J. DIFFERENTIAL GEOMETRY
35 (1992) 103-130

ALMOST CONVEX GROUPS, LIPSCHITZ COMBING, AND π_1^∞ FOR UNIVERSAL COVERING SPACES OF CLOSED 3-MANIFOLDS

V. POÉNARU

Abstract

If $\pi_1 M^3$ is almost convex, then $\pi_1^\infty \widetilde{M}^3 = 0$. Under a mild restriction, the same conclusion holds if $\pi_1 M^3$ admits a Lipschitz combing in the sense of Thurston.

1. Introduction

The main result of this paper is that if M^3 is a closed 3-manifold such that $\pi_1 M^3$ is almost convex, then the universal covering space $\widetilde{M}^3$ is simply-connected at infinity. We start by recalling what "almost convex" means.

We consider a finitely generated group G and a specific finite set of generators $B = B^{-1}$ for G. To this, we can attach the Cayley graph $\Gamma = \Gamma(G, B)$. For each $g \in G$, we will denote by $\|g\|$ the minimal length of a word with letters in B expressing g. We also define $d(g, h) = \|g^{-1}h\| = \|h^{-1}g\|$.

For any positive integer, we can consider the ball of radius n in Γ,

$$B(n) \underset{\text{def}}{=} \{x \in \Gamma \text{ such that } \|x\| \leq n\}, \tag{1.1}$$

and the sphere of radius n in Γ,

$$S(n) \underset{\text{def}}{=} \{x \in \Gamma \text{ such that } \|x\| = n\}. \tag{1.2}$$

Following J. Cannon [3], we will say that the Cayley graph $\Gamma = \Gamma(G, B)$ is *k-almost convex* (for $k \in Z^+$) if there exists an $N = N(k) \in Z^+$ with the property that for any n if $x, y \in S(n)$ are such that $d(x, y) \leq k$, then x and y can be joined in $B(n)$ by a path of length $\leq N(k)$. If, for

Received March 5, 1990 and, in revised form, November 20, 1990.

Key words and phrases. Almost convex groups, Lipschitz combing, Cayley graph, universal covering space, π_1^∞, equivalence relation commanded by the singularities $(\Psi(f))$, equivalence relation commanded by the double points $(\Phi(f) \supset \Psi(f))$.

some B and all k, the Cayley graph $\Gamma = \Gamma(G, B)$ is k-almost convex, we will say that the group G is *almost convex.*

We can state now our main result.

Theorem 1. *Let M^3 be a closed 3-manifold such that $\pi_1 M^3$ has a finite system of generators $B = B^{-1}$, for which $\Gamma = \Gamma(\pi_1 M^3, B)$ is 3-almost convex. Then, for any compact subset of the universal covering space of M^3, $K \subset \widetilde{M}^3$, we can find a simply-connected compact 3-dimensional submanifold $U^3 \subset \widetilde{M}^3$ such that $K \subset U^3$. In other words, if $\pi_1 M^3$ is almost convex, then $\pi_1^\infty \widetilde{M}^3 = 0$.*

In more heuristic language, this means that if the "curvature" of the n-sphere in the Cayley graph of $\pi_1 M^3$ is *bounded from below*, independently of the radius n, then for the universal covering space $\widetilde{M}^3$ of M^3, we have $\pi_1^\infty \widetilde{M}^3 = 0$. The methods by which this kind of connection between the "Gromov geometry" [9], [10] of Γ and $\pi_1^\infty \widetilde{M}^3$ is established in the present paper are very similar to the ones which we have used in [17] for connecting the π_1^∞ of an open simply-connected 3-manifold to issues in infinite simple-homotopy theory.

Cannon's *almost convexity* is a fairly general metric property of groups. It is satisfied, for instance, for small cancellation groups (at least if they are $C'(1/6)$) and more generally by hyperbolic groups in the sense of Gromov ([9], [10]). It is also satisfied by the groups occurring in the 3-dimensional NIL geometries of Thurston and by any group which is the fundamental group of some closed n-manifold with sectional curvature $\leq k < 0$.

It is an open question whether this property is B-independent. But in [5], it is shown that for any cocompact group based on solvgeometry (i.e., coming from the 3-dimensional SOL geometries of Thurston), *almost convexity is violated* for any choice of $B = B^{-1}$.

The second theorem in this paper concerns the groups which admit a *quasi-Lipschitz combing,* in the sense of W. Thurson. We recall this notion.

For any finitely generated group G, with a given system of generators $A = A^{-1}$, a *combing* of G is, by definition, a choice, for each $g \in G$ of a continuous (not necessarily geodesic) path of $\Gamma(G, A)$, joining 1 to g. It will be convenient to think of this path as a function $Z^+ \xrightarrow{s_g} G$ such that $s_g(0) = 1$, $d(s_g(t), s_g(t+1)) \leq 1$, and, for all sufficiently large t, we have $s_g(t) = g$.

Abstracting from the properties of automatic groups [4], W. Thurston calls a combing *quasi-Lipschitz* if there are constants C_1, C_2 such that, for all $g, h \in G$ and $t \in Z^+$, we have

$$d(s_g(t), s_h(t)) \leq C_1 d(g, h) + C_2. \tag{1.3}$$

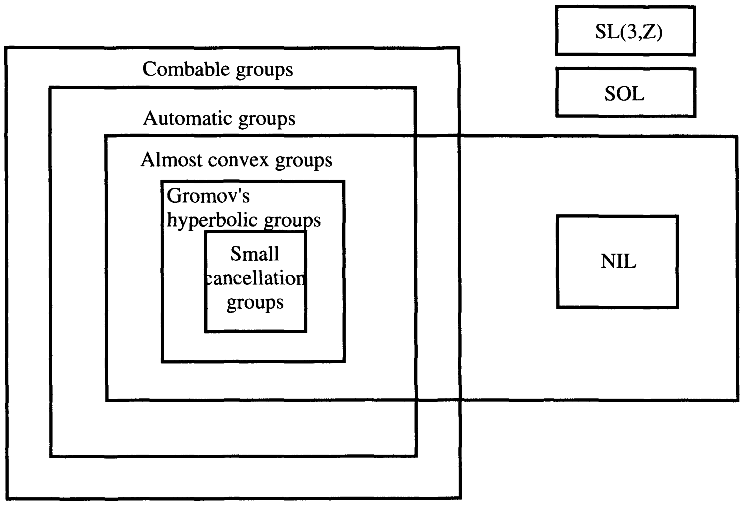


FIGURE 1.1

A group admitting a quasi-Lipschitz combing is said to be *combable*.

Following E. Ghys [8] (and also [2]), we summarize in Figure 1.1 the various inclusions between some well-known classes of groups. We can now state

Theorem 2. *Let M^3 be a closed 3-manifold such that $\pi_1 M^3$ has the following properties:*

(I) *With respect to some finite system of generators $A = A^{-1}$, $\pi_1 M^3$ admits a quasi-Lipschitz combing* (1.3), *i.e., $\pi_1 M^3$ is combable.*

(II) *There exist a system of generators $B = B^{-1}$ and two constants C_3, $\varepsilon > 0$, such that the following occur. In the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$, consider $S(n) \subset B(n) \subset \Gamma$ defined by* (1.1), (1.2). *For any $x, y \in S(n)$ with $d(x, y) \leq 3$, consider a path $\gamma = \gamma(x, y) \subset B(n)$ of minimal length joining x to y. Then*

$$\text{length}(\gamma) \leq C_3 n^{1-\varepsilon}. \tag{1.4}$$

Under these two conditions, for any compact subset $K \subset \widetilde{M}^3$, we can find a simply-connected compact 3-dimensional submanifold $U^3 \subset \widetilde{M}^3$ such that $K \subset U^3$, i.e., $\pi_1^\infty \widetilde{M}^3 = 0$.

Important remarks. (a) Condition (II) is a very mild restriction since on one hand ε is allowed to be arbitrarily close to zero, and on the other hand, for an arbitrary group G and an arbitrary $B = B^{-1}$, we *always* have

$$(1.5) \qquad \text{length}(\gamma) \leq 2n,$$

this estimate being true for an arbitrary pair $x, y \in S(n)$.

(b) We can replace $n^{1-\varepsilon}$ in (1.4) by any function $f(n)$ such that, for any constant C, we have $\lim_{n=\infty}(n - Cf(n)) = \infty$.

Theorems 1 and 2 are partial results concerning the very well-known conjecture in 3-manifold topology which states that for *any* closed 3-manifold with infinite fundamental group, one has $\pi_1 \widetilde{M}^3 = 0$. In contrast with this conjecture, Davis has shown that, in any dimension $n \geq 4$, there are closed manifolds M^n such that $M^n = K(\pi_1 M^n, 1)$ and at the same time $\pi_1^\infty \widetilde{M}^3 \neq 0$ [6].

There are also some other partial results concerning the general conjecture mentioned above. McMillan and Thickstun [12] have shown that contractible open 3-manifolds which are *not* universal covering spaces do necessarily exist. More recently, Myers [14] showed for some very specific open contractible 3-manifolds with $\pi_1^\infty \neq 0$ (namely for the genus one Whitehead manifolds) that they are not universal covering spaces.

The present paper, as already stated is strongly connected with the methods used in[17] and also in [16], but we have tried to write it in such a way that one can read it independently of the other two papers. We will use a Dehn-type lemma, which we state in the next section, the proof of which is contained in [17]. Most of the arguments used in the present paper make sense in any dimension, *except for the Dehn-type lemma.* This makes our results purely 3-dimensional, just as in [17]. Finally, we have a weakening of the *almost convexity* condition: Consider first the 2-dimension complex $Z = Z(G, B, \mathscr{R}) \supset \Gamma(G, B)$ attached to the finitely presented group G, to a finite system of generators (with $B = B^{-1}$), and to a finite system of relations $\mathscr{R}$, via the following procedure. Every time there exists a closed loop $l \subset \Gamma(B, B)$ such that, for some $x \in g$, $xlx^{-1} \in \mathscr{R}$, we add to $\Gamma = \Gamma(G, B)$ a 2-cell along l. (In other words, Z is the universal covering space of the obvious finite 2-complex K with $\pi_1 = G$, constructed with the help of B and $\mathscr{R}$.)

Next, we consider some $k \in Z^+$ and an arbitrary pair $x, y \in S(n)$ with $d(x, y) \leq k$. We join x, y by a path $\gamma_1 \subset \Gamma(G, B)$ of length $\leq k$ and by a path $\gamma_2 \subset B(n)$. This gives us a closed loop Λ in $\Gamma(G, B) \subset Z$.

