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KÄHLER HYPERBOLICITY AND L_2-HODGE THEORY

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0. Basic definitions and results

0.1. **Bounded and d(bounded) forms.** A differential form α on a Riemannian manifold $X = (X, g)$ is called *bounded* with respect to the Riemannian metric g if the L_∞*-norm* of α is finite,

$$\|\alpha\|_{L_\infty} \overset{\text{def}}{=} \sup_{x\in X} \|\alpha(x)\|_g < \infty.$$

We say that α is d(bounded) if α is the exterior differential of a bounded form β, i.e., $\alpha = d\beta$, where $\|\beta\|_{L_\infty} < \infty$.

Remark. It is not required that α is bounded, yet in all our applications the notion "d(bounded)" applies to bounded forms α.

If X is a compact, these notions bring nothing new. Namely, every smooth (or just continuous) form α is bounded, and α is d(bounded) if and only if it is exact. However, if X is noncompact, then an exact bounded form is not necessarily d(bounded).

0.1.A. **Example.** The form

$$\alpha = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \text{ on } \mathbf{R}^n$$

is bounded and exact but not d(bounded).

Proof. Write $\alpha = d\beta$ and apply Stokes formula to a ball B of large radius $R \to \infty$ in $\mathbf{R}^n$. Then $\mathrm{Vol}_n B = \int_B \alpha = \int_{\partial B} \beta \le \|\beta\|_{L_\infty} \mathrm{Vol}_{n-1} \partial B$, and $\|\beta\|_{L_\infty} \ge \mathrm{Vol}_n B / \mathrm{Vol}_{n-1} \partial B = R/n$. This shows that $\|\beta\|_{L_\infty} = \infty$; moreover, β grows at least linearly on $\mathbf{R}^n$, i.e., $\sup_{x\in\partial B} \|\beta(x)\| \ge R/n$.

0.1.B. **Hyperbolic manifolds.** (See [4], [10], [22].) If (X, g) is complete simply connected and has strictly negative sectional curvature $\sup_{x\in X} K_X(X) \le -c < 0$, then every smooth bounded closed form α of degree $i \ge 2$ is d(bounded). This immediately follows from the well-known bound on the volume of the *geodesic cones* over the $(i-1)$-dimensional submanifolds $S \subset X$,

$$(*) \qquad \mathrm{Vol}_i(\mathrm{Cone}\, S) \le (c(i-1))^{-1} \mathrm{Vol}_{i-1} S.$$

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Notice that $(*)$ is vacuous for $i = 1$ as well as for $c = 0$ (e.g., for the Euclidean space where $K = 0$).

0.1.C. **Symmetric spaces.** (See [3], [18].) Let X be a Riemannian symmetric space of noncompact type. One knows that such an X is complete simply connected with $K(X) \leq 0$, but the strict inequality $K(X) \leq -c < 0$ holds true only for rank $X = 1$. Recall that "rank" denotes the dimension of a maximal *flat* in X that is a subspace isometric (for the distance induced from X) to a Euclidean space.

It easily follows from Lemma 6.4 in [18] that every closed bounded form on X of degree $i > \operatorname{rank} X$ is d(bounded), and one has trivial counterexamples for all $i \leq \operatorname{rank} X$.

0.1.C$'$. **Kähler case.** Let X be a *Hermitian* symmetric space, and let ω denote the imaginary part of the Hermitian metric of X. This ω is an exterior 2-form which is well known to be closed for symmetric X. In other words X is *Kähler*, and ω is called the *Kähler form* of X.

It is obvious that ω is bounded for all Hermitian manifolds. Furthermore, one knows that if X is a Hermitian symmetric space with no Euclidean factor, then the Kähler form ω is d(bounded). In fact, X admits a proper positive function (Kähler potential) of the form $f(x) = \varphi(\operatorname{dist}(x_0, x))$, such that $dJdf = \omega$ where df is bounded.

Recall that "no Euclidean factor" condition rules out isometrically split manifolds $X = X' \times \mathbf{R}^k$, $k \geq 1$, but admits manifolds of rank ≥ 2.

0.1.C$''$. It is not hard to generalize the above to all symmetric Riemannian spaces X of noncompact type. Namely, every closed invariant form (of any degree) on X is d(bounded), where "invariant" refers to the isometry group of X.

0.2. $\tilde{d}$(**bounded**) **forms.** A form α on $X = (X, g)$ is called $\tilde{d}$(bounded) if the lift $\tilde{\alpha}$ of α to the universal covering $\widetilde{X} \to X$ is d(bounded) on $\widetilde{X}$ with respect to the lift $\tilde{g}$ of the Riemannian metric g.

If α is d(bounded), then it is, obviously, $\tilde{d}$(bounded). Thus, if α becomes d(bounded) on some (not necessarily universal) covering of X, then α is $\tilde{d}$(bounded). On the other hand a $\tilde{d}$(bounded) form α on X need not be d(bounded). In fact, α need not be even exact.

0.2.A. **Example.** If X is complete and of negative curvature, $K(X) \leq -c < 0$, then every closed bounded form α on X of degree $i \geq 2$ is $\tilde{d}$(bounded) by 0.1.B. In particular, if X is compact and of negative curvature, then every closed form of degree ≥ 2 is $\tilde{d}$(bounded).

0.2.A$'$. **Opposite example.** If α is a $\tilde{d}$(bounded) form on a compact manifold with an abelian fundamental group Γ, then α is exact. This is seen by generalizing the argument in 0.1.A. (In fact that argument applies to all amenable groups Γ.)

0.2.B. **Homotopy invariance.** If X is a compact manifold with or without boundary, then the $\tilde{d}$(boundedness) property of forms on X is obviously independent of the metric g on X. Furthermore, since the exact forms are $\tilde{d}$(bounded), the $\tilde{d}$(boundedness) of a closed form α on X depends only on the cohomology class $[\alpha] \in H^*(X; \mathbf{R})$. Then one sees that $\tilde{d}$(boundedness) is a homotopy invariant property for compact manifolds. In fact, if $f: X \to Y$ is a continuous map, and α is a $\tilde{d}$(bounded) form on Y, then the induced form $f^*(\alpha)$ is $\tilde{d}$(bounded) on X, provided X is compact. (If f is not smooth, then $f^*(\alpha)$ should be thought of as the cohomology class $f^*[\alpha] \in H^*(X; \mathbf{R})$.)

0.2.C. **$\tilde{d}$(bounded) cohomology.** The above discussion leads to the following $\tilde{d}$(boundedness) definition for the cohomology of an abstract group Γ. A cohomology class $h \in H^*(\Gamma; \mathbf{R})$ is called $\tilde{d}$(bounded) if for every compact manifold X and every continuous map f of X into the Eilenberg-MacLane space $K(\Gamma, 1)$, the induced class $f^*(h) \subset H^*(X; \mathbf{R})$ is $\tilde{d}$(bounded) where $H^*(K(\Gamma, 1))$ is identified in the usual way with $H^*(\Gamma)$.

0.2.C$'$. **Example.** If Γ is the fundamental group of a closed (i.e., compact without boundary) manifold of negative curvature, then all cohomology of dimension ≥ 2 is $\tilde{d}$(bounded) by 0.2.A. This remains true for compact manifolds with convex boundary as well as for general hyperbolic groups Γ (see [10], [12]).

0.3. **Kähler hyperbolic manifolds.** A compact complex manifold X without boundary is called *Kähler hyperbolic* if it admits a Kähler metric whose 2-form ω is $\tilde{d}$(bounded).

0.3.A. **Examples.** (a) If X is homotopy equivalent to a compact Riemannian manifold with negative sectional curvature $(K < 0)$ and having convex boundary (if any), then X is Kähler hyperbolic provided it admits some Kähler metric (compare 0.1, 0.2.C$'$).

(b) If the universal covering $\widetilde{X}$ of X is (biholomorphic to) a bounded symmetric domain in $\mathbf{C}^n$, $n = \dim X$, then X is Kähler hyperbolic by 0.7.C$'$. This, probably, remains true for many nonsymmetric bounded domains, e.g., for those where X is compact, and for the Teichmüller space with the Bergman metric. On the other Jean-Pierre Demailly pointed out to the author that all hyperconvex bounded domains are Kähler hyperbolic (compare [7]). Recently, the hyperconvexity of the Teichmüller space was proven by S. L. Kruskal [16].

(c) Every complex submanifold X in a Kähler hyperbolic manifold Y is Kähler hyperbolic. In fact, if X admits a finite (i.e., finite-to-one) morphism $X \to Y$, then Kähler hyperbolicity of Y obviously yields that of X.

(d) Cartesian products of Kähler hyperbolic manifolds are, obviously, Kähler hyperbolic. Probably, if X is a fibered space where the base and the fibers are Kähler hyperbolic, then X is Kähler hyperbolic. (This is not hard to prove in the case where the base is real hyperbolic, for example, if X is a *Kodaira* surface.)

(e) Every projective manifold X of dimension n can be dominated by a Kähler hyperbolic manifold as follows. Start with an arbitrary Kähler hyperbolic manifold Y of dimension n, and then take an n-dimensional submanifold $X' \subset X \times Y$ whose projections to X and Y are finite-to-one. (One obtains such an X' by intersecting n sufficiently ample nonsingular hypersurfaces in $X \times Y$ in general position.) Alternatively, one takes the fiber product of generic morphisms of X and Y to $\mathbf{C}P^n$.

0.3.B. **Remark on different notions of hyperbolicity.** The most general notion of hyperbolicity is due to Kobayashi: A compact complex manifold X is called *Kobayashi(-Broody) hyperbolic* if every holomorphic map $\mathbf{C} \to X$ is constant. A much stronger notion is that of real hyperbolicity for compact Riemannian manifolds X: Every absolutely minimizing conformal map $f\colon \mathbf{R}^2 \to X$ is constant. Here, "absolutely minimizing" means that no homotopy of f fixed outside a compact subset of $\mathbf{R}^2$ can decrease the area of f. It is not hard to see that real hyperbolicity is a purely topological notion. In fact, it is equivalent to the following conditions (i) and (ii):

(i) The fundamental group $\Gamma = \pi_1(X)$ is hyperbolic in the sense of [12].

(ii) $\pi_2(X) = 0$.

It is easy to show (compare 6.4 in [10]) that Kähler hyperbolicity is pinched between "real" and "Kobayashi,"

$$\begin{aligned}(\text{real hyperbolicity} + \text{Kähler}) &\Rightarrow (\text{Kähler hyperbolicity})\\ &\Rightarrow (\text{Kobayashi hyperbolicity}).\end{aligned}$$

0.4. **The sign of $\chi(\Omega^p)$.** Let $\Omega^p = \Omega^p(X)$ denote the sheaf of holomorphic p-forms on X and

$$\chi_p(X) = \chi(\Omega^p) = \sum_{q=0}^{n} (-1)^q h^{p,q},$$

where $h^{p,q}$ are the Hodge numbers $h^{p,q} = \dim H^q(\Omega^p)$.

