map all three subspaces $\mathbb{F}_5 \cdot e_1$, $\mathbb{F}_5 \cdot e_2$, $\mathbb{F}_5 \cdot e_3$ into themselves. Using the bilinear form $\overline{g}$, we see that either $\overline{\zeta}(r_5)$ or $\overline{\varepsilon}_{1234} \cdot \overline{\zeta}(r_5) = -\overline{\zeta}(r_5)$ acts as the identity on $\operatorname{span}\{e_1, e_2, e_3\}$. There are only two possibilities for such an element: the identity map $\mathrm{Id}_{\overline{E}}$, and the reflection $\overline{\varepsilon}_{123} \cdot \overline{\varepsilon}_{1234}$ (cf. (64)). It remains to rule out the case where $\overline{\zeta}(r_5) \in \{\mathrm{Id}_{\overline{E}}, \overline{\varepsilon}_{1234}\}$. The latter are central elements in $O(\overline{E}, \overline{g})$, and therefore we obtain

$$\overline{r}_4 = \overline{r}_4^5 = (\overline{r}_4 \cdot \overline{\zeta}(r_5))^5 \cdot \overline{\zeta}(r_5) = \overline{\zeta}((r_4 r_5)^5) \cdot \overline{\zeta}(r_5) = \overline{\zeta}(r_5) \in \{\mathrm{Id}_{\overline{E}}, \overline{\varepsilon}_{1234}\}, \tag{68}$$

contradicting the fact that r_4 does not lie in the center of Γ_{1234}.

Remark. The reflection r_5 fixes precisely one face of the hyperbolic 120-cell. The map ε_{1234} fixes the line through the center of this particular dodecahedron and the center of the 120-cell, acting as $-\mathrm{Id}$ on the normal space of this line. Hence:

(i) $r_5 \cdot \varepsilon_{123}$ has a fixed point in $\mathbb{H}^4$, the center of the dodecahedron mentioned above, and

(ii) $r_5 \cdot \varepsilon_{123} \cdot \varepsilon_{1234}$ is a transvection on $\mathbb{H}^4$ which identifies two opposite dodecahedral faces of the hyperbolic 120-cell (cf. [8]).

3.3. The hyperbolic 120-cell space and its 625-fold covering. In this subsection we are going to explain how to recover in our setup the hyperbolic 120-cell space constructed by Davis.

Theorem 3.5. (i) $\Gamma' = \Gamma \cap \ker \pi$ *is torsionfree, and so* $\mathbb{H}^4/\Gamma'$ *is the oriented, compact, hyperbolic* 4*-manifold obtained by gluing opposite faces of the hyperbolic* 120*-cell with dihedral angles* $\frac{2\pi}{5}$ *by means of transvections.*

(ii) $\Gamma' \lhd \Gamma$ *is the unique, torsionfree, normal subgroup which acts on* $\mathbb{H}^4$ *with the hyperbolic* 120*-cell as its fundamental domain. In particular,* $\mathbb{H}^4/\Gamma'$ *must be the hyperbolic* 4*-manifold constructed by Davis.*

(iii) *The Euler characteristic* $\chi(\mathbb{H}^4/\Gamma')$ *is* 26. *In fact,* $\mathbb{H}^4/\Gamma'$ *comes with a cell decomposition into one vertex,* 60 *edges,* 144 *regular* 5*-gons,* 60 *dodecahedra, and one cell of dimension* 4 (*the hyperbolic* 120*-cell*).

(iv) $\psi\colon \mathbb{H}^4/\Gamma'' \to \mathbb{H}^4/\Gamma'$ *is a normal covering of degree* 625 *with deck transformation group* $\Gamma'/\Gamma'' \cong (\mathbb{F}_5^4, +)$.

Remark. Evidently, $\overline{\Gamma} \cong \Gamma/\Gamma'$ acts isometrically on $\mathbb{H}^4/\Gamma'$. It can be characterized as the group of those isometries of $\mathbb{H}^4/\Gamma'$, which preserve the cell decomposition from 3.5(iii). However, even the underlying triangulation into 14400 orthoschemes is preserved by a bigger group; the involution $\vartheta \in O(E, g)$ defined in (54) induces an isometry of $\mathbb{H}^4/\Gamma'$ which maps the fundamental orthoscheme described in (53) into itself, switching vertices. This is clear by the following group theoretic considerations: obviously, ϑ lies in the normalizer $N_{O(E,g)}(\Gamma)$, and so $\langle\vartheta\rangle \cdot \Gamma \subset O^+(E, g) = \{\gamma \in O(E, g) \mid g(\gamma \cdot v_5, v_5) < 0\}$ is an extension of degree 2 of Γ, which acts faithfully on $\mathbb{H}^4$. By Proposition 3.4 and formula (63) π maps $\langle\vartheta\rangle \cdot \Gamma$ onto $\overline{\vartheta} \cdot \overline{\Gamma} = O(\overline{E}, \overline{g})$, and hence $\ker \pi \cap (\langle\vartheta\rangle \cdot \Gamma) = \Gamma'$. Altogether:

$(\langle\vartheta\rangle \cdot \Gamma)/\Gamma$ *is the group of those isometries of* $\mathbb{H}^4/\Gamma'$ *which preserve the triangulation into 14400 orthoschemes. As a group, it is isomorphic to* $O_+(4; \mathbb{F}_5)$ *and thus it is an index-*2 *extension of* Γ/Γ'. *In fact* $\langle\vartheta\rangle \cdot \Gamma = O^+(E, g)$.