Consider a (singular) disk of *minimal area* (= minimal number of 2-cells) $D \subset Z$ with $\partial D = \Lambda$. If $\Gamma(G, B)$ is k-almost convex, then clearly

$$\text{Area}(D) \leq \overline{N}(k) \tag{1.6}$$

for some $\overline{N} = \overline{N}(k)$, independent of n. We will relax this condition by asking that there are two positive numbers $C = C(k)$ and $\varepsilon = \varepsilon(k)$ with

$$\text{Area}(D) \leq C(k) n^{1-\varepsilon(k)} \tag{1.7}$$

for *all* $n \in Z^+$. If (once B and $\mathscr{R}$ are given) this holds for some k, we will say that G is *k-weakly almost convex.* A group which is *k-weakly almost convex* for all the k's is said to be *weakly almost convex.* With the methods of this paper, we can also prove the following (mild) extension of Theorem 1.

Theorem 3. *Let M^3 be a closed 3-manifold. If $\pi_1 M^3$ is weakly almost convex, then $\pi_1^\infty \widetilde{M}^3 = 0$.*

2. A Dehn-type lemma

If $A \xrightarrow{F} B$ is any map, we will define $M_2(F) \subset A$ by

$$M_2(F) = \{x \in A \text{ such that } \operatorname{card} F^{-1}F(x) > 1\}.$$

One of the ingredients for Theorem 1 is the following.

Dehn-type Lemma. *Let X and Y be two simply-connected 3-manifolds. We assume X to be compact, connected, with $\partial X \neq \varnothing$ and Y to be open. We are given a commutative diagram*

$$\begin{tikzcd} K \arrow[rr, "g"] \arrow[dr, "f"'] & & \operatorname{int} X \subset X \arrow[dl, "F"] \\ & Y & \end{tikzcd}$$

where K is a compact connected set, g and f are embeddings, and F is a smooth generic immersion. If the condition

$$\boxed{(gK) \cap M_2(F) = \varnothing}$$

is also fulfilled, then $fK \subset Y$ is contained inside a compact simply connected smooth 3-dimensional submanifold $N \subset Y$.

This is proved in [17]; the argument mimics the Shapiro-Whitehead [18] (see also [7]) approach to the Dehn Lemma [15]. For the convenience of the reader, we give an outline of the proof here.

Let Y be a compact, not necessarily connected 3-manifold with non-empty boundary. We will say that Y has "Property S" if ∂Y is a union of spheres. This class of manifolds is closed under the operations of adding handles of index two, splitting along spheres, and splitting along properly embedded disks. In [18], it is shown that a compact 3-manifold with nonempty boundary which does not possess 2-sheeted coverings has Property (S).

Hence, if $FX \subset M^3$ has no 2-sheeted coverings, then FX has Property (S) and by Van Kampen $\pi_1(FX) = 0$, so we can take $N^3 = FX$. If not, we build a *tower* of 2-sheeted coverings starting with FX, as in [18]. This tower has a last floor which has Property (S). In order to climb down the tower, we need the following fact.

Proposition. *Let V have Property (S) and let $K \subset \operatorname{int} V$ be a compact connected subset. We consider a generic immersion of V into some other 3-manifold,*

$$V \xrightarrow{\varphi} W,$$

such that φ has no triple points and $M_2(\varphi) \cap K = \varnothing$. Then K can be engulfed inside a bounded 3-dimensional submanifold of W, having Property (S).

The proof follows by cutting and pasting.

3. A naive theory of universal covering spaces

We start by recalling some of the material presented in more detail in §2 of [16]. We consider the very general situation of a nondegenerate simplicial map $X \xrightarrow{f} M^3$, where M^3 is a closed 3-manifold and X is a simplicial complex of dimension ≤ 3 which is *not* necessarily locally finite. We will denote by $\operatorname{Sing}(f) \subset X$ the set of points $x \in X$ such that the restriction of f to the $\{\text{Star of } x\} \subset X$ is *not* an immersion, i.e., the set of $x \in X$ for which there exist *distinct* simplexes σ_1, σ_2 in X with $\dim \sigma_1 = \dim \sigma_2$, $x \in \sigma_1 \cap \sigma_2$, and $f\sigma_1 = f\sigma_2$ (outside the subcomplex $\operatorname{Sing}(f) \subset X$, our X is locally finite).

There are two interesting equivalence relations connected with this situation: $\Psi(f) \subset \Phi(f) \subset X \times X$. The equivalence relation $\Phi(f)$ is the set of pairs $(x, y) \in X \times X$ with $f(x) = f(y)$, while $\Psi(f)$ is "the smallest equivalence relation, compatible with f, which kills all the singularities." This is supposed to mean that in the natural commutative diagram

$$\text{(3.1)} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & M^3 \\ & X/\Psi(f) \arrow[ur, "f_1"'] & \end{tikzcd}$$

the map f_1 is an *immersion* (i.e., $\operatorname{Sing}(f_1) = \varnothing$) and that no smaller equivalence relation $R \subsetneqq \Psi(f)$ does this job. In §2 of [16], it is shown that there is a unique, conceptually-defined $\Psi(f)$ fulfilling these requirements. We present an easy way to understand $\Psi(f)$.

Let $x \in \operatorname{Sing}(f)$, with σ_1, σ_2 as above. We consider the quotient X' of X obtained by identifying σ_1 to σ_2:

(3.2)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{folding map } \sigma_1=\sigma_2"'] & & M^3 \\ & X' \arrow[ur, "f'"'] & \end{tikzcd}$$

If $\operatorname{Sing} f' \neq \varnothing$, we consider $x' \in \operatorname{Sing}(f')$ and two distinct σ_1', $\sigma_2' \subset X'$ of the same dimension, with $x' \in \sigma_1' \cap \sigma_2'$ and $f'\sigma_1' = f'\sigma_2'$. We have now, in lieu of (3.2), the commutative diagram

(3.3)
$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, "\text{folding map } \sigma_1'=\sigma_2'"'] & & M^3 \\ & X'' \arrow[ur, "f''"'] & \end{tikzcd}$$

etc. If X is finite, this process stops after n steps and $X^{(n)} \xrightarrow{f^{(n)}} M^3$ is an immersion. We then have $X^{(n)} = X/\Psi(f)$, $f^{(n)} = f_1$, and $\Psi(f)$ is just the equivalence relation induced by the n foldings.

If X is *not* finite, then we obtain, to begin with, from (3.2), (3.3), a sequence of equivalence relations

(3.4)
$$\rho_1 \subset \rho_2 \subset \cdots \subset \rho_n \subset \rho_{n=1} \subset \cdots \subset \Phi(f);$$

we can consider $\rho_\omega = \bigcup_1^\infty \rho_n$ and start all over again with $X^{(\omega)}) = X/\rho_\omega \xrightarrow{f^{(\omega)}} M^3$. If $\operatorname{Sing}(f^{(\omega)}) \neq \varnothing$, this gives us a $X^{(\omega+1)}$, etc. so we get a *transfinite* sequence, continuing (3.4):

(3.5)
$$\rho_1 \subset \rho_2 \subset \cdots \subset \rho_n \subset \rho_{n=1} \subset \cdots \subset \rho_\omega \subset \rho_{\omega+1} \subset \cdots \subset \Phi(f).$$

The game stops when we reach an ordinal α such that $\operatorname{Sing}(f^{(\alpha)}) = \varnothing$ and the corresponding equivalence relation ρ_α is $\Psi(f)$. In [16], it is shown that:

(3.6) The $\Psi(f) = \rho_\alpha$ so defined is intrinsic i.e., independent of the various choices which we made.

(3.7) If X is at most countable, then we can choose the succession of folding maps so that already $\rho_\omega = \Psi(f)$. So *no transfinite* sequence (3.5) is needed, just (3.4).

(3.8) As a consequence of (3.7), the map $X \to X/\Psi(f)$ (see (3.1)) induces a *surjection* at the level of fundamental groups. (So contrary to $X/\Phi(f)$ which forgets any topological information, the quotient $X/\Psi(f)$ has some memory.)

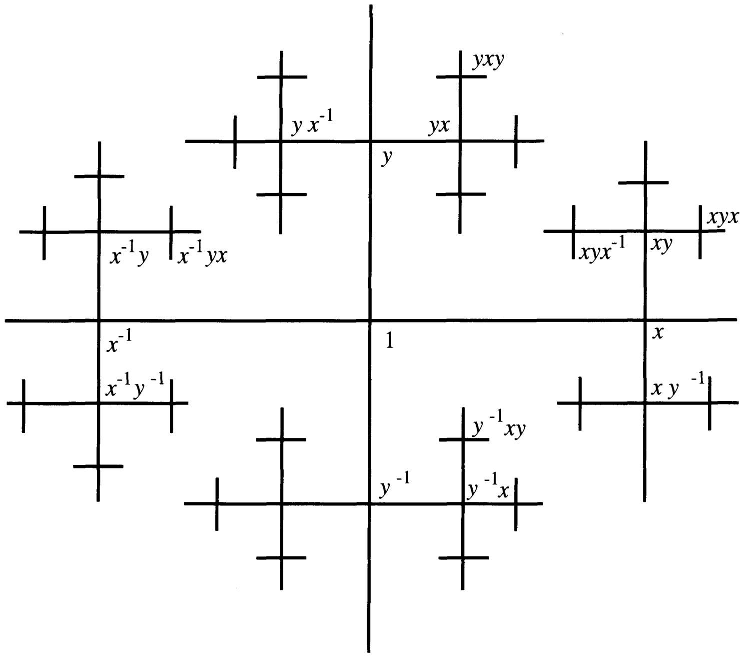


FIGURE 3.1. A PIECE OF THE CAYLEY GRAPH $\Gamma = \Gamma(Z * Z\{x, y\})$

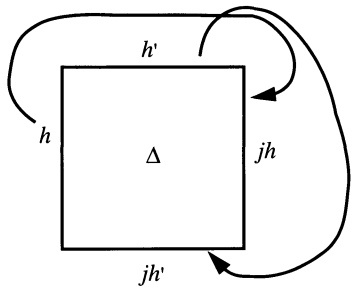
Let us consider now a 3-manifold M^3. We can always represent M^3 as follows. We start with a polyhedral 3-ball Δ with triangulated $\partial\Delta$, containing an even number of triangles $h_1, h_2, \cdots, h_{2p}$. We are also given a fixed-point free involution

$$S \underset{\text{def}}{=} \{h_1, h_2, \cdots, h_{2p}\} \xrightarrow{j} S,$$

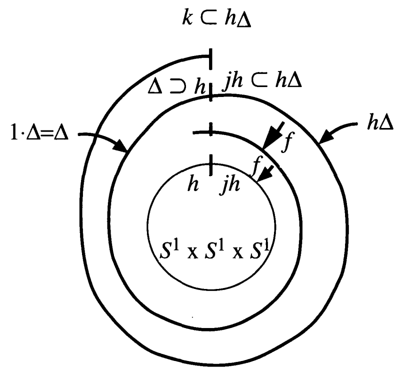
and M^3 is the quotient space Δ/ρ, where the equivalence relation ρ identifies each h_l to jh_l, by an appropriate linear isomorphism.