0.4.A. **Theorem.** *If X is Kähler hyperbolic, then for every $p = 0, 1, \cdots$, $n = \dim X$, the Euler characteristic $\chi_p(X)$ does not vanish and*

$$\operatorname{sign} \chi_p = (-1)^{n-p}. \tag{+}$$

This is proven in §§1 and 2 by passing to the universal covering $\widetilde{X}$ of X and by showing that the *L_2-Hodge number* $h^{p,q}L_2(\widetilde{X})$ vanishes if and only if $p + q < n$.

0.4.B. **Remarks.** The above theorem settles a special (namely the Kähler) case of the Chern conjecture claiming that the topological Euler characteristic of a real $2n$-dimensional manifold X of negative curvature satisfies

$$\operatorname{sign} \chi(X) = (-1)^n. \tag{++}$$

The idea of using here the L_2-Hodge theory was suggested by Atiyah and Singer. Later, Anderson gave an example of a 3-dimensional hyperbolic manifold with nontrivial first L_2-Betti number (see [1]) which has made the L_2-approach look invalid in the real case. (Though there is no serious ground for the belief in the Chern conjecture, one has no counterexample even for the stronger conjecture claiming that *all* aspherical manifolds X satisfy (++) unless $\chi(X) = 0$.)

Our major tool in the Kähler case is the strong L_2-Lefschetz theorem which provides a lower bound on the spectrum of the Laplace operator on the L_2-forms on $\widetilde{X}$ and shows, in particular, that the L_2-Betti numbers $b^i L_2(\widetilde{X})$ vanish for $i \neq n = \dim_{\mathbf{C}} X$ (compare [13]). Then the desired nonvanishing for $i = n$ is achieved with an upper bound on the spectrum which exploits a twisting trick of Vafa and Witten in the ambience of "large manifolds" (see [11], [14], [15], [23]).

The vanishing of $h^{p,q}L_2$ was independently proved by Stern [21] for complete simply connected Kähler manifolds Y with negatively pinched sectional curvature, $-b \leq K(Y) \leq -a < 0$. Then Stern used the positivity of $h^{n,0}L_2$ due to Green and Wu [9] and derived the above (+) for $p = 0$ and $p = n$.

0.4.C. **Quasiampleness of** $\operatorname{Can} X$. Using the existence of holomorphic L_2-forms of degree n on the universal covering $\widetilde{X}$ of X, we obtain in §3 the following:

Corollary. *The canonical bundle of a compact Kähler hyperbolic manifold is quasiample. That is the Kodaira dimension of (the canonical bundle of) X equals $\dim_{\mathbf{C}} X$. It follows that X is Moishezon and, hence, projective algebraic.*

Questions. Is the canonical bundle ample? Is the cotangent bundle of X ample?

Remark. Eckart Viehweg pointed out to the author that since X contains no rational curve (as all hyperbolic curves have genus ≥ 2), the results by Mori [17] and Shokurov [19] imply that the canonical bundle of X is *semipositive* and the canonical map is regular.

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1. L_2-Hodge theory on complete Riemannian manifolds

Basic facts of the Hodge theory for compact manifolds remain valid for complete Riemannian manifolds X (see [8]). In particular, if X is Kähler, one has the L_2-Lefschetz theorem which becomes especially useful if the Kähler form is d(bounded).

1.1. **Cutoff functions and Hodge decomposition.** The standard operators of Hodge theory, d, δ, $*$ as well as $\overline{\partial}$, L, etc., in Kähler geometry are defined locally and thus make sense without the compactness or completeness assumptions. The compactness becomes important when one integrates by parts. For example, one shows that $\delta = d^* =_{\text{def}} \pm * d *$ (here $\pm$ sign is $(-1)^{n+np+1}$, where $n = \dim X$ and p is the degree of the forms in question) is the adjoint of d by first observing the (local!) formula

$$d\varphi \wedge *\psi - \varphi \wedge *\delta\psi = \pm d(\varphi \wedge *\psi),$$

and then deriving the desired relation $\langle d\varphi, \psi\rangle = \langle \varphi, \delta\psi\rangle$ by applying the Stokes formula

$$(*) \qquad \int_X d(\varphi \wedge *\psi) = 0.$$

Notice that $(*)$ is valid for all C^1-smooth $(n-1)$-forms η in place of $\varphi \wedge *\psi$ on closed (i.e., compact without boundary) manifolds X.

If X is noncompact or has a nonempty boundary, then $(*)$ is not true any more. In fact, the integral $\int d\eta$ over X now equals the boundary term, $\int_X d\eta = \int_{\partial X} \eta$, which need not be zero for nonempty ∂X.

However, if X is complete, then $(*)$ remains true for all L_1-forms η on X.

1.1.A. **L_1-Lemma.** *Let η be an L_1-form on X of degree $n-1$, i.e.,*

$$\|\eta\|_{L_1} \overset{\text{def}}{=} \int_X \|\eta(x)\|\, dx < \infty,$$

such that the differential $d\eta$ *is also* L_1. *If* X *is complete, then*

$$(**) \qquad \int_X d\eta = 0.$$

Remark. What is important here is the behavior of the forms η and $d\eta$ at infinity, while the smoothness of η plays no essential role. In fact the relation $(**)$ for C^∞-smooth forms easily yields that for nonsmooth η where $d\eta$ is understood as a distribution.

Proof. The completeness of Y enters the proof via the (obvious) existence of the following:

Cutoff functions. Such a function a_ε on X must satisfy the following conditions:

(i) a_ε is smooth (say C^∞ if X is C^∞) and takes values in the interval $[0, 1]$; furthermore, a_ε has compact support.

(ii) The subsets $a_\varepsilon^{-1}(1) \subset X$ (i.e., of the points $x \in X$ where $a_\varepsilon(x) = 1$) exhaust X as $\varepsilon \to 0$.

(iii) The differential of a_ε everywhere bounded by ε,

$$\|da_\varepsilon\|_{L_\infty} \overset{\text{def}}{=} \sup_{x\in X} \|da_\varepsilon(x)\| \leq \varepsilon.$$

Now, we apply the Stokes formula to the cutoff form $a_\varepsilon \eta$ which has compact support

$$0 = \int_X d(a_\varepsilon \eta) = \int_X da_\varepsilon \wedge \eta + \int_X a_\varepsilon \, d\eta.$$

Then we conclude

$$\left|\int_X a\varepsilon \, d\eta\right| \leq \left|\int_X da_\varepsilon \wedge \eta\right| \leq \varepsilon \|\eta\|_{L_1},$$

and since $d\eta \in L_1$ we have

$$\int_X d\eta = \lim_{\varepsilon \to 0} \int_X a_\varepsilon \, d\eta = 0.$$

1.1.B. The Gaffney cutoff trick can also be applied to one of the forms φ and ψ in the integral $\int d(\varphi \wedge *\psi)$. Thus one obtains another useful

Lemma. *If an* L_2*-form* α *is* Δ*-harmonic,* $\Delta\alpha =_{\text{def}} (d\delta + \delta d)\alpha = 0$, *then* α *is* $(d + \delta)$*-harmonic, i.e.,* $d\alpha = 0$ *and* $\delta\alpha = 0$.

Proof. We want again to justify the integral identity

$$\langle \Delta\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle.$$

If $d\alpha$ and $\delta\alpha$ are L_2 (i.e., square integrable on X), then this follows by 1.1.A. To handle the general case we cutoff α and obtain by a simple computation $0 = \langle \Delta\alpha, a_\varepsilon \alpha\rangle = I_1(\varepsilon) + I_2(\varepsilon)$, where

$$I_1(\varepsilon) = \int_X a_\varepsilon(\|d\alpha\|^2 + \|\delta\alpha\|^2)$$

and

$$|I_2(\varepsilon)| \le \int_X \|da_\varepsilon\| \, \|\alpha\|(\|d\alpha\| + \|\delta\alpha\|),$$

where the norms under the integrals are understood pointwise on X. Then we choose (this is, obviously, possible) a_ε, such that $\|da_\varepsilon\|^2 \le \varepsilon a_\varepsilon$ on X and estimate I_2 by Schwartz inequality. This yields

$$|I_2(\varepsilon)| \le 2\varepsilon\|\alpha\|_{L_2}\left(\int_X \|a_\varepsilon\|(\|d\alpha\|^2 + \|\delta\alpha\|^2)\right)^{1/2},$$

and hence $I_1(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Example. Every harmonic L_2-function on a complete Riemannian manifold X is constant. In particular, if X has finite volume, then it supports no nonzero harmonic L_2-function.

1.1.C. With 1.1.A. and 1.1.B. one concludes, as in the compact case, that the L_2-space $L_2\Omega^p$ of exterior p-forms on a complete manifold X admits Hodge decomposition

$$L_2\Omega^p = \mathscr{H}^p \oplus \overline{d(L_2\Omega^{p-1})} \oplus \overline{\delta(L_2\Omega^{p+1})},$$

where $\overline{d(\dots)}$ is the closure in $L_2\Omega^p$ of the intersection of $L_2\Omega^p$ with the image of d and $\overline{\delta(\dots)}$ has the same meaning.

1.1.C$'$. Let us indicate one simple corollary of the above decomposition.

If a harmonic L_2-form α is $d(L_2)$, i.e., $\alpha = d\beta$ for some L_2-form β, then $\alpha = 0$.

1.1.D. **Some examples.** (a) If $X = \mathbf{R}^n$, then $\mathscr{H}^p = 0$ for all $p = 0, \cdots, n$. Thus the Hodge decomposition contains only $\overline{d(\dots)}$ and $\overline{\delta(\dots)}$. In particular, every L_2-form α on $\mathbf{R}^1$ of degree one can be L_1-approximated by differentials of L_2-functions. This applies, for example, to forms α with compact supports which have $\int_{\mathbf{R}} \alpha \ne 0$, and therefore are not the differentials of L_2-functions.

(b) Let X be the hyperbolic space H^n. It is well known that the space of harmonic forms $\mathscr{H}^p$ on H^n is zero unless $n = 2p$ and $\mathscr{H}^p \ne 0$ for $n = 2p$ (compare 0.4.). Notice that H^n is contractible (in fact, diffeomorphic to $\mathbf{R}^n$) and so the harmonic forms α on H^n are exact, $\alpha = d\beta$, but one can not make $\beta \in L_2$ or even L_2-approximate α by the differentials of L_2-forms.