Proof of Theorem 3.5. (i) and (ii): Let us begin with the uniqueness statement claimed in (ii). Observe that for any torsionfree, normal subgroup $\Gamma' \lhd \Gamma$ which acts on $\mathbb{H}^4$ with the hyperbolic 120-cell $\overline{\Gamma_{1234} \cdot P}$ as its fundamental domain, we have the following split exact sequence:

$$
\begin{array}{ccccccccc}
 & & & & & & & & \\
1 & \to & \Gamma' & \to & \Gamma & \to & \Gamma/\Gamma' & \to & 1 \\
 & & & & \uparrow & \nearrow_{\cong} & & & \\
 & & & & \Gamma_{1234} & & & &
\end{array} \tag{69}
$$

Thus Γ' can be considered as the kernel of a left inverse to the embedding

$\Gamma_{1234} \hookrightarrow \Gamma$, and by Proposition 3.4(iv) there are only two possibilities:

$$\text{(i)}\quad \Gamma' = \ker \check{\zeta} \quad \text{and} \quad \text{(ii)}\quad \Gamma' = \ker \zeta .$$

As explained in the remark below the proof of Proposition 3.4, $\ker \check{\zeta}$ is not torsionfree, i.e., the quotient $\mathbb{H}^4/\ker \check{\zeta}$ is not a manifold.

As a result the only candidate for a torsionfree, normal subgroup $\Gamma' \triangleleft \Gamma$ which acts on $\mathbb{H}^4$ with the required fundamental domain is the group $\ker \zeta$ from Proposition 3.4. But such a normal subgroup or more precisely such a homomorphism $\zeta \colon \Gamma \to \Gamma_{1234}$ is actually the piece of data which Davis constructs in his paper, and so we can refer to his proof that $\ker \zeta$ is torsionfree and read off the existence of a hyperbolic 120-cell space as described in (i) from Davis' paper. All the properties mentioned in part (i) of the theorem also follow directly; we can refer to Davis' paper or to our exposition in the previous sections, in particular to Proposition 3.4 and the remark below.

For the sake of completeness, let us recall that the basic idea in proving Γ' to be torsionfree is to show that

$$(70)\qquad \Gamma' \cap \gamma \cdot \Gamma_{1\cdots\hat{j}\cdots 5} \cdot \gamma^{-1} = 1 \quad \forall \gamma \in \Gamma,\ \forall 1 \le j \le 5 .$$

This suffices, since $\Gamma' \subset O^+(E, g)$ acts faithfully on $\mathbb{H}^4$, so that the torsion elements have fixed points on the walls of the associated triangulation of $\mathbb{H}^4$. Since $\Gamma' \triangleleft \Gamma$ is normal, it is even enough to show that

$$(71)\qquad \Gamma' \cap \Gamma_{1\cdots\hat{j}\cdots 5} = 1 \quad \text{for } 1 \le j \le 5 .$$

The case of Γ_{1234} and, by symmetry, also the case of $\Gamma_{2345} = \vartheta \cdot \Gamma_{1234} \cdot \vartheta^{-1}$ have been handled in Proposition 3.4(i).

The remaining three cases are also easy to handle in our setup. By Corollary 3.2(ii) it is of course sufficient to show that ψ maps the three subgroups $\overline{\overline{\Gamma}}_{1235}$, $\overline{\overline{\Gamma}}_{1245}$, and $\overline{\overline{\Gamma}}_{1345} = \overline{\overline{\vartheta}} \cdot \overline{\overline{\Gamma}}_{1235} \cdot \overline{\overline{\vartheta}}^{-1}$ injectively onto their respective images. Using part (iv), which will be established independently below, a possible kernel of $\psi \mid \overline{\overline{\Gamma}}_{1235}$ must lie in any 5-Sylow subgroup of $\overline{\overline{\Gamma}}_{1235} = \overline{\overline{\Gamma}}_{123} \times \mathbb{Z}/2\mathbb{Z}$, and hence in $\overline{\overline{\Gamma}}_{123} \subset \overline{\overline{\Gamma}}_{1234}$. This reduces the case of $\overline{\overline{\Gamma}}_{1235}$ and similarly that of $\overline{\overline{\Gamma}}_{1345}$ to the two a priori solved cases. The same idea works for $\overline{\overline{\Gamma}}_{1245}$; it has a unique 5-Sylow subgroup, the group

$(\mathbb{Z}/5\mathbb{Z})^2$ generated by

$$\bar{\bar{\rho}}_1^B = \bar{\bar{r}}_1^B \cdot \bar{\bar{r}}_2^B = \begin{pmatrix} 1 & & & 4 & \\ & 1 & & 0 & \\ 2 & & 1 & 4 & \\ & & & 1 & \\ 4 & 0 & 0 & 2 & 1 \end{pmatrix} \tag{72}$$

and

$$\bar{\bar{\rho}}_2^B = \bar{\bar{r}}_5^B \cdot \bar{\bar{r}}_4^B = \begin{pmatrix} 1 & & & 0 & \\ & 1 & & 4 & \\ & 2 & 1 & 4 & \\ & & & 1 & \\ 0 & 1 & 0 & 3 & 1 \end{pmatrix}. \tag{73}$$

On this subgroup the homomorphism ψ, which is given by restricting to the upper left 4×4 submatrices, is clearly injective.