We will consider the free monoid $\overline{G}$ generated by S and 1, and also the space T obtained from the disjoined union $\sum_{x\in\overline{G}} x\Delta$ by glueing, for each $x \in \overline{G}$ and $h_l \in S$, the fundamental domains $x\Delta$ and $(xh_l)\Delta$ along their respective h_l and jh_l faces, in a Cayley graph manner (see Figure 3.1). We do *not* restrict ourselves here to *reduced* words x, which makes T quite complicated.

There is an obvious tautological map $T \xrightarrow{f} M^3$ which sends each fundamental domain $x\Delta \subset T$ *identically* onto $\Delta \to M^3$. This map, which



Schematic representation of $S^1 \times S^1 \times S^1$



We see here $\Delta \cup_h h\Delta \subset T$ and $f \mid (\Delta \cup_h h\Delta)$

FIGURE 3.2

is represented very schematically in Figure 3.2 just unrolls indefinitely the fundamental domain $\Delta \to M^3$, along its faces, like the developing map ([20], [19]). In [16, §2], we prove the following.

Lemma 3.1. *The canonical map* (*see* (3.1))

$$T/\Psi(f) \xrightarrow{f_1} M^3$$

is the universal covering space of M^3.

Let us now look a little closer at the representation $M^3 = \Delta/\rho$, and choose a fundamental domain $\{\overline{g}_1, \cdots, \overline{g}_p\} \subset S$ for the action of j on Γ. This fundamental domain induces a system of generators for $\pi_1 M^3$: we choose as base-point the center $* \in \Delta$ and we associate to each $\overline{g}_i$ the closed loop of M^3 which, in Δ, joins the center of $(j\overline{g}_i)$ to the center of $\overline{g}_i$. Call $g_i \in \pi_1 M^3 = \pi_1(M^3, *)$ the corresponding element. This gives a surjective morphism (in the category of semigroups)

$$\overline{G} \underset{\chi}{\to} \pi_1 M^3,$$

where we note $\overline{G} \ni \overline{g} \to g = \chi(\overline{g}) \in \pi_1 M^3$, which sends $\overline{g}_i$ to g_i and $j\overline{g}_i$ to g_i^{-1}. A complete system of relations for these $\{g_i^{\pm 1}\}$, which generate $\pi_1 M^3$, can be obtained by going around each edge of Δ.

Lemma 3.2. *For any finite system of elements* $\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \cdots, \gamma_l^{\pm 1}\} \subset \pi_1 M^3$, *we can choose a representation* $\Delta/\rho = M^3$ *such that for the*

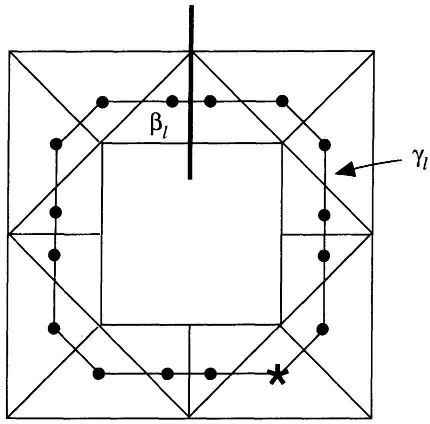


FIGURE 3.3

$\{g_1^{\pm 1}, \cdots, g_p^{\pm 1}\} \subset \pi_1 M^3$ *obtained by the construction above, we have*

$$\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \cdots, \gamma_l^{\pm 1}\} \subset \{g_1^{\pm 1}, g_2^{\pm 1}, \cdots, g_p^{\pm 1}\}.$$

Proof. We consider a triangulation τ for M^3 and the cellular decomposition τ^* dual to it. A general procedure for obtaining $\Delta/\rho = M^3$ is to start by choosing a maximal tree $\theta \subset \{1-\text{skeleton of } \tau^*\}$ and get our Δ by putting together the triangles of τ along the edges in θ. If (τ, τ^*) are fine enough, we can find based embedded loops $\gamma_1, \gamma_2, \cdots, \gamma_l$ in M^3 such that the following hold:

(1) $\gamma_i \cap \gamma_j =$ the base point and $\gamma_i \subset \{1-\text{skeleton of } \tau^*\}$.

(2) We can find a maximal tree θ such that for each γ_i there is an edge $\beta_i \subset \gamma_i$ with $\theta \cap \gamma_i = \gamma_i - \mathring{\beta}_i$ (see Figure 3.3). Of course, we have to allow here the τ-"simplex" which contains the base point, to be a more general 3-cell with triangulated boundary.

The representation $\Delta/\rho = M^3$, obtained with this particular maximal tree θ, will indeed have the desired property.

4. The main argument for Theorem 1

We will prove Theorem 1 in this section. We fix a system of generators $B \subset \pi_1 M^3$ such that the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$ is 3-*almost convex*. As in the previous section, we will represent M^3 as the quotient of some fundamental domain Δ. The set of triangles of

$\partial\Delta$ is $S = \{h_1, h_2, \cdots, h_{2p}\}$ and we choose a fundamental domain $\{\overline{g}_1, \cdots, \overline{g}_p\} \subset S$ for the fixed-point free involution $S \xrightarrow{j} S$. As also explained in the previous section, we can attach a $g_i \in \pi_1 M^3$ to $\overline{g}_i$ (and g_i^{-1} to $j\overline{g}_i$). Lemma 3.2 tells us that we can assume without any loss of generality that $B = \{g_1^{\pm 1}, g_2^{\pm 1}, \cdots, g_q^{\pm 1}\}$ for some $q \le p$. We have a *canonical map*

$$B \to \overline{G}, \tag{4.0}$$

which sends g_i into $\overline{g}_i \in S$ and g_i^{-1} into $j\overline{g}_i \in S$ for $i \le q$.

All the norms $\|g\|$ for $g \in \pi_1 M^3$, in the discussion which follows, will be computed with respect to B, *and not* to the larger system of generators $\mathscr{B} = \{g_1^{\pm 1}, g_2^{\pm 1}, \cdots, g_p^{\pm 1}\}$.

We will choose once and for all a lift of $\Delta \twoheadrightarrow M^3$ to $\widetilde{M}^3$:

$$\begin{tikzcd} & \widetilde{M}^3 \arrow[d, "\pi"] \\ \Delta \arrow[ur, "\Delta_0"] \arrow[r, two heads] & M^3. \end{tikzcd} \tag{4.1}$$

The image of Δ_0 will again be denoted by Δ so that Δ is now a fundamental domain for the action of $\pi_1 M^3$ on $\widetilde{M}^3$. Once $\Delta \xrightarrow{\Delta_0} \widetilde{M}^3$ is fixed by (4.1), we have an obvious commutative diagram, with (T, f) as in the last section:

$$\begin{tikzcd} T \arrow[rr, "F=F(\Delta_0)"] \arrow[dr, "f"'] & & \widetilde{M}^3 \arrow[dl, "\pi"] \\ & M^3 & \end{tikzcd} \tag{4.2}$$

where F sends $\overline{g}\Delta \subset T$ onto $g\Delta \to \widetilde{M}^3$, with $g = \chi(\overline{g})$.

Lemma 4.1. $\Psi(F) = \Phi(F)$.

Proof. Since π is a covering projection, the singularities of f and F have to be killed by the same foldings (3.5), so that $\Psi(f) = \Psi(F) \subset \Phi(F)$. We know, on the other hand, that $T/\Psi(f) = \widetilde{M}^3$, which is also equal to $T/\Phi(F)$, and this implies our result. q.e.d.

We denote by $T_n \subset T = \bigcup_{x \in \overline{G}} x\Delta$ the part of T obtained by keeping only the $x\Delta$'s, where $|x| =_{\text{def}} \{$the length of x as a word in $h_1, \cdots, h_{2p}\}$ $\le n$. We use the notation

$$\Psi_n = \Psi(F|T_n) \quad \text{and} \quad \Phi_n = \Phi(F|T_n) = \Phi(F)|T_n.$$

In general, we only have $\Psi_n \subset \Psi(F)|T_n$, but we also have the following.

Lemma 4.2. *For each $M \in Z^+$, there is an $\overline{M} = \overline{M}(M) \in Z^+$ with $\overline{M} \ge M$, such that $\Psi_{\overline{M}}|T_M = \Phi_M$.*

Proof. Lemma 4.1 tells us that $\Psi(F)|T_M = \Phi_M$, which is the kind of thing we want to prove, but with $\overline{M} = \infty$. On the other hand, $\Psi(F)$ can be exhausted by a sequence of folding maps modelled on the first transfinite ordinal ω (see (3.7)). Since only finitely many of these folding maps involve T_M, we have our result. q.e.d.

For every $\overline{g} \in \overline{G}$, there is a map $T \xrightarrow{\overline{g}} T$ which sends each $x\Delta \subset T$ onto $\overline{g}x\Delta \subset T$ (this is clearly compatible with the incidence relations of T). The map $\overline{g}$ is an isomorphism between T and $\overline{g}T \subset T$. We also have an obvious commutative diagram which connects it to the left action $\pi_1 M^3 \times \widetilde{M}^3 \to \widetilde{M}^3$:

$$\begin{array}{ccc} T & \xrightarrow{\overline{g}} & T \\ {\scriptstyle F=F(\Delta_0)}\downarrow & & \downarrow{\scriptstyle F(\Delta_0)} \\ \widetilde{M}^3 & \xrightarrow{g=\chi(\overline{g})} & \widetilde{M}^3 \end{array}$$

We now fix two positive integers R and M (specific conditions on their sizes will be imposed later on). Depending on R and M, we will construct a certain object $Z^\infty = Z^\infty(R, M)$ which, like T, will be a certain infinite, tree-like union of fundamental domains Δ.

Construction of $Z^\infty = Z^\infty(R, M)$. For any $g \in \pi_1 M^3$, we consider the various *geodesic paths* of the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$ joining $1 \in \Gamma$ to $g \in \Gamma$. For a given g, there are only *finitely many* such paths. We denote their number by $\rho(g)$ and they take the general form

$$(4.3) \qquad \alpha_i(g) = (1, g_{j(1)}, g_{j(1)}g_{j(2)}, \cdots, g_{j(1)}g_{j(2)}\cdots g_{j(n)} = g),$$

where $n = \|g\|$, $g_{j(k)} \in B$, and $i = 1, 2, \cdots, \rho(g)$. By using the canonical map $B \to \overline{G}$ (see (4.0)), we have a *canonical lift* of $\alpha_i(g)$ to $\overline{G}$:

$$(4.4) \qquad \overline{\alpha}_i(g) = (1, \overline{g}_{j(1)}, \overline{g}_{j(1)}\overline{g}_{j(2)}, \cdots, \overline{g}_{j(1)}\overline{g}_{j(2)}\cdots\overline{g}_{j(n)} \underset{\text{def}}{=} \overline{g}).$$

It should be emphasized here that the lift $\overline{g}$ of g, from $\pi_1 M^3$ to $\overline{G}$ given by (4.4), *depends on the specific index* $i = 1, \cdots, \rho(g)$ in (4.3) and (4.4).

