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RIGIDITY OF HOLOMORPHIC MAPS BETWEEN COMPACT HERMITIAN SYMMETRIC SPACES

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The purpose of this note is to prove the following.

Main Theorem. *Let X and Y be two equidimensional irreducible Hermitian symmetric spaces of compact type with* $\text{rank}(Y) \geq 2$. *Then any holomorphic map f from X to Y is either a constant map or a biholomorphism.*

We briefly explain the motivation for studying this problem. Mok [3] studied uniqueness theorems of Hermitian metrics of seminegative curvature on quotients of bounded symmetric domain of rank ≥ 2. An immediate corollary of this theorem is the assertion that any nonconstant holomorphic map from M to N, where M denotes a compact quotient of bounded symmetric domain with rank ≥ 2 and N a Hermitian manifold of seminegative curvature, is in fact an isometric immersion. It is then natural to ask the analogous question in the case of compact type. However, due to the fact that in this case the automorphisms in general are not metric-preserving, we do not have the uniqueness of canonical metrics and therefore we can only try to prove that f is a holomorphic immersion instead of an isometric immersion. The correct formulation for the metric rigidity phenomenon in the case of compact type has been carried out by Mok [4]. As for the mapping rigidity, if we consider a holomorphic map of degree 2 from the quadric $\mathbf{Q}_n$ to $\mathbf{P}_n$ and an imbedding form $\mathbf{P}_n$ to $\mathbf{Q}_{2n}$, then the composite map fails to be an imbedding. To take into account this remark, we formulate our theorem in the equidimensional case. The cases of unequal dimensions remain open.

Our ideas for the proof can be sketched as follows. We look at a class of objects in a Hermitian symmetric space of compact type, i.e., the so-called characteristic spheres as in [4] or minimal rational curves as in [5]. The importance of the roles they play in the theory of Hermitian symmetric spaces has been illustrated before, as can be seen from the articles [4], [5].

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Here, we study the invariant property of these objects under holomorphic maps. It is this very property that leads to our proof. More precisely, we prove (Proposition 3.4) that under the assumption of the Main Theorem the image of each minimal rational curve is still minimal rational. For the proof of this statement we must first of all show that the image is totally geodesic with respect to whatever choice of canonical metrics by making use of a criterion for total geodesy as developed in [4]. Here we need the assumption that $\mathrm{rank}(Y) \geq 2$. Then from the polysphere theorem (see [8] for an account of the proof of Proposition 3.4) one can obtain minimality without too much difficulty. For the rest of the proof we would then produce a minimal rational curve on which the restriction of the holomorphic map is unramified (Proposition 4.1). This uses the Douady space of minimal rational curves and the minimality of their images under holomorphic maps. Having done this, one can readily deduce the Main Theorem.

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1. Background material

In this section we collect some information on Hermitian symmetric spaces of compact type, and refer to [2], [3] for further details.

Let (X, h) be an irreducible Hermitian symmetric space of compact type. The metric h is Kähler and carries semipositive holomorphic bisectional curvature. We call h a canonical metric of X. We know that h is not unique since the isometry group is a proper subgroup of the automorphism group of X. The induced Hermitian metric h^* on $T^*(X)$ carries seminegative curvature. Let $\mathscr{M} \subseteq T(X)$, which is called the characteristic bundle of X, be the subset of all unit tangent vectors of type $(1, 0)$ realizing the maximum of the holomorphic sectional curvature. The metric h induces a map Φ from $T(X)$ to $T^*(X)$, i.e.,

$$\Phi\left(\sum_i a_i \frac{\partial}{\partial z_i}\right) = \sum_{i,j} h^{j\bar{i}} \overline{a_i}\, dz_j.$$

Denote $\Phi(\mathscr{M})$ by $\mathscr{M}^*$, and by $\overline{\mathscr{M}}$ the image of $\mathscr{M}$ under the canonical projection map from $T(X)$ to $\mathbf{P}T(X)$, the projectivization of $T(X)$. For any characteristic vector $\alpha \in \mathscr{M}_p$, there is an orthogonal decomposition

