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DETECTING UNKNOTTED GRAPHS IN 3-SPACE

MARTIN SCHARLEMANN & ABIGAIL THOMPSON

Introduction

Definition. A finite graph Γ is *abstractly planar* if it is homeomorphic to a graph lying in S^2. A finite graph Γ imbedded in S^3 is *planar* if Γ lies on an embedded surface in S^3 which is homeomorphic to S^2.

In this paper we give necessary and sufficient conditions for a finite graph Γ in S^3 to be planar. (All imbeddings will be tame, e.g., PL or smooth.) This can be viewed as an unknotting theorem in the spirit of Papakyriakopolous [12]: a simple closed curve in S^3 is unknotted if and only if its complement has free fundamental group.

[12] can be viewed as a solution for Γ having one vertex and one edge. In [6] or [3, §2.3] this is extended: a figure-eight (bouquet of two circles) in S^3 is planar if and only if its complement has free fundamental group and each circle is unknotted. Gordon [4] generalizes this to all graphs with a single vertex: a bouquet of circles Γ in S^3 is planar if and only if its complement and that of any subgraph of Γ has free fundamental group. If fact, Gordon shows that this generalization of [6] is a fairly direct consequence of Jaco's handle addition lemma [8]. Far more difficult is Gordon's extension to the case in which Γ has two vertices, and no loops. We will require only the solution of the one-vertex case for our proof.

We will show:

7.5. Theorem. *A finite graph* $\Gamma \subset S^3$ *is planar if and only if*

(i) Γ *is abstractly planar,*

(ii) *every graph properly contained in* Γ *is planar, and*

(iii) $\pi_1(S^3 - \Gamma)$ *is free.*

There is an alternative formulation:

Theorem. *A finite graph* $\Gamma \subset S^3$ *is planar if and only if*

(a) Γ *is abstractly planar and*

(b) *for every subgraph* $\Gamma' \subseteq \Gamma$, $\pi_1(S^3 - \Gamma')$ *is free.*

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The equivalence of this formulation follows easily by induction: conditions (a) and (b), if true for Γ, are true for any subgraph of Γ.

Theorem 7.5 has been conjectured by J. Simon [15]. He and Wolcott [16] demonstrated it in two cases (the handcuff and the double-theta-curve) not covered by Gordon's theorem. It is fairly easy to show that no two of the conditions (i), (ii), and (iii) suffice to ensure planarity:

0.1. Example. An embedding of K_5 in S^3 satisfying (ii) and (iii) but not (i).

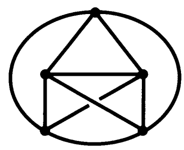

FIGURE 0.1

0.2. Example. An embedding of a theta-curve satisfying (i) and (iii) but not (ii).

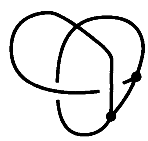

FIGURE 0.2

0.3. Example [10]. An embedding of a theta-curve satisfying (i) and (ii) but not (iii).

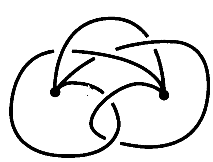

FIGURE 0.3

We have the following corollary, of independent interest.

7.6. Corollary. *There is an algorithm to determine if a graph in* S^3 *is planar.*

There are two other versions of 7.5 available: Condition 7.5(iii) can be replaced with the condition that the complement of a regular neighborhood of Γ is a ∂-reducible. This vastly improves the efficiency of the algorithm of 7.6 (see [18] for details). Alternatively, 7.5(ii) and (iii) can be replaced with the following condition: There is an edge e in Γ not a loop, such that the graph Γ/e obtained by collapsing e and the graph $\Gamma - e$ are both planar (see [14] for some applications).

The bulk of the argument for 7.5 consists of induction lemmas for various types of graphs: e.g., §1 treats graphs Γ containing "cut" edges, with §2 providing a technical lemma needed in that proof. The main theorem is not proven until §7, where the proof consists mostly of references to previous cases.

1. Cut edges

1.1. Definitions. Let Γ be a finite graph in S^3 with vertices $v(\Gamma)$ and edges $e(\Gamma)$. Let $\eta(\Gamma)$ denote a handlebody neighborhood of Γ, with interior $^\circ\eta(\Gamma)$. $\eta(\Gamma)$ is the union of three-cells with disjoint interiors constructed as follows: For each vertex v in Γ let $\eta(v) = B^3$ be a three-cell neighborhood of v in S^3, transverse to the edges of Γ, so that $\eta(v) \cap \Gamma = \operatorname{cone}(\partial\eta(v) \cap \Gamma)$. Let $\eta^0(\Gamma) = \bigcup\{\eta(v) \mid v \in \Gamma\}$. For each edge $e \in \Gamma$ let $\eta(e)$ be a three-cell with a product structure $\eta(e) = B^2 \times I$ such that $\eta(e) \cap \Gamma = e - \eta^0(\Gamma) = \{0\} \times I$, and $\eta(e) \cap \eta^0(\Gamma) = B^2 \times \{\partial I\}$. These latter disks are called the *attaching disks* of $\eta(e)$. Any $B^2 \times \{\text{point}\}$ (or $\partial B^2 \times \{\text{point}\}$) is a *meridian* disk $\overline{\mu}(e)$ (or circle $\mu(e)$) of $\eta(e)$. Let $\eta^1(\Gamma) = \bigcup\{\eta(e) \mid e \in \Gamma\}$. An embedded curve in $\partial\eta(\Gamma)$ is *normal* if its interior intersects meridian circles only transversally and intersects $\partial\eta^0(\Gamma)$ only in arcs essential in $\partial\eta^0(\Gamma) - \eta^1(\Gamma)$. Any curve in $\partial\eta(\Gamma)$ is isotopic rel ∂ to a normal curve, and this isotopy does not increase the intersection number with any meridian.

A handlebody neighborhood $\eta(\Gamma)$ of Γ provides a handlebody neighborhood for any subgraph Γ' of Γ; just take the union of cells associated to vertices or edges in Γ'. If Γ lies in a sphere P (so is planar) one can define similarly a handlebody neighborhood $\nu(\Gamma)$ in P, where 0-handles are disks and 1-handles are homeomorphs of $I \times I$. A *standard* handlebody neighborhood for $\Gamma \subset P \subset S^3$ is a handlebody neighborhood $\eta(\Gamma)$ which is a bicollar $\nu(\Gamma) \times [-1, 1]$ of a handlebody neighborhood $\nu(\Gamma)$ in P. In particular, P bisects each handle of a standard handlebody neighborhood and for any vertex v in Γ, $P \cap \eta(v)$ is the cone to v of $P \cap \partial\eta(v)$.

For M a compact manifold (typically 0 or 1-dimensional), $|M|$ denotes the number of components of M.

1.2. Definitions. Γ is *split* if $S^3 - \Gamma$ is reducible. Γ is *decomposable* if there is a vertex v such that $\partial\eta(v) - \eta^1(\Gamma)$ compresses in $S^3 - {}^\circ\eta(\Gamma))$.

1.3. Lemma. *If Γ is split or decomposable, and every graph properly contained in Γ is planar, then Γ is planar.*

Proof. A reducing sphere for $S^3 - \Gamma$ divides S^3 into two balls, each of which contains a subgraph of Γ. Each subgraph is planar, so can be imbedded in a sphere in the ball. Tube together the spheres to get a sphere containing Γ.