We can associate a continuous path of fundamental domains in T to (4.4):

$$(4.5) \qquad \overline{\alpha}_i(g)\Delta \underset{\text{def}}{=} 1 \cdot \Delta \cup \overline{g}_{j(1)}\Delta \cup \overline{g}_{j(1)}\overline{g}_{j(2)}\Delta \cup \cdots \cup \overline{g}\Delta \subset T.$$

We consider the quotient space Z_1^∞ of the disjoined union

$$\sum_{g\in\pi_1 M^3;\, i=1,2,\cdots,\rho(g)} \overline{\alpha}_i(g)\Delta,$$

obtained by identifying *all* $1 \cdot \Delta \subset \overline{\alpha}_i(g)\Delta$ together; so Z_1^∞ is locally finite, except at $1 \cdot \Delta$.

The fundamental domains $\overline{g}\Delta \subset \overline{\alpha}_i(g)\Delta \subset Z_1^\infty$ which are *end-points* of the corresponding paths $\overline{\alpha}_i(g)\Delta$ will be, by definition, *red fundamental domains.*

Let us consider now some positive integer $r \leq \|g\|$: we will denote by $\alpha_i(g)|r$, $\overline{\alpha}_i(g)|r$, and $(\overline{\alpha}_i(g)\Delta)|r$ the obvious truncations of (4.3), (4.4), and (4.5), respectively. We will also define a quotient space $Z_2^\infty = Z_2^\infty(R)$ of Z_1^∞ as follows. For any $g_1, g_2 \in \pi_1 M^3$, $i_1 \leq \rho(g_1)$, $i_2 \leq \rho(g_2)$, and an $r \leq \inf(R, \|g_1\|, \|g_2\|)$ such that

$$\alpha_{i_1}(g_1)|r = \alpha_{i_2}(g_2)|r,$$

we identify $(\overline{\alpha}_{i_1}(g_1)\Delta)|r$ to $(\overline{\alpha}_{i_2}(g_2)\Delta)|r$ in the obvious manner.

So Z_2^∞ is locally finite except along the sphere of radius R. A fundamental domain of Z_2^∞ which is the image of a *red* fundamental domain of Z_1^∞ will be, by definition, *red.*

So, we now have a family of *red* fundamental domain $\{\overline{g}\Delta\} \subset Z_2^\infty$; here the same $\overline{g} \in \overline{G}$ can appear more than once. On the other hand, every *red fundamental domain* $\overline{g}\Delta \subset Z_2^\infty$ corresponds to a *well-defined* $g \in \pi_1 M^3$ such that $\chi(\overline{g}) = g$. If $n \in Z^+$ is given, there are only *finitely* many *red* fundamental domains $\overline{g}\Delta \subset Z_2^\infty$ with $\|g\| \leq n$.

We are finally in a position to introduce the object $Z^\infty = Z^\infty(R, M)$. For fixed M, Lemma 4.2 provides us with an $\overline{M} = \overline{M}(M) \geq M$, and with this we will set

$$Z^\infty = Z_2^\infty(R) + \sum_{\overline{g}\Delta \text{ red}} \overline{g}T_{\overline{M}}, \tag{4.6}$$

where we sum the $\overline{g}T_{\overline{M}}$'s over all the red fundamental domains $\overline{g}\Delta \subset Z_2^\infty(R)$, and each such $\overline{g}\Delta \subset Z_2^\infty(R)$ is identified to $\overline{g}\Delta \subset \overline{g}T_{\overline{M}}$. The red fundamental domains of $Z_2^\infty \subset Z^\infty$ become, by definition, red fundamental domains of $Z^\infty \supset Z_2^\infty$.

Parenthetical remark. If $X \subset \Gamma(\pi_1 M^3, B)$ is a set of vertices and $g \in B$, then $Xg \subset \Gamma$ is at distance one from X but quite unlike X, while $gX \subset \Gamma$ is isomorphic to X but quite far from X.

There is an obvious tautological map:

$$G\colon Z^\infty \to T \xrightarrow{F} \widetilde{M}^3. \tag{4.7}$$

Remark. We can also consider the quotient $Z_2^\infty(\infty)$ of Z_1^∞, where for any $g_1, g_2 \in \pi_1 M^3$, $i_1 \leq \rho(g_1)$, and $i_2 \leq \rho(g_2)$ if $r \leq \inf(\|g_1\|, \|g_2\|)$

is such that $\alpha_{i_1}(g_1)|r = \alpha_{i_2}(g_2)|r$, we identify $(\overline{\alpha}_{i_1}(g_1)\Delta)|r$ to $(\overline{\alpha}_{i_2}(g_2)\Delta)|r$. This is a quotient of our $Z_2^\infty(R)$ and $Z^\infty(R) \to Z^\infty/\Psi(G)$ factors through $Z_2^\infty(\infty)$. *All the fundamental domains of* $Z_2^\infty(\infty)$ *are red*. We could have worked with $Z_2^\infty(\infty)$ in lieu of $Z_2^\infty(R)$, but the structure of the argument is more transparent in the present version.

We will impose the following conditions on M and R.

Condition C.1. A certain fixed compact $K \subset \widetilde{M}^3$ appears in the statement of Theorem 1. We will require that R be large enough so that, if $g\Delta \cap K \neq \varnothing$ for some $g \in \pi_1 M^3$, then $\|g\| < R$.

Condition C.2. In Theorem 1, we have assumed that $\Gamma(\pi_1 M^3, B)$ satisfies Cannon's 3-*almost convex* condition. This means that if $x, y \in S(n)$ are such that $d(x, y) \leq 3$, then we can join x to y by a path in $B(n)$ of length $\leq N(3)$. The first restriction we will impose on our M is that $M \geq N(3) + 3$.

We will impose the following second lower bound on M. Remember that we had

$$B = \{g_1^{\pm 1}, \cdots, g_q^{\pm 1}\} \subset \{g_1^{\pm 1}, \cdots, g_p^{\pm 1}\} \underset{\text{def}}{=} \mathscr{B},$$

where $\mathscr{B}$ corresponds to *all* the faces of the fundamental domain Δ (and not only to the subset B for which *almost convexity* is verified). We will require that

$$M \geq \sup_{g_i \in \mathscr{B}} \{\|g_i\|\} \underset{\text{def}}{=} \gamma. \tag{4.8}$$

(Remember that all the norms $\|\cdots\|$ are computed with respect to $B \subset \pi_1 M^3$.)

There is finally a third lower bound we will impose on our M. Consider $T_M = 1 \cdot T_M \subset 1 \cdot T_{\overline{M}} \subset Z^\infty$ and $G(T_M) \subset \widetilde{M}^3$. Once R has been fixed by C.1, we will also require that M be large enough so that any $g\Delta \subset \widetilde{M}^3$ with $g \in \pi_1 M^3$ and such that $\|g\| \leq R$ be contained in $G(T_M) \subset \widetilde{M}^3$.

This condition assures us that there is a lifting of K to $T_M/\Phi_M \subset T_{\overline{M}}/\Psi_{\overline{M}}$. This ends condition C.2.

We will now investigate the canonical immersion

$$Z^\infty/\Psi(G) \overset{G_1}{\to} \widetilde{M}^3. \tag{4.9}$$

Lemma 4.3. (1) *As an immediate consequence of Lemma* 4.2, *for any red fundamental domain* $\overline{g}\Delta \subset Z^\infty$, *we have* $\Psi(G)|\overline{g}T_M = \Phi_M$ *and the map*

$$\overline{g}T_M/\Psi(G) = \overline{g}(T_M/\Phi_M) \underset{\approx}{\overset{G_1}{\to}} g(FT_M) \subset \widetilde{M}^3$$

is an isomorphism, where $\overline{g}T_M \subset Z^\infty$.

(2) *Let* $\overline{x}_1\Delta$, $\overline{x}_2\Delta$ *be two red fundamental domains of* Z^∞ *such that* $x_1 = x_2 = x$ *in* $\pi_1 M^3$. *Then the equivalence relation* $\Psi(G)$ *identifies* $\overline{x}_1\Delta$ *to* $\overline{x}_2\Delta$.

(3) *Let* $\overline{x}_1\Delta$, $\overline{x}_2\Delta$ *be two red fundamental domains of* Z^∞ *such that* $x_2 = x_1 g_i^{\pm 1}$ *in* $\pi_1 M^3$ *with* $i \leq q$ *(i.e.,* $g_i^{\pm 1} \in B$*). Then the requivalence relation* $\Psi(G)$ *identifies the* $g_i^{\pm 1}$*-face of* $\overline{x}_1\Delta$ *to the* $g_i^{\mp}$*-face of* $\overline{x}_2\Delta$ *(we write here* $g_i^{\pm 1}$ *for* $\overline{g}_i$ *or* $j\overline{g}_i$ *and* $g_i^{\mp 1}$ *for* $j\overline{g}_i$ *or* $\overline{g}_i$*).*

(3bis) *(Generalization of* (3) *from* B *to* $\mathscr{B}$*) Let* $\overline{x}_1\Delta$, $\overline{x}_2\Delta$ *be two red fundamental domains of* Z^∞ *such that* $x_2 = x_1 g_k^{\pm 1}$ *in* $\pi_1 M^3$ *with* $k \leq p$. *Then the equivalence relation* $\Psi(G)$ *identifies the* $g_k^{\pm 1}$*-face of* $\overline{x}_1\Delta$ *to the* $g_k^{\mp 1}$*-face of* $\overline{x}_2\Delta$.

(4) *As an immediate consequence of* (2) *and* (3bis)*, the subset* $\mathscr{R} \subset Z^\infty/\Psi(G)$ *defined by*

$$\{\textit{the union of the red fundamental domains of } Z^\infty\}/\Psi(G) \subset Z^\infty/\Psi(G)$$

is isomorphic to $\widetilde{M}^3$, *via* G_1.