$T_p(X) = \mathbf{C}\alpha \oplus \mathrm{N}_\alpha \oplus \mathbf{P}_\alpha$ of $T_p(X)$ into the eigenspaces of the Hermitian form $H_\alpha(\xi, \eta) = R_{\alpha\bar{\alpha}\xi\bar{\eta}}$, corresponding to the eigenvalues $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$, 0, $\frac{1}{2}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$, respectively. For any $\beta \in T(X)$, define $\mathrm{N}_\beta = \{v | v \in T_p(X),\ R_{\beta\bar{\beta}v\bar{v}} = 0\}$. Then N_β is a vector space, as can be seen from the semipositivity of R. With this notation, we can give an equivalent definition of $\mathscr{M}$. Namely, $\mathscr{M} = \{\alpha \in T_p(x) | \|\alpha\| = 1,\ \dim \mathrm{N}_\alpha \geq \dim \mathrm{N}_\beta$ for any $\beta \in T_p(X)\}$. We write $d(X) = \dim \mathrm{N}_\alpha$. Then $d(X)$ is independent of the choice of characteristic vector α. Also we know that $d(X) \geq 1$ whenever X is of rank ≥ 2, and that $\dim \overline{\mathscr{M}} = 2\dim X - 1 - d(X)$.

Next we discuss a class of objects in X, the so-called "minimal rational curves", which are very important for our proof of the theorem. Let $\phi\colon (X, h) \to (\mathbf{P}^N$, Fubini-Study) be the first canonical isometric projective imbedding of X [7]. Then in $\mathbf{P}^N$ any projective line contained in X represents a generator of $H_2(X, \mathbf{Z})$. We call such a line a minimal rational curve. If F is a biholomorphism of X, then under the imbedding ϕ, F acts on X by projective linear transformation. It follows that the image of any minimal rational curve under F is also minimal rational. Moreover, for any minimal rational curve L we have $\mathbf{P}(T(L)) \subset \overline{\mathscr{M}}$. The structure of the holomorphic vector bundle $T^*(X)$ over L can be seen by its Grothendieck splitting

$$T^*(X)|_L = \mathscr{O}(-2) \oplus \mathscr{O}^{d(X)} \oplus \mathscr{O}^s(-1),$$

where $s+d(X)+1 = \dim X$. From this it follows that $\dim H^0(L, T^*(X)|_L) = d(X)$ and that any nontrivial holomorphic section w of $T^*(X)|_L$ is nowhere-vanishing. Also note that for any $p \in L$ and $\alpha \in T_p(L)$ we have $w(p) \in \Phi(\mathrm{N}_\alpha)$.

Let X, Y, and f be as above. Denote by ω_X, ω_Y the Kähler forms of X, Y respectively. As a preliminary property of f, we have

Proposition 1.1. *The following conditions are equivalent*:

(i) f *is not constant.*

(ii) $f^*(\omega_Y) \neq 0$ *in* $H^2(X, \mathbf{Z})$.

(iii) f *is not degenerate, i.e.,* df *has maximal rank at a generic point of* X.

Proof. (i) $\Rightarrow$ (ii) Suppose $f^*(\omega_Y) = 0$. Fix a point $q \in f(X)$ and suppose $V = f^{-1}(q) \neq X$. Then V is a proper complex analytic subvariety of X. Take a smooth point p of V. By the irreducibility of X there exists a minimal rational curve L such that $p \in L$ and $T_p(L) \not\subset T_p(V)$. From $f^*(\omega_Y) = 0$ in $H^2(X, \mathbf{Z})$ we obtain $\int_{f(L)} \omega_Y = 0$. This implies that $f(L)$ can be nothing but a point. Hence $L \subset V$, contradicting that $T_p(L) \not\subset T_p(V)$. Thus $V = X$, and f is a constant map.

(ii) $\Rightarrow$ (iii). Suppose not. Since $H^2(X, \mathbf{C}) \cong \mathbf{C}$, we have $f^*(\omega_Y) = c \cdot \omega_X$ for some nonzero constant c. Then $\bigwedge^n f^*(\omega_Y) = c^n \cdot (\omega_X)^n$ in $H^{2n}(X, \mathbf{C})$. It follows from the failure of maximal rank that $\bigwedge^n f^*(\omega_Y) = 0$ in $H^{2n}(X, \mathbf{C})$. This implies $c = 0$.

(iii) $\Rightarrow$ (i). Obvious.

The same reasoning also shows

Proposition 1.2. *Suppose f is not constant. Then f is a finite morphism. Therefore any irreducible component C of $f^{-1}(L)$ for any minimal rational curve L is of complex dimension 1, and $f(C) = L$.*

2. Total geodesy of the inverse image of a generic minimal rational curve

Recall that minimal rational curves are invariant under any biholomorphism F, i.e., if L is minimal rational, then so is $F(L)$. Note also that minimal rational curves are totally geodesic with respect to *any* canonical metric. In this section we are going to prove an analogous result as above. Namely we show that each irreducible component C of the inverse image of a generic minimal rational curve is still totally geodesic with respect to any canonical metric. This fact is crucial for our proof of the Main Theorem. In the next section we will make use of this fact to prove further that C is actually minimal rational. Having done this, we can prove the Main Theorem in some cases immediately.

