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RIGIDITY OF SURFACES WITH NO CONJUGATE POINTS

KEITH BURNS & GERHARD KNIEPER

Abstract

E. Hopf proved that any complete Riemannian metric with no conjugate points on the torus T^2 is flat. We extend Hopf's argument to obtain sufficient conditions for metrics with no conjugate points on a cylinder or the plane to be flat.

0. Introduction

A complete Riemannian manifold has no conjugate points if any two points in its universal cover are joined by a unique geodesic. The no conjugate point property is a natural generalization of nonpositive curvature: any manifold with nonpositive curvature has no conjugate points by the Cartan-Hadamard theorem. In 1943 E. Hopf proved that a Riemannian metric with no conjugate points on the torus T^2 must be flat [8]. The present paper extends Hopf's arguments to obtain sufficient conditions for metrics on the cylinder $S^1 \times \mathbb{R}$ and the plane $\mathbb{R}^2$ to be flat.

In the case of the cylinder, our main result—Theorem 2.2—is that a cylinder with no conjugate points and curvature bounded from below is flat if its ends do not open out, in other words if there is $L > 0$ such that there is a nontrivial loop of length at most L based at every point. This answers affirmatively a question raised in [6], where the result is proved under the stronger assumption that the cylinder has no focal points. Our method also shows that if the cylinder becomes thin as one approaches both ends, then there must be conjugate points. As a consequence, a cylinder with no conjugate points and curvature bounded from below has infinite area. We do not know whether the lower curvature bound can be removed in these results.

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In the case of the plane, we consider a version of Euclid's parallel axiom: we suppose that there is a constant $a \geq 1$ such that for every point p and every geodesic γ in P, there is a geodesic β with $\beta(0) = p$ and

$$\operatorname{dist}(\beta(t'), \gamma) \leq a \operatorname{dist}(\beta(t), \gamma) \qquad \text{for all } t, t'.$$

Our result—Theorem 3.1—is that any metric on the plane satisfying this axiom must be a flat Euclidean metric. An immediate corollary is the result of Green and Gulliver [7] that a flat metric on the plane cannot be changed on a compact set without introducing conjugate points. Other generalizations of Green and Gulliver's result have been obtained by Innami [9], [10] and Croke [3].

For surfaces with nonpositive curvature or no focal points all of the above results are simple corollaries of the following result.

0.1. Flat Strip Theorem [4], [12], [5]. *Let β and γ be geodesics in a simply connected manifold with nonpositive curvature or no focal points. Suppose that β and γ have finite Hausdorff distance. Then β and γ are the edges of a flat strip, i.e., an isometrically and totally geodesically embedded copy of $I \times \mathbb{R}$.*

The obvious generalization of this theorem to manifolds with no conjugate points is false; a compact two dimensional counterexample is constructed in [2]. This example does not, however, contradict the following conjecture.

0.2. Conjecture. *Let S be a simply connected surface with a complete Riemannian metric with no conjugate points. Suppose that S is foliated by a family of geodesics, any two of which have finite Hausdorff distance. Then S is flat.*

All of the results of the present paper would follow easily if this conjecture were true.

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1. Preliminaries

Throughout this paper S will be a smooth surface with a complete Riemannian metric. We shall always measure angles so that they take values in $[0, \pi]$. Let K be the Gaussian curvature and π the projection from TS to S. If $X \subseteq S$, T^1X will denote the set of unit vectors with footpoint in X and $\partial^1 X$ the set of all unit vectors with footpoint in ∂X. Unless otherwise mentioned, geodesics have unit speed. If $v \in T^1S$, γ_v is the geodesic with $\dot{\gamma}_v(0) = v$. The geodesic flow g^t on T^1S is defined by $g^t(v) = \dot{\gamma}_v(t)$. Let ν be the area defined on S by the Riemannian metric and μ the Liouville measure on T^1S. Also λ will denote the measure that is the product of Lebesgue measure on the fibers of T^1S with the Riemannian length measure on a rectifiable one-dimensional subset of S; it will always be clear from the context which one-dimensional set is intended.

We consider the *scalar Jacobi equation* along the geodesic γ_v:

$$y''(t) + K(\gamma_v(t))y(t) = 0. \tag{1.1}$$

If $N(t)$ is a continuous vector field normal to γ_v, then $y(t)$ is a solution to (1.1) if and only if $y(t)N(t)$ is a Jacobi field along γ_v.

1.1. Definition. Two points $\gamma_v(t_1)$ and $\gamma_v(t_2)$ are conjugate along γ_v if there is a solution $y(t)$ of (1.1) that has $y(t_1) = 0 = y(t_2)$ and does not vanish identically. The surface S has no conjugate points if no pair of points is conjugate along any geodesic.

It is well known that this definition is equivalent to the characterization of surfaces with no conjugate points given earlier, that S has no conjugate points if and only if any two points p and q in the universal cover $\widetilde{S}$ are joined by a unique geodesic $\gamma_{p,q}$ with $\gamma_{p,q}(0) = p$ and $\gamma_{p,q}(\mathrm{dist}(p, q)) = q$. In particular, if S has no conjugate points, $\exp_p : T_p\widetilde{S} \to \widetilde{S}$ is a diffeomorphism for every $p \in \widetilde{S}$. Thus a simply connected surface with no conjugate points is diffeomorphic to $\mathbb{R}^2$.

Let $z(v, t)$ be the solution of (1.1) with $z(v, t) = 0$ and $z'(v, t) = 1$. Then S has no conjugate points if and only if, for every v, $z(v, t) \neq 0$ when $t \neq 0$. If there are no conjugate points along γ_v, there is for each $s \neq 0$ a well-defined solution $y(v, s, t)$ of (1.1) with $y(v, s, 0) = 1$ and $y(v, s, s) = 0$. Moreover

$$y_-(v, t) = \lim_{s\to\infty} y(v, s, t) \qquad \text{and} \qquad y_+(v, t) = \lim_{s\to-\infty} y(v, s, t)$$

are well-defined solutions of (1.1) with $y_-'(0) \leq y_+'(0)$; see e.g. [8]. A solution y of (1.1) with $y(0) = 1$ has $y(t) > 0$ for all $t \leq 0$ if and only if $y'(0) \leq y_+'(0)$, and has $y(t) > 0$ for all $t \geq 0$ if and only if $y'(0) \geq y_-'(0)$. We call y_- and y_+ the *stable* and *unstable solutions* respectively of (1.1).

Solving (1.1) by reduction of order shows that if γ_v has no conjugate points, then

$$z(v, t) = y_+(v, t)\int_0^t \frac{dt'}{y_+^2(v, t')}. \tag{1.2}$$

The Riccati equation

$$u'(t) + u^2(t) + K(\gamma_v) = 0 \tag{1.3}$$

is obtained from (1.1) by the change of variable $u = y'/y$. The times $t_1 < t_2$ are consecutive zeroes of a solution y of (1.1) if and only if the corresponding solution u of (1.3) is defined throughout (t_1, t_2) and $u(t) \to \infty$ as $t \searrow t_1$ and $u(t) \to -\infty$ as $t \nearrow t_2$. Thus there are no conjugate points along γ_v if and only if (1.3) has a solution that is defined for all t. If γ_v has no conjugate points, we set $u_+(t) = y'_+/y_+(t)$ and $u_-(t) = y'_-/y_-(t)$. It is clear that u_+ and u_- are the largest and smallest solutions respectively of (1.3) that are defined for all t.

1.2. Proposition. *Suppose that the surface S has no conjugate points. Then the following hold.*

(i) $u_\pm(\cdot, \cdot)$ *are measurable functions.*

(ii) $u_\pm(g^t v, s) = u_\pm(v, s+t)$ *for all s and t.*

(iii) *If in addition $K(p) \geq -b^2$ for all $p \in S$, then $|u_\pm(v, t)| \leq b$ for all (v, t). In particular, if $u(t)$ is a solution of* (1.3) *with $u(0) > b$ (resp. $u(0) < -b$), then the corresponding solution $y(t)$ of* (1.1) *vanishes for some $t < 0$ (resp. some $t > 0$).*

Proof. See [8] or [1].

We set $U = u_+(\cdot, 0)$. Our arguments are based on

1.3. Key Lemma. *Suppose that Q is a compact subset of S whose boundary is a piecewise smooth curve. Then*

$$\int_{T^1Q} U^2(v)\, d\mu(v) \leq -2\pi \int_Q K(p)\, d\nu(p) + 2\int_{\partial^1 Q} |U(v)|\, d\lambda.$$

Proof. Integrating the Riccati equation (1.3) shows that for any $\tau > 0$,

$$\int_{T^1Q} \frac{1}{\tau}\int_0^\tau u'_+(g^t v, 0) + u_+^2(g^t v, 0) + K(\gamma_v(t))\, dt\, d\mu(v) = 0.$$

Since the Liouville measure μ is g^t-invariant, letting $\tau \to 0$ gives

$$\int_{T^1Q} U^2(v)\,d\mu(v) \le -\int_{T^1Q} K(\pi v)\,d\mu(v) + \limsup_{\tau\to 0}\left|\int_{T^1Q} \frac{1}{\tau}\int_0^\tau u_+'(g^t v, 0)\,dt\,d\mu\right|. \tag{1.4}$$

Of course

$$-\int_{T^1Q} K(\pi v)\,d\mu(v) = -2\pi \int_Q K(p)\,d\nu(p).$$

It follows from the invariance property (Proposition 1.2(ii)) of u_+ and the g^t-invariance of μ that

$$\begin{aligned}
\int_{T^1Q}\int_0^\tau u_+'(g^t v, 0)\,dt\,d\mu &= \int_{T^1Q} \{U(g^\tau v) - U(v)\}\,d\mu(v) \\
&= \int_{g^\tau(T^1Q)} U(v)\,d\mu(v) - \int_{T^1Q} U(v)\,d\mu(v) \\
&= \int_{g^\tau(T^1Q)\Delta T^1Q} U(v)\,d\mu(v).
\end{aligned}$$

Observe that $g^\tau(T^1Q)\Delta T^1Q \subseteq G^\tau(\partial^1Q) \overset{\text{def}}{=} \{g^t(v) : v \in \partial^1 Q \text{ and } |t| \le \tau\}$. Define $\varphi : \partial^1Q \times \mathbb{R} \to T^1S$ by $\varphi(v, t) = g^t v$. Then

$$\begin{aligned}
\left|\int_{T^1Q}\int_0^\tau u_+'(g^t v, 0)\,dt\,d\mu\right| &\le \int_{G^\tau(\partial^1Q)} |U(v)|\,d\mu(v) \\
&\le D(\tau)\int_{\partial^1Q}\int_{-\tau}^\tau |U(g^t v)|\,dt\,d\lambda(v),
\end{aligned}$$

where

$$D(\tau) = \sup\{|\det D\varphi(v, t)| : (v, t) \in \partial^1Q \times [-\tau, -\tau] \text{ and } \partial Q \text{ is smooth at } \pi v\}.$$

Since g^t is a unit speed flow, it is easily shown that $\limsup_{\tau\to 0} D(\tau) \le 1$.