(iii) The value of the Euler characteristic $\chi(\mathbb{H}^4/\Gamma')$ is a direct consequence of the data of the cell decomposition. Such a cell decomposition in turn follows directly from the construction. As far as the counting is concerned, notice that the k-dimensional cells correspond bijectively to the points in the orbits of the vertices $[v_{5-k}]$ of the fundamental orthoscheme P under $\bar{\Gamma}$, $0 \leq k \leq 4$, and

$$\begin{aligned} &[\bar{\Gamma} : \bar{\Gamma}_{1234}] = [\bar{\Gamma}_{2345}] = 1, \\ &[\bar{\Gamma} : \bar{\Gamma}_{1235}] = [\bar{\Gamma} : \bar{\Gamma}_{1345}] = \frac{14400}{2 \cdot 120} = 60, \\ &[\bar{\Gamma} : \bar{\Gamma}_{1245}] = \frac{14400}{10 \cdot 10} = 144. \end{aligned} \tag{74}$$

(iv) Clearly $N := \{\bar{\bar{\gamma}} = \mathrm{Id} + \bar{\bar{v}}_5 \otimes \alpha \mid \alpha \in \Lambda^1 E_{\mathbb{F}_5}$ such that $\alpha(\bar{\bar{v}}_5) = 0\}$ is a subgroup in $\ker \psi$. Another obvious subgroup of $\ker \psi$ in $\mathbb{F}_5^*$ which is embedded into $O(E_{\mathbb{F}_5}, \bar{\bar{g}})$ is

$$A := \{\bar{\bar{\gamma}} \in \mathrm{Gl}(E_{\mathbb{F}_5}) \mid \bar{\bar{\gamma}}_{|\operatorname{span}\{b_1, \cdots, b_4\}} = \mathrm{Id},\ \bar{\bar{\gamma}}(\bar{\bar{v}}_5) = a \cdot \bar{\bar{v}}_5,\ a \in \mathbb{F}_5^*\}. \tag{75}$$

It is easy to see that in fact $\ker \psi = A \rtimes N$. However, the image $\varphi(O(E, g))$ is smaller than $\ker \psi$, since $\det \bar{\bar{\gamma}}$ must be ± 1 for any $\bar{\bar{\gamma}} \in \varphi(O(E, g))$. The intersection of this image with $\bar{\bar{\Gamma}} = \varphi(\Gamma)$ is even smaller:

$$\bar{\bar{\Gamma}} \cap \ker \psi \subset N, \tag{76}$$

simply because the generators $\bar{\bar{r}}_1, \cdots, \bar{\bar{r}}_5$ of $\bar{\bar{\Gamma}}$ fix $\bar{\bar{v}}_5$ (cf. formulas (63)).

Of course, N is the group of order 625 isomorphic to $(\mathbb{F}_5^4, +)$ mentioned in the theorem. Thus we just need to establish equality in (76). To this end we compute:

$$\bar{\bar{\sigma}}_1^B := \bar{\bar{r}}_2^B \cdot \bar{\bar{\varepsilon}}_{345}^B = \begin{pmatrix} 4 & & & 4 & \\ & 4 & & 0 & \\ 2 & & 4 & 1 & \\ & & & 4 & \\ 2 & 2 & 4 & 1 & 1 \end{pmatrix}, \tag{77}$$

$$\bar{\bar{\sigma}}_2^B = \bar{\bar{r}}_4^B \cdot \bar{\bar{\varepsilon}}_{123}^B = \begin{pmatrix} 4 & & & 0 & \\ & 4 & & 4 & \\ & 2 & 4 & 1 & \\ & & & 4 & \\ 3 & 3 & 1 & 4 & 1 \end{pmatrix}, \tag{78}$$

and therefore the elements represented by the matrices

(79)

$$\bar{\bar{\rho}}_1^B \cdot (\bar{\bar{\sigma}}_1^B)^{-4} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 3 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\bar{\bar{\sigma}}_1^B)^{-4} \cdot \bar{\bar{\rho}}_1^B = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 3 & 0 & 0 & 2 & 1 \end{pmatrix},$$

$$\bar{\bar{\rho}}_2^B \cdot (\bar{\bar{\sigma}}_2^B)^{-4} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & 2 & 0 & 0 & 1 \end{pmatrix}, \quad (\bar{\bar{\sigma}}_2^B)^{-4} \cdot \bar{\bar{\rho}}_2^B = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & 2 & 0 & 3 & 1 \end{pmatrix}$$

generate a subgroup $N_{125} \subset N$ of index 5. The required fourth generator of N can be obtained by conjugating the element $(\bar{\bar{\sigma}}_1)^4(\bar{\bar{\rho}}_1)^{-1}(\bar{\bar{\sigma}}_1)^{-4}\bar{\bar{\rho}}_1$, which is the same as $(\bar{\bar{\rho}}_2^{-1}(\bar{\bar{\sigma}}_2)^4\bar{\bar{\rho}}_2(\bar{\bar{\sigma}}_2)^{-4}$, with $\bar{\bar{r}}_3^B$. This finishes the proof of (iv) by establishing the isomorphism $N \cong \Gamma'/\Gamma''$. q.e.d.

In the next subsection, we shall come back to the subgroup $N_{125} \subset N$, and make use of the fact that

$$\bar{\bar{\Gamma}}_{1245} \subset N_{\bar{\bar{\Gamma}}}(N_{125}), \tag{80}$$

which is easy to verify by means of formulas (63) and (79).