Let (X, h), (Y, g), and f be as above. Suppose that C is an irreducible algebraic curve in X with its image a minimal rational curve L in Y and that f is local biholomorphic at a smooth point q of C. Denote by U, V $(= f(U))$ the corresponding open neighborhoods of q, $f(q) = p$, respectively. We assert

Proposition 2.1. *The curve $C_U = C \cap U$ is totally geodesic in (X, h).*

The assertion is equivalent to showing that $f(C_U) = L \cap V = L_V$ is totally geodesic in Y with respect to the pushed-forward metric $(f|_U)_*(h)$, also denoted by h. To accomplish our proof we now state a method for proving total geodesy due to Mok.

Proposition 2.2 [4]. *Let (M, g) be an irreducible Hermitian symmetric space of compact type with rank at least two. Let h be a Kähler metric defined on some open set W of M. Suppose L is a minimal rational* *curve. Then* L_W *is totally geodesic at* p *with respect to* h *provided that*

$$\nabla_\alpha h^{q^* \bar{j}}(p) = 0 \quad \textit{for any } \alpha \in \mathcal{M}_p \textit{ and any } q^* \in \Phi(\mathrm{N}_\alpha),$$

where $(h^{i\bar{j}})$ *is the induced metric on* $T^*(M)|_W$, *and the covariant derivatives are taken with with respect to* g.

Returning to Proposition 2.1 it therefore suffices to prove

Proposition 2.1′. *At any point* $p \in L_V$, *we have* $\nabla_\alpha h^{q^* \bar{v}}(p) = 0$ *for any* $\alpha \in \mathcal{M}_p$, $q^* \in \Phi(\mathrm{N}_\alpha)$, *and* $v \in T^*(Y)$.

Proof. First of all we denote by Θ, Θ^1, and Θ^2 the curvature tensors of the holomorphic vector bundle $T^*(Y)|_V$ associated to the metrics g^*, h^*, and $h^* + g^*$, respectively. We would like to show, at p,

$$(*) \qquad \Theta_{q^* \bar{q} \alpha \bar{\alpha}} = 0, \qquad \Theta^1_{q^* \overline{q^*} \alpha \bar{\alpha}} = 0, \qquad \Theta^2_{q^* \overline{q^*} \alpha \bar{\alpha}} = 0.$$

We now deduce $\nabla_\alpha h^{q^* \bar{v}}(p) = 0$ from $(*)$ using the same argument as in [3]. Expressing curvature tensors in terms of local coordinates yields, at p,

$$(1) \qquad -\frac{\partial^2 g^{2\bar{2}}}{\partial z_1 \partial \overline{z_1}} = 0,$$

$$(2) \qquad -\frac{\partial^2 h^{2\bar{2}}}{\partial z_1 \partial \overline{z_1}} + h_{u\bar{v}} \frac{\partial h^{2\bar{v}}}{\partial z_1} \frac{\partial h^{u\bar{2}}}{\partial \overline{z_1}} = 0,$$

$$(3) \qquad -\frac{\partial^2 (g^{2\bar{2}} + h^{2\bar{2}})}{\partial z_1 \partial \overline{z_1}} + (g^* + h^*)_{u\bar{v}} \frac{\partial h^{2\bar{v}}}{\partial z_1} \frac{\partial h^{u\bar{2}}}{\partial \overline{z_1}} = 0,$$

where $\{z_1, z_2, \cdots, z_n\}$ is a complex coordinate system at p with respect to g (i.e., $g_{i\bar{j}}(p) = \delta_{i\bar{j}}$, $dg_{i\bar{j}}(p) = 0$ for any i, j) and has been chosen in such a way that

$$\frac{\partial}{\partial z_1}(p) = \alpha, \qquad dz_2(p) = q^*.$$

From $(1) + (2) - (3)$ it easily follows that

$$\frac{\partial h^{2\bar{j}}}{\partial z_1}(p) = 0 \quad \text{for } 1 \le j \le n.$$

Since we have chosen complex geodesic coordinates at p, this implies $\nabla \alpha h^{q^* \bar{j}}(p) = 0$, proving the desired result.