Proof. We will consider "Statement $2^\circ(n)$" obtained by restriction to x_1, x_2 such that $\|x_1\|$, $\|x_2\| \leq n$, and similarly for 3°. We will prove, to begin with, the implication

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 2^\circ(n)\}.$$

Let $x \in \pi_1 M^3$ be such that $\|x\| = n$, and, as in (4.3), consider $\alpha_1(x)$, $\alpha_2(x)$ (see (4.3)) such that our $\overline{x}_1\Delta$, $\overline{x}_2\Delta$ are the endpoints of the corresponding $\overline{\alpha}_1(x)\Delta, \overline{\alpha}_2(x)\Delta$. So, in Z^∞ we have a continuous path $\overline{\alpha}_1(x)\Delta \cup_{1\cdot\Delta} \overline{\alpha}_2(x)\Delta$ with endpoints $\overline{x}_1\Delta$, $\overline{x}_2\Delta$, and what we want to show is that $\Psi(G)$ forces this path to close.

We will consider the elements $x(\varepsilon) \in \pi_1 M^3$ (with $\varepsilon = 1, 2$) which are the last ones in $\alpha_\varepsilon(x)$ before x; so $\|x(\varepsilon)\| = n-1$. For $x(\varepsilon)$, there is in Γ a geodesic path $\alpha(x(\varepsilon))$ *isomorphic* to $\alpha_\varepsilon(x)|(n-1)$. Clearly, $\Psi(G)$ forces the identification of $\overline{\alpha}(x(\varepsilon))\Delta$ to $(\overline{\alpha}_\varepsilon(x)\Delta)|(n-1)$.

Now $d(x(1), x(2)) \leq 2$ so that, by the 2-*almost convexity* of $\Gamma = \Gamma(\pi_1 M^3, B)$, we can join $x(1)$ to $x(2)$ by a path γ of length $\leq N(2) \leq N(3)$ in the ball of radius $(n-1)$, $B(n-1) \subset \Gamma$. This path γ is represented as a fat polygonal line in Figure 4.1 (next page). Each of the straight lines we see in Figure 4.1, joining 1 respectively to $x(1), y_1, y_2, \cdots, y_{m-1}$, $x(2)$ is a geodesic of Γ.

In Figure 4.1, we also see a *closed* path $\lambda \subset \Gamma$ of length $\leq N(2) + 2$, namely $(x, x(1), y_1, y_2, \cdots, y_{m-1}, x(2), x)$. Since every directed edge of λ corresponds canonically to a generator $g_i^{\pm 1} \in B$, by (4.0) λ defines

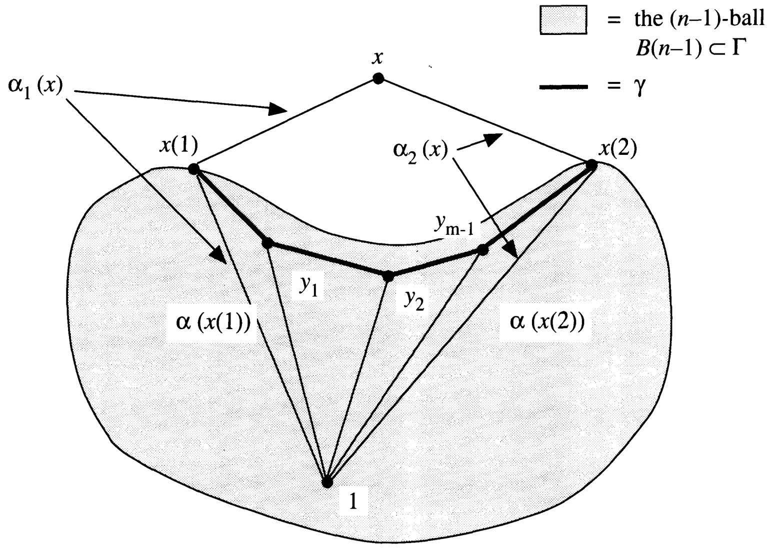


FIGURE 4.1. THIS FIGURE REPRESENTS, SYMBOLICALLY, A PIECE OF THE CAYLEY GRAPH Γ. THE FAT POINTS REPRESENT ELEMENTS OF $\pi_1 M^3$. THE NUMBER m IS $\leq N(2)$, WHERE $N(2)$ IS THE CONSTANT FROM CANNON'S 2-ALMOST CONVEXITY

a continuous path of fundamental domains in T, starting at $1 \cdot \Delta$, which we will denote by $\bar{\lambda}\Delta \subset T$.

To each vertex $v \in \pi_1 M^3$ of λ (with the two endpoints x counting as *distinct* vertices), we will attach a *red fundamental domain* in $Z^\infty/\Psi(G)$, which we will denote by $\bar{v} \cdot \Delta$ in the following manner.

(i) For the endpoints, we take simply the images of $\bar{x}_1\Delta, \bar{x}_2\Delta \subset Z^\infty$ in $Z^\infty/\Psi(G)$.

(ii) As far as the other points $y_i \in B(n-1)$ are concerned, we will use the fact that the inductive hypothesis $2^\circ(n)$ implies that for *any* $w \in B(n-1) \subset \pi_1 M^3$, there is a *unique red* representative in $Z^\infty/\Psi(G)$, which we will denote by $\bar{w}\Delta^3 \subset Z^\infty/\Psi(G)$. (*Caution.* Here $\bar{w}$ is *not* umambiguously defined as an element in the monoid $\bar{G}$; it is only the *red* fundamental domain $\bar{w}\Delta^3$ which is well defined in $Z^\infty/\Psi(G)$.)

As we have already remarked, if v_{l-1} and v_l are two consecutive vertices of λ, then $[v_{l-1}, v_l]$ corresponds to a well-defined element in

$S \subset \overline{G}$, which we will denote by $h[v_{l-1}, v_l]$. With this, we claim that in $Z^\infty/\Psi(G)$, the $h[v_{l-1}, v_l]$-face of $\overline{v}_{l-1}\Delta$ is *identified* to the $j(h[v_{l-1}, v_l])$-face of $\overline{v}_l\Delta$, the glueing pattern being exactly the same as for the path $\overline{\lambda}\Delta \subset T$. For the extremal edges $[x, x(1)]$ and $[x(2), x]$, our claim follows directly from the way in which $\alpha_\varepsilon(x)$ and $\alpha(x(\varepsilon))$ were constructed. For all the other edges, the triangle $(v_{l-1}, v_l, 1)$ (see Figure 4.1) is completely contained in the ball $B(n-1)$, and out claim follows from the inductive hypothesis $3^\circ(n-1)$.

So, we get a continuous path Λ of fundamental domains in $Z^\infty/\Psi(G)$, isomorphic to $\overline{\lambda}\Delta$, joining $\overline{x}_1\Delta$ to $\overline{x}_2\Delta$. We have $\text{length}(\Lambda) \leq N(2)+2$, which by the first part of our condition C.2 imposed on M is certainly *less* than $M \geq N(3)+3 \geq N(2)+2$.

Now, in Z^∞, the *red* fundamental domains $\overline{x}_1\Delta$ and $\overline{x}_2\Delta$ come from Z_2^∞, but Z^∞ *also* contains a piece $\overline{x}_1 T_{\overline{M}} \supset \overline{x}_1 T_M$ whose own $\overline{x}_1\Delta$ is glued (in Z^∞) to our original red $\overline{x}_1\Delta$. If we consider the obvious commutative diagram

(*)
$$\begin{tikzcd} \overline{x}_1 T_{\overline{M}} \cup_{\overline{x}_1\Delta} \Lambda \arrow[rr, "\zeta"] \arrow[dr] & & Z^\infty/\Psi(G) \arrow[dl, "G_1"] \\ & \widetilde{M}^3 & \end{tikzcd}$$

we can make the following remarks. Our Λ which is of length $\leq M$ corresponds canonically to a piece of $\overline{x}_1 T_M \subset \overline{x}_1 T_{\overline{M}}$, and at the level of $Z^\infty/\Psi(G)$ the two corresponding ζ-images *have to be identified* to each other. Point (1) of our lemma tells us, on the other hand, that $G_1|\zeta(\overline{x}_1 T_M)$ is *injective*. This means that in $Z^\infty/\Psi(G)$ the image $\zeta(\Lambda)$ is a *closed path*.

This proves the implication

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 2^\circ(n)\},$$

and the same line of argument can be used to prove that

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 3^\circ(n)\},$$

(The analog of Λ will have now a length $\leq N(3)+3$.)

We leave it to the reader to complete the proof of (3) following this line of argument. We show now how (2) and (3) together imply (3bis).

So we consider the red fundamental domains $\overline{x}_1\Delta$, $\overline{x}_2\Delta \subset Z^\infty$ with $x_2 = x_1 g_k$ $(q < k \leq p)$ in $\pi_1 M^3$. Notice that at the level of $\widetilde{M}^3$, the fundamental domains $G_1(\overline{x}_1\Delta)$ and $G_1(\overline{x}_2\Delta)$ touch along their respective g_k- and g_k^{-1}-faces. We want to show that $\Psi(G)$ forces them actually to be *glued* together at that site, at the level of $Z^\infty/\Psi(G)$, source of the map G_1.

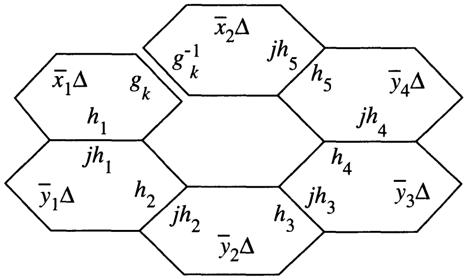


FIGURE 4.2. A CONTINUOUS PATH OF FUNDAMENTAL DOMAINS, WHICH PROJECTS ONTO THE PATH Λ' OF $Z^\infty/\Psi(G)$. HERE $x_2 = x_1 g_k$ AND $g_k = y_1 y_2 y_3 y_4 y_5$.

Now, in $\pi_1 M^3$, we can write $g_k = h_1 h_2, \cdots, h_c$ with $h_j \in B$ and $c \le \gamma$ (see (4.8)). The $h_1, h_2, \cdots, h_c$ lift via (4.0) to $\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_c \in S \subset \overline{G}$.

We consider in $\pi_1 M^3$ the sequence of elements

$$y_0 \underset{\text{def}}{=} x_1, y_1 = x_1 h_1, y_2 = x_1 h_1 h_2, \cdots, y_c = x_1 h_1 h_2 \cdots h_c = x_2.$$

It follows from (2) that, for each $j = 0, \cdots, c$, there is a *unique* red fundamental domain $\bar{y}_j\Delta$ in $Z^\infty/\Psi(G)$. For $j = 0$ and $j = c$, the corresponding $\bar{y}_j\Delta$ are exactly our original $\bar{x}_1\Delta$, $\bar{x}_2\Delta$ (actually their images in $Z^\infty/\Psi(G)$).

It follows from (3) that we can define in $Z^\infty/\Psi(G)$ a continuous path of fundamental domains

$$\Lambda' = \bar{x}_1\Delta \cup \bar{y}_1\Delta \cup \bar{y}_2\Delta \cup \cdots \cup \bar{y}_{c-1}\Delta \cup \bar{x}_2\Delta,$$

with $\bar{y}_{l-1}\Delta$ and $\bar{y}_l\Delta$ glued together along their respective $(\bar{h}_l, j\bar{h}_l)$-faces; this path is suggested in Figure 4.2.