It follows that

$$\limsup_{\tau\to 0}\left|\int_{T^1Q}\frac{1}{\tau}\int_0^{\tau} u'_+(g^t v\,,0)\,dt\,d\mu(v)\right|$$
$$\le \int_{\partial^1 Q}\limsup_{\tau\to 0}\frac{1}{\tau}\int_{-\tau}^{\tau}|U(g^t)|\,dt\,d\mu(v)\le 2\int_{\partial^1 Q}|U(v)|\,d\lambda,$$

since $U(g^t v)=u_+(v\,,t)$ is a continuous (actually C^∞) function of t for each fixed v. q.e.d.

When we have the lower curvature bound $K\ge -b^2$, we shall often compare solutions of (1.1) with solutions of the scalar Jacobi equation in constant curvature $-b^2$ using

1.4. Lemma. *Suppose $K_1(t)\ge K_2(t)$ for all t and $y_i(t)$ is a solution of*

$$y_i''(t)+K_i(t)y_i(t)=0\,,\qquad i=1\,,2.$$

If $0\le y_1(0)=y_2(0)$, $0\le y_1(0)\le y_2(0)$ and $y_1(t)\ge 0$ for $0\le t\le t_0$, then $y_1(t_0)\le y_2(t_0)$.

In particular, if $K(\gamma_v(t))\ge -b^2$ for all t and there are no conjugate points along γ_v, then

$$z(v\,,t)\le\frac{1}{b}\sinh(bt)\,,\qquad t\ge 0. \tag{1.5}$$

We also use two fundamental results of Leon Green; they are Theorems 2.1 and 3.1 of [6].

1.5. Lemma. *Suppose there are no conjugate points along γ_v and $K(\gamma_v(t))$ is bounded from below. Then $z(v\,,t)\to\infty$ as $t\to\infty$.*

1.6. Proposition. *Let S be a complete simply connected surface with no conjugate points and curvature bounded from below. Let γ be a geodesic in S. Suppose $\{p_n\}$ is a sequence such that* $\operatorname{dist}(p_n\,,\gamma(0))\to\infty$ *and* $\operatorname{dist}(p_n\,,\gamma)$ *is bounded. Then*

$$\dot\gamma_{\gamma(0)\,,p_n}(0)\to\dot\gamma(0).$$

In particular if β and γ are two geodesics with $\beta(0)=\gamma(0)$ and $\dot\beta(0)\ne\dot\gamma(0)$, then $\operatorname{dist}(\beta(t)\,,\gamma(t))\to\infty$ *as $t\to\infty$.*

If S is simply connected and has no conjugate points, we can consider the following notions of parallelism and asymptoticity for two geodesics β and γ in S:

(i) $\operatorname{dist}(\beta(t)\,,\gamma(t))$ is constant;

(ii) there is $b \geq 1$ such that $\operatorname{dist}(\beta(t'), \gamma(t')) \leq b \operatorname{dist}(\beta(t), \gamma(t))$ for all t and t';

(iii) $\sup\{\operatorname{dist}(\beta(t), \gamma(t)) : t \in \mathbb{R}\} < \infty$;

(iv) β and γ have finite Hausdorff distance;

(v) β and γ do not intersect;

(vi) β is *asymptotic* to γ, i.e., $\dot\beta(0) = \lim_{t\to\infty} \dot\gamma_{\beta(0), \gamma(t)}(0)$.

Note that for every geodesic γ of S, there is exactly one geodesic asymptotic to γ starting from each point of S [5, Proposition 1]. Our definition of asymptoticity follows [5] and is not the definition usually used in the theory of manifolds with nonpositive curvature, namely that β and γ are asymptotic if $\sup\{\operatorname{dist}(\beta(t), \gamma(t)) : t \geq 0\} < \infty$. The two definitions are equivalent in the context of manifolds with nonpositive curvature or with no focal points. For such manifolds, the flat strip theorem shows that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). It is not difficult to see that (iii) $\Leftrightarrow$ (iv) in any simply connected manifold with no conjugate points, and it is trivial that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). For surfaces with no conjugate points and curvature bounded from below, Proposition 1.6 shows that (iii) $\Rightarrow$ (iv), (v) and that γ is asymptotic to β. However neither (ii) nor (iii) implies (i), even if the curvature is bounded from below [2].

Property (vi) characterizes surfaces with no conjugate points in the following way.

1.7. Proposition. *Assume that S is simply connected. Then S has no conjugate points if and only if, for any geodesic γ and any point p not on γ, there is a geodesic β with $\beta(0) = p$ that does not intersect γ.*

Proof. Suppose that S has no conjugate points. Let $v^{\pm}$ be the vectors in $T_p^1 S$ such that γ_{v^+} is asymptotic to γ and γ_{v^-} is asymptotic to the geodesic $t \mapsto \gamma(-t)$. Let v be the unit vector pointing from p towards $\gamma(0)$. Let A be the connected open arc in the circle $T_p^1 S$ that contains v and is bounded by v^- and v^+. If $u \in T_p^1 S$, the ray $\gamma_u|[0, \infty)$ intersects γ if and only if $u \in A$. Note that $-u \notin A$ if $u \in A$, for otherwise γ_u would intersect γ twice. Thus the open arc A lies in the interior of a semicircle in $T_p^1 S$, and we can find $w \in T_p^1 S$ with $w \notin A$ and $-w \notin A$. The geodesic $\beta = \gamma_w$ does not intersect γ.

Conversely suppose that S has conjugate points. Choose a point $p \in S$ that lies between a pair of conjugate points along some geodesic. Let $\rho = \inf\{r : \delta(-r)$ and $\delta(r)$ are conjugate along a geodesic δ with $\delta(0) = p\}$. Note that, for every $u \in T_p^1 S$, we have $z(u, t) > 0$ for $0 < t \leq \rho$. Since z is continuous and $T_p^1 S$ is compact, there is $R > \rho$ such that

$z(u, t) > 0$ for all $(u, t) \in T_p^1 S \times (0, R]$. Hence $\exp_p$ is nonsingular and injective on the ball $B = \{v \in T_p S : \|v\| < R\}$. Let g be the metric on B that is the pullback by $\exp_p$ of the metric on S. Choose a geodesic δ_0 in S with $\delta_0(0) = p$ and $\delta_0(-\rho)$ conjugate to $\delta_0(\rho)$ along δ_0. Then $\tilde{\delta}_0(t) \stackrel{\text{def}}{=} t\dot{\delta}_0(0)$ is a geodesic of (B, g). Since $\tilde{\delta}_0(-\rho)$ and $\tilde{\delta}_0(\rho)$ are conjugate along $\tilde{\delta}_0$, a geodesic of (B, g) that passes through $\tilde{\delta}_0(-\rho)$ and makes a small (but nonzero) angle with $\tilde{\delta}_0$ will intersect $\tilde{\delta}_0$ near $\tilde{\delta}_0(\rho)$; see [11, 2.1.13]. Such a geodesic cannot pass through 0, since this would contradict injectivity of $\exp_p$ on B. Thus there is a geodesic segment $\tilde{c}$ in (B, g) that joins $\tilde{\delta}_0(-\rho)$ to $\tilde{\delta}_0(\tau)$ for some $\tau \in (0, R)$ and does not pass through 0. Every geodesic in (B, g) that passes through 0 crosses $\tilde{c}$. Hence every geodesic in S that passes through p must cross the geodesic segment $c = \exp_p \circ \tilde{c}$. The extension γ of c to a complete geodesic may contain p. Choose a point p' that lies on δ_0 between the endpoints of c and does not lie on γ. We can choose p' so close to p that $\exp_{p'}$ is nonsingular on $B' = \{v \in T_{p'} S : \|v\| < R\}$ and c lies in $\exp_{p'} B'$. Then every geodesic that passes through p' intersects γ.

2. Cylinders with bounded cross section

Let C be a cylinder ($S^1 \times \mathbb{R}$) with a complete Riemannian metric. Let φ be a generator of $\pi_1(C)$ thought of as the group of covering transformations acting as isometries on the Riemannian universal cover $\widetilde{C}$ of C.

2.1. Definition. C has bounded cross section if there is L such that $\operatorname{dist}(p, \varphi p) \leq L$ for all $p \in \widetilde{C}$.

An equivalent statement is that there be a non-null-homotopic loop with length $\leq L$ based at each point of C.

2.2. Theorem. *Suppose C has no conjugate points, curvature bounded from below and bounded cross section. Then C is flat.*

Proof. Choose $b, L > 0$ so that $K(p) \geq -b^2$ and $\operatorname{dist}(p, \varphi p) \leq L$ for all $p \in \widetilde{C}$. We shall show below that the function $U = u_+(\cdot, 0)$ is in $L^2(T^1 C)$ and satisfies $\int_{T^1 C} U^2(v)\, d\mu(v) = 0$. This implies that U vanishes almost everywhere. It then follows from the Riccati equation (1.3) that the curvature K vanishes almost everywhere. Since K is continuous, C must be flat.