We proceed to the proof of $(*)$. Being a characteristic vector in $T_p(Y)$, α is a tangent vector of some minimal rational curve $L_\alpha \subset Y$ passing through p. Also q^*, as a vector in $\Phi(\mathrm{N}_\alpha)$, corresponds to a holomorphic section w of $T^*(Y)|_{L_\alpha}$, i.e., $w(p) = q^*$. The section w has no zero and can generate a holomorphic line subbundle Λ of $T^*(Y)|_{L_\alpha}$. w itself becomes a holomorphic section of Λ. We assert

Proposition 2.3. *$\|w\|$ is a constant on $L_\alpha \cap V$ with respect to any one of the metrics g^*, h^*, and $g^* + h^*$.*

Note that for a holomorphic line bundle over a complex manifold the curvature is locally given by $\Theta = -\sqrt{-1}\partial\bar{\partial}\log\|s\|_e$, where s is a local holomorphic section, and e is some given Hermitian metric on this line bundle. The claim $(*)$ follows from this and the above proposition since the curvatures of $(T^*(Y), g^*)$, $(T^*(Y), h^*)$, and $(T^*(Y), g^* + h^*)$ are seminegative, and the curvatures of their subbundles would decrease.

Proof of Proposition 2.3. $\|w\|_{g^*}$ is constant for the following reasons. We write

$$\Theta_\Lambda = -\sqrt{-1}\partial\bar{\partial}\log\|w\|_{g^*}, \qquad c_1(\Lambda) = [\Theta/(2\pi)]$$

for the curvature form and the first Chern form of Λ. Also recall the formula

$$\deg(\Lambda) = \int_{L_\alpha} c_1(\Lambda),$$

where $\deg(\Lambda)$ counts the number of zeros of any holomorphic section of Λ. Since w is nowhere-vanishing, we have $\deg(\Lambda) = 0$. Moreover Λ, as a line subbundle of $T^*(Y)$, carries seminegative curvature. Hence it follows that $\log\|w\|_{g^*}$ is a harmonic function on L_α. This implies that $\|w\|_{g^*}$ is constant.

Next we consider $\|w\|_{h^*}$. Note that the pushed-forward metric $f_*(h)$ is only defined locally. Thus we consider the pulled-back form $f^*(w)$ instead. We would like to show that $\|f^*(w)\|$ is constant with respect to the original metric h of X. This would give us the desired result. The arguments are essentially the same as in previous case. First we assume that the irreducible component C of L is smooth. The 1-form $f^*(w)$ is a holomorphic section of $T^*(X)|_C$. It may have zeros a priori. Nonetheless it can generate a holomorphic line subbundle of $T^*(X)|_C$, also denoted by Λ, and becomes a holomorphic section of this bundle. Thus $\deg(\Lambda) \geq 0$. On the other hand we know that $c_1(\Lambda)$ is seminegative. It follows that $\deg(\Lambda) = 0$ and $\Theta_\Lambda = 0$. Therefore $\|f^*(w)\|$, hence $\|w\|_{h^*}$, is constant, as desired.

Since $\|w\|_{g^*+h^*} = \|w\|_{g^*} + \|w\|_{h^*}$, we have also shown that $\|w\|_{g^*+h^*}$ is constant.

As for the singular case let $\psi\colon \tilde{C} \to C$ be a normalization of C and $\phi = i \circ \psi$, where i is the injection map $i\colon C \to X$. With this smooth curve

$\widetilde{C}$ we can pull back all the holomorphic objects with associated metrics defined on C to $\widetilde{C}$ using the map ϕ. By exactly the same argument as before one can show that $\|\psi^*(f^*(w))\|_{\psi^*(h^*)}$ is constant, which will yield the desired results. Our proof of Proposition 2.1 is now complete.

Remarks. From the irreducibility of C and the total geodesy of $C\backslash\{\text{finite many points}\}$ one can see that C can actually be obtained by exponentiating its tangent space at a smooth point. It is therefore a totally geodesic submanifold of the symmetric space X, and in particular it is smooth everywhere.

As a consequence of Proposition 2.1 and the above remarks we have

Corollary 2.4. *C is a holomorphically embedded Riemann sphere.*

Proof. The Riemann surface C is totally geodesic in (X, h), hence a Hermitian symmetric space. Since its curvature is positive (X has strictly positive holomorphic sectional curvatures), it must be isomorphic to $\mathbf{P}^1$.

3. Minimality of the inverse image of a generic minimal rational curve

Let us adopt the same notation as before. We have seen in §2 that each irreducible component C of $f^{-1}(L)$ in X for a generic minimal rational curve L is a totally geodesic Riemann sphere with respect to *any* canonical metric of X. In this section we will show that C is actually minimal rational. Before doing so, we first draw a corollary of Proposition 2.1, which proves the Main Theorem in the case $d(X) < d(Y)$.