Now, by (4.8), we have $\text{length}(\Lambda') \le \gamma \le M$ and we can consider the obvious commutative diagram, analogous to $(*)$ above:

$$(**) \qquad \begin{tikzcd} \bar{x}_1 T_{\overline{M}} \cup_{\bar{x}_1\Delta} \Lambda' \arrow[r, "\zeta'"] \arrow[dr] & Z^\infty/\Psi(G) \arrow[d, "G_1"] \\ & \widetilde{M}^3 \end{tikzcd}$$

An analysis which is completely similar to our treatment of $(*)$ shows that $\zeta'(\Lambda') \subset Z^\infty/\Psi(G)$ is a *closed path*, and this finishes the proof. q.e.d.

As a consequence of Lemma 4.3, we now have

Lemma 4.4. *For Z^∞, we have $\Psi(G) = \Phi(G)$, and hence G_1 (see (4.9)) is an isomorphism between $Z^\infty/\Psi(G)$ and $\widetilde{M}^3$.*

Proof. We consider the commutative diagram

$$(*)\qquad \begin{tikzcd} \mathscr{R} \arrow[rr, "i"] \arrow[dr, "\approx"'] & & Z^\infty/\Psi(G) \\ & \widetilde{M}^3 & \arrow[l, "G_1=\text{immersion}"'] \end{tikzcd}$$

where i is the obvious inclusion map. It suffices to show that i is *surjective.* If not, we could find a fundamental domain $\Delta \subset Z^\infty/\Psi(G)$ such that $(\operatorname{int}\Delta)\cap \operatorname{Im} i = \emptyset$. But Z^∞ is connected and hence so is $Z^\infty/\Psi(G)$. This means that we could also find a Δ with $(\operatorname{int}\Delta)\cap \operatorname{Im} i = \emptyset \neq (\partial\Delta)\cap \operatorname{im} i$. But since $\mathscr{R} \underset{\approx}{\rightarrow} \widetilde{M}^3$ is a homeomorphism, any $x \in (\partial\Delta)\cap \operatorname{Im} i$ would be a singularity for G_1, which is absurd. q.e.d.

Consider now an arbitrary riemannian metric on M^3 and lift it to $\widetilde{M}^3$. It follows from [13] that for fixed $x \in \widetilde{M}^3$, we have

$$\lim_{\|g\|\to\infty} d(x, gx) = \infty. \tag{4.10}$$

This estimate is *uniform* as long as x moves inside a compact subset of $\widetilde{M}^3$, since

$$d(y, gy) \geq d(x, gx) - 2d(x, y).$$

For fixed n, there are only *finitely* many red fundamental domains $\overline{g}\Delta \subset Z_2^\infty \subset Z^\infty$ such that $\|g\| \leq n$; estimate (4.10) then tells us that only *finitely* many of the $\overline{g}T_{\overline{M}}$ appearing in (4.6) are such that $G(\overline{g}T_{\overline{M}})\cap K \neq \emptyset$. On the other hand, condition C.1 which we have imposed on R tells us also that there are only *finitely* many fundamental domains (*red or not*) $\Delta \subset Z_2^\infty \subset Z^\infty$ such that $G(\Delta)\cap K \neq \emptyset$.

All these things having been said, the proof of Theorem 1 proceeds now as follows.

Step I. Consider any exhaustion of Z^∞ by *collapsible finite* unions of fundamental domains

$$Y_1 \subset Y_2 \subset \cdots \subset Z^\infty.$$

According to our previous discussion, there is a Y_n such that

$$G(Z^\infty - Y_n)\cap K = \emptyset.$$

Without any loss of generality, we can assume n large enough so that $1\cdot T_M \subset Z^\infty$ lives already in Y_n. Condition C.2 then implies that K lifts to $Y_n/\Phi(G|Y_n)$.

Step II. It follows from Lemma 4.4 that we can find an $m > n$ such that

$$\Psi(G|Y_m)|Y_n = \Phi(G|Y_n).$$

Once we know that $\Psi(G) = \Phi(G)$, this fact can be proved exactly as in Lemma 4.2 (where we made use of $\Psi(F) = \Psi(F)$ in a similar context).

Step III. We consider the inclusion map

$$K \subset Y_n/\Phi(G|Y_n) \subset Y_m/\Psi(G|Y_m)$$

and the commutative diagram

$$\begin{tikzcd} K \arrow[r, hook] \arrow[dr] & Y_m/\Psi(G|Y_m) \arrow[d, "g=\text{immersion}"] \\ & \widetilde{M}^3 \end{tikzcd}$$

This diagram has the following properties:

III.1 $Y_m/\Psi(G|Y_m)$ is a *simply-connected* finite 3-dimensional polyhedron (see (3.8)).

III.2 The map g is an immersion.

III.3 If we denote by $M_2(g) \subset Y_m/\Psi(G|Y_m)$ the double points of g, then

$$\boxed{K \cap M_2(g) = \varnothing}.$$

If we replace $Y_m/\Psi(G|Y_m)$ with a very thin 3-dimensional regular neighborhood, compatible with g, we are exactly in the conditions of our Dehn-type lemma, at least if K is connected.

So the Dehn-type lemma tells us that any compact connected $K \subset \widetilde{M}^3$ can be engulfed by a *compact, simply-connected* 3-*dimensional* submanifold $N^3 \subset \widetilde{M}^3$.

Since any compact subset of $\widetilde{M}^3$ is contained inside a compact connected subset, Theorem 1 follows.

5. The proof of Theorem 2

We now have two systems of generators for $\pi_1 M^3$, namely $A = \{a_i\}$ and $B = \{b_j\}$. If $g \in \pi_1 M^3$, then

$$C^{-1}\|g\|_B \leq \|g\|_A \leq C\|g\|_B \tag{5.1}$$

for some universal constant C.

If $x = x(u)$, $u \in Z^+$, is some continuous path in the Cayley graph $\Gamma(\pi_1 M^3, A)$ (with $x(u) \in \pi_1 M^3$, $d(x(u), x(u+1))_A \leq 1$), we denote

$$\|x\|_A = \sup_{t_1, t_2} d(x(t_1), x(t_2))_A, \tag{5.2}$$

and a similar notation $\|y\|_B$ will be used for a continuous path $y = y(v)$ of $\Gamma(\pi_1 M^3, B)$.

For typographical convenience, our given *quasi-Lipschitz combing* will be denoted by $s(g)$ instead of s_g. The current point on the curve $s(g)$ will be denoted by $s(g, t)$ with $t \in Z^+$. Without any loss of generality, as a consequence of the obvious estimate

$$d(s(g, 0), s(g, t))_A = d(1, s(g, t))_A \leq d(s(1, t), s(g, t))_A + \|s(1)\|_A,$$

there are constants C_1, C_2 such that, for all $g, h \in \pi_1 M^3$ and all t, we have

$$d(s(g, t), s(h, t))_A \leq C_1 d(g, h)_A + C_2, \tag{5.3}$$

and also

$$\|s(g)\|_A \leq C_1 \|g\|_A + C_2. \tag{5.4}$$

A priori the curves $s(g, t) \in \pi_1 M^3$ are defined for $t = n \in Z^+$. We will define new curves $\sigma(g, t) \in \pi_1 M^3$, which are *continuous* polygonal paths, when considered in $\Gamma(\pi_1 M^3, B)$, by the following procedure.

(5.a) Our $\sigma(g, t)$ is defined for all $t = n \in Z^+$ and then $\sigma(g, n) = s(g, n)$, *but* also for some nonintegral values, intermediate between each n and $n+1$.

(5.b) More precisely, let $s(g, n+1) = s(g, n)\cdot a$, with $a = a(g, n) \in A$; we consider a minimal word expressing a in terms of B-generators $a = b_1 b_2 \cdots b_\alpha$, and we define $\sigma(g, n + 1/\alpha) = s(g, n) \cdot b_1$, $\sigma(g, n + 2/\alpha) = s(g, n) \cdot b_1 b_2$, etc.

Clearly $\alpha \leq C$ and hence

$$\|\sigma(g)\|_B \leq C \|s(g)\|_A. \tag{5.5}$$

Exactly as in §§3 and 4 above, *for our given system* $B = B^{-1}$, we construct a representation $M^3 = \Delta/\rho$ with $S = \{h_1, \cdots, h_{2p}\}$, $B \subset \mathscr{B} \approx S$, etc.; in particular we have the T form §4. By analogy with $Z^\infty = Z^\infty(R, M)$, we will construct now another infinite, tree-like union of fundamental domains Δ, which will be denoted by Y^∞, which Y^∞ will depend on two given nonnegative integers R, M_1.

Construction of $Y^\infty = Y^\infty(R, M_1)$**.** We start with some preliminaries concerning the Cayley graph $\Gamma(\pi_1 M^3, B)$. Let $x(1), x(2) \in \pi_1 M^3$ be such that $\|x(1)\|_B = \|x(2)\|_B = p$ and $d(x(1), x(2))_B \leq 3$. We consider an arc $\lambda = \lambda(x(1), x(2))$ of length ≤ 3 joining $x(1)$ to $x(2)$ and another arc $\gamma = \gamma(x(1), x(2)) \subset \{\text{ball of radius } p\} \subset \Gamma(\pi_1 M^3, B)$ joining $x(1)$ to $x(2)$; it is assumed that γ is of minimal length. Let γ_1 be the closed path of $\Gamma(\pi_1 M^3, B)$ which is built from γ and λ. We parametrize it by

$$\gamma_1(0) = x, \qquad \gamma_1(1), \cdots, \gamma_1(N) = x \in \lambda, \tag{5.5.1}$$

where x is some fixed vertex belonging to λ. *Starting at this* $x = \gamma_1(0)$, *we will comb* γ_1 by the arcs

$$\tau(\gamma_1(u)) \underset{\text{def}}{=} x \cdot \sigma(x^{-1} \cdot \gamma_1(u)), \tag{5.6}$$

where $u = 1, \cdots, N-1$. The current point on the path $\tau(\gamma_1(u))$ corresponding to t (where t assumes some values in Q^+) will be denoted by $\tau(u, t) \underset{\text{def}}{=} \tau(\gamma_1(u), t)$. In Figure 5.1, we see this combing; we fix our attention there on integral values $t \in Z^+$, like $m, m+1$.

Condition (II) from Theorem 2 tells us that

$$\|\gamma_1\|_B \le C_3 p^{1-\varepsilon} + 3, \tag{5.7}$$

and it is not hard to find a C_4 such that *for the combing arc* $\tau(\gamma_1(u))$, we have

$$\|\tau(\gamma_1(u))\|_B \le C_4 \|\gamma_1\|_B + C_4, \tag{5.8}$$

independently of $u \in Z^+$.