1.1.E. **Homotopy invariance.** The above example shows that the space $\mathscr{H}^p = \mathscr{H}^p(X)$ is not a homotopy invariant of X, not even a diff-invariant. Yet, one recaptures the invariance if one restricts to bi-Lipschitz homeomorphism. More generally, let $f: X \to Y$ be a Lipschitz map between Riemannian manifolds, i.e.,

$$\mathrm{dist}_Y(f(x_1), f(x_2)) \leq \mathrm{const}\,\mathrm{dist}_X(x_1, x_2)$$

for all pairs of points x_1 and x_2 in X. (If f is C^1-smooth, this is equivalent to $\|df\| \leq \mathrm{const}$.) Then the induced map on forms, called f° sends L_2-forms on Y to X. The composition of f° on $\mathscr{H}^*(Y)$ with the orthogonal projection $h: L_2\Omega^*(X) \to \mathscr{H}^*(X)$ defines a linear map, between the harmonic spaces

$$f^*: \mathscr{H}^*(Y) \to \mathscr{H}^*(X).$$

An easy (and well-known) argument shows that f^* is Lipschitz homotopy invariant. That is, if f_1 and f_2 can be joined by a homotopy $F: X \times [0, 1] \to Y$, which is a Lipschitz map for the product metric in $X \times [0, 1]$, then $f_1^* = f_2^*$.

Remark. If X and Y are compact, one gets this way the usual homotopy invariance of $\mathscr{H}^*$ as all maps can be approximated by Lipschitz maps. A more interesting case is that where X and Y are infinite coverings of compact manifolds, say, X_0 and Y_0 respectively, and pertinent maps $f: X \to Y$ are lifts of continuous maps $f_0: X_0 \to Y_0$. Here again we may assume f_0 and f Lipschitz, and then we can see that if f_0 is a homotopy equivalence, then the induced map

$$f^*: \mathscr{H}^*(Y) \to \mathscr{H}^*(X)$$

is an isomorphism.

1.2. **Strong L_2-Lefschetz theorem.** Let X be a Kähler manifold, and let ω denote the Kähler form. From the Riemannian point of view, ω is a closed 2-form, such that

(i) ω is nonsingular,
(ii) ω is parallel for the Riemannian connection on X.

Notice that condition (i) needs no Riemannian metric on X. It means, in effect, that X is even dimensional, $\dim X = n = 2m$, and the top exterior power ω^m of ω does not vanish on X.

On the contrary, condition (ii) indicates a (very strong) relation between ω and the Riemannian metric on X.

Since ω is parallel, the operator $L^k: \Omega^p \to \Omega^{p+2k}$ defined by $L^k(\varphi) = \omega^k \wedge \varphi$ for all p-forms $\varphi \in \Omega^P$ commutes with d and Δ (the commutation with d only needs ω to be closed, but the commutation with δ

requires the parallelism of ω). Thus L^k sends harmonic form to harmonic forms, $L^k : \mathscr{H}^p \to \mathscr{H}^{p+2k}$.

1.2.A. **Theorem.** (Lefschetz.) *The map L^k is injective on harmonic forms for $2p + 2k \leq n = \dim X$ and surjective for $2p + 2k \geq n$.*

Proof. The core of the proof lies in linear algebra.

1.2.A$'$. **Lemma.** *Let ω_0 be an exterior 2-form on a linear space T_0 of dimension $n = 2m$ and let $L_0^k \colon \wedge^p T_0 \to \wedge^{p+2k} T_0$ denote the linear map $\alpha \to \omega_0^k \wedge \alpha$ on the exterior algebra $\wedge^* T_0$. If ω_0 is nonsingular, then L_0^k is injective for $2p + 2k \leq n$ and surjective for $2p + 2k \geq n$.*

The proof can be found in any textbook on Kähler geometry (e.g., see [23]).

This lemma immediately yields the "injective" part of the theorem. To prove the surjectivity we invoke the adjoint operator

$$\Lambda_k = (L^k)^* : \Omega^{p+2k} \to \Omega^p .$$

Since L is parallel, Λ also is parallel and, hence, sends $\mathscr{H}^{p+2k} \to \mathscr{H}^p$.

Now, the algebraic lemma easily implies that the corresponding (algebraic) operator adjoint to L_0^k, say

$$(\Lambda_k)_0 \colon \wedge^{p+2k} T_0 \to \wedge^p T_0 ,$$

is injective for $2p + 2k \leq n$. Hence Λ_k is also injective and therefore L^k has dense image. To conclude the proof of the surjectivity of L, we notice that $L^{k+l} = L^k \circ L^l$ and thus reduce the general case to that where $2p + 2k = n$, and where L^k is (as we already know) injective. In fact, our proof of the injectivity shows that L^k is a *quasi-isometry*, i.e.,

$$(\text{const})^{-1} \|\varphi\|_{L_2} \leq \|L^k \varphi\|_{L_2} \leq \text{const}\, \|\varphi\|_{L_2}, \qquad \text{for all } \varphi \in \Omega^p .$$

This and the dense image property show that L^k is bijective for $2p+2k = n$ and, hence, surjective for $2p + 2k \geq n$.

1.2.A$''$. **Remarks.** (a) In the classic case where the manifold X in question is compact, surjectivity follows directly from injectivity, since the spaces of harmonic forms are finite dimensional and

$$\dim \mathscr{H}^{p+2k} = \dim \mathscr{H}^p , \qquad \text{for } 2p + 2k = n ,$$

by Poincaré duality.

(b) What makes the theorem really interesting is the topological interpretation of the spaces $\mathscr{H}^p$ as the real cohomology of X. If X is compact, then the standard corollary is the Lefschetz inequalities for the Betti numbers b_p of X, $b_{p+2} \geq b_p$, for $p < n/2$.

We are mostly interested in the noncompact case where $\dim \mathscr{H}^p = \infty$ and the topological interpretation of $\mathscr{H}^p$ is not so simple (compare [6]).

1.2.B. **Lefschetz vanishing theorem.** *If the Kähler form* ω *is* d(bound-ed), *then* $\mathscr{H}^p = 0$, *unless* $p = n/2$.

Proof. Let $\omega = d_\eta$, where η is a bounded 1-form,

$$\|\eta\|_{L_\infty} = \sup_{y \in Y} \|\eta(y)\| < \infty.$$

Then for every closed L_2-form φ, the form $L^k\varphi = \omega^k \wedge \varphi$ is $d(L_2)$, $L^k\varphi = d\psi$, for $\psi = \eta \wedge (d\eta)^{k-1} \wedge \varphi$, where ψ is L_2 since φ is L_2 and $\eta \wedge (d\eta)^{k-1}$ is bounded. In particular, if φ is harmonic, then $L^k\varphi = 0$ for $k > 0$. This implies, by 1.2.A., that $\varphi = 0$, unless $\deg \varphi = n/2$.

1.2.B$'$. **Remark.** We shall see in §2 that $\mathscr{H}^p \neq 0$, for $p = n/2$.

1.3. **Von Neumann dimension and the L_2-index theorem.** A Hilbert space $\mathscr{H}$ with a unitary action of a countable group Γ is called a Γ-*module* if $\mathscr{H}$ is isomorphic to a Γ-invariant subspace in the space of L_2-functions on Γ with values in some Hilbert space H. To each Γ-module Γ, one assigns the *Von Neumann dimension*, also called Γ-*dimension*, $0 \leq \dim_\Gamma \mathscr{H} \leq \infty$, which is a nonnegative real number or $+\infty$ (see [2], [6] and references therein). The precise definition is not important for the moment but the following properties (i) to (iv) convey the idea of $\dim_\Gamma \mathscr{H}$ as some kind of size of the "quotient space" $\mathscr{H}/\Gamma$:

(i) $\dim_\Gamma \mathscr{H} = 0 \Leftrightarrow \mathscr{H} = 0$.

(ii) If Γ is a *finite* group, then $\dim_\Gamma \mathscr{H} = \dim \mathscr{H} / \operatorname{card} \Gamma$.

(iii) $\dim_\Gamma \mathscr{H}$ is additive. Given $0 \to \mathscr{H}_1 \to \mathscr{H}_2 \to \mathscr{H}_3 \to 0$, one has $\dim_\Gamma \mathscr{H}_2 = \dim_\Gamma \mathscr{H}_1 + \dim_\Gamma \mathscr{H}_3$.

(iv) If $\mathscr{H}$ equals the (whole) space of L_2-functions $\Gamma \to H$, then $\dim_\Gamma \mathscr{H} = \dim H$.

In particular, if $H = \mathbf{R}^n$, then $\dim_\Gamma \mathscr{H} = n$.

Here we are interested in the situation where Γ is a discrete faithful group of isometrics of a Riemannian manifold X. One can easily show (see [2]) that the spaces $\mathscr{H}^p$ of harmonic L_2-forms are Γ-moduli for all degrees p, and then one defines the L_2-Betti numbers $b^p(X\colon \Gamma) \stackrel{\text{def}}{=} \dim_\Gamma \mathscr{H}^p$. The most interesting case is where X/Γ is compact. Then the L_2-Betti numbers are finite $b^p(X\colon \Gamma) < \infty$, for all p, and the L_2-Euler characteristic

$$\chi(X\colon \Gamma) \underset{\text{def}}{=} \sum_{b=0}^{n} (-1)^p b^p(X\colon \Gamma)$$

equals the ordinary Euler characteristic of the orbifold X/Γ (see [2]).

For example, if Γ acts freely on X, and X/Γ is a manifold, then the "ordinary characteristic" satisfies the usual formula

$$\chi(X/\Gamma) = \Sigma(-1)^p b^p(X/\Gamma).$$

If Γ has fixed points, one defines χ by taking some Γ-invariant triangulation of X, and setting

$$\chi(X/\Gamma) = \sum_{\Delta}(-1)^{\dim \Delta}(\operatorname{card}\Gamma_\Delta)^{-1},$$

where Δ runs over a (finite) set of simplices Δ in the triangulation containing exactly one representative (simplex) in each Γ-orbit of simplices, and Γ_Δ denotes the isotropy subgroup of Δ (compare [6]).

Now we return to the case where X is a complete Kähler manifold of dimension $n = 2m$, and combine the above theorem of Atiyah on the equality of the two characteristics,

$$\chi_{L_2}(X\colon \Gamma) = \chi_{\text{orb}}(X/\Gamma),$$

with the Lefschetz vanishing theorem. Thus we obtain the following property of the ordinary Euler characteristic $\chi(X/\Gamma)$.