3.4. Surfaces in $\mathbb{H}^4/\Gamma''$, $\mathbb{H}^4/\Gamma'$, or some intermediate covering. Recall that $\rho_1 = r_1 r_2$ and $\rho_2 = r_5 r_4$ are two commuting rotations by $\frac{2\pi}{5}$. Their fixed point sets $\Sigma_1 = \operatorname{Fix}\rho_1$ and $\Sigma_2 = \operatorname{Fix}\rho_2$ are 2-dimensional hyperbolic subspaces in $\mathbb{H}^4$, which intersect each other perpendicularly at the point $[v_3]$. The involution ϑ interchanges Σ_1 and Σ_2.

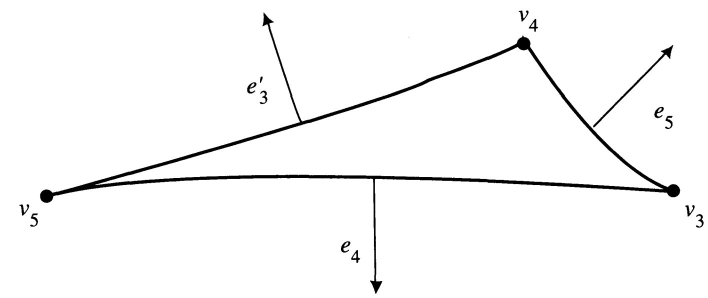


FIGURE 4

Lemma 3.6. *The faces* $P_{345} := [v_3][v_4][v_5]$ *and* $P_{321} := [v_3][v_2][v_1]$ *of the fundamental orthoscheme* P *lie on* Σ_1 *and* Σ_2, *respectively. The interior angles in these triangles are* $\frac{\pi}{5}$, $\frac{\pi}{2}$, *and* $\frac{\pi}{10}$.

Proof. Since the two triangles in question are interchanged by the isometry ϑ, it is sufficient to consider the triangle P_{345}. The edges $[v_3][v_4]$, $[v_4][v_5]$, and $[v_5][v_3]$ lie on the lines $\mathrm{Fix}(r_1, r_2, r_5)$, $\mathrm{Fix}(r_1, r_2, r_3)$, and $\mathrm{Fix}(r_1, r_2, r_4)$. The hyperplane reflections $r_v\colon x \mapsto x-2(x, v)/g(v, v)\cdot v$, which fix such a line and map Σ_1 into itself, are given by

$$r_5 = r_{e_5}, r_{e_3'} \quad \text{and} \quad r_4 = r_{e_4}, \tag{81}$$

where $e_3' := (1+2\tau)\cdot e_1 + (2+2\tau)\cdot e_2 + (2+\tau)\cdot e_3$ is a vector in the intersection $\mathrm{span}\{e_1, e_2, e_3\} \cap \mathrm{span}\{e_1, e_2\}^\perp$. The size of the various angles can now be read off from the scalar products

$$\begin{aligned} &g(e_4, e_4) = 2, && g(e_4, e_5) = -\tau, \\ &g(e_5, e_5) = 2, && g(e_3', e_5) = 0, \\ &g(e_3', e_3') = 2\cdot(2+\tau), && g(e_3', e_4) = -(2+\tau), \end{aligned} \tag{82}$$

using the identities $\tau = 2\cdot\cos\frac{\pi}{5}$ and $\sqrt{2+\tau} = 2\cdot\cos\frac{\pi}{10}$. q.e.d.

The reflection $r_{e_3'}$ does not lie in Γ. However, ε_{123} does, and its action on Σ_1 coincides with the action of $r_{e_3'}$. Thus the subgroup $G_1 := \langle r_5, r_4, \varepsilon_{123}\rangle \subset \Gamma$ acts as a discrete reflection group on Σ_1, having the triangle P_{345} as its fundamental domain. By formulas (77), (78), and Lemma 3.6, both the elliptic isometries $r_5\varepsilon_{123}$ and $r_5 r_{e_3'}$ have order 2, and both the elliptic isometries $r_4\varepsilon_{123}$ and $r_4 r_{e_3'}$ have order 10. Hence the action of G_1 on Σ_1 is *faithful*. Since P_{345} is a fundamental domain, the embedding $\Sigma_1 \hookrightarrow \mathbb{H}^4$ is a simplicial map with respect to the triangulations induced by G_1 and Γ, respectively. Therefore

$$\mathrm{Stab}_\Gamma(\Sigma_1) := \{\gamma \in \Gamma \mid \gamma(\Sigma_1) = \Sigma_1\} = G_1 \times \langle r_1, r_2\rangle, \tag{83}$$

where

$$\langle r_1, r_2\rangle = \{\gamma \in \Gamma \mid \gamma([v]) = [v] \text{ for all } [v] \in \Sigma_1\} := \mathrm{Fix}_\Gamma(\Sigma_1).$$

Because of the involution ϑ, there are corresponding results for the surface Σ_2, and the above considerations yield:

Proposition 3.7. (i) $\mathrm{Stab}_\Gamma(\Sigma_1)/\mathrm{Fix}_\Gamma(\Sigma_1) = N_\Gamma(\langle r_1, r_2\rangle)/\langle r_1, r_2\rangle$ *acts as a reflection group on* Σ_1, *having the triangle* $P_{345} = [v_3][v_4][v_5]$ *as a fundamental domain. This action is represented faithfully by the Coxeter group* $G_1 = \langle r_5, r_4, \varepsilon_{123}\rangle$, *whose diagram is:*

$$\underset{r_5}{\bullet} \overset{5}{\text{---}} \underset{r_4}{\bullet} \overset{10}{\text{---}} \underset{\varepsilon_{123}}{\bullet}$$