A decomposing disk $(D, \partial D) \subset (S^3 - {}^\circ\eta(\Gamma), \partial\eta(v) - \eta^1(\Gamma))$ divides $S^3 - \eta(v)$ into two balls B_1 and B_2. Then $\eta(\Gamma) \cup B_2$ and $\eta(\Gamma) \cup B_1$ can be viewed as handlebody neighborhoods of subgraphs Γ_1 and Γ_2 of Γ, with $\Gamma_1 \cap \Gamma_2 = v$. Since Γ_1 and Γ_2 are planar, there are disjoint disks D_1 and D_2 in $S^3 - \eta(v)$ containing $\Gamma_1 - \eta(v)$ and $\Gamma_2 - \eta(v)$. Piping these together produces a single disk containing $\Gamma - \eta(v)$; coning the boundary of the disk gives a sphere containing Γ.

1.4. Definitions. For e an edge of a graph Γ, let $\Gamma - e$ denote the graph obtained from Γ by removing the interior of e. Let $\hat{e}$ denote the subgraph of Γ which is the union of e and its incident vertex (or vertices). Let Γ/e denote the graph obtained from Γ by identifying $\hat{e}$ to a point, which is then a vertex of Γ/e. If $\Gamma \subset S^3$ and e is not a loop, then $\hat{e}$ is a tame arc in S^3. Hence $S^3/\hat{e} \cong S^3$, so the imbedding $\Gamma \subset S^3$ gives rise to an imbedding $\Gamma/\hat{e} \subset S^3$.

A vertex v in a connected graph Γ is a *cut vertex* if Γ is the union of two subgraphs Γ_0 and Γ_1, each containing at least one edge, such that $\Gamma_0 \cap \Gamma_1 = v$ [19]. An edge e in a connected graph Γ is a *cut* edge if e is not a loop and the vertex $\hat{e}/e$ is a cut vertex of Γ/e. Equivalently, e is a cut edge if it is not a loop and $\Gamma - \hat{e}$ is a disconnected topological space. (The graph $\Gamma - e$ may still be connected.)

1.5. Examples. (a) If Γ is a connected graph properly containing a two-cycle (i.e., a bigon), then either edge of the two-cycle is a cut edge.

(b) suppose Γ is a connected graph containing a cut vertex v and at least one other vertex, of valence > 1. Then there is an edge e with one end on the cut vertex and the other at another vertex of valence > 1. Then e is a cut edge. In particular:

(c) If a connected graph Γ contains a loop and a vertex with valence ≥ 2 other than the base of the loop, then Γ contains a cut edge.

1.6. Proposition. *Suppose e is a cut edge in a connected graph $\Gamma \subset S^3$ such that*

(a) *the graph $\overline{\Gamma} = \Gamma/e$ is planar,*

(b) *every graph properly contained in Γ is planar.*

Then Γ is planar.

Proof. Let $v_\pm$ be the distinct vertices incident to e. Denote by v_0 the cut vertex $\hat{e}/e$, the image of $\hat{e}$ in $\overline{\Gamma}$. Since $\overline{\Gamma}$ is planar, there is a compressing disk for $\partial\eta(v_0) - \overline{\Gamma}$ in $S^3 - \eta(\overline{\Gamma})$. Hence there is a compressing disk for $\partial\eta(\hat{e})$ in $S^3 - \eta(\Gamma)$. Choose D to be such a compressing disk for which ∂D is normal in $\eta(\Gamma)$ and $|\partial D \cap \mu(e)|$ is minimal for every meridian circle $\mu(e)$ of $\eta(e)$.

If $|\partial D \cap \mu(e)| = 0$, then ∂D lies entirely in $\partial\eta(v_+)$, say. Then Γ is decomposable, hence planar. So henceforth we will

1.6.1. Assume $|\partial D \cap \mu(e)| > 0$.

By assumption, $\Gamma' = \Gamma - e$ is planar, so lies in a two-sphere $P \subset S^3$. We may take each $\eta(v_\pm)$ to be a ball which is bisected by P. In particular, $P - \eta(v_+ \cup V_-)$ is an annulus in $S^3 - \eta(v_+ \cup v_-)$ containing all of $\Gamma' - \eta(v_+ \cup v_-)$. Let W denote the closure of $S^3 - \eta(v_+ \cup v_-)$ with boundary components the spheres $V_\pm = \partial\eta(v_\pm)$. Let Γ_W denote the one-complex $\Gamma \cap W$, and similarly for Γ'_W and P_W. Note that $D \subset W$, with part of ∂D lying on each of $V_\pm$ (since $|\partial D \cap \mu(e)| > 0$) and part on $\partial\eta(e)$. The interior of D is disjoint from Γ (not just Γ').

Let Q be a properly imbedded finite union of disks and at most a single annulus in W, in general position with respect to e and D, chosen so that

(a) $\Gamma'_W \subset Q$,

(b) no component of Q is disjoint from Γ', and

(c) $|D \cap Q|$ is minimal among all Q satisfying (a) and (b).

Note that P_W, for example, satisfies (a) and (b).

Claim. If e is disjoint from Q or any disk component of Q, Γ is planar.

Proof of Claim. If Q contains a disk q disjoint from e, push $\Gamma_W \cap q$ slightly into the component of $W - q$ not containing e. Since $\partial q \subset V_+$, say, q is a decomposing disk Γ at v_+ and the proof concludes as above. So suppose Q is an annulus and e is disjoint from Q. Let W_0 be the component of $W - Q$ in which e does not lie. Γ'_W has at least two components, since e is a cut edge.

If any component Γ_0 of Γ'_W is incident to only one of $V_\pm$, say V_+, push Γ_0 slightly into W_0. Then a disk with boundary in V_+ can be imbedded between Γ_0 and Q, hence between Γ_0 and the result of Γ. This is a decomposing disk for Γ, so, by 1.3, Γ is planar.

On the other hand, if every component of Γ'_W is incident to both of $V_\pm$, then there is a path γ in Γ'_W from V_+ to V_-. Since Γ is connected, every component of $Q - \Gamma'_W \subset Q - \gamma$ must be a disk. Since Γ'_W is disconnected, there is a component Q_0 of $Q - \Gamma'_W$ whose boundary ∂Q_n intersects more than one component of Γ'_W. Each arc component of $\partial Q_0 \cap \Gamma'_W$ must be a spanning arc of the annulus Q, since every component of Γ'_W is incident to both of $V_\pm$. Pushing the interior of one of these arcs slightly into Q_0 gives a spanning arc α in Q disjoint from Γ'_W. Now $\gamma \cup e$ is a subgraph of Γ, hence is unknotted. γ is parallel to α in the annulus Q, so $\alpha \cup e$ is unknotted. An unknotting disk can be found disjoint from Q, and provides an isotopy from e to α. After the isotopy, $\Gamma \subset Q$, so Γ is planar. This proves the claim.

Following the claim, it suffices to derive a contradiction if we

1.6.2. Assume that $|e \cap Q| = p > 0$, and that e intersects every disk component of Q.

Label the points in $e \cap Q$ by $e_1, \cdots e_p$ in order from V_- to V_+. Q can be isotoped so it intersects $\eta(e)$ in meridia, whose boundary circles we similarly denote $\mu_1, \cdots, \mu_p$.

$Q \cap D$ is a one-manifold. If it contained a simple curve, then an innermost such curve c would bound a disk F in D. Consider the union U of a collar neighborhood of Q on the side away from F and a bicollar neighborhood of F. ∂U is the union of a surface parallel to Q and a surface $Q' \supset \Gamma'_W$. If c is essential in Q, then Q' is now a union of disks. Discard any disjoint from Γ'_W. Q' still satisfies (a) and (b) in our definition of Q, but has at least one fewer component of intersection with D. Since D was chosen to minimize $|Q \cap D|$, this is impossible. If c is inessential in Q, then Q' is homeomorphic to Q union a sphere. The sphere is the union of a disk parallel to F and the disk F' in Q which c bounds. Since $c \cap \Gamma = \varnothing$ and each component of Γ' contains either v_+ or v_-, it follows that $\Gamma' \cap F' = \varnothing$. Then Γ' is disjoint also form the sphere, so discard it. Again we get the contradiction that Q' still satisfies (a) and (b), but has at least one fewer component of intersection with D. We conclude that $Q \cap D$ contains no simple closed curves. In particular

1.6.3. $|Q \cap \partial D| = 2|Q \cap D|$ must be minimal.

