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LIE'S THIRD THEOREM FOR INTRANSITIVE LIE EQUATIONS

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Introduction

In [4], H. Goldschmidt used the formalism developed by B. Malgrange [9] to prove Lie's third theorem in the context of transitive Lie algebras: "If $L_{k+1} \subset J_{k+1}T\mathbf{R}_0^m$, where $k > 0$, is a $(k+1)$-truncated transitive Lie algebra such that the symbol of $L_k = \pi_k L_{k+1}$ is 3-acyclic, then there exists a formally integrable analytic Lie equation $R_k \subset J_k T\mathbf{R}^m$ such that $R_{k+1,0} = L_{k+1}$."

In this paper, we show that the above R_k can be constructed without using the Cartan-Kähler theorem; our proof only requires Frobenius' theorem. Consequently, in the intransitive case, we are able to prove a version of E. Cartan's results [1] without assuming that the structure functions c_{ijk} and $a_{ij\lambda}$ are analytic.

Our main result is the following theorem, which we state here only in the transitive case for simplicity.

Theorem. *Suppose $L_{k+2} \subset J_{k+2}T\mathbf{R}_0^m$, where $k > 0$, is a $(k+2)$-truncated transitive Lie algebra. Then there exists a C^∞ vector sub-bundle $R_{k+1} \subset J_{k+1}T\mathbf{R}^m$ such that*:

(i) $R_k = \pi_k(R_{k+1})$ *is a vector sub-bundle of* $J_k T\mathbf{R}^m$;
(ii) $[R_{k+1}, R_{k+1}] \subset R_k$;
(iii) $R_{k+1,0} = L_{k+1}$;
(iv) $R_{k+1} \subset (R_k)_{+1}$

If the symbol of $L_k = \pi_k L_{k+1}$ is 3-acyclic, then L_{k+1} can be prolonged to L_{k+2}. We know that all its prolongations are isomorphic, thus the assumption in Goldschmidt's theorem gives us a $(k+2)$-truncated transitive Lie algebra.

The equation R_k in the Theorem may not be formally integrable (we only know that $\pi_k\colon (R_k)_{+1} \to R_k$ is surjective). However, when the symbol of L_k is 2-acyclic, Theorem 4.1 of Goldschmidt [2] implies that R_k

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is formally integrable. Therefore Goldschmidt's theorem can be obtained as a consequence of our theorem.

To prove our result, we first consider the flat connection ∇ on $J_{k+2}T\mathbf{R}^m$, as in [4], defined by a section

$$\omega = \sum dx^i \otimes j^{k+3}\frac{\partial}{\partial x^i},$$

i.e., $\nabla\xi = [\tilde{\omega}, \xi]^{\sim}$ for $\xi \in \mathscr{J}_{k+2}\mathscr{T}\mathbf{R}^m$. We construct R_{k+2} by taking the parallel transport of L_{k+2}. Then $R_{k+1,0} = L_{k+1}$, and $[R_{k+1}, R_{k+1}] \subset R_k$. Now we twist R_{k+1} by a section $\phi \in \widetilde{\mathscr{Q}}_{k+2}$, as in [4], so that the new R_{k+1} satisfies our condition (iv). To achieve this, we must solve the equation

$$(*) \qquad \mathscr{D}\phi = -\pi_{k+1}\omega \mod T^* \otimes R_{k+1}.$$

In [4], the sophisticated Spencer operator is used. However, the first nonlinear Spencer operator $\mathscr{D}$ seems to us to be more appropriate for this problem because the bracket in L_{k+2} is defined pointwise.

We associate to $(*)$ the submanifold $S^{k+2} \subset Q_{(1,k+2)}$. We prove that: (1) the symbol of S^{k+2} is the tensor product of T^* and a vector bundle, (2) the mapping $\pi_1: (S^{k+2})_{+1} \to S^{k+2}$ is surjective. Then our equation may be solved using Frobenius' theorem, as is shown in the Appendix.

To prove statement (2), we consider a section $X \in \mathscr{S}^{k+2}$, and lift it to $\tilde{F} \in \mathscr{Q}_{(2,k+3)}$ with $\pi_{1,k+3}\tilde{F} \in \mathscr{S}^{k+3}$, where S^{k+3} is defined in the same way as S^{k+2}, replacing k by $k+1$. We show that

$$p_1(\mathscr{D})\tilde{F} = j^1(-\pi_{k+2}\omega) + y - x,$$

where $y \in J_1(T^* \otimes R_{k+2})$ and $x \in \ker\sigma(\mathscr{D}_1)$. The sequence