1.3.A. **Proposition.** *If the Kähler form of X is d(bounded), then $\chi(X/\Gamma)$ is nonnegative for m even and nonpositive for m odd.*

1.3.A$'$. **Remarks.** (a) The above proposition does not tell us whether $\chi(X/\Gamma)$ vanishes or not. In fact the major point of the remaining part of this paper is to prove nonvanishing of $\chi(X/\Gamma)$ by showing that $\mathscr{H}^m(X) \neq 0$.

(b) The Lefschetz-type vanishing theorem and the resulting equality $\dim_\Gamma \mathscr{H}^m = \chi(X/\Gamma)$, for $2m = \dim X$, is known to be true for many non-Kähler manifolds X, such, for example, as symmetric spaces of noncompact type. In fact one may conjecture on the basis of known examples (or rather on the lack of counterexamples) that the vanishing theorem for $\mathscr{H}^p$ for $p \neq m$ holds true for all contractible manifolds with X/Γ compact.

(c) We shall see later how to extend 1.3.A. to the characteristics $\chi_p(X/\Gamma)$ defined in 0.4.

1.4. **Lower bound on the spectrum.** We want to sharpen the Lefschetz vanishing theorem by giving a lower bound on the spectrum of the Laplace operator Δ on L_2-forms Ω^p for $p \neq n/2$. Namely, we shall prove in this section the following:

1.4.A. **Theorem.** *Let (X, ω) be a complete Kähler manifold of dimension $n = 2m$ and $\omega = d\eta$ where η is a bounded 1-form on X. Then* *every L_2-form ψ on X of degree $p \neq m$ satisfies the inequality*

$$(*) \qquad \langle \psi , \Delta\psi \rangle \geq \lambda_0^2 \langle \psi , \psi \rangle ,$$

where λ_0 is a strictly positive constant which depends only on $n = \dim X$ and the bound on η,

$$\lambda_0 \geq \mathrm{const}_n \|\eta\|_{L_\infty}^{-1} .$$

Furthermore, inequality $(*)$ is satisfied by the L_2-forms of degree m which are orthogonal to the harmonic m-forms.

1.4.A$'$. **Remark.** Inequality $(*)$ makes sense, strictly speaking, if $\Delta\psi$ (as well as ψ) is in L_2. In this case $(d+\delta)\psi$ also is in L_2 by the proof of 1.1. and $(*)$ is equivalent to

$$(+) \qquad \|(d+\delta)\psi\|_{L_2} \geq \lambda_0 \|\psi\|_{L_2} .$$

Moreover the cutoff argument in §1.1 shows that the general case of $(+)$, where we only assume ψ and $(d+\delta)\psi$ in L_2, follows from that where ψ is a smooth function with compact support. In particular, inequality $(*)$ with ψ in L_2 implies the general case of $(+)$.

Proof. To simplify notation we shall write $a \lesssim b$, for $a \leq \mathrm{const}_n b$, and $a \approx b$, for $b \lesssim a \lesssim b$. Then we recall the operator $L^k : \Omega^p \to \Omega^{p+2k}$ for a given $p < m$ and $2p + 2k = n$. By the Lefschetz theorem L^k is a bijective quasi-isometry and so every L_2-form ψ of degree p is the product $\psi = L^k \varphi = \omega^k \wedge \varphi$, where $\varphi = L^{-k}\psi$ and $\|\varphi\|_{L_2} \approx \|\psi\|_{L_2}$. Since L^k commutes with Δ, we also have

$$\langle \Delta\varphi , \varphi \rangle \approx \langle \Delta\psi , \psi \rangle .$$

In particular,

$$\langle \Delta\varphi , \varphi \rangle \lesssim \langle \Delta\psi , \psi \rangle .$$

Then we write $\psi = d\theta - \psi'$, for $\theta = \eta \wedge \omega^{k-1} \wedge \varphi$ and $\psi' = \eta \wedge \omega^{k-1} \wedge d\varphi$, and observe that

$$\|\theta\|_{L_2} \lesssim |\eta| \|\varphi\|_{L_2} \lesssim |\eta| \|\psi\|_{L_2} ,$$

where we used the abbreviation $|\eta| = \|\eta\|_{L_\infty}$. Next, since

$$\|d\varphi\|_{L_2}^2 \leq \langle \Delta\varphi , \varphi \rangle \lesssim \langle \Delta\psi , \psi \rangle ,$$

we have $\|\psi'\|_{L_2} \lesssim |\eta| \langle \Delta\psi , \psi \rangle^{1/2}$. Now,

$$\|\psi\|_{L_2}^2 = \langle \psi , \psi \rangle = \langle \psi , d\theta - \psi' \rangle \lesssim |\langle \psi , d\theta \rangle| + |\langle \psi , \psi' \rangle| ,$$

where

$$\begin{aligned}|\langle\psi, d\theta\rangle| &= |\langle\delta\psi, \theta\rangle| \leq \|\delta\psi\|_{L_2}\|\theta\|_{L_2} \\ &\leq \langle\Delta\psi, \psi\rangle^{(1/2)}\|\theta\|_{L_2} \lesssim |\eta|\langle\Delta\psi, \psi\rangle^{(1/2)}\|\varphi\|_{L_2} \\ &\approx |\eta|\langle\Delta\psi, \psi\rangle^{(1/2)}\|\psi\|_{L_2},\end{aligned}$$

and

$$|\langle\psi, \psi'\rangle| \leq \|\psi\|_{L_2}\|\psi'\|_{L_2} \lesssim |\eta|\|\psi\|_2\langle\Delta\psi, \psi\rangle^{(1/2)}.$$

This yields the desired estimate

$$(**) \qquad \|\psi\|_{L_2} \lesssim |\eta|\langle\Delta\psi, \psi\rangle^{(1/2)}$$

for the forms ψ of degree $p > m$. The case $p < m$ follows by the Poincaré duality as the operator $*: \Omega^p \to \Omega^{n-p}$ commutes with Δ and is isometric for the L_2-norms.

According to 1.4.A$'$. the above inequality $(**)$ shows that the *closed* L_2-forms ψ of degree $p \neq m$ satisfy $\|\psi\|_{L_2} \lesssim \|\delta\psi\|_{L_2}$ and the *coclosed* forms satisfy $\|\psi\|_{L_2} \lesssim \|d\psi\|_{L_2}$. Therefore the operators $\delta: L_2\Omega^p \to L_2\Omega^{p-1}$ and $d: L_2\Omega^p \to L_2\Omega^{p+1}$ have *closed* images for $p \neq m$ since the orthogonal complement of $\operatorname{Ker}\delta \subset L_2\Omega^p$ consists of closed forms, and the complement of $\operatorname{Ker} d$ consists of coclosed forms (see 1.1.C.). In particular, we have the Hodge decomposition in the middle dimension without taking the closures of the images of d and δ (compare 1.1.C.).

$$L_2\Omega^m = \mathscr{H}^m \oplus dL_2\Omega^{m-1} \oplus \delta L_2\Omega^{m+1}.$$

Now we are able to prove the theorem (i.e., inequality $(**)$) for the form ψ of degree m orthogonal to $\mathscr{H}^m$. We have $\psi = d\alpha + d\beta$, where $d\alpha$ is orthogonal to $\delta\beta$, and the L_2-forms α and β of degrees $m-1$ and $m+1$ correspondingly satisfy $\delta\alpha = 0$, $d\beta = 0$. This implies

$$\langle\psi, \psi\rangle = \langle d\alpha, d\alpha\rangle + \langle\delta\beta, \delta\beta\rangle = \langle\Delta\alpha, \alpha\rangle + \langle\Delta\beta, \beta\rangle,$$

as well as $\Delta\alpha = \delta\psi$ and $\Delta\beta = d\psi$. On the other hand, applying inequality $(**)$ to α and β yields, in consequence of Schwartz inequality, the following estimates

$$\langle\Delta\alpha, \alpha\rangle \lesssim |\eta|^2\langle\Delta\alpha, \Delta\alpha\rangle$$

and

$$\langle\Delta\beta, \beta\rangle \lesssim |\eta|^2\langle\Delta\beta, \Delta\beta\rangle.$$

Thus

$$\langle\psi, \psi\rangle \lesssim |\eta|(\langle\delta\psi, \delta\psi\rangle + \langle d\psi, d\psi\rangle) = |\eta|^2\langle\Delta\psi, \psi\rangle.$$

2. Twisted operators and an upper bound on $\operatorname{spec}(d+\delta)$

We show in this section that under certain conditions the spectrum of the operator $d+\delta\colon L_2\Omega^* \to L_2\Omega^*$ contains zero. This, together with Theorem 1.4.A., ensures the desired nonvanishing of the space $\mathscr{H}^m$ of middle dimensional harmonic L_2-forms on X.

2.1. **Tensoring differential operators with connections.** Let E and E' be C^∞-vector bundles over a smooth manifold X, and $D\colon C^\infty(E) \to C^\infty(E')$ be a differential operator between C^∞-smooth sections of these bundles. If F is a trivial k-dimensional bundle over X with a given trivialization, then one can define the (twisted) operator

$$D^\otimes\colon C^\infty(E\otimes F) \to C^\infty(E'\otimes F),$$

by

$$(*)\qquad D^\otimes\left(\sum_{i=1}^k e_i\otimes f_i\right) = \sum_{i=1}^k (De_i)\otimes f_i,$$

where $(f_1, \cdots, f_k)$ is the frame of (parallel) sections trivializing F, and e_i are arbitrary sections of E. In other words $D^\otimes$ equals the "direct sum" of k copies of D.

Next, let F be a (not necessary trivial) bundle with a *flat* connection ∇. Then one can define

$$D\otimes\nabla\colon C^\infty(E\otimes F) \to C^\infty(E'\otimes F)$$

by applying $(*)$ to (locally defined) frames $(f_1, \cdots, f_k)$ of ∇-parallel sections of F. The linearity of D shows that the right-hand side of $(*)$ is independent of the choice of the (parallel!) frame, and hence

$$(D\otimes\nabla)\left(\sum_{i=1}^k e_i\otimes f_i\right) = \sum_{i=1}^k (De_i)\otimes f_i$$

is globally defined on $C^\infty(E\otimes F)$.

Now, we turn to the case where ∇ is an arbitrary (not flat) linear connection in E, and we want to construct a linear operator

$$D\otimes\nabla\colon C^\infty(E\otimes F) \to C^\infty(E'\otimes F)$$

with the following property:

$(**)$ If a frame $(f_1, \cdots, f_k)$ satisfies $\nabla f_i(x) = 0$, for $i = 1, \cdots, k$, at some point $x\in X$, then

$$(D\otimes\nabla)\left(\sum_{i=1}^k e_i\otimes f_i\right)(x) = \sum_{i=1}^k (De_i)\otimes f_i(x),$$

for arbitrary C^∞-sections $e_1, \cdots, e_k$ of E.