A point in $\partial D \cap Q$ either lies in $\partial_\pm Q = \partial Q \cap V_\pm$ or on one of the meridian circles μ_i. Consider an outermost arc α of $\partial D \cap Q$ in D. α cuts off from D a disk F such that interior(F) is disjoint from Q and $\partial F = \alpha \cup \beta$, for β some subarc of ∂D. The ends of α either both lie in $\partial_\pm Q$, or one end lies in $\partial_\pm Q$ and one end on a μ_i, or both ends lie on the μ_i. Consider each possibility in turn:

If both ends of α lie in ∂Q, then β must not intersect any of the meridia of $\eta(e)$, so we may assume β lies entirely in V_+, say. Consider the union U of a collar neighborhood of Q on the side away from F and a bicollar neighborhood of F. ∂U is the union of a surface parallel to Q

and a surface $Q' \supset \Gamma'_W$. Discard any component of Q' which is disjoint from Γ'_W. Q' still satisfies (a) and (b) in our definition of Q, but has at least one fewer component of intersection with D. This contradicts (c).

Suppose α has one end on V_+ say, and other end on a meridian μ_i. The arc β is then an arc, disjoint from all other meridian, running between μ_i and $\partial Q \subset V_+$. Hence $i = p$. The arc β it self consists of two arcs, β_e running from μ_p to the end of $\eta(e)$ in V_+ and β_+ running from $\eta(e)$ to ∂Q in V_+. The arc β_e and the subarc of e lying between e_p and V_+ are parallel in $\eta(e)$; attach to F the rectangle in $\eta(e)$ lying between them, replacing β_e in ∂F with the parallel section of e. As above, consider the union U of a collar neighborhood of Q on the side away from F and a bicollar neighborhood of F. ∂U is the union of a surface parallel to Q and a surface $Q' \supset \Gamma'_W$ (in fact $U \cong Q \times I$). But e no longer intersects Q' at e_p. This eliminates $|\partial D \cap \mu(e_p)|$ points of intersection of ∂D with Q. ∂D intersects Q' in at most $|\partial D \cap \mu(e)| - 1$ points near the end of e in V_+, since $Q' \cap \beta_+ = \varnothing$, and no longer intersects Q' at the end of α in V_+. Hence $|Q' \cap \partial D| \leq |Q \cap \partial D| - 2$, contradicting 1.6.3.

We conclude that α has one end on μ_i and the other end on μ_j for some $1 \leq i, j \leq p$. The arc β is disjoint from the meridia and connects μ_i to μ_j. Hence $|i - j| \leq 1$. If $i = j \pm 1$, then proceed much as above: Attach to F a rectangle in $\eta(e)$ lying between β and the subarc of e lying between e_i and e_j. Consider the union U of a collar neighborhood of Q on the side away from F and a bicollar neighborhood of F. ∂U is the union of a surface parallel to Q and a surface $Q' \supset \Gamma'_W$ (again, $U \cong Q \times I$). But e does not intersect Q' at e_i or e_j, so $|Q \cap \partial D|$ has been reduced by at least $2|\partial D \cap \mu(e)|$, contradicting 1.6.3. Hence $i = j$. If $i \neq 1$ or p, then β must lie entirely between μ_i and $\mu_{i\pm 1}$ on $\partial\eta(e)$, and so be inessential in that annulus. Then ∂D is not in normal form (alternatively, an isotopy of ∂D near β reduces $|Q \cap \partial D|$ by 2, violating 1.6.3).

Hence $i = j = 1$ (or p). Moreover, the argument shows that β must contain a subarc lying in V_- (or V_+). This subarc must be essential in $V_- - \eta(\Gamma)$, hence in $V_- - \partial Q$, since ∂D is normal in $\partial\eta(\Gamma)$. Therefore ∂Q must have more than one component on V_-; in particular

1.6.4. Q contains disk components.

[Note that the contradiction is now complete if Γ is a graph in which all edges have one end incident to each of $V_\pm$.]

Let $\Lambda \subset D$ be the set of arcs $D \cap Q$. An end of such an arc either lies in $\partial_\pm Q = \partial Q \cap V_\pm$ or it lies in some μ_i. To any end of an arc of Λ lying in μ_i assign the label i, $1 \leq i \leq p$, and to an end lying in $\partial_{\pm} Q$ assign the label $\pm$. We have seen that

1.6.5. Any outermost arc in D has both ends labelled 1 or both ends labelled p.

1.6.6. Claim. For every label i, $1 \leq i \leq p$, there is a component of Λ which has both ends labelled i.

Proof. This is the main point of §2; we defer the proof to Lemma 2.3.

The arcs $\Lambda = D \cap Q$, when viewed in Q, are the edges of a graph Λ' in Q whose edges are disjoint from Γ' and whose vertices are $\{v_+, v_-\} \cup \{e_1, \cdots, e_p\}$. The latter p-vertices $\{e_1, \cdots, e_p\} = e \cap Q$ are called ε-*vertices.* An arc in Λ with ends labelled i and j corresponds in Λ' to an edge running from e_i to e_j. We have from 1.6.6 that every ε-vertex is the base of a loop in $\Lambda' \subset Q$.

Let q be a disk component of Q (one exists by 1.6.4). By 1.6.2 there are ε-vertices on q. Choose an innermost loop in q based at an ε-vertex e_i. The interior of the loop is disjoint from Γ' since Γ is connected and from e since any ε-vertex is the base of a loop. Hence the interior of the loop is an empty disk E. The union of D and E along the arc of Λ forming the loop has a regular neighborhood whose boundary consists of three disks, one parallel to D and the others D' and D'' each having boundaries intersecting the meridian of $\eta(e)$ at e_i in fewer points than did D. At least one of D' or D'' must be a compressing disk for $\partial\eta(\hat{e})$ in $S^3 - \eta(\Gamma)$ since D was. This contradicts our choice of D.

2. Outermost forks

2.1. Definitions [13]. Let T be a finite tree. An *outermost vertex* of T is a vertex of valence one. A *fork* is a vertex of valence ≥ 3. If T has forks, let F be the collection of forks of T, and remove from T all components of $T - F$ which contains an outermost vertex of T. An outermost vertex of the resulting tree (possibly just a vertex) is called an *outermost fork* of T. If ν is an outermost fork, then all but at most one component of $T - \nu$ contains no forks. Call each of these components a *tine* of T. By a *tine* of T we mean either a tine of an outermost fork, or all of T if T is linear and an end of T is specified. Define the *distance* between two vertices in T to be the number of edges in the path between them. Define the *distance* from a vertex v to an edge ε to be the distance from v to the nearest end of ε. Hence if ε is incident to v, the distance is zero.

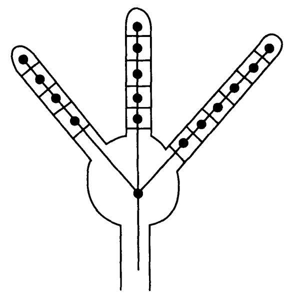

FIGURE 2.1

Suppose the tree T is imbedded in a disk. If ν is an outermost fork, then two tines are *adjacent* if a small circle around ν in the plane contains an arc intersecting only those two components of $T - \nu$.

Now let Q and D be as in §1 and consider the tree T in D constructed from $\Lambda = Q \cap D$ as follows. For vertices of T take a single point ν in the interior of each component of $D - \Lambda$. Connect with edges those vertices representing components of $D - \Lambda$ which have a common component of Λ in their closures. To each $\lambda \in \Lambda$ there then corresponds a dual edge in T (see Figure 2.1).