(ii) *Similarly,* $\mathrm{Stab}_\Gamma(\Sigma_2)/\mathrm{Fix}_\Gamma(\Sigma_2) = N_\Gamma\langle r_4, r_5\rangle/\langle r_4, r_5\rangle$ *acts as a discrete reflection group on* Σ_2, *having the triangle* $P_{321} = [v_3][v_2][v_1]$ *as a fundamental domain. This action is represented faithfully by the Coxeter group* $G_2 = \langle r_1, r_2, \varepsilon_{345}\rangle$; *its diagram is*

$$\underset{r_1}{\bullet} \overset{5}{\text{---}} \underset{r_2}{\bullet} \overset{10}{\text{---}} \underset{\varepsilon_{345}}{\bullet}$$

Remark. It is of course always true that the triangulation of $\mathbb{H}^4$ induced by Γ restricts to triangulations of the walls Σ_1 and Σ_2. However, in general, these triangulations are not associated with reflection groups on the surfaces. One may get several nonisometric, top-dimensional simplices, and the (dihedral) angles may even be irrational. Such an example arises for instance when starting out from the Coxeter diagram $\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \equiv \bullet$ rather than $\bullet \equiv \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \equiv \bullet$; the antipodal map ε_{123} does not lie in the subgroup $\langle r_1, r_2, r_3\rangle$, which in this context is the tetrahedral group, and indeed, the triangle $P_{345} = [v_3][v_4][v_5]$ on $\Sigma_1 = \mathrm{Fix}(r_1, r_2)$ has angles $\frac{\pi}{5}$, $\frac{\pi}{2}$, and $\arccos\sqrt{\frac{3}{8}} \approx 0.29\cdot\pi$, i.e., it has one irrational angle at the vertex $[v_5]$.

Return to the hyperbolic 120-cell space, and recall that by Theorem 0.2 the images $\overline{\Sigma}_1 = \overline{\mathrm{pr}}(\Sigma_1)$ and $\overline{\Sigma}_2 = \overline{\mathrm{pr}}(\Sigma_2)$ under the projection $\mathbb{H}^4 \to \mathbb{H}^4/\Gamma'$ are compact, totally geodesically embedded surfaces, which intersect each other in simple normal crossings. The next proposition will therefore establish Theorem 0.1 for the end depicted in Figure 1(b).

Proposition 3.8. $\overline{\Sigma}_1$ *and* $\overline{\Sigma}_2$ *are orientable surfaces of genus* 2 *in the hyperbolic* 120-*cell space* $\mathbb{H}^4/\Gamma'$. *Their isometry groups* $\overline{G}_1 = \pi(G_1)$ *and* $\overline{G}_2 = \pi(G_2)$ *are isomorphic to the dihedral group* D_{10}, *and their intersection* $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$ *consists of precisely two points. It is the orbit of* $\overline{\mathrm{pr}}(\Sigma_1 \cap \Sigma_2)$

under the isometry $\overline{\sigma}_1^5 = \overline{\sigma}_2^5$ *of order* 2, *which generates the center of* $\overline{G}_1$ *as well as the center of* $\overline{G}_2$.

Proof. Notice that the isometries $\rho_2 = r_5 r_4$ and $\sigma_2 = r_4 \varepsilon_{123}$ (resp. $\rho_1 = r_1 r_2$ and $\sigma_1 = r_2 \varepsilon_{345}$) act on Σ_1 (resp. Σ_2) in an orientation-preserving way. We set

$$(84) \qquad G_1^0 := \langle \rho_2, \sigma_2 \rangle, \qquad G_2^0 := \langle \rho_1, \sigma_1 \rangle,$$

which are the index-2 subgroups of G_1 and G_2 obtained by intersecting with $\ker(\det)$. In group theoretic terms the orientability of $\mathbb{H}^4/\Gamma'$ means that Γ' is a subgroup of $\ker(\det) \cap O^+(E, g)$; thus $\Gamma' \cap G_i^0 = \Gamma' \cap G_i$ for $i = 1, 2$. Now Proposition 3.7 implies that

$$(85) \qquad \overline{\Sigma}_i \equiv \overline{\mathrm{pr}}(\Sigma_i) = \Sigma_i/(\Gamma' \cap G_i^0) \quad \text{for } i = 1, 2,$$

hence the *orientability* of the images $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ in $\mathbb{H}^4/\Gamma'$. Their *genus* will be obtained by computing the Euler characteristic from the Gauss-Bonnet Theorem. The required angle information is supplied by Lemma 3.6:

$$(86) \qquad \begin{aligned} -2\pi \cdot \chi(\overline{\Sigma}_1) &= \#\overline{G}_1 \cdot \mathrm{Area}(P_{345}) \\ &= \#\overline{G}_1 \cdot \left(\pi - \frac{\pi}{2} - \frac{\pi}{5} - \frac{\pi}{10}\right) = \#\overline{G}_1 \cdot \frac{\pi}{5}. \end{aligned}$$