If such $D \otimes \nabla$ exists, it is clearly unique. Now we prove the existence in the case where D is a first-order operator. For such D, we define a homomorphism $S = S_D \colon T^* \otimes E \to E'$, where T^* is the cotangent bundle of E, by the following condition $S(df \otimes e) = D(fe) - fDe$ for all C^∞-functions f on X and sections e of E. Then we view our connection ∇ as a differential operator, $\nabla \colon C^\infty(F) \to C^\infty(F \otimes T^*)$, and we consider the operator $S^\otimes \colon E \otimes (F \otimes T)^* \to E' \otimes F$ defined by $S^\otimes(e \otimes f \otimes t^*) = S(t^* \otimes e) \otimes f$. Finally we define $D \otimes \nabla \colon C^\infty(E \otimes F) \to C^\infty(E' \otimes F)$ by

$$(D \otimes \nabla)(\Sigma e_i \otimes f_i) = \Sigma(De_i) \otimes f_i + S^\otimes(\Sigma e_i \otimes \nabla f_i).$$

2.1.A. **Connection as potentials.** Suppose we have two connections ∇_0 and ∇ on the same E. Then the difference $A = \nabla - \nabla_0$ is a homomorphism $A \colon E \to E \otimes T^*$, which is sometimes called the *connection* form. We observe that the difference $D \otimes \nabla - D \otimes \nabla_0 = D \otimes A$ is a *zero-order* operator $C^\infty(E \otimes F) \to C^\infty(E' \otimes F)$ for

$$D \otimes A(\Sigma e_i \otimes f_i) = S^\otimes(\Sigma e_i \otimes Af_i).$$

In particular, if ∇_0 is a trivial connection and

$$D \otimes \nabla = k^\oplus D \stackrel{\text{def}}{=} \underbrace{D \oplus \cdots \oplus D}_{k},$$

then $D \otimes \nabla$ is a *perturbation* of $k^\oplus D$ by a *potential*, $D \otimes \nabla = k^\oplus D + P$, where P is the homomorphism (vector-potential) $E \otimes F \to E' \otimes F$ corresponding to the (zero-order!) operator $D \times A$.

2.1.B. **Hermitian line bundles.** Let us specialize the above discussion to the case where F is the trivial complex line bundle $F = \mathbf{C} \times X \to X$ with the trivial connection ∇_0, and let ∇ be a Hermitian connection in F. Then the connection form $A \colon F \to F \otimes T^*$ reduces to an ordinary 1-form a on X, such that $Af = \sqrt{-1} f \otimes a$, and da equals the curvature of ∇.

Conversely, let (F, ∇) be a complex line bundle with a Hermitian connection, such that the curvature form $\omega(\nabla)$ is exact, i.e., $\omega(\nabla) = da$. Then, there exists a flat Hermitian connection ∇_0 in F, such that $(\nabla - \nabla_0)f = \sqrt{-1} f \otimes a$. Namely, one defines ∇_0 by $\nabla_0 f = \nabla_f - \sqrt{-1} f \otimes a$. Furthermore, if the underlying manifold X is simply connected, then the bundle F is trivial, and ∇_0 is (isomorphic to) a trivial connection.

2.2. **Perturbation of strictly positive operators D.** We assume here that the fibrations E and E' in question are given Hermitian (or Euclidean) structures and that the manifold X comes with some measure. For example, if X is a Riemannian manifold, we shall use the Riemannian measure. The Hermitian structure in the bundles gives us the L_∞-norms on the spaces of sections of E and E'. Then the measure on X leads to L_2-norms.

Now we define the *lower spectral bound* $\lambda_0 = \lambda_0(D) \geq 0$ as the upper bound of the nonnegative numbers λ, such that $\|De\|_{L_2} \geq \lambda \|e\|_{L_2}$ for all those (distribution) sections e of E where De is in L_2.

Next we assume X is a Riemannian manifold, and then we have the pointwise norm of the homomorphism $S = S_D \colon T^* \otimes E \to E'$. The supremum of this norm over X is denoted by $|S| = \|S\|_{L_\infty}$. Notice that this can be infinite, but we assume below that $|S| < \infty$.

Finally, we take a complex line bundle with a Hermitian connection, denoted by (F, ∇), and we assume that the curvature $\omega(\nabla)$ is d(bounded). Namely, $\omega(\nabla) = da$, for $a \in L_\infty \Omega^1$, where the L_∞-norm of a is denoted by $|a| = \|a\|_{L_\infty} < \infty$.

2.2.A. **Proposition.** *If $|a|$ is bounded by $|a| \leq \lambda_0 |CS|^{-1}$, where $C = C(\dim X, \dim E) > 0$ is a universal constant, then* $\operatorname{Ker} D \otimes \nabla = 0$, *provided the connection $\nabla_0 = \nabla - \sqrt{-1}a$ is trivial* (*e.g., X is simply connected*).

Proof. The operator $D \otimes \nabla$ is the perturbation of D (or of $D \otimes D$ if D is real) by P, where $|P| \leq C|a|\,|S|$. Since $\|(D+P)e\|_{L_2} \leq \|De\|_{L_2} + |P| \|e\|_{L_2}$, the operator $D + P = D \otimes \nabla$ has trivial kernel for $|P| < \lambda_0$.

2.2.A$'$. **Remark.** The triviality of ∇_0 is essential. For example, let X be the circle S^1, and (F, ∇) a flat Hermitian line bundle with nontrivial holonomy. Take the twisted differential on functions on S^1 for $D = d \otimes \nabla$ and observe that $d = D \otimes \nabla^{-1}$, where ∇^{-1} refers to the connection in the *reciprocal bundle* F^{-1} (i.e., $F^{-1} \otimes F =$ trivial bundle, compare 2.2.B. below). Then $\lambda_0 > 0$, $|S| = 1$, $|a| = 0$, yet $\operatorname{Ker} d \neq 0$ as $d(\text{const}) = 0$.

2.2.B. Hermitian line bundles with connections form an abelian group for the **C**-tensor product, where $\nabla_1 \otimes \nabla_2$ on $F_1 \otimes F_2$ is defined by

$$\nabla_1 \otimes \nabla_2 (f_1 \otimes f_2) = (\nabla_1 f_1) \otimes f_2 + f_1 \otimes \nabla_2 f_2 .$$

If F is (topologically) isomorphic to some power

$$F = (F_0)^k = \underbrace{F_0 \otimes \cdots \otimes F_0}_{k},$$

then $F_0 = F^{(1/k)}$ carries a connection, say $\nabla^{(1/k)}$, such that

$$(F^{(1/k)}, \nabla^{(1/k)})^k = (F, \nabla),$$

and $(F^{(1/k)}, \nabla^{(1/k)})$ is unique up to isomorphism if $H^1(X, \mathbb{Z}_k) = 0$. Furthermore, if F is topologically trivial, one defines in an obvious way $(F^\alpha, \nabla^\alpha)$ for all real α, and this Hermitian bundle with connection is again unique up to isomorphism if $H_1(X) = 0$. Notice that the curvature form of ∇^α satisfies $\omega(\nabla^\alpha) = \alpha\omega(\nabla)$. As $\omega(\nabla) = da$ and $\nabla = \nabla_0 + \sqrt{-1}a$ for a flat connection ∇_0, one could define ∇^α by $\nabla^\alpha = \nabla_0^\alpha + \alpha\sqrt{-1}a$, where ∇_0^α is the flat connection in F^α corresponding to ∇_0 in F. (Notice that $(F_0^\alpha, \nabla_0^\alpha)$ is isomorphic to (F_0, ∇_0) for all $\alpha \in \mathbf{R}$ if $H_1(X) = 0$.)

2.2.C. We have with the above discussion the following corollary to Proposition 2.2.A.

2.2.C$'$. **Corollary.** *Let $\lambda_0 = \lambda_0(D) > 0$, $|S| < \infty$ and $|a| < \infty$, and let the connection ∇_0 be trivial. Then* $\operatorname{Ker} D \otimes \nabla^\alpha = 0$ *for all α in a sufficiently small interval $\alpha \in [-\varepsilon, \varepsilon] \subset \mathbf{R}$ for some $\varepsilon > 0$.*

2.3. **Twisted L_2-index theorem.** Let X be a Riemannian manifold, and Γ a discrete group of isometries of X, such that the differential operator D in question commutes with the action of Γ. This presupposes that the action of Γ lifts to the pertinent bundles E and E', and then the commutation between the actions of Γ on sections of E and E' and $D\colon C^\infty(E) \to C^\infty(E')$ makes sense. A typical example is that of Galois action for a covering map $X \to X_0$, where D is pulled back from an operator on X_0.

Next we consider a Γ-invariant Hermitian line bundle (F, ∇) on X, we assume X/Γ is compact, and we state Atiyah's L_2-index theorem for $D \otimes \nabla$.

2.3.A. **Theorem.** *Let D be a first-order elliptic operator. Then there exists a closed nonhomogeneous form*

$$\begin{aligned}\widehat{I} = \widehat{I}(D) &= I^0 + I^1, + \cdots + I^n \\ &\in \Omega^*(X) = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^n, \qquad n = \dim X,\end{aligned}$$

invariant under Γ, such that the L_2-index of the twisted operator $D \otimes \nabla$ satisfies

$$(*) \qquad \operatorname{Ind}_\Gamma D \otimes \nabla = \int_{X/\Gamma} \widehat{I} \exp \hat{\omega},$$

where $\hat{\omega} = (2\pi)^{-1}\omega(\nabla)$ *is the Chern form of* ∇, *and*

$$\exp\hat{\omega} = 1 + \hat{\omega} + \frac{\hat{\omega}\wedge\hat{\omega}}{2!} + \frac{\hat{\omega}\wedge\hat{\omega}\wedge\hat{\omega}}{3!} + \cdots .$$

2.3.A$'$. **Remarks.** (a) This theorem (as well as the generalization which follows) remains true for some cases where X/Γ is noncompact of finite volume (see [5]).

(b) As we noticed earlier, the precise definition of Ind_Γ is not important for our applications. What is relevant here is the implication

$$\mathrm{Ind}_\Gamma > 0 \Rightarrow \mathrm{Ker}\, D\otimes\nabla \neq 0,$$

for Ind_Γ *defined* by $(*)$.