We first construct a geodesic γ_0 that has no self-intersections and joins the two ends of C. To do this, choose an increasing sequence $\{K_n\}$

of compact subsets of C such that $C = \bigcup_{n=1}^{\infty} K_n$ and each $C \backslash K_n$ has precisely two components A_n and B_n. Let g_n be the shortest geodesic segment joining a point in A_n to a point in B_n. Let γ_0 be a geodesic that is a limit of $\{g_n\}$. Then γ_0 is minimal and in particular has no self-intersections. For any $n \geq m$, g_n contains points of both the compact subsets ∂A_m and ∂B_m. Hence γ_0 contains points of both ∂A_m and ∂B_m for every m. Since γ_0 is minimal, it follows easily that $\gamma_0(s)$ approaches one end of C as $s \to \infty$ and the other end as $s \to -\infty$.

Choose a lift γ_1 of γ_0 to $\widetilde{C}$. Let $\gamma_2 = \varphi \circ \gamma_1$. Since γ_0 has no self-intersections, γ_1 and γ_2 do not intersect. They bound a strip Σ that is a fundamental domain for $\pi_1(C)$.

Let $l(s) = \operatorname{dist}(\gamma_1(s), \gamma_2(s))$. Note that $l(s) \leq L$ for all s. Let σ_s be the geodesic of $\widetilde{C}$ with $\sigma_s(0) = \gamma_1(s)$ and $\sigma_s(l(s)) = \gamma_2(s)$. We shall call the segment of σ_s between $\gamma_1(s)$ and $\gamma_2(s)$ the *cross section* of Σ at s. Two different cross sections of Σ cannot intersect: if $s \neq s'$, the points $\gamma_1(s')$ and $\gamma_2(s')$ lie on the same side of σ_s, and so the geodesic segment joining them cannot intersect σ_s. Let c_s be the projection to C of the cross section of Σ at s and let $T^1(s)$ be the set of all unit vectors based at points on c_s. Note that

$$\lambda(T^1(s)) \leq 2\pi L \qquad \text{for all } s. \tag{2.1}$$

Let $\alpha(s) = \sphericalangle(\dot{\sigma}_s(0), \dot{\gamma}_1(s))$ and $\beta(s) = \sphericalangle(-\dot{\sigma}_s(l(s)), \dot{\gamma}_2(s))$. Let $\hat{\alpha}(s) = \sphericalangle(\dot{\sigma}_s(0), -\dot{\gamma}_1(s)) = \pi - \alpha(s)$ and $\hat{\beta}(s) = \sphericalangle(-\dot{\sigma}_s(l(s)), -\dot{\gamma}_2(s)) = \pi - \beta(s)$.

Assume that $s' < s''$ and consider the set $Q(s', s'') \subseteq C$ consisting of points that lie on or between $c_{s'}$ and $c_{s''}$. The lift of $Q(s', s'')$ to Σ is a quadrilateral with geodesic sides, whose interior angles are $\hat{\alpha}(s'')$, $\hat{\beta}(s'')$, $\alpha(s')$ and $\beta(s')$. It follows from the Gauss-Bonnet theorem that

$$-\int_{Q(s',s'')} K(p)\,d\nu(p) = \{2\pi - \hat{\alpha}(s'') - \hat{\beta}(s'') - \alpha(s') - \beta(s')\},$$

which together with Lemma 1.3 implies that

$$\int_{T^1Q(s',s'')} U^2(v)\,d\mu(v) \leq 2\pi\{2\pi - \hat{\alpha}(s'') - \hat{\beta}(s'') - \alpha(s') - \beta(s')\} + 2\int_{T^1(s')\cup T^1(s'')} |U(v)|\,d\lambda(v). \tag{2.2}$$

Note that $-2\pi \leq \{2\pi - \hat{\alpha}(s'') - \hat{\beta}(s'') - \alpha(s') - \beta(s')\} \leq 2\pi$ and $|U(v)| \leq b$

for all v by (iii) of Proposition 1.2. It follows from this and (2.1) that

$$\int_{T^1Q(s',s'')} U^2(v)\,d\mu(v) \le 4\pi^2 + 4\pi bL \qquad \text{whenever } s' < s''.$$

Thus $U \in L^2(T^1C)$.

2.3. Lemma. *There is $\Theta > 0$ such that every cross section of Σ makes angle at least Θ with γ_1 and γ_2.*

Proof. In fact one can take

$$\Theta = \frac{bL}{3\sinh bL}.$$

We give the proof for the angle with γ_1. For a given s, let θ be the angle between σ_s and γ_1, and set $l = l(s)$. We have $\operatorname{dist}(\sigma_s(l), \gamma_1) \le \operatorname{dist}(\sigma_s(l), \gamma_1(s+l))$, which in turn is less than the distance between these points along the circle with radius l centered at $\sigma_s(0)$. Comparison with an arc subtending angle θ at the center of a circle of radius l in constant curvature $-b^2$ (using (1.5)) shows that

$$\operatorname{dist}(\sigma_s(l), \gamma_1) \le \frac{\theta}{b}\sinh bl = l\theta\frac{\sinh bl}{bl} \le l\theta\frac{\sinh bL}{bL},$$

since $l \le L$ and $x^{-1}\sinh x$ is an increasing function. Thus if $\theta \le \Theta$, we have

$$\operatorname{dist}(\gamma_2(s), \gamma_1) = \operatorname{dist}(\sigma_s(l), \gamma_1) \le l/3.$$

Choose s' so that $\operatorname{dist}(\gamma_2(s), \gamma_1(s')) \le l/3$. Projecting from $\widetilde{C}$ to C shows that $\gamma_0(s)$ and $\gamma_0(s')$ are joined by a curve of length $\le l/3$. Since γ_0 is a minimal geodesic, $|s - s'| \le l/3$. Thus

$$l = \operatorname{dist}(\gamma_1(s), \gamma_2(s)) \le \operatorname{dist}(\gamma_1(s), \gamma_1(s')) + \operatorname{dist}(\gamma_1(s'), \gamma_2(s)) \le \frac{2l}{3},$$

which is impossible. Hence $\theta \ge \Theta$.

2.4. Lemma. *There are $m, M > 0$ such that, if $s' < s''$ and $f : T^1Q(s', s'') \to [0, \infty)$ is integrable, then*

$$m\int_{s'}^{s''}\int_{T^1(s)} f(v)\,d\lambda(v)\,ds \le \int_{T^1Q(s^-,s^+)} f(v)\,d\mu(v) \le M\int_{s'}^{s''}\int_{T^1(s)} f(v)\,d\lambda(v)\,ds.$$

Proof. Set $\psi(s, t) = \sigma_s(t)$, and let $y_s(t)$ be the length the projection of $(\partial\psi/\partial s)(s, t)$ onto the direction orthogonal to σ_s. Then y_s is a scalar

Jacobi field along σ_s. Since two different cross sections of Σ cannot intersect, $y_s(t) > 0$ for $0 < t < l(s)$.

To prove the lemma, it suffices to choose m and M so that $m \le y_s(t) \le M$ whenever $0 \le t \le l(s)$. Since $\dot\gamma_1(s) = (\partial\psi/\partial s)(s, 0)$ and

$$\dot\gamma_2(s) = \frac{d}{ds}\{\psi(s, l(s))\} = \frac{\partial\psi}{\partial s}(s, l(s)) + l'(s)\dot\sigma_s(l(s)),$$

$y_s(0)$ and $y_s(l(s))$ are the components orthogonal to σ_s of $\dot\gamma_1(s)$ and $\dot\gamma_2(s)$ respectively. It follows from Lemma 2.3 that

(2.3) $$\sin\Theta \le y_s(0), y_s(l(s)) \le 1 \qquad \text{for all } s.$$

Let

$$m = \frac{\sin\Theta}{\cosh bL}.$$

Suppose that for some s we have $y_s(t) < m$ for $t \in [0, l(s)]$. Since $m < \sin\Theta$, the function $y_s|[0, l(s)]$ must attain its infimum at a time $t_0 \in (0, l(s))$ where its derivative vanishes. Comparison with constant curvature $-b^2$ gives $y_s(0) \le y_s(t_0)\cosh bt_0 < m\cosh bL < \sin\Theta$, which is impossible. Thus $y_s(t) > m$ whenever $0 \le t \le l(s)$.

Let $u_s(t) = y_s'(t)y_s^{-1}(t)$. If $u_s(0) > b$, then $y_s(t)$ vanishes for some $t < 0$ by (iii) of Proposition 1.2. Similarly if $u_s(l(s)) < -b$, then $y_s(t)$ vanishes for some $t > l(s)$. But $y_s(t)$ vanishes at most once. From this and (2.3) we see that

(i) $y_s'(0) \le by_s(0) \le b$, or
(ii) $y_s'(l(s)) \ge -by_s(l(s)) \ge -b$.

Suppose (i) holds. Comparing with constant curvature $-b^2$ and using Lemma 1.4 show that $y_s(t) \le e^{bt}$ for $t \ge 0$, since e^{bt} is the solution of the initial value problem $y''(t) - b^2y(t) = 0$, $y(0) = 1$, $y'(0) = b$. Hence

$$y_s(t) \le e^{bt} \le e^{bl(s)} \le e^{bL},$$

for $0 \le t \le l(s)$. A similar argument leads to that if (ii) holds and $0 \le t \le l(s)$, then

$$y_s(l(s) - t) \le e^{bt} \le e^{bL}.$$

Thus we can take $M = e^{bL}$. q.e.d.