By the same method used to change $\alpha_i(g)$ (see (4.3)) into $\overline{\alpha}_i(g)$ (see (4.4)) starting at $1 \in \overline{G}$, we can change $\tau(\gamma_1(u))$ into $\overline{\tau}(\gamma_1(u)) \subset \overline{G}$, starting at some $\overline{x} \in \chi^{-1}(x)$. This allows us to define a continuous path of fundamental domains $\overline{\tau}(\gamma_1(u))\Delta \subset T$ starting at $\overline{x}\Delta$. It is assumed that both $\overline{\tau}(\gamma_1(u))$ and $\overline{\tau}(\gamma_1(u))\Delta$ are *finite*, i.e., they stop when $\tau(u, t)$ starts becoming constant.

In Figure 5.1, we see a closed curve $\mu = \mu(x(1), x(2), u, m)$ bounding the hatched area; we will parametrize it by

$$\mu(0) = \tau(u, m), \mu(1), \cdots, \mu(r) = \mu(0).$$

It is not hard to find a *uniform bound* β (independent of p, $x(1)$, $x(2)$, u, m) such that, for all μ's, we have

$$\|\mu\|_B \le \beta. \tag{5.9}$$

All this having been said, our first intermediary step towards Y^∞ is to consider again $Z_2^\infty(R)$ from the last section. Next, for each red fundamental domain $\overline{x}\Delta \subset Z_2^\infty(R)$, we consider $\chi(\overline{x}) = x \in \pi_1 M^3$, and denote $n = \|x\|_B$. For each $y \in \pi_1 M^3$ such that $d(x, y)_B \le C_3 n^{1-\varepsilon} + 3$ (see (5.7)), we consider the *combing arc* $\tau(y) =_{\text{def}} x\sigma(x^{-1}y)$ and the continuous path of fundamental domains $\overline{\tau}(y)\Delta \subset T$, starting at $\overline{x} \subset T$. We consider next

$$Z_3^\infty = Z_2^\infty + \sum_{\text{RED}\,\overline{x}\Delta} \overline{\tau}(y)\Delta, \tag{5.10}$$

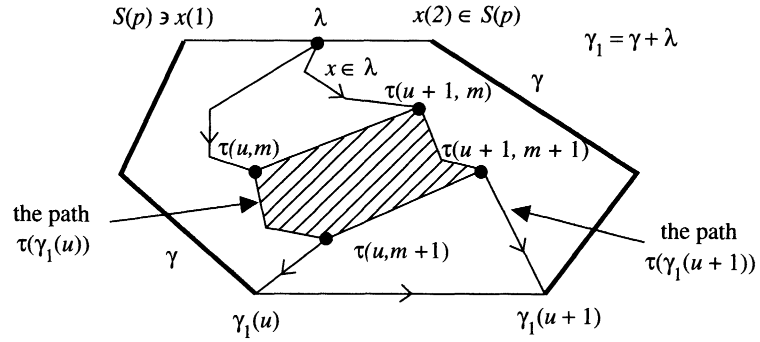


FIGURE 5.1. COMBING; THE ACTION DESCRIBED HERE TAKES PLACE IN $\Gamma(\pi_1 M^3, B)$. THE FAT LINE IS $\gamma = \gamma(x(1), x(2))$. THE PARAMETERS u, m BELONG TO Z^+ AND THE ARCS $(\tau(u, m), \tau(u+1, m))$, $(\tau(u+1, m), \tau(u+1, m+1))$ ARE GEODESIC.

where each $\mathrm{RED}\,\overline{x}\Delta \subset Z_2^\infty$ is identified to the initial fundamental domain $\overline{x}\Delta$ of each of the corresponding $\overline{\tau}(y)\Delta$. For given $\overline{x}\Delta$, *all* the $y \in \pi_1$ with $d(x, y) \le C_3 n^{1-\varepsilon} + 3$ are supposed to appear in (5.10). The path $\overline{\tau}(y)\Delta$ is made out of fundamental domains $\overline{\tau}(y, t)\Delta$, where the parameter t takes not only (finitely many) integral values $t = m$, but also some intermediary rational values $m+1/\alpha$, $m+2/\alpha$, $\cdots$. For each $\overline{\tau}(u, m)\Delta$ with $m \in Z^+$, we consider $\overline{\tau}(u, m)T_{\overline{M}_1} \subset T$, where $\overline{M}_1$ is such that, via Lemma 4.2, we have $\Psi_{\overline{M}_1} | T_{M_1} = \Phi_{M_1}$. Here M_1 is the quantity appearing in our definition of Y^∞.

Finally, we define

$$(5.11) \qquad Y^\infty = Y^\infty(R, M_1) = Z_3^\infty + \sum_{\overline{\tau}(y, m)} \overline{\tau}(y, m)T_{\overline{M}_1} + \sum_{\mathrm{RED}\,\overline{g}\Delta} \overline{g}T_{\overline{M}_1},$$

where, for each $\overline{\tau}(y, m)$, we identify $\overline{\tau}(y, m)\Delta \subset Z_3^\infty$ to the initial $\overline{\tau}(y, m)\Delta \subset \overline{\tau}(y, m)T_{\overline{M}_1}$ and for each $\mathrm{red}\,\overline{g}\Delta \subset Z_2^\infty$, we identify this $\overline{g}\Delta \subset Z_2^\infty$ to the initial $\overline{g}\Delta \subset \overline{g}T_{\overline{M}_1}$ (just as we did for Z^∞ in formula (4.6)).

By analogy to (4.7), we have an obvious tautological map

$$(5.12) \qquad Y^\infty \xrightarrow{H} \widetilde{M}^3,$$

which induces the *immersion*

$$(5.13) \qquad Y^\infty / \Psi(H) \xrightarrow{H_1} \widetilde{M}^3.$$

The quantity R will be subjected, as in the proceeding section, to condition C.1 (with respect to the compact $K \subset \widetilde{M}^3$ from Theorem 2), while M_1 will be sufficiently large so that it verifies the following condition.

Condition C.3. If β is as in (5.9), any path of length $\leq \beta$ of T, starting at $1 \cdot \Delta$, is contained in T_{M_1}. Also, in complete analogy with C.2, we have

$$M_1 \geq \sup_{g_i \in \mathscr{B}} \{\|g_i\|_B\}, \tag{5.14}$$

and any $g\Delta \subset \widetilde{M}^3$, with $g \in \pi_1 M^3$ such that $\|g\| \leq R$, is contained in $H(T_{M_1}) \subset \widetilde{M}^3$.

Exactly as in the proceeding section, only *finitely* many $\overline{g}T_{\overline{M}_1}$ appearing in (5.11) are such that $H(\overline{g}T_{\overline{M}_1}) \cap K \neq \varnothing$. We also have

Lemma 5.1. (1) *Let us consider $x \in \pi_1 M^3$ with $\|x\|_B = n$ and any $y \in \pi_1 M^3$ with $d(x, y)_B \leq C_3 n^{1-\varepsilon} + 3$. For any parameter value t which makes sense, we consider the corresponding point on the combing arc $\tau(y) = x\sigma(x^{-1}y)$, i.e.,*

$$g = \tau(y, t) \in \pi_1 M^3.$$

There is a constant C_5 such that

$$\|g\|_B \geq n - C_5(n^{1-\varepsilon} + 1). \tag{5.15}$$

(2) *As a consequence of* (5.15), *there are only finitely many $\overline{\tau}(y)\Delta \subset Z_3^\infty$ appearing in* (5.10) *such that $H(\overline{\tau}(y)\Delta) \cap K \neq \varnothing$, and hence only finitely many $\overline{\tau}(y, m)T_{\overline{M}_1} \subset Y_3^\infty$ (appearing in formula* (5.11)) *such that $H(\overline{\tau}(y, m)T_{\overline{M}_1}) \cap K \neq \varnothing$.*

Proof. Using the estimates (5.7) and (5.8), we can find a C_5 such that $d(x, g)_B \leq C_5(n^{1-\varepsilon} + 1)$, from which (5.15) follows immediately. q.e.d.

We also have the analogue of Lemma 4.3, but for the map H_1 (given by (5.13)).

Lemma 5.2. (1) *For any red fundamental domain $\overline{g}\Delta \subset Z_2^\infty \subset Y^\infty$, we have $\Psi(H_1)|\overline{g}T_{M_1} = \Phi_{M_1}$ and the map*

$$\overline{g}T_{M_1}/\Psi(H_1) = \overline{g}(T_{M_1}/\Phi_{M_1}) \underset{\approx}{\xrightarrow{H_1}} g(FT_{M_1}) \subset \widetilde{M}^3$$

is an isomorphism. The same kind of thing is true for the pieces $\overline{\tau}(y, m)T_{\overline{M}_1}$ appearing in (5.11), *and these pieces use the fundamental domains $\overline{\tau}(y, m)\Delta \subset Z_3^\infty \subset Y^\infty$, in lieu of the red $\overline{g}\Delta$.*

(2) *Let* $\overline{x}_1\Delta, \overline{x}_2\Delta$ *be two red fundamental domains of* Y^∞ *such that* $x_1 = x_2 = x$ *in* $\pi_1 M^3$. *Then the equivalence relation* $\Psi(H)$ *identifies* $\overline{x}_1\Delta$ *to* $\overline{x}_2\Delta$.

(3) *Let* $\overline{x}_1\Delta, \overline{x}_2\Delta$ *be two red fundamental domains of* Y^∞ *such that* $x_2 = x_1 g_i^{\pm 1}$ *in* $\pi_1 M^3$ *with* $g_i^{\pm 1} \in B$. *The equivalence relation* $\Psi(H)$ *identifies the* $g_i^{\pm 1}$*-face of* $\overline{x}_1\Delta$ *to the* $g_i^{\mp}$*-face of* $\overline{x}_2\Delta$.

(3bis) *The analogue of* (3) *is true for all* $g_k^{\pm 1} \in \mathscr{B} \supset B$.

Proof. We will show the implication

$$\{\text{Statements } 2^\circ(n-1) \text{ and } 3^\circ(n-1)\} \Rightarrow \{\text{Statement } 2^\circ(n)\},$$

trying to imitate the similar step in the proof of Lemma 4.3. As in Figure 4.1, we consider $\|x(1)\|_B = \|x(2)\|_B = n-1$ which are to be seen also in Figure 5.1 (we take $p = n-1$). The fat polygonal line from Figure 4.1 is to be replaced now by our $\gamma \subset B(n-1)$, *and* γ *is no longer of bounded length* (i.e., the fat polygonal line from Figure 5.1).