Similarly $-2\pi \cdot \chi(\overline{\Sigma}_2) = \#\overline{G}_2 \cdot \frac{\pi}{5}$. As the homomorphism $\psi : \overline{\overline{\Gamma}} \to \overline{\Gamma}$ is given by passing to the upper left 4×4 submatrix, it is a direct consequence of formulas (79) that $\overline{\rho}_1 = \overline{\sigma}_1^4$ and $\overline{\rho}_2 = \overline{\sigma}_2^4$. Hence, computing the order of the $\overline{\sigma}_i$ from the matrices given in (77) and (78), we get that

$$(87) \qquad \overline{G}_i^0 = \langle \overline{\sigma}_i \rangle \cong \mathbb{Z}/10\mathbb{Z} \qquad \text{for } i = 1, 2.$$

Substituting this into (86) yields

$$(88) \qquad \begin{aligned} \chi(\overline{\Sigma}_i) &= -\frac{1}{2\pi} \cdot \#\overline{G}_i^0 \cdot [\overline{G}_i : \overline{G}_i^0] \cdot \frac{\pi}{5} \\ &= -\frac{1}{2\pi} \cdot 10 \cdot 2 \cdot \frac{\pi}{5} = -2, \end{aligned}$$

as required.

In view of (87) the groups $\overline{G}_i$ are *identified* with the *dihedral group* D_{10} by verifying the relations $\overline{r}_1 \cdot \overline{\sigma}_1 \cdot \overline{r}_1^{-1} = \overline{\sigma}_1^{-1}$ and $\overline{r}_5 \cdot \overline{\sigma}_2 \cdot \overline{r}_5^{-1} = \overline{\sigma}_2^{-1}$. For this purpose we use (63), (77), and (78), to compute

$$(89) \quad (\overline{\overline{r}}{}_1^B \cdot \overline{\overline{\sigma}}{}_1^B)^2 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 0 & 0 & 3 & 1 \end{pmatrix}, \quad (\overline{\overline{r}}{}_5^B \cdot \overline{\overline{\sigma}}{}_2^B)^2 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & 4 & 0 & 2 & 1 \end{pmatrix}.$$

The *intersection* $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$ is evidently contained in $\overline{\Gamma}\cdot\overline{\mathrm{pr}}([v_3])$, and more precisely, it is the set $(\overline{G}_1 \times \langle \overline{r}_1, \overline{r}_2\rangle \cap \overline{G}_2 \times \langle \overline{r}_4, \overline{r}_5\rangle)\cdot\overline{\mathrm{pr}}([v_3])$. We can easily calculate

$$\text{(90)}\quad \overline{\sigma}_1^5 = \overline{\sigma}_2^5, \qquad \overline{G}_1 \times \langle \overline{r}_1, \overline{r}_2\rangle \cap \overline{G}_2 \times \langle \overline{r}_4, \overline{r}_5\rangle = \langle \overline{r}_1, \overline{r}_2, \overline{r}_4, \overline{r}_5\rangle \times \langle \overline{\sigma}_1^5\rangle .$$

Since $\langle \overline{r}_1, \overline{r}_2, \overline{r}_4, \overline{r}_5\rangle$ is the stabilizer of $\overline{\mathrm{pr}}([v_3])$ in $\overline{\Gamma}$, we see that $\langle \overline{\sigma}_1^5\rangle = \langle \overline{\sigma}_2^5\rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts faithfully and simply transitive on $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$, hence the claim.

Remarks. (i) Using the matrices from (63), it is not hard to identify the stabilizer of the intersection point $\overline{\mathrm{pr}}([v_3])$ as follows:

$$\text{(91)}\quad \begin{aligned} H &:= \{\overline{\gamma} \in O(\overline{E}, \overline{g}) \mid \overline{\gamma} \circ \overline{\mathrm{pr}}([v_3]) = \overline{\mathrm{pr}}([v_3])\} \\ &= \langle \overline{r}_1, \overline{r}_2, \overline{r}_4, \overline{r}_5, \overline{\vartheta}\rangle \\ &= \{\overline{\gamma} \in O(\overline{E}, \overline{g}) \mid \overline{\gamma}(\overline{b}_3) = \overline{b}_3\} \cong O_+(2; \mathbb{F}_5) \rtimes \mathbb{F}_5^2 . \end{aligned}$$

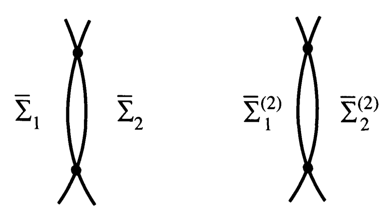
The normalizer of this group $H \subset O(\overline{E}, \overline{g})$ is a centro-affine group:

$$\text{(92)}\quad \begin{aligned} N_{O(\overline{E}, \overline{g})}(H) &= \{\overline{\gamma} \in O(\overline{E}, \overline{g}) \mid \overline{\gamma}(\overline{b}_3) \in \mathbb{F}_5 \cdot \overline{b}_3\} \\ &\cong (\mathbb{F}_5^* \times O_+(2; \mathbb{F}_5)) \rtimes \mathbb{F}_5^2 . \end{aligned}$$

Notice that $N_{O(\overline{E}, \overline{g})}(H)/H = \langle \overline{\eta}\rangle$, where $\overline{\eta} \in O(\overline{E}, \overline{g})$ is the element of order 4 given by the matrix

$$\text{(93)}\quad \overline{\eta}^B = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & 4 \\ & & 0 & 3 \end{pmatrix} .$$