We are now ready to prove that $\int_{T^1C} U^2\, d\mu(v) = 0$. In fact we shall show that, if $s_0 > 0$ and $0 < \varepsilon < 10\pi$, there are $s^- < -s_0$ and $s^+ > s_0$ such that

(2.4) $$\int_{T^1Q(s^-, s^+)} U^2(v)\, d\mu(v) \le \varepsilon.$$

Choose $s_1 > s_0$ large enough so that $s_1 > 1$,

$$\int_{T^1Q(-s_1, s_1)} U^2(v)\, d\mu(v) > \int_{T^1C} U^2(v)\, d\mu(v) - \frac{m^2\varepsilon^2}{1600\pi LM}, \tag{2.5}$$

and

$$\frac{\varepsilon^2}{50\pi^4}\left(s_1 - \sqrt{s_1}\right) - 2\sqrt{s_1} > L. \tag{2.6}$$

It is convenient to say that s is *good* if

$$\int_{T^1(s)} |U(v)|\, d\lambda(v) \leq \frac{\varepsilon}{20}.$$

We choose $s^+ \in [s_1, 2s_1]$ and $s^- \in [-2s_1, -s_1]$ so that they are good. Since $s_1 > 1$, the next lemma shows that this is possible.

2.5. Lemma. *Both of the sets* $\{s \in [s_1, 3s_1] : s \text{ is not good}\}$ *and* $\{s \in [-3s_1, -s_1] : s \text{ is not good}\}$ *have length at most* $\sqrt{s_1}$.

Proof. We consider the first set; the other case is similar.

$$\begin{aligned}
m \cdot \frac{\varepsilon}{20} &\cdot \text{length}\{s \in [s_1, 3s_1] : s \text{ is not good}\} \\
&\leq m \int_{s_1}^{3s_1} \int_{T^1(s)} |U(v)|\, d\lambda(v)\, ds \\
&\leq \int_{T^1Q(s_1, 3s_1)} |U(v)|\, d\mu(v) \quad \text{by Lemma 2.4} \\
&\leq \left\{ \int_{T^1Q(s_1, 3s_1)} U^2(v)\, d\mu(v) \right\}^{1/2} \left\{ \int_{T^1Q(s_1, 3s_1)} 1\, d\mu(v) \right\}^{1/2} \\
&\leq \frac{m\varepsilon}{40\sqrt{\pi LM}} \left\{ M \int_{s_1}^{3s_1} \int_{T^1(s)} 1\, d\lambda(v)\, ds \right\}^{1/2} \quad \text{by (2.5) and Lemma 2.4} \\
&\leq \frac{m\varepsilon}{40\sqrt{\pi LM}} \sqrt{4\pi LM s_1} \quad \text{by (2.1)} \\
&= m \cdot \frac{\varepsilon}{20} \cdot \sqrt{s_1}. \qquad \text{q.e.d.}
\end{aligned}$$

Since s^+ and s^- are good, (2.2) gives

$$\int_{T^1Q(s^-,s^+)} U^2(v)\,d\mu(v) \le 2\pi\{2\pi - \hat{\alpha}(s^+) - \hat{\beta}(s^+) - \alpha(s^-) - \beta(s^-)\} + \frac{\varepsilon}{5}.$$

Thus (2.4) will hold if

$$\text{(2.7)} \qquad \pi - \hat{\alpha}(s^+) - \hat{\beta}(s^+) \le \frac{\varepsilon}{5\pi} \qquad \text{and} \qquad \pi - \alpha(s^-) - \beta(s^-) \le \frac{\varepsilon}{5\pi}.$$

Suppose that $\pi - \hat{\alpha}(s^+) - \hat{\beta}(s^+) > \varepsilon/(5\pi)$, or equivalently that

$$\pi - \alpha(s^+) - \beta(s^+) < -\frac{\varepsilon}{5\pi}.$$

Under this assumption, we shall show that

$$\int_{s^+}^{3s_1} l'(s)\,ds > L,$$

which is impossible, since $0 \le l \le L$. Suppose that $s \ge s^+$ is good. Since s^+ is also good, (2.2) with $s' = s^+$ and $s'' = s$ gives that

$$0 \le 2\pi\{2\pi - \hat{\alpha}(s) - \hat{\beta}(s) - \alpha(s^+) - \beta(s^+)\} + 2\int_{T^1(s^+)\cup T^1(s)} |U(v)|\,d\lambda(v)$$

$$< 2\pi\left\{\pi - \hat{\alpha}(s) - \hat{\beta}(s) - \frac{\varepsilon}{5\pi}\right\} + \frac{\varepsilon}{5},$$

so that

$$\text{(2.8)} \qquad \hat{\alpha}(s) + \hat{\beta}(s) < \pi - \frac{\varepsilon}{5\pi} + \frac{\varepsilon}{10\pi} = \pi - \frac{\varepsilon}{10\pi}.$$

Since $\hat{\alpha}(s)$ and $\hat{\beta}(s)$ are both nonnegative, (2.8) implies

$$\text{(2.9)} \qquad |\hat{\alpha}(s) - \hat{\beta}(s)| < \pi - \frac{\varepsilon}{10\pi}.$$

It follows from the first variation of arclength formula that

$$\text{(2.10)} \qquad l'(s) = \cos\hat{\alpha}(s) + \cos\hat{\beta}(s) = 2\cos\frac{\hat{\alpha}(s) + \hat{\beta}(s)}{2}\cos\frac{\hat{\alpha}(s) - \hat{\beta}(s)}{2}.$$

It is clear from (2.8), (2.9) and (2.10) that if $s \ge s^+$ is good, then

$$l'(s) \ge 2\sin^2\frac{\varepsilon}{20\pi} \ge 2\left\{\frac{2}{\pi}\frac{\varepsilon}{20\pi}\right\}^2 = \frac{\varepsilon^2}{50\pi^4},$$

since $0 < \varepsilon < 10\pi$, and $\sin x$ is convex for $0 < x < \pi/2$. Even if s is not good, we have $l'(s) \ge -2$. From these estimates and Lemma 2.5 we see that if $0 < \varepsilon < 10\pi$, then

$$\begin{aligned}\int_{s^+}^{3s_1} l'(s)\,ds &\geq \frac{\varepsilon^2}{50\pi^4}\,\text{length}\{s \in [s^+, 3s_1] : s \text{ is good}\} \\ &\quad - 2\,\text{length}\{s \in [s^+, 3s_1] : s \text{ is not good}\} \\ &\geq \frac{\varepsilon^2}{50\pi^4}\left(s_1 - \sqrt{s_1}\right) - 2\sqrt{s_1} \\ &> L,\end{aligned}$$

by (2.6). Since this is impossible, $\pi - \hat{\alpha}(s^+) - \hat{\beta}(s^+) \leq \varepsilon/(5\pi)$. A similar argument shows that $\pi - \alpha(s^-) - \beta(s^-) \leq \varepsilon/(5\pi)$. Thus (2.7) is true, which completes the proof of the theorem.

We now show that the above theorem can be viewed as a special case of Conjecture 0.2.

2.6. Lemma. *Suppose C has no conjugate points, curvature bounded from below and bounded cross section. Then every point of $\widetilde{C}$ lies on a geodesic whose Hausdorff distance from γ_1 is bounded.*

Proof. It is enough to show that every point of Σ lies on a geodesic that does not leave Σ. Since C has bounded cross section, γ_1 and γ_2 have finite Hausdorff distance. It is obvious from this and Proposition 1.6 that a geodesic that starts at a point in Σ stays in Σ for all time if and only if it does not cross γ_1. Proposition 1.7 implies that every point of Σ lies on a geodesic that does not intersect γ_1.

2.7. Definition. The cylinder C is constricted if there are sequences p_n and q_n such that p_n diverges to one end of C, q_n diverges to the other end of C and $\lim_{n\to\infty} l(p_n) = 0 = \lim_{n\to\infty} l(q_n)$.

2.8. Theorem. *Let C be a cylinder with curvature bounded from below that is constricted. Then C has conjugate points.*

Proof. Assume that C has no conjugate points. We use the same notation as in the proof of Theorem 2.2. Since C is constricted, $\liminf_{s\to\infty} l(s) = 0$. Either there is s_0 such that $l'(s) \leq 0$ for all $s \geq s_0$ or there is not. In the first case, $\lim_{s\to\infty} l(s) = \limsup_{s\to\infty} l'(s) = 0$. In the latter case, we can choose $s_n^+ \to \infty$ such that l has a local minimum at each s_n^+ and $l(s_n^+) \to 0$ as $n \to \infty$. In either case we obtain a sequence $s_n^+ \to \infty$ such that $l(s_n^+) \to 0$ and $l'(s_n^+) \to 0$. In a similar way we can choose a sequence $s_n^- \to -\infty$ such that $l(s_n^-) \to 0$ and $l'(s_n^-) \to 0$. It is clear from (2.10) that $\hat{\alpha}(s_n^+) + \hat{\beta}(s_n^+) \to \pi$ and $\alpha(s_n^-) + \beta(s_n^-) \to \pi$ as $n \to \infty$. By (2.2), we have

$$\int_{T^1Q(s_n^-, s_n^+)} U^2(v)\,d\mu(v) \le 2\pi\{2\pi - \hat{\alpha}(s_n^+) - \hat{\beta}(s_n^+) - \alpha(s_n^-) - \beta(s_n^-)\}$$
$$+ 4\pi b\{l(s_n^-) + l(s_n^+)\} \to 0,$$

as $n \to \infty$. Hence $\int_{T^1C} U^2(v)\,d\mu(v) = 0$. It follows as in the proof of Theorem 2.2 that C is flat. But then $l(s)$ is constant, which is impossible if C is constricted.

2.9. Corollary. *Let C be a cylinder with curvature bounded from below and finite area. Then C has conjugate points.*

3. Planes with "parallel" geodesics

This section contains the proof of

3.1. Theorem. *Let P be the plane $\mathbb{R}^2$ with a Riemannian metric. Suppose that there is a constant $a \ge 1$ such that for every point p and every geodesic γ in P, there is a geodesic β with $\beta(0) = p$ and*

$$(3.1) \qquad \operatorname{dist}(\beta(t'), \gamma) \le a \operatorname{dist}(\beta(t), \gamma) \qquad \textit{for all } t, t'.$$

Then g is flat.