Let Φ be a tine of T. The outermost edge of T is dual to an outermost arc of Γ in D, which has both ends labelled either both 1 or both p. In the former case, say, Φ is a 1-tine, in the latter, a p-tine.

We have the following:

2.2. Lemma. *Let Φ be a* 1*-tine* (*resp. p-tine*) *of Φ. Then the component of Λ dual to the edge ε in Φ a distance $d < p$ from the end of Φ has both ends labelled $d + 1$* (*resp. $p - d$*).

Proof. Let F be the cell corresponding to the end of the 1-tine containing ε, and λ be the component of Λ dual to ε. According to the remarks preceding 1.6.4, $\partial F = \alpha \cup \beta$, where β is an arc running from e_1 to V_-, around V_-, and back up to e_1. α has both ends labelled 1. Now the arc λ divides D into two disks; let D' be the one which contains F. $\partial D'$ is the union of λ, β, and two other arcs β' and β'', each of which runs from an end of β at e_1 to an end of λ (see Figure 2.2, next page). Since $|\partial D \cap Q|$ has been minimized (1.6.3), both β' and β'' must run straight up $\partial\eta(e)$, crossing in order $e_2, e_3, \cdots$. By assumption, the path from the end of Φ to ε contains $d + 1 \le p$ edges, if we include ε. Hence each of β' and β'' begin at e_1 and end at e_{d+1}, so both ends of λ are labelled $d + 1$.

2.3. Lemma. *For every label i, $1 \le i \le p$, there is a component of Λ which has both ends labelled i.*

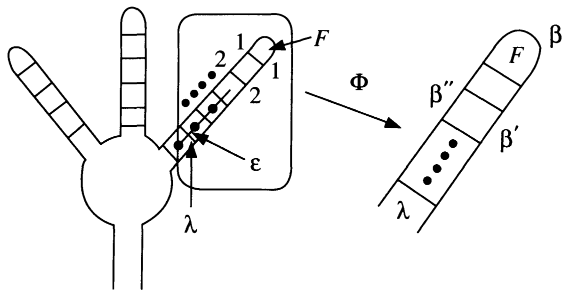

FIGURE 2.2

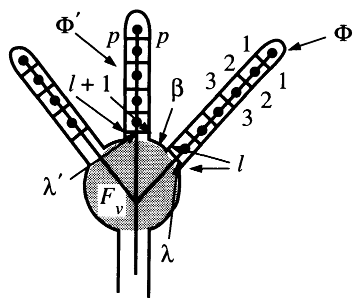

FIGURE 2.3

Proof. If some tine has length $\geq p$ (e.g., T itself if T is linear), then 2.2 shows the outermost p edges of T correspond in Λ to arcs with both ends having the same label, and with all labels from 1 to p included.

If all tines have length $< p$, then T is not linear. Consider two adjacent tines Φ and Φ' of an outermost fork ν, and suppose they both have lengths d and $d' < p$. Let F_ν be the component of $D - \Lambda$ corresponding to ν. The edges of Φ and Φ' incident to ν correspond to subarcs λ and λ' of ∂F which are component of Λ; we know from 2.2 that both ends of λ (λ') have the same label l (l') (see Figure 2.3).

We know from above that $l = d$ or $p - d + 1$, and similarly for l'. Since the tines are adjacent, there is a component β of $\partial F \cap \partial D$ running from an end of λ to an end of λ'. Hence (with no loss of generality) $l' = l + 1$, and β runs along $\partial\eta(\varepsilon)$ from e_l to e_{l+1}. This means that Φ must be a 1-tine, Φ' must be a p-tine, $d = l$, and $d' = p - l$. Then the d edges in Φ (resp. Φ') are dual to arcs in Λ each having the same label on both ends, with labels running from 1 to d (resp. $d + 1$ to p).

This completes the proof of Lemma 2.3, hence of the proofs of Claim 1.6.6 and Proposition 1.6.

3. The tetrahedral graph

We begin with a familiar observation from "tangle theory" [2].

3.1. Lemma. *Suppose $\gamma \subset S^3$ is the unlink of two components, $S \subset S^3$ is a two-sphere dividing S^3 into two three-balls $B_\pm$, and γ intersects each of $B_\pm$ in an unknotted pair of arcs. Then there is a unique essential simple closed curve in $S - \gamma$ which bounds a disk in $B_+ - \gamma$. It also bounds a disk in $B_- - \gamma$.*

Proof. This is best seen by considering the two-fold branched cover $S^1 \times S^2$ of γ. The link γ lifts to $\tilde{\gamma}$, a pair of curves each of which is an equator of sphere fiber. S lifts to a Heegaard splitting F of $S^1 \times S^2$ into solid tori $T_\pm = S^1 \times D^2_\pm$. A proper disk D in B_+ is essential if and only if D separates the strands of $\gamma \cap B_+$. Such a disk lifts to a meridian of T_+ disjoint from $\tilde{\gamma}$. The same is true for disks in B_-. But a curve in F bounds a meridian of T_+ disjoint from $\tilde{\gamma}$ if and only if it bounds a meridian of T_- disjoint from $\tilde{\gamma}$.

3.2. Corollary. *Let S be a two-sphere in S^3 dividing S^3 into two balls $B_\pm$. Suppose τ is an unknotted pair of arcs in B_+. Then, up to isotopy rel end points, there is a unique imbedded pair of curves σ in S such that $\partial\sigma = \partial\tau$ and $\sigma \cup \tau$ is the unlink of two components.*

Proof. Since τ is unknotted, it is isotopic rel end points to some pair of curves σ in S; then $\sigma \cup \tau$ is clearly the unlink. Suppose σ' is another pair of curves in S such that $\partial\sigma' = \partial\tau$ and $\sigma' \cup \tau$ is the unlink of two components. There is a simple closed curve c (c') in S separating the pair of curves σ (σ'). Push σ slightly into B_- and apply 3.1: the curve c bounds an essential disk in $B_- - \sigma$, hence c bounds an essential disk in $B_+ - \tau$. Similarly c' bounds an essential disk in $B_+ - \tau$. But a standard innermost disk, outermost arc argument shows that such a disk is unique up to isotopy in $B_+ \operatorname{rel} \tau$. Hence c and c' are isotopic rel $\partial\tau$. But then $c = c'$ divides S into two disks, each of them containing a single arc of σ and σ'. Since in a disk any two imbedded arcs with the same end points are isotopic rel ∂, σ' is isotopic to $\sigma \operatorname{rel} \partial\sigma$ (via an isotopy disjoint from $c = c'$).

3.3. Theorem. *Let $\Gamma \subset S^3$ be homeomorphic to the one-skeleton of a tetrahedron, and let e be an edge of Γ. If Γ/e and $\Gamma - e$ are planar, so is Γ.*

Proof. Let $\overline{\Gamma} = \Gamma/e$ and $\Gamma' = \Gamma - e$. Denote the end vertices of e by w_l and w_r. Let f be the edge of Γ which is disjoint from e, with end vertices $v_\pm$. Denote by $\varepsilon_{l\pm}$ $(\varepsilon_{r\pm})$ the four other edges, with ends

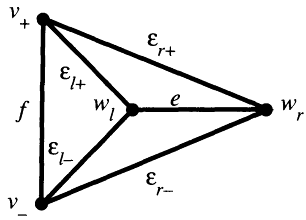

FIGURE 3.1

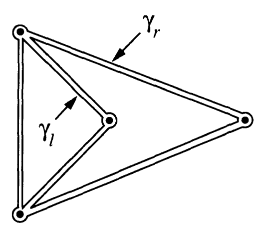

FIGURE 3.2

respectively at w_l (w_r) and $v_\pm$ (see Figure 3.1).