Proposition 3.1. *Let X, Y be as stated above. Suppose further that $d(X) < d(Y)$. Then there is no nontrivial holomorphic map from X to Y.*

Proof. Suppose not. Let f be a nontrivial holomorphic map. Then f is nondegenerate by Proposition 1.1. Fix a point p at which f has maximal rank. As before we can find $C \ni p$ so that $f(C) = L$ is minimal rational. For each holomorphic section w of $T^*(Y)|_L$ we know $\Theta_\Lambda = 0$, where Λ is the holomorphic line bundle induced by $f^*(w)$. Since $\Theta_\Lambda \leq \Theta \leq 0$, where Θ denotes the curvature of $T^*(Y)|_C$, we have

$$\Theta_{q^*\overline{q^*}\alpha\bar{\alpha}}(p) = 0,$$

where $q^* = f^*(w)(p)$ and $\alpha \in T_p(C)$. This shows that $\dim \mathrm{N}_\alpha \geq d(Y)$ since $d(Y) = H^0(L, T^*(Y)|_L)$. In view of the definition of $d(X)$ and the assumption $d(X) < d(Y)$, the contradiction follows.

The above argument also shows that in the case $d(X) = d(Y)$, α is a characteristic vector. Since C is totally geodesic, one immediately obtains the minimality of C in this case. In order to deal with the general situation we resort to some general theory of symmetric spaces.

Let us begin with notation and some fundamentals. Suppose M is a Hermitian symmetric space of compact type, G_c its largest connected group of holomorphic isometries, and K the isotropy subgroup at some point $p \in M$. Then $M \cong G_c/K$, and the semisimple Lie algebra $\mathfrak{g}_c$ of G_c has a Cartan decomposition $\mathfrak{g}_c = \mathfrak{k} + \mathfrak{m}_c$. Moreover there is a central element $z \in \mathfrak{k}$ such that $J = \mathrm{ad}(z)|_{\mathfrak{m}_c}$ is the complex structure of M. Denote by G and $\mathfrak{g}$ the complexified Lie group and Lie algebra of G_c and $\mathfrak{g}_c$ respectively. Choose a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{k}$. Write $\mathfrak{b}^c$ for the complexification of $\mathfrak{b}$. Then $\mathfrak{b}^c$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by Δ the $\mathfrak{b}^c$-root system of $\mathfrak{g}$, and by Δ_M the set of noncompact roots, i.e., roots ϕ with $\mathfrak{g}^\phi \subset \mathfrak{m} = \mathfrak{m}_c \otimes_{\mathbf{R}} \mathbf{C}$. Let $\mathfrak{m}^+$ and $\mathfrak{m}^-$ be the $(\pm\sqrt{-1})$-eigenspace decomposition of J. Furthermore we can choose an ordering of Δ such that

$$\mathfrak{m}^+ = \sum_{\phi\in\Delta_M^+} \mathfrak{g}^\phi \quad \text{and} \quad \mathfrak{m}^- = \sum_{\phi\in\Delta_M^+} \mathfrak{g}^{-\phi}.$$

Also we can find the unit vector $e_\phi \in \mathfrak{g}^\phi$, with respect to the Killing metric B, such that the following hold:

(i) $e_{-\phi} = \overline{e_\phi}$, where the conjugate is taken with respect to the decomposition $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$.

(ii) $[e_\phi, e_{-\phi}] = H_\phi \in \mathfrak{b}^c$, where $\phi(H) = B(H_\phi, H)$ for $H \in \mathfrak{b}$.

Recall that a set $\{\phi_1, \phi_2, \cdots, \phi_r\}$ is called strongly orthogonal if none of the $\phi_i \pm \phi_j$ is a root for $i \neq j$. Let us find such a maximal set Π by using Harish-Chandra's construction. Write $\mathbf{V} = \sum_{\phi\in\Pi} \mathbf{R}e_\phi$. Then we have the following standard facts.

Lemma 3.2 (cf. [2]). $\mathfrak{m}^+ = \bigcup_{k\in K} \mathrm{Ad}(k)\mathbf{V}$.

Lemma 3.3 [6]. *All H_ϕ have the same length for $\phi \in \Pi$.*

Returning to our situation we would like to show

Proposition 3.4. *C is a minimal rational curve in X.*

Before proceeding further, let us make a few remarks. If the metric h is fixed, then not every totally geodesic Riemann sphere with respect to h is minimal. For example, one can have a totally geodesic isometric embedding $i: \mathbf{P}^1 \times \mathbf{P}^1 \to X$ using the "Polysphere Theorem" (see the following discussion). Consider the diagonal embedding $\Delta: \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$. Then the image of $\mathbf{P}^1$ under $i \circ \Delta$ is still totally geodesic. However it is not minimal since it is of degree 2.