We shall say that the geodesic β of P is "weakly parallel" to γ if (3.1) holds. Note that if β is "weakly parallel" to γ, then either β is a reparametrization of γ or β does not intersect γ.

3.2. Lemma. *Let P be as in Theorem* 3.1. *Then the following hold*:

(i) *P has no conjugate points;*

(ii) *$y_- = y_+$ along every geodesic;*

(iii) *$y_-(v, t) \le a y_-(v, t')$ for all v and all t and t'. In particular P has the bounded asymptote property of* [5].

Proof. (i) follows easily from Proposition 1.7.

We now show that (iii) holds whenever $t' \le t$. If not, there are $t_3 > t_2 > t_1 > 0$ and an orthogonal Jacobi field along a geodesic γ with $\|Y(t_2)\| > a\|Y(t_1)\|$ and $Y(t_3) = 0$. Thus there is a geodesic δ with $\delta(t_3) = \gamma(t_3)$ and $\operatorname{dist}(\delta(t_2), \gamma) > a \operatorname{dist}(\delta(t_1), \gamma)$. But this is impossible, because one of the geodesics "weakly parallel" to γ would have to cross δ twice.

Next we show that $z(v, t) \to \infty$ as $t \to \infty$ for every $v \in T^1P$. Since $y_+(v, t) = y_-(-v, -t)$, it follows from the above that

$$(3.2) \qquad y_+(v, t) \ge a^{-1} y_+(v, t') \qquad \text{whenever } t \ge t'.$$

In particular, $y_+(v, t) \geq a^{-1}$ whenever $t \geq 0$. It is clear from this and (1.2) that $z(v, t) \to \infty$ as $t \to \infty$ if $\int_0^\infty (y_+(v, t))^{-2} dt$ diverges. If this integral converges, there is a sequence $t_n \to \infty$ such that $y_+(v, t_n) \to \infty$. Then it is clear from (3.2) that $y_+(v, t) \to \infty$ as $t \to \infty$, and it follows easily from (1.2) that $z(v, t) \to \infty$.

Now suppose that (ii) is false. Then we can find v and a solution of the scalar Jacobi equation (1.1) such that $y(0) = 1$ and $y'_-(v, 0) < y'(0) < y'_+(v, 0)$. It is clear that $y(t) > 0$ for all t and $y(t) \to \infty$ as $t \to \pm\infty$. Thus we can choose $t_- < 0 < t_+$ and a geodesic δ such that $\delta(t_-)$, $\delta(0)$ and $\delta(t_+)$ are all on the same side of γ_v and $\operatorname{dist}(\delta(t_\pm), \gamma_v) > a \operatorname{dist}(\delta(0), \gamma_v)$. But this means that one of the geodesics "weakly parallel" to γ_v crosses δ twice, which is impossible. Thus (ii) holds.

It follows easily from (ii) and the fact that (iii) holds when $t' \leq t$ that (iii) holds for all t and t'.

3.3. Lemma.

(i) $a^{-1} \leq y_-(v, t) \leq a$ *for all* v *and* t;

(ii) $a^{-3} t \leq z(v, t) \leq a^3 t$ *for all* v *and* t;

(iii) *there is* $B > 1$ *such that, if* y *is a Jacobi field along a geodesic* γ *of* P *and* $y(t) > 0$ *for* $t_1 \leq t \leq t_2$, *then*

$$B^{-1} \min\{y(t_1), y(t_2)\} \leq y(t) \leq B \max\{y(t_1), y(t_2)\},$$

for $t_1 \leq t \leq t_2$;

(iv) *if* γ *and* δ *are unit speed geodesics with the same initial point and* $\sphericalangle(\dot\gamma(0), \dot\delta(0)) = \psi$, *then*

$$\frac{\psi t}{2\pi a^3} \leq \operatorname{dist}(\gamma(t), \delta(t)) \leq a^3 \psi t,$$

for all $t \geq 0$.

Proof. (i) This is immediate from Lemma 3.2.

(ii) This follows from (i) and (1.2).

(iii) Let z_i be the scalar Jacobi field along γ with $z_i(t_i) = 0$ and $z'_i(t_i) = (-1)^{i-1}$. The sign of $z'_i(t_i)$ is chosen so that $z_i(t) > 0$ for $t_1 < t < t_2$. Let j denote the element of $\{1, 2\}$ that is not i. Then

$$y(t) = \frac{y(t_i)}{y_-(t_i)} y_-(t) + \frac{y(t_j) - \{y(t_i)/y_-(t_i)\} y_-(t_j)}{z_i(t_j)} z_i(t) \qquad \text{for } i = 1, 2,$$

where y_- is the stable scalar Jacobi field along γ. We can choose i so that the coefficient of $z_i(t)$ is nonnegative. Then we have, for $t_1 < t < t_2$,

$$y(t) \geq \frac{y(t_i)}{y_-(t_i)} y_-(t) \geq a^{-2} \min\{y(t_1), y(t_2)\},$$

by (i). Also (i) and (ii) imply that for $t_1 < t < t_2$,

$$\begin{aligned} y(t) &\le \frac{y(t_i)}{y_-(t_i)} y_-(t) + \frac{y(t_j)}{z_i(t_j)} z_i(t) \\ &\le a^2 y(t_i) + \frac{y(t_j)}{a^{-3}(t_2 - t_1)} a^3 |t - t_i| \\ &\le 2a^6 \max\{y(t_1), y(t_2)\}. \end{aligned}$$

Thus we can choose $B = 2a^6$.

(iv) The upper bound follows from (ii). For the lower bound, let r and θ be polar coordinates about $\gamma(0)$, so that $\partial/\partial r$ is the unit vector field pointing away from $\gamma(0)$ and, along each ray starting from $\gamma(0)$, $\partial/\partial\theta$ is the perpendicular Jacobi field with $\|\partial/\partial\theta\| = z$. By (ii), we have $\|\partial/\partial\theta\| \ge a^{-3}t/2$ outside the circle about $\gamma(0)$ with radius $t/2$. Thus any curve from $\gamma(t)$ to $\delta(t)$ that lies outside this circle has length at least $a^{-3}\psi t/2$. But any curve from $\gamma(t)$ to $\delta(t)$ that goes inside this circle has length at least t. Since $\psi/\pi \le 1$, we see that $\mathrm{dist}(\gamma(t), \delta(t)) \ge \psi t/(2a^3\pi)$.

3.4. Lemma. *Suppose that β is "weakly parallel" to γ. Then there is a unique $\varepsilon \in \{-1, 1\}$ such that*

$$\mathrm{dist}(\beta(t'), \gamma(\varepsilon t')) \le 3a\, \mathrm{dist}(\beta(t), \gamma(\varepsilon t)) \qquad \textit{for all } t, t'. \tag{3.3}$$

The proof is based on

3.5. Lemma. *Suppose that $\tilde\beta$ is "weakly parallel" to $\tilde\gamma$. Then there is a unique $\tilde\varepsilon \in \{-1, 1\}$ such that* $\mathrm{dist}(\tilde\beta(s), \tilde\gamma(\tilde\varepsilon s))$ *is uniformly bounded. Furthermore*

$$\mathrm{dist}(\tilde\beta(s), \tilde\gamma(\tilde\varepsilon s)) \le 3a\, \mathrm{dist}(\tilde\beta(0), \tilde\gamma(0)) \qquad \textit{for all } s.$$

Proof. Lemma 3.2(i) tells us that every geodesic in P is minimizing. It follows that $\mathrm{dist}(\tilde\beta(s), \tilde\gamma(s))$ and $\mathrm{dist}(\tilde\beta(s), \tilde\gamma(-s))$ cannot both be uniformly bounded, for then we would have $\mathrm{dist}(\tilde\gamma(s), \tilde\gamma(-s))$ uniformly bounded, which is impossible for a minimizing geodesic.

Since $\tilde\beta$ is "weakly parallel" to $\tilde\gamma$, there is a function $\tau : \mathbb{R} \to \mathbb{R}$ with $\tau(0) = 0$ such that for all s,

$$\mathrm{dist}(\tilde\beta(s), \tilde\gamma(\tau(s))) \le a\, \mathrm{dist}(\tilde\beta(0), \tilde\gamma) \le a\, \mathrm{dist}(\tilde\beta(0), \tilde\gamma(0)). \tag{3.4}$$

For any s_1 and s_2, we have

$$\begin{aligned}|\tau(s_1)-\tau(s_2)| &= \operatorname{dist}(\tilde{\gamma}(\tau(s_1)), \tilde{\gamma}(\tau(s_2)))\\ &\le \operatorname{dist}(\tilde{\gamma}(\tau(s_1)), \tilde{\beta}(s_1)) + \operatorname{dist}(\tilde{\beta}(s_1), \tilde{\beta}(s_2))\\ &\quad + \operatorname{dist}(\tilde{\beta}(s_2), \tilde{\gamma}(\tau(s_2)))\\ &\le |s_1 - s_2| + 2a\operatorname{dist}(\tilde{\beta}(0), \tilde{\gamma}(0)),\end{aligned}$$

and similarly

$$|s_1 - s_2| \le |\tau(s_1) - \tau(s_2)| + 2a\operatorname{dist}(\tilde{\beta}(0), \tilde{\gamma}(0)).$$

It follows easily that there are $\tilde{\varepsilon} \in \{-1, 1\}$ and $c \in \mathbb{R}$ such that

$$(3.5) \qquad |\tilde{\varepsilon}s + c - \tau(s)| \le a\operatorname{dist}(\tilde{\beta}(0), \tilde{\gamma}(0)) \qquad \text{for all } s.$$