Recall that $\hat{e}$ denotes the subgraph $\{e \cup w_l \cup w_r\}$ of Γ. Choose an imbedding of Γ' in the sphere P and a standard handlebody neighborhood $\eta(\Gamma')$ of Γ' in S^3. $\eta(\Gamma') \cap P$ is a three punctured sphere and $\partial\eta(\Gamma') \cap P$ consists of three simple closed curves. Let ∂_l (resp. γ_r) be the curve which runs along $\partial\eta(f)$ and $\partial\eta(\varepsilon_{l\pm})$ (resp. $\partial\eta(\varepsilon_{r\pm})$) and let γ be the unlink $\gamma_l \cup \gamma_r$ (see Figure 3.2). The disks on which $\eta(e)$ is attached to $\eta(\Gamma')$ can be taken to be disjoint from the curves γ_l and γ_r, so henceforth we will regard γ as lying in $\partial\eta(\Gamma)$.

Consider now the planar graph $\overline{\Gamma}$. There vertices of $\overline{\Gamma}$ are $v_\pm$ and a third vertex $W_0 = \hat{e}/e$. The edges of $\overline{\Gamma}$ are f, $\varepsilon_{l\pm}$, and $\varepsilon_{r\pm}$. Choose an imbedding of $\overline{\Gamma}$ in a sphere P and a standard handlebody neighborhood $\eta(\overline{\Gamma})$ of $\overline{\Gamma}$ in S^3. Since $\overline{\Gamma} = \Gamma/e$, the three-manifolds $\eta(\overline{\Gamma})$ and $\eta(\overline{\Gamma})$ are isotopic in S^3, for $\eta(\Gamma)$ is a handlebody neighborhood of $\eta(\overline{\Gamma})$ if we set $\eta(\hat{e}) = \eta(w_0)$. Then identify corresponding handles in $\eta(\Gamma - \hat{e})$ and $\eta(\overline{\Gamma} - w_0)$. In particular, $\eta(\Gamma) = \eta(\overline{\Gamma})$ and so we can regard γ as lying on $\eta(\overline{\Gamma})$.

$\eta(\overline{\Gamma}) \cap P$ is a four-punctured sphere, and $\partial\eta(\overline{\Gamma}) \cap P$ consists of four simple closed curves. Let $\overline{\gamma}_l$ (resp. $\overline{\gamma}_r$) be the curve which runs along $\partial\eta(f)$ and $\partial\eta(\varepsilon_{l\pm})$ (resp. $\partial\eta(\varepsilon_{r\pm})$), and let $\overline{\gamma} = \overline{\gamma}_l \cup \overline{\gamma}_r$, also the unlink (see Figure 3.3). The curves $\overline{\gamma}$ and γ both intersect the four-punctured

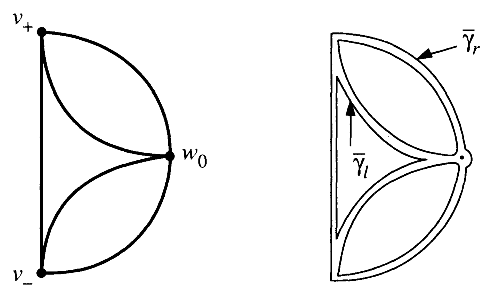

FIGURE 3.3

sphere $\partial\eta(\overline{\Gamma})-\eta(w_0)$ in two arcs, one running between the attaching disks of each of $\eta(\varepsilon_{r\pm})$ on $\eta(w_0)$ and the other running between the attaching disks of each of $\eta(\varepsilon_{l\pm})$ on $\eta(w_0)$. Each component of γ and of $\overline{\gamma}$ also intersects a meridian $\mu(f)$ of $\eta(f)$ in exactly one point. Hence γ and $\overline{\gamma}$ differ in $\partial\eta(\overline{\Gamma})-\eta(w_0)$ by at most some twists around $\mu(f)$ and some twists around the attaching disks. The latter twists can be pushed into $\eta(w_0)$ and so off of $\partial\eta(\overline{\Gamma})-\eta(w_0)$. Now consider the choice of imbedding of $\overline{\Gamma}$ in the sphere P: Each bigon of $\overline{\Gamma}$ with one end at w_0 and other end at $v_\pm$ may be rotated about w_0 and $v_\pm$. The effect is to alter $\overline{\gamma}$ by a twist around $\mu(f)$. It follows that the imbedding of $\overline{\Gamma}$ in P can be chosen so that $\gamma=\overline{\gamma}$ on $\partial\eta(\overline{\Gamma})-\eta(w_0)$ and the from 3.2 that also $\gamma\cap\partial\eta(w_0)$ is isotopic in $\partial\eta(w_0)$ to $\overline{\gamma}\cap\partial\eta(w_0)$ rel end points. (Note that this last isotopy absorbs a difference in twists around the attaching disks of $\eta(\varepsilon_{l\pm})$ and $\eta(\varepsilon_{r\pm})$ in $\partial\eta(w_0)$ because the isotopy may sweep across these attaching disks. In particular, this isotopy does not necessarily lie entirely on $\partial\eta(\overline{\Gamma})$.)

Return now to $\eta(\Gamma')$ and $\eta(\Gamma)$. A graph G' isotopic to Γ' can be recovered from the unlink $\gamma\subset\eta(\Gamma)$ as follows: Remove the arc $\gamma_r\cap\partial\eta(f)$ and attach arcs which connect the points of intersection of γ_r and γ_l in each of the two attaching disks of $\eta(f)$ at $\eta(v_\pm)$. A graph G isotopic to Γ can then be recovered from G' by attaching an unknotted arc in the ball $\eta(\hat{e})$ with one end on each of $\gamma_l\cap\partial\eta(\hat{e})$ and $\gamma_r\cap\partial\eta(\hat{e})$.

Let us view how this construction appears in $\eta(\overline{\Gamma})$, using the facts that $\gamma=\overline{\gamma}$ outside of $\eta(w_0)=\eta(\hat{e})$, and that the pair of arcs $\gamma\cap\partial\eta(w_0)$ is isotopic in $\partial\eta(w_0)$ to $\overline{\gamma}\cap\partial\eta(w_0)$ rel end points. First note that $\eta(f)$ intersects P in a rectangle $I\times I$, with $\partial I\times I$ corresponding to $\eta(f)\cap\overline{\gamma}=\eta(f)\cap\gamma$ and with $I\times\partial I$ corresponding to two arcs, one in each of the attaching disks of $\eta(f)$, connecting the points of intersection of $\overline{\gamma}_l$ and $\overline{\gamma}_r$ in each of the attaching disks. Thus a graph G isotopic to

Γ can be obtained from $\overline{\gamma}$ by replaced the arc $\{1\} \times I$ with $I \times \partial I$ in $I \times I = \eta(f) \cap P$, and attaching an unknotted arc in $\eta(w_0)$ connecting the two arc components of $\overline{\gamma} \cap \eta(w_0$. But $\overline{\gamma} \cap \eta(w_0)$ consists of two arcs in the boundary of the disk $P \cap \eta(w_0)$, so they can be connected by an unknotted arc α in the disk $P \cap \eta(v_0)$. Since $G = \overline{\gamma} \cup \alpha \subset P$, Γ is planar.

3.4. Remark. The argument above, while apparently God-given for the proof of the tetrahedral graph, in fact generalizes. Indeed, our original proof of 7.5 consisted of two parts: Graphs with cut edges were covered much as in §1. Graphs without cut edges, but with ≥ 4 vertices, were covered by a generalization of Lemma 2.1 above to braids of many strands. This generalization, in turn, can be proven from 7.5. Details appear in [14].

4. Special three-cycles

4.1. Definition. Let Γ be a graph in S^3 and σ a cycle in Γ. If there is an imbedded disk D is S^3 for which $D \cap \Gamma = \partial D = \sigma$ we say σ is *flat*. D is called a *flattening disk* for σ.