For the proof of Proposition 3.4 we start with the following lemma.

Lemma 3.5. *Each e_ϕ is a characteristic vector.*

Proof. Fix a characteristic vector α of unit length at $[K] = p$. By Lemma 3.3 we can write without loss of generality

$$\alpha = \sum_{\phi\in\Pi} a_\phi e_\phi.$$

Now the holomorphic bisectional curvature generated by e_ϕ, e_ψ is given by

$$R_{\phi\bar{\phi}\psi\bar{\psi}} = \|[e_\phi, e_{-\psi}]\|^2$$

for any two roots ϕ, ψ. In particular $R_{\phi\bar{\phi}\psi\bar{\psi}} = 0$ for two distinct roots ϕ, ψ in Π. It follows from the seminegativity of R in the dual Nakano sense that $R_{\phi\bar{\psi}u\bar{v}} = 0$ for all u, v in $\mathfrak{m}^+$. Therefore we have

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \sum_{\phi\in\Pi} |a_\alpha|^4 R_{\phi\bar{\phi}\phi\bar{\phi}}.$$

By the maximality of $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ this implies $\alpha = a_\phi \cdot e_\phi$ for some $\phi \in \Pi$. Hence e_ϕ is a characteristic vector. Since

$$R_{\psi\bar{\psi}\psi\bar{\psi}} = \|[e_\psi, e_{-\psi}]\|^2 = \|H_\psi\|^2 = \|H_\phi\|^2 \quad \text{(by Lemma 3.3)}$$

for any other root $\psi \in \Pi$, $R_{\psi\bar{\psi}\psi\bar{\psi}}$ is also maximal. Hence our lemma is proved.

We proceed to prove Proposition 3.4. Without loss of generality we can assume that K is the isotropy subgroup of some point $p \in C$ and that $T_p(C) \subset \mathbf{V}$. It therefore suffices to show

Proposition 3.4′. $T_p(C) = \mathbf{C} \cdot e_\phi$ *for some* $\phi \in \Pi$.

Proof. Throughout the proof we fix a canonical metric h of X. Take a nonzero vector $v \in T_p(C)$. Write

$$v = \sum_{\phi\in\Pi} c_\phi e_\phi = \sum_{\psi\in\Pi_0} c_\psi e_\psi,$$

where Π_0 is the subset of Π such that $c_\psi \neq 0$ for $\psi \in \Pi_0$. Since C is totally geodesic, the complexified tangent space is a Lie triple system (cf. [2]), i.e.,

$$u_1, u_2, u_3 \in E \quad \text{implies} \quad [u_1, [u_2, u_3]] \in E,$$

where $E = T_p(C) \oplus \overline{T_p(C)}$. Let $u_1 = u_2 = v$, $u_3 = \bar{v}$. We compute, using

$[e_\phi, e_{-\phi}] = H_\phi$ and $[e_\phi, e_{\pm\psi}] = 0$ for $\psi \neq \phi$,

$$\begin{aligned}[v, [v, \overline{v}]] &= \Big[\sum_{\phi \in \Pi_0} c_\phi e_\phi , \sum_{\phi\in\Pi_0} |c_\phi|^2 H_\phi \Big] \\ &= - \sum_{\phi\in\Pi_0} |c_\phi|^2 \cdot c_\phi \cdot \phi(H_\phi) \cdot e_\phi \in E.\end{aligned}$$

By our choice of h_ϕ one has

$$\phi(H_\phi) = B(H_\phi, H_\phi) = \|H_\phi\|^2.$$

This is independent of ϕ by Lemma 3.3. Hence we must have

$$\sum_{\phi\in\Pi_0} |c_\phi|^2 \cdot c_\phi \cdot e_\phi \quad \text{is proportional to} \quad \sum_{\phi\in\Pi_0} c_\phi \cdot e_\phi ,$$

or equivalently

$$(**) \qquad |c_\phi|^2 = |c_\psi|^2 \quad \text{for any } \psi, \phi \text{ in } \Pi_0.$$

This holds for any totally geodesic Riemann sphere with tangent sphere contained in $\mathbf{V}$. Suppose that the cardinality of Π_0 is at least 2. We would like to reach a contradiction. Let us proceed as follows. If $\phi \in \Pi_0$, we define a Lie subalgebra by

$$\mathfrak{g}[\phi] = \mathbf{C} \cdot H_\phi + \mathfrak{g}^\phi + \mathfrak{g}^{-\phi} \subset \mathfrak{g}.$$