Setting $s = 0$ gives

$$(3.6) \qquad |c| \le a\operatorname{dist}(\tilde{\beta}(0), \tilde{\gamma}(0)).$$

Combining (3.4), (3.5) and (3.6) shows that for all s,

$$\begin{aligned}\operatorname{dist}(\tilde{\beta}(s), \tilde{\gamma}(\tilde{\varepsilon}s)) &\le \operatorname{dist}(\tilde{\beta}(s), \tilde{\gamma}(\tau(s)))\\ &\quad + \operatorname{dist}(\tilde{\gamma}(\tau(s)), \tilde{\gamma}(\tilde{\varepsilon}s + c)) + \operatorname{dist}(\tilde{\gamma}(\tilde{\varepsilon}s + c), \tilde{\gamma}(\tilde{\varepsilon}s))\\ &\le 3a\operatorname{dist}(\tilde{\beta}(0), \tilde{\gamma}(0)).\end{aligned}$$

Proof of Lemma 3.4. By Lemma 3.5, there is a unique $\varepsilon \in \{-1, 1\}$ such that

$$(3.7) \qquad \operatorname{dist}(\beta(t), \gamma(\varepsilon t)) \le 3a\operatorname{dist}(\beta(0), \gamma(0)) \qquad \text{for all } t.$$

Now consider the geodesics $\tilde{\beta}_t$ and $\tilde{\gamma}_t$ defined by $\tilde{\beta}_t(s) = \beta(t+s)$ and $\tilde{\gamma}_t(s) = \gamma(\varepsilon t + \varepsilon s)$. The geodesic $\tilde{\beta}_t$ is "weakly parallel" to $\tilde{\gamma}_t$ and it is clear from (3.7) that, for each t, $\operatorname{dist}(\tilde{\beta}_t(s), \tilde{\gamma}_t(s))$ is uniformly bounded for all s. It follows from Lemma 3.5 that for all s and t,

$$\operatorname{dist}(\tilde{\beta}_t(s), \tilde{\gamma}_t(s)) \le 3a\operatorname{dist}(\tilde{\beta}_t(0), \tilde{\gamma}_t(0)) = \operatorname{dist}(\beta(t), \gamma(\varepsilon t)).$$

Setting $s = t' - t$ gives us (3.3).

3.6. Definition. The geodesics α and β are "parallel" if

$$\operatorname{dist}(\alpha(t'), \beta(t')) \le 3a\operatorname{dist}(\alpha(t), \beta(t)) \qquad \text{for all } t, t'$$

and "antiparallel" if α is "parallel" to $t \mapsto \beta(-t)$.

It follows from Lemma 3.4 that for every point p and every geodesic γ of P there is a geodesic β with $\beta(0) = p$ that is "parallel" to γ. Lemma 3.3(iv) shows that this uniquely determines β. Moreover β is (up to reparametrization) the unique geodesic through p that does not cross γ

transversally. We see that "weak parallelism" is an equivalence relation (in particular symmetric) and two geodesics are "weakly parallel" if and only if they are "parallel" or "antiparallel". We also see that "parallelism" is a continuous relation in the following sense: if $v_n \to v$ in T^1P and each γ_{v_n} is "parallel" to a given geodesic γ, then γ_v is "parallel" to γ.

We use this to introduce coordinates on P. Choose a point p_0 to be the origin. Choose geodesics γ_1 and γ_2 with $\gamma_1(0) = p_0 = \gamma_2(0)$. Let $\Gamma_2(r, \cdot)$ be the geodesic "parallel" to γ_2 with $\Gamma_2(r, 0) = \gamma_1(r)$. Let $\Gamma_1(s, \cdot)$ be the geodesic "parallel" to γ_1 with $\Gamma_1(s, 0) = \gamma_2(s)$ and $\dot{\Gamma}_1(s, 0)$ on the same side of γ_2 as $\dot{\gamma}_1(0)$. Define the coordinates x^1 and x^2 so that $\Gamma_2(r, \cdot)$ and $\Gamma_1(s, \cdot)$ meet at the point with coordinates (r, s).

These coordinates are C^1. To see this, note that geodesics asymptotic (in the sense defined in §1) to γ_2 do not intersect γ_2 transversally. It follows easily that $\Gamma_2(r, \cdot)$ is the geodesic through $\gamma_1(r)$ that is asymptotic to γ_2. Since P has bounded asymptote by Lemma 3.2(iii), Proposition 5 and Theorem 1 of [5] imply that $\dot{\Gamma}_2(r, 0)$ is a C^1 function of r; similarly $\dot{\Gamma}_1(s, 0)$ is a C^1 function of s. It follows easily that the coordinates are C^1.

3.7. Lemma. *There is* η *such that* $0 < \eta \le \pi/2$ *and*

$$\eta \le \sphericalangle(\partial/\partial x^1, \partial/\partial x^2) \le \pi - \eta$$

everywhere in P.

Proof. Let $\psi(p)$ be the angle between $\partial/\partial x^1(p)$ and $\partial/\partial x^2(p)$. For $i = 1, 2$ and $t > 0$, let $q_i(p, t)$ be the point obtained by moving distance t in the x^i-direction from p. Note that $\psi(p_0) = \pi/2$ and $q_i(p_0, t) = \gamma_i(t)$. Let

$$d_0(t) = \mathrm{dist}(\gamma_1(t), \gamma_2(t)) \quad \text{and} \quad d_p(t) = \mathrm{dist}(q_1(p, t), q_2(p, t)).$$

It follows from Lemma 3.3(iv) that

$$d_0(t) \ge \frac{t}{4a^3} \quad \text{and} \quad d_p(t) \le a^3\psi(p)t \qquad \text{for all } t > 0.$$

On the other hand, $|d_0(t) - d_p(t)|$ is uniformly bounded for all t, since

$$|d_0(t) - d_p(t)| \le \mathrm{dist}(\gamma_1(t), q_1(p, t)) + \mathrm{dist}(\gamma_2(t), q_2(p, t))$$

and $\gamma_i(\cdot)$ and $q_i(p, \cdot)$ are "parallel" geodesics. Thus $a^3\psi(p) \ge 1/(4a^3)$, and therefore $\psi(p) \ge 1/(4a^6)$. A similar argument can be applied to the angle between $\partial/\partial x^1$ and $-\partial/\partial x^2$ to show that $\pi - \psi(p) \ge 1/(4a^6)$.

3.8. Lemma. *Let $A = a/\sin\eta$, and y_i be the length of the projection of $\partial/\partial x^i$ onto the direction orthogonal to the curves $x^i = const$. Then the following hold for all $p \in P$ and $i = 1, 2$.*

(i) $A^{-1} \le y_i(p) \le a$;
(ii) $a^{-1} \le \|\partial/\partial x^i(p)\| \le A$;
(iii) $a^{-1} \le \|\mathrm{grad}_p x^i\| \le A$.

Proof. Let j be the element of $\{1, 2\}$ that is not i. Observe that if β_j is a geodesic "parallel" to γ_j, then $y_i \circ \beta_j$ is a scalar Jacobi field along β_j that never vanishes. Since $y_+ = y_-$ along β_j, $y_i \circ \beta_j$ is a multiple of the stable solution y_-. It follows from Lemma 3.2(ii) that if t_0 is the time when β_j crosses γ_i, then

$$a^{-1} y_i \circ \beta_j(t_0) \le y_i \circ \beta_j(t) \le a y_i \circ \beta_j(t_0) \qquad \text{for all } t. \tag{3.8}$$

Thus by Lemma 3.7 we have

$$\sin\eta \|\partial/\partial x^i(p)\| \le y_i(p) \le \|\partial/\partial x^i(p)\| \qquad \text{for all } p \in P. \tag{3.9}$$

Since we chose x^i so that $\|\partial/\partial x^i\| \equiv 1$ along γ_i, part (i) of the lemma follows from (3.8) and (3.9). Part (ii) follows from (i) and (3.9). Part (iii) follows from (i), since $\|\mathrm{grad}_p x^i\| = y_i(p)^{-1}$.

3.9. Definition. Let $\mathscr{S}(r) = \{p : -r \le x^1(p), x^2(p) \le r\}$ be the "square" defined by the coordinates.

3.10. Lemma. *For any $r \ge 0$, the "square" $\mathscr{S}(r)$ satisfies*

$$\lambda(\partial^1 \mathscr{S}(r)) \le 16\pi A r.$$

If $0 \le r' \le r''$ and $\phi : T^1\mathscr{S}(r'') \to [0, \infty)$ is integrable, then

$$\frac{1}{A}\int_{r'}^{r''} \int_{\partial^1\mathscr{S}(r)} \phi(v)\, d\lambda(v)\, dr \le \int_{T^1(\mathscr{S}(r'')\setminus\mathscr{S}(r'))} \phi(v)\, d\mu(v)$$
$$\le A\int_{r'}^{r''} \int_{\partial^1\mathscr{S}(r)} \phi(v)\, d\lambda(v)\, dr.$$

Proof. This follows easily from the previous lemma.