(c) The operators D used in the present paper are the signature operator (i.e., "one half" of $d+\delta$) and $\bar\partial+\bar\partial^*$. In these cases the I_0-component of $\widehat{I}(D)$ is nonzero. Hence $\int_{X/\Gamma}\widehat{I}\exp\alpha\hat{\omega}\neq 0$, for almost all α, provided the curvature form $\omega=\omega(\nabla)$ is "homologically nonsingular" $\int_{X/\Gamma}\omega^n\neq 0$, for $n=\dim X$.

2.3.B. We want to indicate here a generalization of the L_2-index theorem to the situation where the group Γ does not act on (F, ∇), but the curvature form $\omega(\nabla)$ on X is still Γ-invariant. For example, we may start with Γ acting on (F, ∇) and then pass (if the topology allows) to the kth root $(F, \nabla)^{(1/k)}$ of (F, ∇) for some $k\geq 2$. Since the bundle $(F, \nabla)^{(1/k)}$ is only defined up to an isomorphism, the action of Γ does not necessarily lift to F. Yet there is a larger group Γ_k acting on (F, ∇), where $0\to \mathbf{Z}/k\mathbf{Z}\to\Gamma_k\to\Gamma\to 1$. In the general case where $\omega(\nabla)$ is Γ-equivariant, the action of Γ on (F, ∇) is defined up to the automorphism group of (F, ∇) which is the circle group $S^1=\mathbf{R}/\mathbf{Z}$ as we assume X is connected. Thus we have a nondiscrete group, say $\overline{\Gamma}$, such that $1\to S^1\to\overline{\Gamma}\to\Gamma\to 1$, and such that the action of Γ on X lifts to that of $\overline{\Gamma}$ on (F, ∇). This gives us the action of $\overline{\Gamma}$ on the spaces of sections of $E\otimes F$ and $E'\otimes F$, and we can speak of the $\overline{\Gamma}$-dimension of $\mathrm{Ker}\, D\otimes\nabla$ and $\mathrm{Coker}\, D\otimes\nabla$. The proof by Atiyah of the L_2-index theorem does not change a bit, and the formula $(*)$ remains valid with $\overline{\Gamma}$ in place of Γ,

$$(*)\qquad \mathrm{Ind}_{\overline{\Gamma}} D\otimes\nabla = \int_{X/\Gamma}\widehat{I}\exp\hat{\omega}.$$

Here again, the relevant fact is the implication

$$(**)\qquad \int_{X/\Gamma}\widehat{I}\exp\hat{\omega} > 0 \Rightarrow \mathrm{Ker}\, D\otimes\nabla\neq 0.$$

2.3.B$'$. **Remark.** Suppose we are given no bundle F at all but rather a closed Γ-invariant 2-form ω on X. If the cohomology class $[\omega]$ of ω is integral (e.g., if ω is exact), then there exists (this is well known and easy to prove) a bundle (F, ∇), such that $\omega(\nabla) = \omega$, and this (F, ∇) is unique up to an isomorphism if $H_1(X) = 0$.

2.4. **Vanishing of $\lambda_0(D)$.** Let D be a Γ-equivariant elliptic operator on X of the first order, and let $\widehat{I} = \widehat{I}(D) = I^0 + I^1 + \cdots + I^n$ be the corresponding (index) form on X. Let ω be a closed Γ-invariant 2-form on X and denote by $\widehat{I}^n_\alpha$ the top component of the product $\widehat{I}(D)\exp\alpha\omega$, for $\alpha \in \mathbf{R}$. This $\widehat{I}^n_\alpha$ is a Γ-invariant n-form on X, $\dim X = n$, depending on parameter α.

2.4.A. **Theorem.** *Let $H_1(X) = 0$, and let X/Γ be compact and $\int_{X/\Gamma} I^n_\alpha \neq 0$, for some $\alpha \in \mathbf{R}$. If the form ω is d(bounded), i.e., $\omega = d(a)$, where $\sup_{x\in X}\|a(x)\| < \infty$, then either $\lambda_0(D) = 0$ or $\lambda_0(D^*) = 0$, where D^* is the adjoint operator. (Notice that $\lambda_0(D) = 0$ if and only if D^{-1} is unbounded and that $\lambda_0(D) = \lambda_0(D^*)$, if $\operatorname{Ker} D = \operatorname{Ker} D^* = 0$.)*

Proof. We consider the Hermitian line bundle (F, ∇^α) for $\omega(\nabla^\alpha) = (2\pi)^{-1}\alpha\omega$ (see 2.3.B$'$.), and we observe with 2.3.B. that the $\overline{\Gamma}$-index $\operatorname{Ind}_{\overline{\Gamma}} D\otimes\nabla_\alpha = \int_{X/\Gamma} I^n_\alpha$ is nonzero for all but finitely many α, as the integral on the right-hand side is a nonzero polynomial in α. If this polynomial is positive for some α close to zero, then $\operatorname{Ind}_{\overline{\Gamma}} D \otimes \nabla^\alpha > 0$ and, consequently, $\operatorname{Ker} D \otimes \nabla^\alpha \neq 0$. It thus follows from 2.2.C$'$. that $\lambda_0(D) = 0$.

Now, if the polynomial $\int I^n_\alpha$ is negative for all small α, we pass to the adjoint operator D^* which clearly satisfies $\lambda_0(D^*) = \lambda_0(D)$ and $\widehat{I}(D^*) = -\widehat{I}(D)$. Hence the above arguments show that $\lambda_0(D^*) = 0$.

2.4.A$'$. **Remark.** The condition $H_1(X) = 0$ is only needed to insure that the (closed!) forms $\gamma^*(a) - a$ for all $\gamma \in \Gamma$ represent the integral classes in $H^1(X; \mathbb{R})$.

2.4.B. **Examples:** *the signature operator and $\overline{\partial}+\overline{\partial}^*$.* The space of exterior forms on a Riemannian manifold X admits a splitting $\Omega^*(X) = E \oplus E'$ according to eigenvalues of the Hodge operator $*$ on forms, such that $d+\delta$ interchanges E and E'. The form $\widehat{I}$ for $d+\delta\colon C^\infty(E) \to C^\infty(E')$ corresponds to the (full) L-class of X, $\widehat{I}(d+\delta) = 1+L_1+\cdots$. Since this $\widehat{I}$ starts with a nonzero term in degree zero, we have $\int_{X/\Gamma}\widehat{I}\exp\alpha\omega \neq 0$. If ω is homologically nonsingular, i.e., if $\int_{X/\Gamma}\omega^m \neq 0$, $2m = \dim X$. Therefore, $\lambda_0(d+\delta) = 0$, provided X/Γ is compact and X admits a closed Γ-invariant d(bounded) 2-form ω which is homologically nonsingular.

Now, let us assume X is a complex manifold, and $E_0 \to X$ is a Γ-invariant complex vector bundle. Then we have the operator

$$\bar{\partial}: E_0 \otimes \Omega^{0,*} \to E_0 \otimes \Omega^{0,*},$$

where $\Omega^{0,*}$ is the sum of the sheaves of $(0, q)$-forms on X:

$$\Omega^{0,*} = \bigoplus_{q=0}^{n} \Omega^{0,q}, \qquad n = \dim_{\mathbf{R}} X.$$

Next, with Γ-equivariant metrics in X and E_0 we define $\bar{\partial}^*$ and therefore obtain $\bar{\partial} + \bar{\partial}^*$ acting on sections of $E_0 \otimes \Omega^{0,*}$. This operator interchanges the parity of the E_0-valued forms, and thus we have our operator

$$D = \bar{\partial} + \bar{\partial}^* : E_0 \otimes \Omega^{\mathrm{ev},*} \to E_0 \otimes \Omega^{\mathrm{odd},*}.$$

Here again $\hat{I}(D)$ starts from a nonzero term and so

$$\lambda_0(\bar{\partial} + \bar{\partial}^*) = 0,$$

if the form ω in question is homologically nonsingular.

2.5. **Main Theorem.** *Let X be a complete simply connected Kähler manifold whose Kähler form ω is d*(bounded), *and let Γ be a discrete group of isometries of X, such that X/Γ is compact. Then the space $\mathscr{H}^{p,q}$ of harmonic L_2 forms on X of bidegree (p, q) satisfies $\mathscr{H}^{p,q} = 0$ for $p + q \neq m = \dim_{\mathbf{C}} X$ and $\mathscr{H}^{p,q} \neq 0$ for $p + q = m$.*

Proof. The vanishing of $\mathscr{H}^{p,q}$ for $p+q \neq m$ follows from 1.2.B. and the L_2-version of the Hodge decomposition $\bigoplus_{p+q=i} \mathscr{H}^{p,q} = \mathscr{H}^i$, which holds true for all complete manifolds by a straightforward generalization of the compact Hodge theory (compare §1).

Now, to prove $\mathscr{H}^{p,q} \neq 0$ for $p + q = m$, we apply the $(\bar{\partial} + \bar{\partial}^*)$-discussion in 2.4.B. to the holomorphic vector bundle $E_0 = \Lambda^p(X)$ that is the pth exterior power of the complex cotangent bundle of X. Thus we obtain the vanishing of λ_0 for the $\bar{\partial} + \bar{\partial}^* = d + \delta$ on $\Omega^{p,*}$. According to 1.4., the spectrum of $d + \delta$ lies away from zero apart from possible harmonic forms in the middle dimension. Since the vanishing of λ_0 amounts to the inclusion $0 \in \operatorname{spec} d + \delta | \Omega^{p,*}$, the space of harmonic (p, q)-form for $p + q = m$ is necessarily nonzero.

2.5.A. **Remarks.** (a) The conclusion of the theorem remains valid if X/Γ is noncompact of finite volume, provided X has bounded geometry. That is, the sectional curvature of X is bounded, i.e., $\sup_{x\in X} |K(x)| < \infty$, and the injectivity radius is bounded away from zero, i.e., $\inf_{x \in X} \operatorname{Rad}_x X > 0$. This follows from the L_2-index theorem in [5].

(b) The simply connectedness of X can be relaxed to $H^1(X; S^1) = 0$, and even this weaker condition does not seem truly necessary.

3. Holomorphic forms

One of the most interesting consequences of the Main Theorem is the existence of holomorphic L_2-forms on X. In fact, the Main Theorem gives us Δ-harmonic forms of type $(m, 0)$ as it claims $\mathscr{H}^{m,0} \neq 0$. Then the local Hodge theory tells us that $\Delta = 2(\overline{\partial} + \overline{\partial}^*)^2$ for all forms on X and by the cutoff argument (see §1), "Δ-harmonic" implies $(\overline{\partial} + \overline{\partial}^*)$-harmonic for the L_2-forms on X. Finally, we observe that the following three conditions are (obviously) equivalent for forms φ of type $(p, 0)$ (in particular, of type $(m, 0)$), $(\overline{\partial}+\overline{\partial}^*)\varphi = 0 \Leftrightarrow \overline{\partial}\varphi = 0 \Leftrightarrow \varphi$ is holomorphic.