4.2. Definition. An imbedded three-cycle σ in a graph Γ is *special* if at least one of its vertices (called the *apex*) has valence $= 3$. The edge not incident to the apex is called the *base* of the three-cycle.

4.3. Lemma. *Suppose Γ is a graph in S^3 containing a special three-cycle σ with base e. If $\Gamma - e$ is planar and σ is flat, then Γ is planar.*

Proof. Let $\eta(\Gamma')$ be a standard handlebody neighborhood for the graph $\Gamma' = \Gamma - e$ imbedded in a sphere P. Let v denote the apex of σ, $w_\pm$ the other two vertices, and $f_\pm$ the edges of σ with ends on v and w_+ respectively. Let $(D, \partial D) \subset (S^3, \sigma)$ be a flattening disk for σ, so $D \cap \Gamma = \partial D = \sigma$. We can isotope D near ∂D so that $\gamma = D \cap \eta(\Gamma')$ is a normal curve running from $e \cap \partial\eta(w_+)$ to $e \cap \partial\eta(w_-)$.

Let N be the three-holed sphere in $\eta(\Gamma')$ constructed by attaching the annuli $\partial\eta(f_\pm) - \eta^0(\Gamma')$ to the three-holed sphere $\partial\eta(v) - \eta^1(\Gamma')$. The normal curve γ consists of three arcs: $\gamma_0 = \gamma \cap N$ and the two arcs $\gamma_\pm = \gamma \cap [\partial\eta(w_\pm) - \eta(f_\pm)]$. Since; $\eta(\Gamma')$ is a standard handlebody neighborhood of Γ', $P \cap N$ also contains a proper arc $\overline{\gamma}_0$ running from $\partial N \cap \eta(w_+)$ to $\partial N \cap \eta(w_+)$. Since N is a three-holed sphere, we may isotope γ_0 (perhaps changing $\gamma_\pm$ by some twists about the attaching disks of $\eta(f_\pm)$ to $\eta(w_\pm)$) so that $\gamma_0 = \overline{\gamma}_0$. We can now view the disk $D' = D - \eta(\Gamma')$ as giving an isotopy from the arc $e - \eta(w_\pm)$ to the arc $\overline{\gamma}_0$. During the course of this isotopy the end points $e \cap \partial(w_\pm)$ move along $\gamma_\pm$ to the end points of $\overline{\gamma}_0$.

This motion of the end points can be coned in $\eta(w_\pm)$, extending it to an isotopy from e to the union of $\overline{\gamma}_0$ and the arcs in $\eta(w_\pm)$ obtained by coning the end of γ_0. The latter lies in P, so, after the isotopy, $\Gamma \subset P$.

4.4. Proposition. *Suppose Γ is a graph in S^3 containing a special three-cycle σ with base e. Suppose f is an edge of Γ such that f is not a loop and is not incident to σ. If Γ/f and $\Gamma - e$ are planar, then so is Γ.*

Proof. Let P be a two-sphere containing $\overline{\Gamma} = \Gamma/f$. The image of σ remains a three-cycle $\overline{\sigma}$ in $\overline{\Gamma}$, which divides P into two disks. Push the interior of one of them slightly off of P. Since f and its end points are disjoint from σ, the preimage of the disk before f is shrunk remains a disk D with boundary σ, whose interior is disjoint from Γ. This shows that σ is flat. Apply 4.3.

5. Two-separable graphs

5.1. Definitions [19]. If Γ is connected and has a cut vertex we say Γ is *one-separable.* If Γ is connected but not one-separable it is two-connected. A pair of vertices $v_\pm$ in a two-connected graph Γ is *two-separating* if Γ is the union of two subgraphs Γ_0 and Γ_1, each containing at least two edges, such that $\Gamma_0 \cap \Gamma_1 = \{v_+, v_-\}$. If Γ is one-connected and has a two-separating pair of vertices, Γ is *two-separable.* A two-connected graph which is not two-separable is called *three-connected.*

5.2. Definition. Let M be a three-manifold with boundary, and let $(\alpha, \partial\alpha) \subset (M, \partial M)$ be a properly imbedded arc in M. A *flange* φ from α is an imbedding $\varphi: I \times I \to M$ such that $\varphi^{-1}(\alpha) = I \times \{0\}$ and $\varphi^{-1}(\partial M) = \partial I \times I$.

5.3. Lemma. *The image of any two flanges from the same arc in M are isotopic in M rel α, via an isotopy fixed outside a neighborhood of the images.*

Proof. Suppose φ and ψ are two flanges based at α. By a small isotopy of ψ whose support lies near α we can make $\psi = \varphi$ on a neighborhood of $I \times \{0\}$. Let $f_t: I \times I \to I \times I$ be the map $f_t(u, v) = (u, tv)$, and let $\varphi_t: I \times I \to S^3$ ($\psi_t: I \times I \to S^3$) be the map $\varphi_t = \varphi f_t$ (resp. $\psi_t = \psi f_t$), which is an imbedding as long as $t > 0$. then for $\varepsilon > 0$ sufficiently small, $\varphi_\varepsilon = \psi_\varepsilon$. The required isotopy is then obtained by following the isotopy ψ_t, $1 \geq t \geq \varepsilon$, by φ_s, $\varepsilon \leq s \leq 1$. q.e.d.

Suppose $\{v_\pm\}$ are a two-separating pair of vertices in a two-connected graph Γ. Let Γ_0 and γ_1 be the subgraphs of the two-separation. Suppose

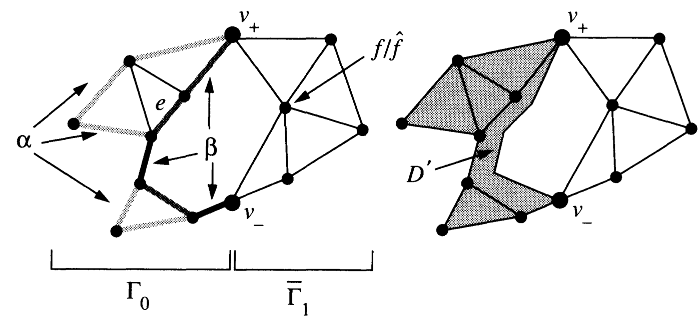

FIGURE 5.1

Γ_1 contains an edge f with distinct end vertices, neither of which are $v_\pm$, and suppose Γ_0 contains an edge e for which $\Gamma_0 - e$ is connected.

5.4. Lemma. *If Γ/f and $\Gamma - e$ are planar, so is Γ.*

Proof. The idea will be to show that there is an arc in Γ_0 such that all of Γ_0 lies inside a flange on that arc.

Let $\overline{\Gamma} = \Gamma/f$, $\overline{\Gamma}_1 = \Gamma_1/f$, and P be a two-sphere that contains $\Gamma \supset \Gamma_0$. Since Γ is two-connected, Γ_0 and Γ_1 are connected. Since Γ_1 is connected, $\overline{\Gamma}_1 - v_\pm$ lies entirely in one component of $P - \Gamma_0$ whose boundary contains both $v_\pm$. Since Γ_0 is connected, that component is a disk D. Though $\partial D \subset \Gamma_0$ may not be an imbedded circuit, it follows from the two-connectivity of Γ that ∂D is the union of two imbedded arcs α and β in Γ_0, each running from v_+ to v_-. One of them, α say, does not contain e, since $\Gamma_0 - e$ is connected. Remove from D a collar of β disjoint from $\overline{\Gamma}_1$, so that D is an imbedded disk in P, $\overline{\Gamma}_1 - \{v_\pm\} \subset \text{interior}(D)$, and $\alpha \subset \partial D$. The other disk D' (see Figure 5.1) which ∂D bounds in the sphere P then has the following properties:

(a) $\Gamma_0 \subset D'$,

(b) $D' \cap \overline{\Gamma}_1 = \{v_\pm\}$, and

(c) $\alpha \subset \partial D'$.