3.11. Proposition. $U \in L^2(P)$.

Proof. Let $f(r) = \int_{\partial^1\mathscr{S}(r)} U^2(v)\, d\lambda(v)$. Then by Lemma 3.10 and Lemma 1.3 applied to $\mathscr{S}(R)$ we see that

$$A^{-1}\int_0^R f(r)\,dr \le \int_{T^1\mathcal{S}(R)} U^2(v)\,d\mu(v)$$
$$\le -2\pi \int_{\mathcal{S}(R)} K(p)\,d\nu(p) + 2\int_{\partial^1\mathcal{S}(R)} |U(v)|\,d\lambda(v).$$

Since $\mathcal{S}(R)$ has geodesic sides and four interior angles between 0 and π,

$$-\int_{\mathcal{S}(R)} K(p)\,d\nu(p) \le 2\pi,$$

by the Gauss-Bonnet theorem. On the other hand, Hölder's inequality and Lemma 3.10 imply that

$$\int_{\partial^1\mathcal{S}(R)} |U(v)|\,d\lambda(v) \le \left\{\int_{\partial^1\mathcal{S}(R)} 1\,d\lambda(v)\right\}^{1/2} \left\{\int_{\partial^1\mathcal{S}(R)} U^2(v)\,d\lambda(v)\right\}^{1/2}$$
$$\le \sqrt{16\pi A R}\sqrt{f(R)}.$$

Hence

$$\int_0^R f(r)\,dr \le 4\pi^2 A + 8A\sqrt{\pi A}\sqrt{Rf(R)}.$$

Lemma 3.12 below now shows that $f \in L^1([0,\infty))$. It follows from Lemma 3.10 that

$$\int_{T^1P} U^2(v)\,d\mu(v) \le A\int_0^\infty f(r)\,dr < \infty.$$

3.12. Lemma. *Let $f:[0,\infty)\to[0,\infty)$. Suppose there are $C_1, C_2 > 0$ such that*

$$\int_0^R f(r)\,dr \le C_1 + C_2\sqrt{Rf(R)} \qquad \textit{for all } R \ge 0.$$

Then

$$\int_0^R f(r)\,dr \le C_1 \qquad \textit{for all } R \ge 0.$$

Proof. Let $F(R) = \int_0^R f(r)\,dr - C_1$. Suppose that $F(R_0) > 0$. Then

$$F^2(R) \le C_2^2 R F'(R) \quad \text{and} \quad F(R) > 0 \qquad \text{for all } R \ge R_0,$$

and hence

$$\ln R \le \ln R_0 + \frac{C_2^2}{F(R_0)} - \frac{C_2^2}{F(R)} \le \ln R_0 + \frac{C_2^2}{F(R_0)} \qquad \text{for all } R \ge R_0,$$

which is impossible.

3.13. Lemma. $|x^2(\Gamma_2(r, s))| \ge A^{-1}|s|$ *for all* r *and* s.

Proof. This follows from Lemma 3.8(ii), since $\Gamma_2(r, \cdot)$ is parametrized by arclength and is always tangent to $\partial/\partial x^2$.

3.14. Lemma. *Suppose* $r' < r''$. *Then for all* s,

$$\operatorname{dist}(\Gamma_2(r', s), \Gamma_2(r'', s)) \le 3a(r'' - r').$$

Proof. $\Gamma_2(r', \cdot)$ is "parallel" to $\Gamma_2(r'', \cdot)$ and $\operatorname{dist}(\Gamma_2(r', 0), \Gamma_2(r'', 0)) = r'' - r'$.

3.15. Lemma. *There is* $\Theta > 0$ *such that for any* r', r'' *and* s *with* $r' < r''$, *the geodesic through* $\Gamma_2(r', s)$ *and* $\Gamma_2(r'', s)$ *crosses* $\Gamma_2(r', \cdot)$ *and* $\Gamma_2(r'', \cdot)$ *with angle at least* Θ.

Proof. Let σ be the geodesic segment from $\Gamma_2(r', s)$ to $\Gamma_2(r'', s)$. Then length$(\sigma) \le 3a(r'' - r')$ by the previous lemma. Suppose that σ makes angle less than $1/(6a^4A)$ with $\Gamma_2(r', \cdot)$ or $\Gamma_2(r'', \cdot)$. Then by Lemma 3.3(iv), there are points p' on $\Gamma_2(r', \cdot)$ and p'' on $\Gamma_2(r'', \cdot)$ with $\operatorname{dist}(p', p'') \le (1/2A)(r'' - r')$. From this and Lemma 3.8(iii) it follows that

$$|x^2(p'') - x^2(p')| \le A\frac{1}{2A}(r'' - r') = \frac{1}{2}(r'' - r'),$$

which is impossible, since $x^2(p'') = r''$ and $x^2(p') = r'$.

3.16. Proposition. *Given* $r_0 > 0$ *and* $\varepsilon \in (0, 10\pi)$, *there is a set* $\mathcal{Q}$ *such that* $\mathcal{Q} \supseteq \mathcal{S}(r_0)$ *and* $\int_{T^1\mathcal{Q}} U^2(v)\,d\mu(v) \le \varepsilon$.

Proof. The set $\mathcal{Q}$ will be the convex hull of four points, $\Gamma_2(r^+, s^+)$, $\Gamma_2(r^-, s^+)$, $\Gamma_2(r^-, s^-)$ and $\Gamma_2(r^+, s^-)$, with $r^- < -r_0$, $r^+ > r_0$, $s^+ > Ar_0$ and $s^- < -Ar_0$. It is obvious from the first two inequalities that $\mathcal{Q}$ will contain $\mathcal{S}(r_0)$ if neither the side joining $\Gamma_2(r^+, s^+)$ to $\Gamma_2(r^+, s^-)$ nor the side joining $\Gamma_2(r^-, s^+)$ to $\Gamma_2(r^-, s^-)$ intersects $\mathcal{S}(r_0)$. But this is clear, because the coordinate x^2 is a monotone function along these edges (in fact along any geodesic), and it follows from Lemma 3.13 that $|x^2| > r_0$ at all four corners of $\mathcal{Q}$.

Choose $\rho > 1$ large enough so that

$$\frac{\varepsilon^2}{50\pi^4}(\rho - 3) - 2 > 12. \tag{3.10}$$

Choose $r_1 \geq r_0$ large enough so that

$$\int_{T^1\mathscr{S}(r_1/2)} U^2(v)\,d\mu(v) > \int_{T^1P} U^2(v)\,d\mu(v) - \frac{\varepsilon^2}{6400A^4\pi\rho} \tag{3.11}$$

and

$$\int_{T^1\mathscr{S}(r_1/2)} U^2(v)\,d\mu(v) > \int_{T^1P} U^2(v)\,d\mu(v) - \frac{\varepsilon^2\sin^2\Theta}{1600\cdot 24B^3 a\pi\rho}, \tag{3.12}$$

where the constants A, B and Θ are defined in Lemma 3.8, Lemma 3.3(iii) and Lemma 3.15 respectively.

We now choose r^+. For $r' \leq r''$, let

$$\mathscr{R}(r', r'') = \{p : r' \leq x^1(p) \leq r'' \text{ and } -\rho r_1 \leq x^2(p) \leq \rho r_1\},$$

and let $\mathscr{V}(r)$ be the set of all unit vectors with footpoint on the geodesic segment $\mathscr{R}(r, r)$ which joins $\Gamma_2(r, -\rho r_1)$ to $\Gamma_2(r, \rho r_1)$. We see from Lemma 3.8 that

$$\lambda(\mathscr{V}(r)) \leq 4\pi\rho A r \qquad \text{for all } r,$$

and, if $r' \leq r''$ and $\phi : T^1\mathscr{R}(r', r'') \to [0, \infty)$ is integrable, then

$$\frac{1}{A}\int_{r'}^{r''}\int_{\mathscr{V}(r)} \phi(v)\,d\lambda(v)\,dr \leq \int_{T^1\mathscr{R}(r', r'')} \phi(v)\,d\mu(v) \leq A\int_{r'}^{r''}\int_{\mathscr{V}(r)} \phi(v)\,d\lambda(v)\,dr.$$

Choose $r^+ \in [r_1, 2r_1]$ so that $\int_{\mathscr{V}(r^+)} |U(v)|\,d\lambda(v)$ is as small as possible. Since

$$\mathscr{R}(r_1, 2r_1) \cap \mathscr{S}(r_1/2) = \varnothing,$$

we obtain

$$\begin{aligned}
\int_{\mathscr{V}(r^+)} |U(v)|\, d\lambda(v) &\le \frac{1}{r_1}\int_{r_1}^{2r_1}\int_{\mathscr{V}(r)} |U(v)|\, d\lambda(v)\, dr \\
&\le \frac{A}{r_1}\int_{\mathscr{R}(r_1,2r_1)} |U(v)|\, d\mu(v) \\
&\le \frac{A}{r_1}\left\{\int_{\mathscr{R}(r_1,2r_1)} U^2(v)\, d\mu(v)\right\}^{1/2}\left\{\int_{\mathscr{R}(r_1,2r_1)} 1\, d\mu(v)\right\}^{1/2} \\
&\le \frac{A}{r_1}\left\{\int_{\mathscr{R}(r_1,2r_1)} U^2(v)\, d\mu(v)\right\}^{1/2}\left\{A\int_{r_1}^{2r_1}\int_{\mathscr{V}(r)} 1\, d\lambda(v)\, dr\right\}^{1/2} \\
&\le \frac{A}{r_1}\frac{\varepsilon}{80A^2\sqrt{\pi\rho}}\sqrt{A\cdot r_1\cdot 4\pi\rho A r_1} \qquad \text{by (3.11)} \\
&= \frac{\varepsilon}{40}.
\end{aligned}$$

In a similar way, we choose $r^- \in [-2r_1, -r_1]$ so that

$$\int_{\mathscr{V}(r^-)} |U(v)|\, d\lambda(v) \le \frac{\varepsilon}{40}.$$

Now we prepare to choose s^+ and s^-. Let $\gamma_- = \Gamma_2(r^-, \cdot)$ and $\gamma_+ = \Gamma_2(r^+, \cdot)$. Let $l(s) = \operatorname{dist}(\gamma_-(s), \gamma_+(s))$ and let c_s be the geodesic segment from $\gamma_-(s)$ to $\gamma_+(s)$, parametrized by arclength so that $c_s(0) = \gamma_-(s)$ and $c_s(l(s)) = \gamma_+(s)$. Let $\mathscr{T}(s)$ be the set of all unit vectors with footpoint on c_s. Observe that, by Lemma 3.14 we have

$$l(s) \le 3a\cdot(r^+ - r^-) \le 12ar_1 \quad\text{and}\quad \lambda(\mathscr{T}^1(s)) \le 24\pi a r_1. \tag{3.13}$$

We shall say that s is *good* if

$$\int_{\mathscr{T}^1(s)} |U(v)|\, d\lambda(v) \le \frac{\varepsilon}{40}.$$

3.17. Definition. If $s' \leq s''$, let $\mathscr{Q}(s', s'')$ be the closed set bounded by γ_-, γ_+, $c_{s'}$ and $c_{s''}$.