Exactly on the same lines as we constructed the path $\Lambda \subset T$ in the proof of Lemma 4.3, we construct here a continuous path of fundamental domains of $Y^\infty/\Psi(H)$ going from $\overline{x}_1\Delta$ to $\overline{x}_2\Delta$, and we continue to call it Λ; of course the inductive hypotheses $2^\circ(n-1)$ and $3^\circ(n-1)$ have to be used here. To be explicit, Λ takes the form

$$\Lambda = \overline{\gamma}_1(0)\Delta \cup \overline{\gamma}_1(1)\Delta \cup \cdots \cup \overline{\gamma}_1(N)\Delta, \tag{5.16}$$

with $\overline{\gamma}_1(0) = \overline{x}_1$, $\overline{\gamma}_1(N) = \overline{x}_2$, and $\chi(\overline{\gamma}_1(u)) = \gamma_1(u)$ for each u (see Figure 5.1).

We have to show that actually $\Psi(H)$ *forces* Λ *to close.*

Now, for our red $\overline{x}_1\Delta$, formula (5.10) gives us various *combing arcs* $\overline{\tau}\Delta$ corresponding to $y = \gamma_1(1)$, $y = \gamma_1(2), \cdots, y = \gamma_1(N-1)$ (see (5.1)). Notice that the norms of the extreme arcs, namely $\|\overline{\tau}(\gamma_1(1))\Delta\|_B$, $\|\overline{\tau}(\gamma_1(N-1))\Delta\|_B$, are smaller than a certain uniform bound, which without any loss of generality we can assume $\ll \beta$ (see condition C.3).

For $u = 1, \cdots, N-1$, each $\overline{\tau}(\gamma_1(u))\Delta$ goes from $\overline{x}_1\Delta$ to some *endpoint* which we denote by $\overline{g}(u)\Delta \subset Y^\infty$. A priori, $\overline{g}(u)\Delta \neq \overline{\gamma}_1(u)\Delta$ (see (5.16)). The situation is schematized in Figure 5.2.

Unlike Figure 5.1 which lives in $\Gamma(\pi_1 M^3, B)$, Figure 5.2 supposedly lives in Y^∞.

Claim. The equivalence relation $\Psi(H)$ forces all the identifications $\overline{g}(u)\Delta = \overline{\gamma}_1(u)\Delta$ for $u = 1, \cdots, N-1$.

We will proceed by induction on u, the case $u = 1$ being taken care of by the inequality $\|\bar{\tau}(\gamma_1(1))\Delta\|_B \ll \beta$.

In Figure 5.2, the straight lines starting at $\bar{\tau}(u, m)$, $\bar{\tau}(u, m+1)$ and not quite making it to $\bar{\tau}(u+1, m)$, $\bar{\tau}(u+1, m+1)$ are T-lifts of the corresponding geodetic arcs from Figure 5.1. We will assume *inductively* that the identification of $\bar{\gamma}_1(u)\Delta$ to $\bar{g}(u)\Delta$ has already been forced by $\Psi(H)$.

In Figure 5.1, the triangle $(x, \tau(u, 1), \tau(u+1, 1))$ and then the rectangles $(\tau(u, m), \tau(u, m+1), \tau(u+1, m+1), \tau(u+1, m), \tau(u, m))$ are all of length $\leq \beta$. In view of (the first part of) condition C.3, this means that inductively (induction on m this time), the corresponding paths of fundamental domains are forced to close, by $\Psi(H)$. When we reach the last rectangle, resting on $(\bar{\gamma}_1(u), \bar{\gamma}_1(u+1))$ (see Figure 5.2), *this forces the identification of* $\bar{\gamma}_1(u+1)\Delta$ to $\bar{g}(u+1)$. All this part of the argument uses the pieces $\bar{\tau}(y, m)T_{\overline{M}_1} \subset Y^\infty$.

So, the claim is proved, and this means in particular that $\Psi(H)$ forces the identification of $\bar{g}(N-1)\Delta$ to $\bar{\gamma}_1(N-1)$. Since $\|\bar{\tau}(\gamma_1(N-1))\Delta\|_B \ll \beta$, it also forces the identification of $\bar{x}_1\Delta$ to $\bar{x}_2\Delta$, which proves $2^\circ(n)$.

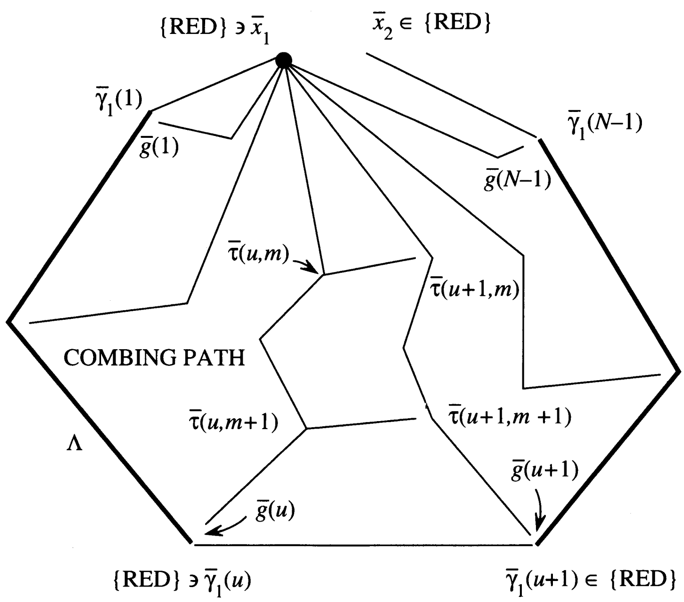


FIGURE 5.2. WE SEE HERE Λ AND THE VARIOUS COMBING PATHS (OF FUNDAMENTAL DOMAINS), STARTING AT $\bar{x}_1$. THIS SYMBOLICAL FIGURE IS SUPPOSED TO LIVE IN Y^∞.

The rest of the proof of Lemma 5.2 follows exactly on the same lines as the proof of Lemma 4.3; the proof of (3bis) uses the piece $\sum \overline{g} T_{\overline{M}_1} \subset Y^\infty$ (see (5.11) and (5.14)). q.e.d.

Also, once we have Lemmas 5.1 and 5.2, we can finish the proof of Theorem 2 on the same lines as for Theorem 1.

Theorem 3 can be proved by arguments similar to the ones which we just used for Theorem 2; we leave this to the reader.

Final comments. (a) It is not a priori clear what happens to the quasi-Lipschitz estimate (1.3) when we change our system of generators $A = A^{-1}$ to another one. So we propose to weaken (1.3) by passing from the *uniform* distance to the *Hausdorff* distance. So we will think from now on of a combing as being a continuous map $[0, 1] \xrightarrow{s_g} \Gamma(G, A)$ defined for each $g \in G$ and such that $s_g(0) = 1$, $s_g(1) = g$. The combing will be said to be *Hausdorff* if for given $g, h \in G$, we can find an orientation preserving homeomorphism $[0, 1] \underset{u=u_{g,h}}{\rightarrow} [0, 1]$ such that for all $t \in [0, 1]$ we have

$$(5.17) \qquad d(s_g(u(t)), s_h(t)) \le c_1 d(g, h) + C_2.$$

Clearly Hausdorff combing (5.17) is implied by quasi-Lipschitz combing (1.3). On the other hand, Hausdorff combing is an invariant notion (i.e., it is independent of the choice of A). We can also substitute it in Theorem 2 instead of the quasi-Lipschitz combing, and get the same conclusion. This will be the object of a subsequent paper.

(b) Very loosely speaking, the classification given in Figure 1.1 corresponds to increasing price, in terms of relators, for killing a word of given length; typically this price is high for groups with unsolvable word problem [11].

(c) Any effective estimate about the growth rate of the function $\overline{M} = \overline{M}(M)$, from Lemma 4.2, would be quite valuable.

Acknowledgments

We are very grateful to F. Laudenbach who went very carefully through a first draft of this paper; his suggestions helped simplify and improve the proofs. We are also quite indebted to M. Brin, E. Ghys, M. Gromov, A. Marin, and H. Short for very helpful conversations, and to D. Sullivan from whom we have learned a lot of mathematics during the years. Finally, we also want to thank Bernadette Barbichon for her typing.

Bibliography

[1] J. M. Alonso, *Combing of groups*, preprint.

[2] W. Ballmann, E. Ghys, A. Haefliger, P. de la Harpe, E. Salem, R. Strebel & M. Troyanov, *Sur les groupes hyperboliques, d'après M. Gromov* (E. Ghys and P. de la Harpe, eds.), to appear in book form.

[3] J. W. Cannon, *Almost convex groups*, Geometricae Dedicata **22** (1987) 197–210.

[4] J. W. Cannon, D. B. A. Epstein, D. F. Holt, M. S. Paterson & W. P. Thurston, *Word processing and group theory*, preprint.

[5] J. W. Cannon, W. J. Floyd, M. A. Grayson & W. P.Thurston, *Solvable groups are not almost convex*, preprint.

[6] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean spaces*, Ann. of Math. (2) **117** (1983) 293–324.

[7] M. Domergue, *Extension du lemme de Dehn*, C. R. Acad. Sci. Paris Sér. I Math. **286** (1978) 885–887.

[8] E. Ghys, Private communication.

[9] M. Gromov, *Infinite groups as geometric objects*, Proc. ICM, Warszewa 1983, 385–392.

[10] ——, *Hyperbolic groups in Essays in group theory* (S. M. Gersten, ed.), Math. Sci. Res. Inst. Publ., Vol. 8, Springer, Berlin, 1987, 75–263.

[11] ——, Private communication.

[12] D. R. McMillan & T. L. Thickstun, *Open 3-manifolds and the Poincaré conjecture*, Topology **19** (1980) 313–320.

[13] J. Milnor, *A note on curvature and the fundamental group*, J. Differential Geometry **2** (1968) 1–7.

[14] R. Myers, *Contractible 3-manifolds which are not covering spaces*, Topology **27** (1988) 27–35.

[15] C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **1** (1957), 1–26.

[16] V. Poénaru, *Infinite processes and the 3-dimensional Poincaré conjecture*, I (*the collapsible pseudo-spine representation theorem*), Prépubl. d'Orsay, 1989; to appear in Topology.

[17] ——, *Killing handles of index one stably and* π_1^∞ , Duke Math. J. **63** (1991), 431–447.

[18] A. Shapiro & J. H. C. Whitehead, *A proof and extension of Dehn's Lemma*, Bull. Amer. Math. Soc. **64** (1958) 174–178.

[19] D. Sullivan & W. Thurston, *Manifolds with canonical coordinate charts, some examples*, Enseignement Math. **29** (1983) 15–25.

[20] W. P. Thurston, *The geometry and topology of 3-manifolds*, to appear.

UNIVERSITÉ DE PARIS-SUD, ORSAY