In particular, $(\overline{\eta})^2 = \overline{\sigma}_1^5 \cdot \overline{r}_1 \cdot \overline{r}_5$ interchanges the two points of intersection of $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$, whereas the element $\overline{\eta}$ itself maps both $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ onto different components of $\mathrm{Fix}\,\overline{\rho}_1$ and $\mathrm{Fix}\,\overline{\rho}_2$, possibly interchanging the indices, as $\overline{\vartheta}$ does. Our reasoning also shows that *each* $\overline{\Sigma}_i$ *is precisely one out of two isometric connected components of* $\mathrm{Fix}\,\overline{\rho}_i$. The total intersection pattern is:



(ii) Observe that $\mathrm{Iso}(\overline{\Sigma}_i) \cong D_{10}$ contains a unique 5-Sylow subgroup, which is of course cyclic of order 5. Conversely, there is—up to conformal equivalence—*precisely one Riemann surface of genus* 2 *which admits an automorphism* ρ *of order* 5:

$$\Sigma = \{[z_0 : z_1 : z_2] \mid z_2^5 = (z_1 - z_0) \cdot z_0^2 z_1^2\} \subset \mathbb{C}\mathrm{P}^2 . \tag{94}$$

For a proof one considers the branched normal covering $\Sigma \to \Sigma/\langle \rho \rangle$. The Hurwitz theorem enforces that $\Sigma/\langle\rho\rangle \approx \mathbb{C}\mathrm{P}^1$ and that there are precisely three branch points over, say, 0, 1, and ∞; thus Σ can be reconstructed from the homomorphism $\pi_1(\Sigma \backslash \{\textit{branch points}\}) \to \mathbb{Z}/5\mathbb{Z}$ into the deck transformation group. In particular, it is sufficient to know the images a_0, a_1, and a_∞—all of them are nonzero—of the simple loops around the branch points. Of course $a_0 + a_1 + a_\infty = 0$. Up to permuting 0, 1, and ∞ and up to an automorphism of $\mathbb{Z}/5\mathbb{Z}$, there is only one solution: $a_0 = a_\infty = 2$, $a_1 = 1$, hence formula (94).

This choice singles out the branch point $[1 : 1 : 0]$, whereas the other two branch points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ can still be interchanged by means of the involution $[z_0 : z_1 : z_2] \mapsto [z_1 : z_0 : -z_2]$. When identifying Σ and $\overline{\Sigma}_i$ the latter two branch points must therefore correspond to the two points in the intersection $\overline{\Sigma}_1 \cap \overline{\Sigma}_2$.

Finally, we are going to investigate the preimages of $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ in some covering spaces. By Theorem 3.5 the projection $\overline{\mathrm{pr}}\colon \mathbb{H}^4 \to \mathbb{H}^4/\Gamma'$ factors over some 625-sheeted normal covering space:

$$\begin{tikzcd} \mathbb{H}^4 \arrow[r, "\overline{\overline{\mathrm{pr}}}"] \arrow[dr, "\overline{\mathrm{pr}}"'] & \mathbb{H}^4/\Gamma'' \arrow[d, "\psi"', "625:1"] \\ & \mathbb{H}^4/\Gamma' \end{tikzcd} \tag{95}$$

The group of covering transformations Γ'/Γ'' is the group $N \cong (\mathbb{F}_5^4, +)$. Our next result explains Figure 1(d) in the context of Theorem 0.1.

Proposition 3.9. *The images* $\overline{\overline{\Sigma}}_i = \overline{\overline{\mathrm{pr}}}(\Sigma_i)$, $i = 1, 2$, *are compact orientable, totally geodesically embedded surfaces in* $\mathbb{H}^4/\Gamma'$ *of genus* 26, *which intersect each other orthogonally at five points. They are* 25*-fold coverings of the* $\overline{\Sigma}_i$, *and so each total preimage* $\psi^{-1}(\overline{\Sigma}_i)$ *consists of* 25 *components, each of which is isometric to* $\overline{\overline{\Sigma}}_i$.

Proof. This proposition is almost a direct consequence of Proposition 3.8 and diagram (95). All we need to show is that (i) the degree of the covering $\psi\colon \overline{\overline{\Sigma}}_i \to \overline{\Sigma}_i$ is indeed 25 and not another divisor of 625, and (ii) $\#(\overline{\overline{\Sigma}}_1 \cap \overline{\overline{\Sigma}}_2) = 5$.

By Proposition 3.7 these questions are reduced to questions about the group $\overline{\overline{G}}_i = G_i/(\Gamma'' \cap G_i)$ or, even simpler, about $\overline{\overline{G}}{}_i^0 := G_i^0/(\Gamma'' \cap G_i^0)$. These are of course extensions of $\overline{G}_i$ and $\overline{G}{}_i^0$, respectively. Since $\overline{\overline{G}}{}_i^0 = \langle \overline{\overline{\rho}}_i, \overline{\overline{\sigma}}_i \rangle$, it can be seen from (79) that

$$(96)\quad \ker(\psi_|\colon \overline{\overline{G}}{}_i^0 \to \overline{G}{}_i^0) = \langle \overline{\overline{\rho}}_i \cdot \overline{\overline{\sigma}}{}_i^{-4}, \overline{\overline{\sigma}}{}_i^{-4} \cdot \overline{\overline{\rho}}_i \rangle \cong (\mathbb{F}_5^2, +) \quad \text{for } i = 1, 2,$$

which show that $\deg(\psi\colon \overline{\overline{\Sigma}}_i \to \overline{\Sigma}_i) = \#\ker(\psi\colon \overline{\overline{G}}{}_i^0 \to \overline{G}{}_i^0) = 25$ as claimed in (i).