The sum

$$\mathfrak{g}[\Pi_0] = \sum_{\phi\in\Pi_0} \mathfrak{g}[\phi]$$

is also a Lie subalgebra as can be seen from the strong orthogonality of Π_0. Let the group $G[\Pi_0]$ be the corresponding Lie subgroup of G for $\mathfrak{g}[\Pi_0]$. Then we have obtained the "Polysphere Theorem". More precisely the orbit $G[\Pi_0](p) = \mathbf{S}$ is a totally geodesic complex submanifold, and is a product of the $[\Pi_0]$ Riemann sphere (see [8] for further details). Moreover we have an isomorphism of the local direct product:

$$G[\Pi_0] \approx \prod_{\phi\in\Pi_0} G[\phi].$$

One can also see that $G[\Pi_0]$ cannot act on $\mathbf{S}$ as an isometry group. Pick an element $\gamma \in G[\phi_1]$ near the identity of G such that $\gamma(p) = p$ and that it is not an isometry on $\mathbf{S}$, i.e., $\|\gamma_*(e_{\phi_1})\| \neq \|e_{\phi_1}\|$. Let Γ be the element $(\gamma, 1, \cdots, 1) \in G[\Pi_0]$. Since C is totally geodesic with respect to any canonical metric, in particular $\Gamma^*(h)$, $\Gamma(C)$ is also totally geodesic in (X, h). However by $(*)$ we can write the tangent of $\Gamma(C)$ at p as

$$\gamma_*(v) = \sum_{\phi \in \Pi_0} c_\phi \cdot \Gamma_*(e_\phi) = \sum_{\phi \in \Pi_0} c'_\phi \cdot e_\phi,$$

where $|c_{\phi_1}| \neq |c_{\phi_1}|$ and $c'_\psi = c_\psi$ for $\psi \neq \phi_1$. This contradicts $(**)$. Hence $|\Pi_0| = 1$, as desired.

The following corollary gives a proof of the Main Theorem in the case $d(X) > d(Y)$.

Corollary 3.6. *With one more condition of $d(X) > d(Y)$ in the Main Theorem there is no nontrivial holomorphic map from X to Y.*

Proof. Otherwise given the map f, at the point p with $\operatorname{rank} df(p) = \dim X$ we have, by the above proposition,

$$f_*(\mathscr{M}_p(X)) \supseteq \mathscr{M}_q(Y).$$

Hence $\dim \mathscr{M}(X) \geq \dim \mathscr{M}(Y)$. However due to the condition $d(X) > d(Y)$ together with the formula (see §1) $\dim \mathscr{M}(X) = 2\dim X - d(X)$, this is impossible.

4. Proof of the Main Theorem

In this section we would like to give a complete proof of the Main Theorem. A key step is to prove

Proposition 4.1. *Adopt the same notation as before. Then there exists a minimal rational curve L in X such that the restriction $f|_L$ is a biholomorphism.*

Our proof of Proposition 4.1 needs the use of the Douady space of minimal rational curves. Let us start with a description of it. Let $\mathscr{D}$ denote the Douady space for the set of minimal rational curves in X (cf. [1]). Then it is a compact complex space. It is also homogeneous since the induced biholomorphism group by that of X acts on $\mathscr{D}$ transitively. Hence it is a compact complex manifold. One relation between $\overline{\mathscr{M}}$ and $\mathscr{D}$ can be seen as follows. We first of all identify L with $\widetilde{L} = \{(x, \mathbf{P}T_x(L)) | x \in L\} \subset \overline{\mathscr{M}}$, where L is minimal rational. By the total geodesy of L one knows that $\widetilde{L}_1 \neq \widetilde{L}_2$ if $L_1 \neq L_2$. Since any given point in $\overline{\mathscr{M}}(X)$ is contained in one and only one lifting curve, one can define a holomorphic map from $\overline{\mathscr{M}}$ to $\mathscr{D}$:

$$\pi : p \to [L_{x,\alpha}],$$

where $p = (x, \alpha)$ and $[L_{x,\alpha}]$ represents as a point in $\mathscr{D}$ the minimal rational curve L_x passing through x in the direction of α. In conclusion, we have $\dim \overline{\mathscr{M}} = \dim \mathscr{D} + 1$.

Return to Proposition 4.1. If f has maximal rank everywhere, there is nothing to prove. So assume that

(i) the ramification divisor R is not empty;

(ii) $f|_L$ is ramified somewhere for every minimal rational curve L.