Let $\Gamma' = \Gamma_1 \cup \alpha$. Since $\Gamma' \subset \Gamma - e$, Γ' is planar, so lies in a sphere Q. Let $\eta(\Gamma_1)$ be a standard handlebody neighborhood of $\Gamma_1 \subset Q$ and $W = S^3 - {}^\circ\eta(\Gamma_1)$. A neighborhood of the arc $\alpha \cap W$ in Q contains a flange F on $\alpha \cap W$. $D' \cap W$ is also a flange on $\alpha \cap W$ and contains $\Gamma_0 \cap W$. Both flanges F and D' intersect $\partial W = \partial\eta(\Gamma')$ on arcs lying in $\partial\eta(v_\pm)$. By 5.3, D' can be isotoped rel α onto F, forcing $\Gamma_0 \cap W$ onto Q as well. Coning the isotopy of the points $\Gamma_0 \cap \partial\eta(v_\pm)$ to $v_\pm$ extends the isotopy to $\Gamma_0 - W = \Gamma_0 \cap \eta(\Gamma_1)$, after which $\Gamma \subset Q$.

6. Three-connected graphs

6.1. Definition. Let Ω_n, $n \geq 3$, be the wheel with n spokes. Its vertices are the central vertex w and vertices $\{w_1, \cdots, w_n\}$ lying in order on a cycle C_n. Its edges are those of C_n together with the n spokes, each incident to w and one of the w_i. Denote by σ_i, $i \in \mathbf{Z}_n$, the circuit $w - w_i - w_{i+1}$ in Ω_n. Note each σ_i is a special three-cycle, with at least two apexes w_i and w_{i+1}.

6.2. Lemma. *Let Γ be the graph obtained by adjoining to Ω_n, $n \geq 3$, an edge e with (perhaps new) distinct end vertices $v_\pm \subset C_n$. Either Γ contains a two-cycle or Γ contains a special three-cycle σ and an edge f not incident to σ.*

Proof. Let $\overline{C}_n \subset \Gamma$ be $C_n \cup \{v_\pm\}$. The vertices w_i and w_{i+1} in σ_i are adjacent in $\overline{C}_n$. If $\{w_i, w_{i+1}\} = \{v_+, v_-\}$ for some $i \in \mathbf{Z}_n$, then Γ contains a two-cycle. If not, then at least one apex of each σ_i persists as a valence three vertex in Γ. It follows that each σ_i remains a special three-cycle in Γ unless $v_\pm$ is in the interior of the edge of σ_i on C_n. Hence at least $n-2 \geq 1$ of the σ_1 remain as special three-cycles in Γ. Also, since $n \geq 3$, there must be at least four vertices in $\overline{C}_n$ or Γ would contains a two-cycle. Hence in $\overline{C}_n$ there is an edge disjoint from one of the remaining special three-cycles.

6.3. Definition. A graph is *strict* if it has no loops or two-cycles and every vertex is of valence ≥ 3.

6.4. Lemma. *Suppose Γ is a three-connected strict graph lying in a sphere P, and F is a face of Γ in P. Either there is an edge of Γ not incident to ∂F or Γ is a wheel Ω_n, $n \geq 3$, whose circuit $C_n = \partial F$.*

Proof. Since Γ is strict it contains at least one vertex not in ∂F. Suppose Γ contains exactly one vertex w not in ∂F. Since Γ is strict, every edge incident to w is incident to a vertex in ∂F and every edge incident to ∂F but not in ∂F is incident to w. Hence Γ is a wheel Ω_n with $n = \text{valence}(w) \geq 3$ and circuit ∂F.

Suppose Γ contains more than one vertex not in ∂F. Let F' be a face of Γ whose boundary contains vertices w and w' not in ∂F. If any edge of Γ is not incident to ∂F we are done. If every edge is incident to ∂F, then there is an arc α properly imbedded in F', separating w from w', whose boundary lies on vertices w_i and w_j of ∂F. There is also an arc β in F with $\partial\beta = \partial\alpha$. Then the circle $\alpha \cup \beta$ shows that w_i and w_j two-separate Γ, contradicting the hypothesis that Γ is three-connected.

6.5. Proposition. *Let $\Gamma \subset S^3$ be a three-connected strict graph contained in a sphere $P \subset S^3$. Let F be a face of Γ in P. Suppose $\overline{F} \subset S^3$*

is a disk such that $\partial\overline{F} = \overline{F} \cap \Gamma = \partial F$. *Then there is a sphere* $\overline{P} \subset S^3$ *so that* $\Gamma \subset \overline{P}$ *and* $\overline{F} \subset \overline{P}$ *is a face of* Γ *in* $\overline{P}$.

Proof. Applying general position and isotopies which taper as they approach Γ, we can assume that P intersects the interior of $\overline{F}$ in a properly imbedded one-manifold Λ, and that the closure in $\overline{F}$ of any arc component of Λ is either an imbedded properly imbedded arc in $\overline{F}$ with ends at vertices of ∂F, or a circle containing a vertex of ∂F. Let $\overline{\Lambda}$ denote this closure of $\overline{\Lambda}$ in F, and call the circles of $\overline{\Lambda}$ which contains a vertex of ∂F *loops.*

We will induct on $|\Lambda|$. If $|\Lambda| = 0$ so $\overline{F} \cap P = \emptyset$, just replace F with $\overline{F}$, yielding a new sphere $\overline{P}$. So we suppose $\overline{F} \cap P \neq \emptyset$.

Suppose first that there were a simple closed curve in Λ, and let D be a disk in $\overline{F}$ cut off by an innermost such curve. Since Γ is connected, ∂D also bounds a disk D' in $P - \Gamma$. Replace D' by a slight push-off of D to eliminate ∂D (and perhaps more) from Λ, reducing $|\Lambda|$. So henceforth assume Λ consists of arcs. Then $\overline{\lambda}$ consists of arcs and loops.

In each case below, we will replace some disk in P with a slight push-off of a disk in $\overline{F}$, obtaining a new two-sphere P' containing Γ and intersecting $\overline{F}$ in at least one fewer component.

An arc of Λ outermost in $\overline{F}$ cuts off a disk D in $\overline{F}$ such that the interior of D is disjoint from P. Among all such outermost arcs, choose α to be one for which ∂D contains as few edges in ∂F as possible.

Case 1: *α is not a loop.* The ends of α are two vertices w_1 and w_2 of ∂F. $\{w_1, w_2\}$ separates ∂F into two arcs d_1 and d_2 with $\partial D = \alpha \cup d_1$, say, and d_1 having no more edges than d_2. If $\alpha \subset F$, then α also cuts F into two disks, one of which also has boundary $\alpha \cup d_1$. Replace that disk in F with a copy of D, then push F slightly rel ∂F to eliminate α (and perhaps more) from Λ.

If $\alpha \subset P - F$, then consider a slight push-off β of d_1 onto F. $\alpha \cup \beta$ is a simple closed curve in P intersecting Γ in the vertices $w_1 \cup w_2$ and containing edges of Γ on both sides. Since Γ is three-connected, one side must contain precisely one edge. Hence α lies in a face F' of Γ adjacent to F and $F' \cap F$ is a single edge, either d_1 or d_2. If $F' \cap F = d_1$ proceed as above using F' instead of F. If $F' \cap F = d_2$, then d_1 can have no more than one edge. But then ∂F would have no more than two-edges, contradicting the assumption that Γ is strict.

Case 2: *α is a loop.* The ends of α lie on a vertex w in ∂F. Let D be the disk in $\overline{F}$ bounded by the loop $\alpha \cup w$.