3.18. Lemma. *If* $s' \leq s''$ *and* $\phi : T^1\mathscr{Q}(s', s'') \to [0, \infty)$ *is integrable, then*

$$B^{-1} \sin\Theta \int_{s'}^{s''} \int_{\mathscr{T}^1(s)} \phi(v)\, d\lambda(v) ds \leq \int_{T^1\mathscr{Q}(s', s'')} \phi(v)\, d\mu(v)$$

$$\leq B \int_{s'}^{s''} \int_{\mathscr{T}^1(s)} \phi(v)\, d\lambda(v) ds,$$

where B *is the constant defined in Lemma* 3.3(iii), *and* Θ *is the angle defined in Lemma* 3.15.

Proof. Set $\psi(s, t) = c_s(t)$ and let $y_s(t)$ be the length of the projection of $(\partial\psi/\partial s)(s, t)$ onto the direction orthogonal to c_s. Then y_s is a scalar Jacobi field along c_s that does not vanish for $0 \leq t \leq l(s)$. It suffices to show that for every s, we have $B^{-1}\sin\Theta \leq y_s(t) \leq B$ for $0 \leq t \leq l(s)$.

Since $\dot{\gamma}_-(s) = (\partial\psi/\partial s)(s, 0)$ and

$$\dot{\gamma}_+(s) = \frac{d}{ds}(\psi(s, l(s)) = \frac{\partial\psi}{\partial s}(s, l(s)) + l'(s)\dot{c}_s(l(s)),$$

$y_s(0)$ and $y_s(l(s))$ are the components orthogonal to c_s of the unit vectors $\dot{\gamma}_-(s)$ and $\dot{\gamma}_+(s)$. It follows from Lemma 3.15 that

$$\sin\Theta \leq y_s(0) \leq 1 \quad \text{and} \quad \sin\Theta \leq y_s(l(s)) \leq 1.$$

Thus by Lemma 3.3(iii), $B^{-1}\sin\Theta \leq y_s(t) \leq B$ for $0 \leq t \leq l(s)$. q.e.d.

Choose s_1 so that

$$s_1 > Ar_1. \tag{3.14}$$

It is clear from Lemma 3.13 that $\mathscr{Q}(-s_1, s_1) \supseteq \mathscr{S}(r_1)$. Hence $\mathscr{Q}(s_1, \rho s_1)$, $\mathscr{Q}(-\rho s_1, -s_1)$ and $\mathscr{S}(r_1/2)$ are pairwise disjoint.

3.19. Lemma. *Both of the sets* $\{s \in [s_1, \rho s_1] : s \text{ is not good}\}$ *and* $\{s \in [-\rho s_1, -s_1] : s \text{ is not good}\}$ *have length less than* s_1.

Proof. We consider the first set. Since $\mathscr{Q}(s_1, \rho s_1) \cap \mathscr{S}(r_1/2) = \varnothing$,

$$\begin{aligned}
&B^{-1} \sin\Theta \cdot \frac{\varepsilon}{40} \cdot \operatorname{length}\{s \in [s_1, \rho s_1] : s \text{ is not good}\} \\
&\leq B^{-1} \sin\Theta \int_{s_1}^{\rho s_1} \int_{\mathscr{T}^1(s)} |U(v)|\, d\lambda(v)\, ds \qquad \text{by Lemma 3.18} \\
&\leq \int_{T^1\mathscr{Q}(s_1, \rho s_1)} |U(v)|\, d\mu(v) \\
&\leq \left\{ \int_{T^1\mathscr{Q}(s_1, \rho s_1)} U^2(v)\, d\mu(v) \right\}^{1/2} \left\{ \int_{T^1\mathscr{Q}(s_1, \rho s_1)} 1\, d\mu(v) \right\}^{1/2} \\
&< \frac{\varepsilon B^{-1} \sin\Theta}{40\sqrt{24\pi\rho a B}} \sqrt{B \cdot \rho s_1 \cdot 24\pi a s_1} \qquad \text{by (3.12), (3.13) and Lemma 3.18} \\
&= B^{-1} \sin\Theta \cdot \frac{\varepsilon}{40} \cdot s_1.
\end{aligned}$$

This completes the proof for the first set; the other case is similar.

The previous lemma shows that we can choose $s^+ \in [s_1, 2s_1]$ and $s^- \in [-2s_1, -s_1]$ so that they are good; our desired set $\mathscr{Q}$ is $\mathscr{Q}(s^-, s^+)$.

Let $\alpha(s) = \sphericalangle(\dot{c}_s(0), \dot{\gamma}_-(s))$ and $\hat{\alpha}(s) = \sphericalangle(\dot{c}_s(0), -\dot{\gamma}_-(s)) = \pi - \alpha(s)$. Let $\beta(s) = \sphericalangle(-\dot{c}_s(l(s)), \dot{\gamma}_+(s))$ and $\hat{\beta}(s) = \sphericalangle(-\dot{c}_s(l(s)), -\dot{\gamma}_+(s)) = \pi - \alpha(s)$. Since $\mathscr{Q}(s', s'')$ is a geodesic quadrilateral with interior angles $\hat{\alpha}(s'')$, $\hat{\beta}(s'')$, $\alpha(s')$ and $\beta(s')$, it follows from the Gauss-Bonnet theorem that

$$-\int_{\mathscr{Q}(s', s'')} K(p)\, d\nu(p) = \{2\pi - \hat{\alpha}(s'') - \hat{\beta}(s'') - \alpha(s') - \beta(s')\}.$$

Note that $\partial^1\mathscr{Q}(s', s'') \subseteq \mathscr{V}(r'') \cup \mathscr{V}(r') \cup \mathscr{T}^1(s') \cup \mathscr{T}^1(s'')$. We see from our choice of r^- and r^+, that if s' and s'' are both good, then

$$\int_{\partial^1\mathscr{Q}(s', s'')} |U(v)|\, d\lambda(v) \leq \frac{\varepsilon}{10},$$

which together with Lemma 1.3 implies that if s' and s'' are both good, then

$$\int_{T^1\mathscr{Q}(s', s'')} U^2(v)\, d\mu(v) \leq 2\pi\{2\pi - \hat{\alpha}(s'') - \hat{\beta}(s'') - \alpha(s') - \beta(s')\} + \frac{\varepsilon}{5}. \tag{3.15}$$

In particular, we will have $\int_{T^1\mathscr{Q}} U^2(v)\,d\mu(v) \le \varepsilon$ if

$$\pi - \hat\alpha(s^+) - \hat\beta(s^+) \le \frac{\varepsilon}{5\pi} \quad \text{and} \quad \pi - \alpha(s^-) - \beta(s^-) \le \frac{\varepsilon}{5\pi}. \tag{3.16}$$

Suppose that $\pi - \hat\alpha(s^+) - \hat\beta(s^+) > \varepsilon/(5\pi)$, or equivalently that

$$\pi - \alpha(s^+) - \beta(s^+) < -\frac{\varepsilon}{5\pi}.$$

Under this assumption, we shall show that

$$\int_{s^+}^{\rho s_1} l'(s)\,ds > 12ar_1,$$

which contradicts (3.13). Suppose that $s \in [s^+, \rho s_1]$ is good. Since s^+ is also good, (3.15) with $s' = s^+$ and $s'' = s$ shows that

$$\begin{aligned} 0 &\le 2\pi\{2\pi - \hat\alpha(s) - \hat\beta(s) - \alpha(s^+) - \beta(s^+)\} + \frac{\varepsilon}{5} \\ &< 2\pi\left\{\pi - \hat\alpha(s) - \hat\beta(s) - \frac{\varepsilon}{5\pi}\right\} + \frac{\varepsilon}{5} \end{aligned}$$

and hence

$$\hat\alpha(s) + \hat\beta(s) > \pi + \frac{\varepsilon}{5\pi} - \frac{\varepsilon}{10\pi} = \pi + \frac{\varepsilon}{10\pi}. \tag{3.17}$$

As in §2, we have $l'(s) = \cos\hat\alpha(s) + \cos\hat\beta(s)$ and it follows from this and (3.17) that if $s \in [s^+, \rho s_1]$, then $l'(s) \ge \varepsilon^2/(50\pi^4)$ if s is good and $l'(s) \ge -2$ even if s is not good. We see from these estimates and Lemma 3.19 that, if $0 < \varepsilon < 10\pi$,

$$\begin{aligned} \int_{s^+}^{\rho s_1} l'(s)\,ds &\ge \frac{\varepsilon^2}{50\pi^4}\,\text{length}\{s \in [s^+, \rho s_1] : s \text{ is good}\} \\ &\quad - 2\,\text{length}\{s \in [s^+, \rho s_1] : s \text{ is not good}\} \\ &\ge \frac{\varepsilon^2}{50\pi^4}\{(\rho - 2)s_1 - s_1\} - 2s_1 \\ &> 12s_1 \qquad \text{by (3.10)} \\ &> 12Ar_1 \qquad \text{by (3.14)} \\ &\ge 12ar_1. \end{aligned}$$

Since this contradicts (3.13), we must have $\pi - \hat\alpha(s^+) - \hat\beta(s^+) \le \varepsilon/(5\pi)$. A similar argument shows that $\pi - \alpha(s^-) - \beta(s^-) \le \varepsilon/(5\pi)$. Thus (3.15) is true, which completes the proof of the proposition.

Proposition 3.13 implies that U vanishes almost everywhere. It now follows in the same way as in the proof of Theorem 2.2 that the curvature vanishes everywhere.

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Northwestern University

Universität Göttingen