Define a subset B of $\mathscr{D} \times R$ by

$$B = \{([L], x) | [L] \in \mathscr{D}, \ x \in L \text{ such that } f|_L \text{ is ramified at } x\}.$$

Then B is a complex analytic subvariety of $\mathscr{D} \times R$. Let π_1 and π_2 be the projections from B to $\mathscr{D}$ and R respectively. By (i) we have $\pi_1(B) = \mathscr{D}$. We also have the following lemma, which will be used in the proof of Proposition 4.1.

Lemma 4.2. *In the notation above one has*

(i) $\dim \pi_2(B) = \dim X - 1$,

(ii) $\pi_1(\pi_2^{-1}(x)) = \{[L] | x \in L\}$ *for every* $x \in \pi_2(B)$.

Proof. Fix a point $[L]$ in $\mathscr{D}$. Then $\pi_1^{-1}([L]) \neq \emptyset$ consists of finite many points since f is a finite map. By counting dimension one has

$$\dim B = \dim \mathscr{D} + \dim(\text{generic fiber}) = \dim \mathscr{D}.$$

It is also clear that $\dim \overline{\mathscr{M}}_x \geq \dim \pi_2^{-1}(x)$. Therefore we have

$$\begin{aligned} \dim \pi_2^{-1}(x) &\geq \dim B - \dim \pi_2(B) \geq \dim B - \dim R \\ &= \dim \mathscr{D} - (\dim X - 1) = \dim \overline{\mathscr{M}} - \dim X \\ &= \dim \overline{\mathscr{M}}_x \geq \dim \pi_2^{-1}(x). \end{aligned}$$

Hence we have equalities everywhere. In particular $\dim R = \dim \pi_2(B)$ and $\dim \pi_2^{-1}(X) = \dim \overline{\mathscr{M}_x}$. Our lemma follows.

We can now give the proof of Proposition 4.1.

Proof of Proposition 4.1. By (i) of the preceding lemma there exists a point $x \in R \cap \pi_2(B)$ such that x is a smooth point of R. By (ii) of the same lemma $f|_L$ is ramified at x for every L passing through x. Hence we have

$$df(x)(v) = 0 \quad \text{for every } L \text{ and } v \in T_x(L).$$

But $\overline{\mathscr{M}_x} = \bigcup_{L \ni x} \mathbf{P}T_x(L)$ and $\langle \mathscr{M}_x \rangle =$ the linear span of $\mathscr{M}_x = T_x(X)$ by the irreducibility of X. Therefore we have

$$df(x)(v) = 0 \quad \text{for any } v \in T_x(X).$$

Fix a minimal rational curve L passing through x. We have known $f(L)$ is also minimal rational. Since Y is of rank at least 2, there exists a nontrivial holomorphic section w of $T^*(Y)|_{f(L)}$. Pulling back w

one obtains a holomorphic section $f^*(w)$ of $T^*(X)|_L$. Since $T^*(X)$ is seminegative, $f^*(w)$ can be either nowhere-vanishing or trivial. As we have just seen that $df(x) = 0$, $f^*(w)$ is therefore trivial. This shows, from the nondegeneracy of f, that

$$L \subset R \quad \text{and} \quad T_x(L) \subset T_x(R).$$

The above holds for every $L \ni x$. Hence $\mathscr{M}_x \subset T_x(R)$. This contradicts that $\langle \mathscr{M} \rangle = T_x(X)$ because x is a smooth point of R and $\dim R = \dim X - 1$. The contradiction results from the assumption that $f|_L$ is ramified somewhere for every minimal rational curve L. We have thus proved Proposition 4.1.

We can now finish the proof of the Main Theorem.

Proof of the Main Theorem. By Proposition 4.1, f induces an isomorphism $f_* H_2(X, \mathbf{Z}) \cong H_2(Y, Z)$. Since any minimal rational curve generates $H_2(X, \mathbf{Z})$ and its image under f is also minimal rational, we conclude that $f|_L$ is biholomorphic for every minimal rational curve L. Fix a point p in X and a minimal rational curve L containing $f(p) = q$. Then $C = f^{-1}(L)$ is minimal rational. Since $f|_C$ is biholomorphic as just seen, one has $df(T_p(C)) = T_q(L)$. Therefore $df(T_p(X))$ contains $\mathscr{M}_q$. By the irreducibility of Y, f thus has maximal rank at p. This shows that f is a covering map. Since Y is simply connected, f is actually a biholomorphism, completing the proof of the Main Theorem.

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