If $\alpha \subset F$, then $\alpha \cup w$ also bounds a disk D' in F. Replace D' with

D, then push F slightly rel ∂F. This eliminates α (and perhaps more) from Λ.

If $\alpha \subset P - F$, then the interior of the loop α in P must be disjoint from Γ, since Γ is two-connected. Hence $\alpha \cup w$ also bounds a disk D' in a face F'. Proceed as above, using F' instead of F.

7. Criteria for planarity

7.1. Lemma. *Let Γ be a finite graph in S^3 with handlebody neighborhood $\eta(\Gamma)$. Then $\pi_1(S^3 - \Gamma)$ is free if and only if $S^3 - {}^\circ\eta(\Gamma)$ is a connected sum of handlebodies, one for each component of Γ.*

Proof. Stallings theorem [17] shows that a submanifold of S^3 with free fundamental group is either the solid torus, or a connected sum, or a boundary connected sum of other submanifolds of S^3 with free fundamental group. By induction, such a manifold must then be a connected sum of handlebodies. Each handlebody summand has connected boundary.

7.2. Lemma. *Let Γ be a finite graph in S^3 such that $\pi_1(S^3 - \Gamma)$ is free and every graph properly contained in Γ is planar. If Γ is not connected, it is planar.*

Proof. Let $\eta(\Gamma)$ be a handlebody neighborhood of Γ. Since Γ is not connected, $S^3 - \eta(\Gamma)$ has more than one boundary component, and so is a connected sum. In particular, Γ is split, and so by 1.3 is planar. q.e.d.

We are now ready to prove the main theorem. We will need the following theorem, due to Barnette and Grünbaum [1, Theorem 1]. If e is an edge in a strict graph Γ, let $\Gamma \sim e$ denote the graph obtained from $\Gamma - e$ by amalgamating any newly-created valence two vertices at the ends of e.

7.3. Theorem. *Suppose Γ is a three-connected strict graph other than the tetrahedral graph. There is an edge e in Γ such that $\Gamma \sim e$ is also a three-connected strict graph.*

We will also need the following special case of a theorem due to Mason [11]:

7.4. Theorem. *Suppose that Γ_1 and Γ_2 are planar graphs in S^3. Any homeomorphism $g: \Gamma_1 \to \Gamma_2$ extends to a homeomorphism $H: S^3 \to S^3$ isotopic to the identity.*

If follows that if $\Gamma \subset S^3$ is planar and $g: \Gamma \to S^2$ is *any* imbedding of Γ in the two-sphere, then there is a sphere $P \subset S^3$ such that Γ lies in P just as it lies in S^2. That is, there is a homeomorphism $h: S^2 \to P$ so that $hg: \Gamma \to P \subset S^3$ is the inclusion.

7.5. Theorem. *A finite graph* $\Gamma \subset S^3$ *is planar if and only if*

(i) Γ *is abstractly planar,*

(ii) *every graph properly contained in* Γ *is planar, and*

(iii) $\pi_1(S^3 - \Gamma)$ *is free.*

Proof. Clearly, if Γ is planar it satisfies (i)–(iii); the interest is in the other direction. So we will assume Γ satisfies (i)–(iii) and try to show it is planar. Nothing is lost by assuming every vertex of Γ has valence ≥ 3.

The proof will be by induction on the number of vertices. In particular, we can assume that if f is an edge in Γ which is not a loop, then the graph $\Gamma/f \subset S^3$ is planar.

Following 7.2 assume that Γ is connected. The case in which Γ has a single vertex is [4, Theorem 1], so we will assume Γ has more than one vertex. By 1.6 we can assume Γ has no cut edge. Hence, by 1.5 we can assume Γ is strict and two-connected.

Suppose Γ is two-separable, with two-separating vertices $\{v_\pm\}$. Let Γ_0 and Γ_1 be the connected subgraphs of the two-separation. Consider first Γ_1. Since Γ contains no two-cycles, at most one edge of Γ_1 is incident to both $v_\pm$. Since Γ_1 contains more than one edge, it must contains at least one $v \neq v_\pm$. Since Γ contains no two-cycles, at most two edges incident to v have their other end on $v_\pm$. Since v has valence ≥ 3, some edge f incident to v is not incident to $v_\pm$. Now consider Γ_0. Γ_0 contains more than one edge, so it has at least one other vertex v, of valence ≥ 3 since Γ is strict. Thus, Γ_0 is not a tree, for it can have at most two ends, $v_\pm$. Since Γ_0 is not a tree, it contains an edge e with $\Gamma - e$ connected. Then by 5.4 Γ is planar.

If Γ is not two-separable it is a three-connected strict graph. If it is the tetrahedral graph, then by 3.3 it is planar. If it is not tetrahedral, then by 7.3 there is an edge e in Γ such that $\Gamma \sim e$ is also a three-connected strict graph.

If $\Gamma \sim e$ is a wheel, then by 6.2 Γ contains a special three-cycle σ and an edge not incident to σ. Then by 4.4 Γ is planar. So assume $\Gamma \sim e$ is not a wheel.

Γ is abstractly planar, so imbed Γ in a sphere Q. By hypothesis, $\Gamma' = \Gamma - e$ is also planar, and by Mason's theorem (7.4) we can assume that $\Gamma' = \Gamma - e$ lies in a sphere $P \subset S^3$ exactly as Γ lies in Q. In particular, the ends of e lie on the boundary of some face F of P. Since $\Gamma \sim e$ is not a wheel, there is, by 6.4, an edge f of Γ not incident to ∂F. Let $\overline{\Gamma} = \Gamma/f$. By hypothesis, $\overline{\Gamma}$ lies in a sphere $\overline{P} \subset S^3$. By 7.4, we can assume that $\overline{\Gamma}$ lies in $\overline{P}$ exactly as Γ/f lies in Q. In particular, e lies in a face $\overline{F}$ of $\overline{\Gamma}$ in $\overline{P}$ with $\partial\overline{F} = \partial F$. Since f is not incident to ∂F, $\overline{F}$ persists when we "unshrink" f. That is, $\overline{F}$ is a disk in S^3 such that $\partial\overline{F} = \overline{F} \cap \Gamma = \partial F$. Then by 6.5 applied to Γ' and $\overline{F}$, there is a sphere $P'' \subset S^3$ containing Γ' and $\overline{F}$. But $e \subset \overline{F}$, so $\Gamma \subset P''$.

7.6. Corollary. *There is an algorithm to determine if a graph* $\Gamma \subset S^3$ *is planar.*

Proof. Kuratowski's theorem provides an algorithm to determine abstract planarity. In fact, abstract planarity of graphs can be determined in linear time [7].

It suffices to have, then, an algorithm to determine if the fundamental group of the complement of graph Γ is free. According to 7.1 this is equivalent to showing $S^3 - {}^\circ\eta(\Gamma)$ is the connected sum of handlebodies, one for each component of Γ. Haken's original algorithm [5] can be used to determine if a three-manifold contains a two-sphere separating its boundary components. This reduces the problem to the case in which Γ is connected. Then $M = S^3 - {}^\circ\eta(\Gamma)$ is irreducible. A variant of Haken's algorithm suffices to determine if an irreducible three-manifold is ∂-reducible, and gives a ∂-reducing disk (cf. [9, 4.1]). Cut M open along a ∂-reducing disk, if one exists. Continue this process until M does not have a ∂-reducing disk. If ∂M is then a union of spheres, $S^3 - {}^\circ\eta(\Gamma)$ was a handlebody. If not, then a nonspherical component of ∂M was an incompressible closed surface in $S^3 - {}^\circ\eta(\Gamma)$, so $S^3 - {}^\circ\eta(\Gamma)$ was not a handlebody.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA
UNIVERSITY OF CALIFORNIA, DAVIS