In order to get the second claim, we observe that the points in $\overline{\overline{\Sigma}}_1 \cap \overline{\overline{\Sigma}}_2$ are in one-to-one correspondence with the elements in

$$(97)\qquad N_5 := (\psi_{|\overline{\overline{G}}{}_1^0})^{-1}(\overline{\sigma}_1^5) \cap (\psi_{|\overline{\overline{G}}{}_2^0})^{-1}(\overline{\sigma}_2^5).$$

Since

$$(\overline{\overline{\sigma}}{}_1^B)^5 = \begin{pmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 4 & \\ 1 & 2 & 4 & 0 & 1 \end{pmatrix}, \qquad (\overline{\overline{\sigma}}{}_2^B)^5 = \begin{pmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & 4 & \\ 3 & 4 & 1 & 0 & 1 \end{pmatrix},$$

it follows from formulas (79) and (96) that N_5 is a cyclic group of order 5 generated by $(\overline{\overline{\rho}}_1)^{-1}(\overline{\overline{\sigma}}_1)^4\overline{\overline{\rho}}_1(\overline{\overline{\sigma}}_1)^{-4}$, which represents the same element of $O(E_{\mathbb{F}_5}, \overline{\overline{g}})$ as the word $(\overline{\overline{\sigma}}_2)^4(\overline{\overline{\rho}}_2)^{-1}(\overline{\overline{\sigma}}_2)^{-4}\overline{\overline{\rho}}_2$. q.e.d.

The chain of subgroups $\Gamma' = \varphi^{-1}(N) \supset \varphi^{-1}(N_{125}) \supset \varphi^{-1}(N_5) \supset \ker\varphi = \Gamma''$, where $N_{125} \subset N \subset \overline{\overline{\Gamma}}$ has been defined in the proof of Theorem 3.5, defines intermediate coverings. The projection $\psi\colon \mathbb{H}/\Gamma'' \to \mathbb{H}/\Gamma'$ therefore factors as

$$(98)\qquad \mathbb{H}^4/\Gamma'' \xrightarrow{\psi_1} \mathbb{H}^4/\varphi^{-1}(N_5) \xrightarrow{\psi_2} \mathbb{H}^4/\varphi^{-1}(N_{125}) \xrightarrow{\psi_3} \mathbb{H}^4/\Gamma'.$$

Clearly, $\deg\psi_1 = \deg\psi_3 = 5$ and $\deg\psi_2 = 25$. Using formulas (79) in the same manner as above, it is now easy to determine the behavior of the corresponding surfaces in these intermediate coverings.

Proposition 3.10. (i) *The images* $\psi_1(\overline{\overline{\Sigma}}_i)$, $i = 1, 2$, *are compact, oriented, totally geodesically embedded surfaces in* $\mathbb{H}^4/\varphi^{-1}(N_5)$ *of genus* 6, *which intersect each other precisely at one point.*

(ii) *The images* $\psi_2 \circ \psi_1(\overline{\overline{\Sigma}}_i)$, $i = 1, 2$, *are compact, oriented, totally geodesically embedded surfaces in* $\mathbb{H}^4/\varphi^{-1}(N_{125})$ *of genus* 2, *which intersect* *each other precisely at one point. Each total preimage* $\psi_3^{-1}(\overline{\Sigma}_i)$ *consists of five such components, whose intersection pattern has been depicted in Figure* 2.

This proposition finishes off the proof of Theorem 0.1; the two parts correspond precisely to the remaining possibilities for the end structure of M^4 listed in Figures 1(a) and 1(c). More precisely, the end of $M^4 = \mathbb{H}^4/\varphi^{-1}(N_{125})\backslash(\psi_3^{-1}(\overline{\Sigma}_1) \cup \psi_3^{-1}(\overline{\Sigma}_2))$ is modelled on the graph manifold defined in Figure 1(a). Of course, it is also possible to delete just some components of $\psi_3^{-1}(\overline{\Sigma}_1) \cup \psi_3^{-1}(\overline{\Sigma}_2)$, and the end is modelled on a subgraph along the lines of Theorem 0.1. The simplest subgraph which we obtain in this way is ②—②. It corresponds to the tubular neighborhood of two genus-2 surfaces, which intersect perpendicularly precisely at one point and have an automorphism of order 10 each (cf. Remark (ii) below the proof of Proposition 3.8). In Figures 1(c) and 1(d) we have already restricted to the simplest possible subgraph for the intermediate covering with genus-6 surfaces and the original genus-26 surfaces, respectively.

This discussion explains that we have explicitly investigated only a few very basic examples associated to the hyperbolic 120-cell space $\mathbb{H}^4/\Gamma'$. Intersecting Γ' with an arbitrary congruence subgroup of $O(E, g)$, which is of course not necessarily associated to the prime number 5, one obtains plenty of normal covering spaces of $\mathbb{H}^4/\Gamma'$; they come in towers. Moreover, one gets many more intermediate coverings as well. Hence even the single Coxeter diagram • ≡ •— • —• ≡ • leads to a long and rich list of possible genera and intersection patterns.

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