Our major application of holomorphic $(m, 0)$-forms on X is the construction of meromorphic functions on X and X/Γ.

3.1. **Holomorphic sections and meromorphic functions on X.** Let F be a holomorphic vector bundle over X, and let us recall the standard construction of meromorphic functions on X using holomorphic sections of F. We fix a linear space $\mathscr{H}$ of holomophic sections of X, and assign to each point $x \in X$ the subspace $\mathscr{H}_x \subset \mathscr{H}$ of the sections vanishing at x. We assume X is connected and then observe that the codimension of $\mathscr{H}_x$ is constant on the complement of a proper subvariety $Z \subset X$, say $\operatorname{codim} \mathscr{H}_x = k$ for $x \in X - Z$. Thus we obtain a meromorphic map g of X into the Grassmannian of k-codimensional subspaces in $\mathscr{H}$:

$$g\colon X \to \mathrm{Gr} = \mathrm{Gr}^k(\mathscr{H}),$$

for $g(x) = \mathscr{H}_x$. Notice that the fibers $g^{-1}(p)$ for $p \in \mathrm{Gr}$ are analytic subvarieties which form a partition of $X - Z$. The closures of the fibers in $X \subset X - Z$ are also analytic subvarieties, but now they may have common points in $Z \subset X$.

One can equivalently describe the fibers using the field $\mathscr{M} = \mathscr{M}(\mathscr{H})$ of meromorphic functions on X associated to $\mathscr{H}$ as follows. Let $h_0, h_1, \cdots, h_k$ be nonzero sections of $\mathscr{H}$. Then there exist linear relations between h_i: $\sum_{i=0}^k \mu_i h_i = 0$ for some meromorphic functions μ_i on X. The field generated by the ratios μ_i/μ_j for all $(k+1)$-tuples $(h_0, \cdots, h_k)$ is, by definition, our $\mathscr{M}$.

Example. If $k \leq 1$, e.g., if F is a line bundle, then $\mathscr{M}$ is generated by the ratios h_0/h_1 of the sections in $\mathscr{H}$. Observe that two generic points

x_1 and x_2 lie in the same fiber of g if and only if $\mu(x_1) = \mu(x_2)$ for all $\mu \in \mathscr{M}$.

3.1.A. Let the bundle F be endowed with a Hermitian metric, and let us give a condition for separation of points in X by g. Assume $\|h(x)\| \to 0$, for $x \to \infty$, for all $h \in \mathscr{H}$, and observe the following is trivial:

Lemma. *For each point $x_0 \in X$ where* $\operatorname{codim} \mathscr{H}_{x_0} = k$ *(notice that these points are generic) there exists a constant $C_0 = C_0(x_0) > 0$, such that the equality $g(x) = g(x_0)$ implies $\|h(x)\| \le C_0 \|h(x_0)\|$ for all $x \in X$ and $h \in \mathscr{H}$.*

Proof. Let h be a section in $\mathscr{H}$, such that $\|h(x_1)\|$ is much greater than $\|h(c_0)\|$ for some $x_1 \in X$. Then the section $h' = h/\|h(x_1)\|$ nearly vanishes at x_0 while $\|h^1(x_1)\| = 1$. It follows that there exists a small perturbation $h'' \in \mathscr{H}$ of h', which vanishes at x_0 but not at x_1. Hence, $g(x_1) \ne g(x_0)$ and the proof follows by contradiction.

Corollary. *Suppose X and F are acted upon by a discrete group Γ preserving the norm in F and the subspace $\mathscr{H}$. If $h(x) \ne 0$, then $g(\gamma(x)) \ne g(x_0)$ for all but finitely many $\gamma \in \Gamma$. Moreover, if some section h in $\mathscr{H}$ does not vanish anywhere on a compact subset $B \subset X$, then the g-fiber through x_0 intersects the subset $\Gamma(B) = \bigcup_{\gamma\in\Gamma} \gamma(B) \subset X$ over a compact subset. In particular if X/Γ is compact, then generic fibers of g are compact.*

3.1.B. The above conditions are satisfied if $\mathscr{H}$ consists of L_2-sections and X/Γ is compact. Moreover, assume that X contains no compact submanifold of positive dimension. Notice that this is so for Kähler manifolds X where the Kähler form is exact (e.g., d(bounded)). Now, under these conditions we have the following:

Separation lemma. *If the space $\mathscr{H}$ of L_2-sections of F is nonempty, then the generic fiber of g is finite.*

Corollary. *For a generic point $x \in X$ (i.e., $x \in X - Z$ for some proper subvariety Z) there are at most finitely many points x' such that $\mu(x') = \mu(x)$ for all meromorphic functions μ on X.*

3.2. **Γ-Invariant forms and functions.** We assume, as in the previous section, that Γ acts on X and the bundle $F \to X$, and we want to construct a nonconstant Γ-invariant meromorphic function on X using (noninvariant) L_2-sections of F. If f is an L_1-section of F (with respect to some Γ-invariant norm on F given beforehand), then one can average f over Γ and thus obtain an invariant section $f \mapsto \overline{f} = \sum_{\gamma\in\Gamma} \gamma f$. Furthermore, if f is in L_2 but not in L_1, one can pass to the tensor power

$L \otimes L$ and to the section $f \otimes f$ of $L \otimes L$. If f is in L_2, then $f \otimes f$ is in L_1, and one can average over Γ, i.e.,

$$f \otimes f \mapsto \overline{f \otimes f} = \sum_{\Gamma} \gamma(f \otimes f).$$

Moreover, in many interesting cases the higher powers of an L_2-section, $f \otimes f \otimes f$, $f \otimes f \otimes f \otimes f, \cdots$, are also L_1, and so these powers can be averaged over Γ. This is possible, for instance if X/Γ is compact. In fact, if X/Γ is compact and f is an L_p-section, then $\bigotimes^k f$ is in L_1 for all $k \geq p$.

We assume below for simplicity's sake that F is a line bundle, and we write F^k for $\bigotimes^k F$ and f^k for $\bigotimes^k f$. Now, besides the averages $\overline{f^k} = \sum_\Gamma \gamma(f^k)$, we consider products of these $\prod_{i=1}^j \overline{f}^k i$, for $\sum_{i=1}^j k_i = k$, which are also Γ-invariant sections of F^k.

3.2.A. Proposition. *If X is connected and Γ is infinite, then for each nonzero L_1-section f of F there exist an integer k and two partitions $k = \sum_{k_i}$ and $k = \sum_{k'_i}$, such that the ratio of the corresponding products $\mu = \Pi/\Pi'$ for $\Pi = \Pi \overline{f}^k i$ and $\Pi' = \Pi \overline{f}^{k'} i$ is a nonconstant meromorphic Γ-invariant function on X.*

Proof. We start with the following trivial lemma.

Lemma. *Let $\{a_\gamma\}$ and $\{b_\gamma\}$ be countable sets of complex numbers indexed by $\gamma \in \Gamma$, such that $\sum_\Gamma |a_\gamma| < \infty$ and $\sum |b_\gamma| < \infty$. If $\sum_\Gamma a_\gamma^k = \sum_\Gamma b_\gamma^k$ for all $k = 1, 2, \cdots$, then $\{a_\gamma\} = \{b_\gamma\}$ up to a permutation of a_γ.*

We apply the lemma to the values of f at the pairs of points in X, $a_\gamma = \gamma(f(x))$, $b_\gamma = \gamma(f(y))$, and conclude that if μ is constant for all k and $\sum_{k_i} = k$, then the section f is "constant" in the following sense. There exists a function $C(x, \gamma)$ on $X \times \Gamma$, which is locally constant on X and

$$(*) \qquad \gamma(f(x)) = C(x, \gamma) f(x) \qquad \text{for all } \gamma \in \Gamma.$$

Since X is connected, C is constant in x

$$C(x, \gamma) = C(\gamma),$$

and as $\gamma(f(x)) \overset{\text{def}}{=} f(\gamma x)$ and f is nonzero, the map $\gamma \mapsto C(\gamma)$ is a homorphism by $(*)$ $C: \Gamma \to \mathbf{C}^\times$. Since Γ is infinite, $\sum_\Gamma |C(\gamma)| = \infty$, and since $f(\gamma x) = C(\gamma) f(x)$, the function f cannot lie in L_1 unless it is identically zero.

3.2.A$'$. **Remark.** Instead of the sums $\sum_\Gamma \gamma f^k$ one can use the elementary symmetric functions

$$\sum_\Gamma \gamma f^k, \sum_\Gamma \sum_\Gamma (\gamma_1 f)(\gamma_2 f), \cdots,$$

and these can be seen altogether in the infinite product

$$\Pi(x, z) = \prod_{\gamma\in\Gamma}(1 - zf(\gamma x)).$$

This product must be viewed as a holomorphic function on the total space of the line bundle F^{-1}, and one can characterize Π geometrically as follows. Let $Y \subset F^{-1}$ be the union of the Γ-translates of the graph (or image) of the section f^{-1}, $Y = \bigcup_{\gamma\in\Pi} \gamma f^{-1}(X)$. This Y meets every fiber $F_x^{-1} \subset F^{-1}$ over a countable set indexed by Γ say $\Gamma_x \subset F_x^{-1}$. Since

$$\sum_{\gamma_x\in\Gamma_x} |\gamma_x|^{-1} < \infty,$$

there exists a unique entire holomorphic function $\Pi_x(z)$ on the fiber $F_x^{-1} = \mathbf{C}$ of order < 1 whose zero set equals Γ_x and such that $\Pi_x(0) = 1$. Clearly, this is our product, $\Pi_x(z) = \Pi(x, z)$.

3.2.B. **Corollary.** *Assume X/Γ is compact, and let X contain no compact submanifold of positive dimension. If F admits a nonzero L_2-section, then the* (Γ*-invariant!*) *meromorphic functions μ of the form Π/Π' quasiseparate points in X* (*compare* 3.1.B.). *Namely, for a generic point $x \in X$ there exist at most finitely many Γ-orbits $\Gamma x' \subset X$, such that $\mu(x') = \mu(x)$ for the above functions μ.* (*Since X/Γ is compact, this is equivalent to the equality* $\operatorname{trans.deg}\mathscr{M} = \dim X$ *for the field $\mathscr{M}$ generated by μ.*)

Proof. Let $X' \subset X$ be a (generic) Γ-invariant subvariety on which all μ are constant. By applying 3.2.A. to a connected component X_0' of X', we conclude that X_0' is compact and hence finite.

3.2.B$'$. **Remarks.** (a) If the action of Γ is free, then $X \to X/\Gamma$ is the covering map, and the above corollary implies that the bundle $F/\Gamma \to X/\Gamma$ is quasiample. This applies, for example, to the canonical bundle of a Kähler hyperbolic manifolds X/Γ, where X supports nonzero holomorphic m-forms.

(b) The above construction of meromorphic functions as ratios of averaged sections of a line bundle goes back to Poincaré. In the modern times this was used by Kodaira who proved that if a compact complex manifold V can be covered by a bounded domain in a Stein manifold, or more generally, if the Bergmann metric of some covering X of V is nondegenerate, then the canonical bundle of V is ample. In particular, V is projective algebraic.

3.2.C. We can generalize 3.2.A. and 3.2.B. to the case of rank $F \geq 2$ by reducing the general case to that of rank=1 as follows. The L_2-sections of F span at a generic point $x \in X$ a k-dimensional subspace, $L_x^k \subset F_x \subset F$. Then by passing to kth exterior power we get 1-dimensional subspaces

$$L_x = \bigwedge^k L_x^k \subset \bigwedge^k F_x \subset \bigwedge^k F,$$

which define in a usual way a line bundle $\widetilde{L} \to X$ together with a homomorphism $\widetilde{L} \to \bigwedge^k F$ whose image equals the union $\bigcup_{x \in X} L_x$. Then 3.2.A. and 3.2.B. apply to $\widetilde{L}$ and yield the corresponding statement for F.

3.2.C$'$. **Corollary.** *Let V be a compact Kähler manifold whose arithmetic genus does not vanish, i.e.,*

$$T(V) = \chi(\mathscr{O}(V)) = \sum_{p=0}^{n} (-1)^p h^{p,0}(V) \neq 0.$$

If the universal covering X of V contains no compact submanifold of positive dimension, then V is Moishezon and hence projective algebraic.

Proof. The relation $\chi \neq 0$ implies via the L_2-index theorem the existence of nonzero holomorphic L_2-form on X of some degree d. Then the above applies to $F = \bigwedge^d T^*(X)$.

3.3. **Holomorphic 1-forms.** We prove here a result stated in [13] concerning holomorphic L_2-forms φ of degree one on a complete Kähler manifold of bounded geometry. We start with the simple case (compare 1.2. in [13]) where φ is exact and nonzero, i.e., where $\varphi = df$ for a nonconstant holomorphic function $f: X \to \mathbf{C}$ on X. We recall the *coarea formula* for holomorphic functions,

$$\int_{\mathbf{C}} \operatorname{Vol} f^{-1}(z)\, dz = \|df\|_{L_2},$$

where Vol refers to the $(n-2)$-dimensional volume of the levels $f^{-1}(z)$ for $n = \dim_{\mathbf{R}} X$. As $\|df\|_{L_2} = \|\varphi\|_{L_2} < \infty$, we conclude that $\operatorname{Vol} f^{-1}(z) < \infty$ for almost all $z \in \mathbf{C}$, and since X has bounded geometry, the complex hypersurfaces $f^{-1}(z) \subset X$ are necessarily compact for almost all $z \in \mathbf{C}$. This is immediate with the following:

Standard fact. *Let Σ be a k-dimensional complex subvariety in a Kähler manifold X with bounded geometry. Then for each $x \in \Sigma$ the $2k$-dimensional volume of the part of Σ contained in the unit ball around x*

is bounded from below

$$\mathrm{Vol}_{2k}(\Sigma \cap B_x(1)) \geq C > 0\,,$$

where C *depends on the implied bound on the geometry of* X.

Now, since

$$\mathrm{Vol}\, f^{-1}(z) = \int_{f^{-1}(z)} \omega^{m-1}\,, \qquad m = \dim_{\mathbf{C}} X\,,$$

for the Kähler form ω on X, we see that $\mathrm{Vol}\, f^{-1}(z)$ is constant in z and so *all* levels $f^{-1}(z)$ are compact. Thus by the Stein factorization theorem (see [20]) there exist a Riemann surface S and a holomorphic map $\overline{f}\colon X \to S$ such that each level $\overline{f}^{-1}(s)$ is a connected component of some level $f^{-1}(z)$. Finally we notice that if we take another function, say g, with dg in L_2, we obtain the same factorization, as g is constant on each (compact!) level of f. Therefore, the factorization $X \to S$ is compatible with the isometries of X. Thus we arrive at the following:

3.3.A. **Theorem.** *If* X *admits an exact nonzero* L_2*-form of degree one, then there exists a smooth Riemann surface* S *which is acted upon by the isometry group* Γ *of* X *and a proper holomorphic* Γ*-equivariant map* $\sigma\colon X \to S$, *such that the induced map* σ^* *is a bijection of the space of holomorphic* L_2*-forms of degree one on* S *to that on* X.

3.3.A$'$. **Remark.** If $H^1(X; \mathbf{R}) = 0$ (for example, if X is simply connected), then holomorphic L_2-forms of degree one one-to-one correspond to real harmonic L^2-forms. In particular, the existence of such a (nonzero!) form on the universal covering X of a compact manifold V is a purely topological property of V. In fact this depends on the fundamental group $\pi_1(V)$ alone (see [13] for further discussion).

3.3.B. Now, we turn to a more general case where the holomorphic 1-form φ on X is not necessarily exact, but the real part of φ is exact, i.e., $\varphi = \alpha + i\beta$ for $\alpha = dh$, where $h\colon X \to \mathbf{R}$ is a (necessarily) pluriharmonic function on X. Since φ is closed, it is locally exact, i.e., $\varphi = df$, where f is defined on each small neighborhood up to an additive constant. Then the levels of these local functions fit into a (singular) holomorphic foliation on X whose leaves need not be compact or closed subsets in X.

The following trivial lemma relates these leaves to the levels of h:

3.3.B$'$. **Lemma.** *The leaf through the point* $x \in X$ *is the maximal connected complex subvariety in* X, *which is contained in the level* $h^{-1}(y) \in X$ *for* $y = h(x)$.

Now we want to show that the above holomorphic foliation associated to φ is, in fact, independent of φ. We start with the following:

3.3.C. **Cup-product Lemma.** *The cup-product is trivial on the exact part of the* 1*-dimensional* L_2*-cohomology. Namely, if* φ *and* ψ *are closed* L_2*-forms where* $\varphi = df$ *for some function* $f\colon X \to \mathbf{R}$, *then the product* $\varphi \wedge \psi$ *is in the* L_2*-closure of the differentials of* L_2*-forms, i.e.,* $\varphi \wedge \psi \in \overline{d(L_2\Omega^i)}$, *for* $i = \deg \psi$.

Proof. For every $c \geq 0$, there exists a unique continuous function f_c on X, such that $f(x) = f_c(x)$ for $|f(x)| \leq c$ and $|f_c(x)| = c$ for $|f(x)| \geq c$. Clearly, $df_c \underset{L_2}{\to} f$ for $c \to \infty$ and since f_c is bounded,

$$df_c \wedge \psi = d(f_c \wedge \psi) \subset d(L_2\Omega^i).$$

3.3.D. Let us apply the cup-product lemma to holomorphic 1-forms $\varphi = \alpha + i\beta$ and $\varphi' = \alpha' + i\beta'$, where α and α' are exact. Then $\varphi \wedge \varphi' = a + ib$, for $a = \alpha \wedge \alpha' - \beta \wedge \beta'$ and $b = \alpha \wedge \beta' + \beta \wedge \alpha'$. According to the lemma $b \in \overline{dL_2\Omega^1}$. On the other hand, since $a + ib$ is holomorphic, b is harmonic; thus $b = 0$ and therefore $a = 0$. Now we observe that the relation $\varphi \wedge \varphi' = 0$ implies that the foliations respectively defined by φ and φ' coincide.

Conclusion. *The holomorphic foliation defined by* φ *is independent of* φ.

This conclusion sharply contrasts with what happens to the foliations corresponding to the real parts of holomorphic forms.

3.3.D$'$. **Lemma.** *Let* α *and* α' *be real exact forms, such that* $\alpha = dh$ *and* $\alpha' = dh'$, *where* h *and* h' *are pluriharmonic functions on* X. *If the* (*real of codimension one*) *foliation corresponding to* α *is equal to that for* α', *then* $\alpha = \operatorname{const} \alpha'$.

Proof. If α' is not identically zero, then there are local coordinates at almost every point in X, say

$$x_1, y_1, x_2, y_2, \cdots, x_m, y_m,$$

such that

$$h'(x_1, y_1, \cdots, x_m, y_m) = x_1.$$

Now, the equality of the foliations means that h depends only on x_1, $h = h(x_1)$ and since h is harmonic, $h = h(x_1) = \operatorname{const} x_1 + \operatorname{const}_0$.

Now, suppose that our forms $\varphi = \alpha + i\beta$ and $\varphi' = \alpha' + i\beta'$, where $\alpha = dh$ and $\alpha' = dh'$ are linearly independent. Then the generic leaves of the foliation corresponding to φ (and to φ') equal the connected components of the levels of the map $H\colon (h, h')\colon X \to \mathbf{R}^2$.

Here we are again in a position to apply the coarea inequality

$$\int_{\mathbf{R}^2} \operatorname{Vol} H^{-1}(y)\, dy \leq \|dH\|_{L_2} < \infty,$$

which implies, as in 3.3.A., that there exists an equivariant factorization $\sigma: X \to S$, $\dim_{\mathbf{C}} S = 1$, which is bijective on the space of those holomorphic L_2-forms of degree one whose real parts are exact.

Remark. The above argument uses two independent forms rather than a single nonzero form φ as in 3.3. However, if the isometry group Γ is noncompact, then $\gamma\varphi$ is linearly independent of φ for $\gamma \to \infty$, and so again a single φ does the job.

3.3.D$''$. The above factorization $X \to S$ applies to a Galois coverings X of a compact Kähler manifold V insofar as the Galois group Γ is infinite and has nontrivial L_2-cohomology in degree one, i.e., $L_2H^1(\Gamma) \neq 0$. In this case S/Γ is a compact Riemann surface, and the map $X \to S$ factors to a surjective holomorphic map $V \to S/\Gamma$.

For example, if $\pi_1(V)$ admits a surjective homomorphism onto a free group $\Gamma = F_k$ for $k \geq 2$, then V admits a holomorphic map onto a Riemann surface, since $L_2H^1(F_k) \neq 0$ for $k \geq 2$. We refer to [13] for further examples of this kind.

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