$$S^2T^* \otimes VQ_{k+3} \xrightarrow{\sigma_1(\mathscr{D})} T^* \otimes T^* \otimes J_{k+2}T \xrightarrow{\sigma(\mathscr{D}_1)} \bigwedge^2 T^* \otimes J_{k+1}T$$

is not exact, but

$$\pi_{k+1}(\ker\sigma(\mathscr{D}_1)) = \sigma_1(\mathscr{D})(S^2T^* \otimes V\mathscr{Q}_{k+2});$$

hence there exists $h \in \mathscr{S}^2\mathscr{T}^* \otimes \mathscr{V}Q_{k+2}$ such that $\sigma_1(\mathscr{D}_1)h = \pi_{k+1}x$. This explains why we must start from a $(k+2)$-truncated Lie algebra L_{k+2} instead of one of order $k+1$. Then $\tilde{X} = \pi_{2,k+2}\tilde{F} + h$ is a section of $(S^{k+2})_{+1}$ which proves (2).

The proof in the intransitive case follows the same lines. We only have to add the hypothesis: L_{k+2} is defined on a submanifold N transverse to the orbits, and the restriction of the linear Spencer operator D to $\mathcal{TN}$ sends $\mathcal{L}_{k+2}$ into $\mathcal{TN}^* \otimes \mathcal{L}_{k+1}$.

In a separate paper, we shall define the intransitive Lie algebras, a notion of isomorphism, and prove realization theorems analogous to those of Guillemin-Sternberg [6].

Preliminaries

Throughout this paper, we shall use the notation of Malgrange [9] or of Goldschmidt-Spencer [5], unless it is stated otherwise.

All the results are local. Let M be an open subset of $\mathbf{R}^m$ containing 0, let (x^i, y^j) be coordinates on M, and let H, V be sub-bundles of $T = TM$ such that H (resp. V) is generated by $\{\partial/\partial x^i\}$ (resp. $\{\partial/\partial y^j\}$).

We denote by $J_k V$ the sub-bundle of $J_k T$ of k-jets of sections of V. Then

$$D: \mathcal{J}_{k+1}\mathcal{V} \to \mathcal{T}^* \otimes \mathcal{J}_k\mathcal{V}.$$

is defined by $D\xi = [\psi, \xi]$ (see [9, Proposition 3.7]), where $\psi = \psi_H + \psi_V$ and

$$\psi_H = \sum dx^i \otimes \frac{\partial}{\partial x^i}, \qquad \psi_V = \sum dy^j \otimes \frac{\partial}{\partial y^j}.$$

The decomposition $T = H \oplus V$ induces a decomposition $D = D_H \oplus D_V$, with $D_H(\mathcal{J}_{k=1}\mathcal{V}) \subset \mathcal{H}^* \otimes \mathcal{J}_k\mathcal{V}$. It is easily verified that $D_H\xi = [\psi_H, \xi]$ and $D_V\xi = [\psi_V, \xi]$. We can extend D_H to a mapping

$$D_H: \bigcap \mathcal{T}^* \otimes \mathcal{J}_{k+1}\mathcal{V} \to \mathcal{H} \wedge \left(\bigwedge \mathcal{T}^*\right) \otimes \mathcal{J}_k\mathcal{V}$$

by

$$D_H(\alpha \otimes \xi) = d_H\alpha \otimes \pi_k\xi + (-1)^{\deg\alpha}\alpha \wedge d_H\xi, \tag{1}$$

where again $d = d_H + d_V$. Also, D_V extends in a similar way.

We denote by $Q_k(V)$ the manifold of k-jets of diffeomorphisms f of M, which are equal to the identity mapping in the variables x, i.e., of the form $f(x, y) = (x, g(x, y))$. So $Q_k(V)$ is a submanifold of Q_k, and we denote by $\tilde{Q}_k(\mathcal{V})$ the sheaf of invertible sections of $Q_k(V)$.

The first nonlinear Spencer operator

$$\mathcal{D}: \tilde{\mathcal{Q}}_{k+2}(\mathcal{V}) \to \mathcal{T}^* \otimes \mathcal{J}_{k+1}\mathcal{V}$$

acts on $\tilde{\mathcal{Q}}_{k+2}(\mathcal{V})$ by

$$\mathcal{D}F = \psi - F^{-1}(\psi) \tag{2}$$

(see [9, p. 520]). The formula (6.8) of [9] tells us that

$$(\mathscr{D}F)_x = (\lambda^1 F(x))^{-1} \cdot j_x^1 \pi_{k+1} F - j_x^1 I_{k+1}, \tag{3}$$

where I_{k+1} is the identity section of $Q_{k+1}(V)$. We identify I_{k+1} with M. We can interpret this formula in the following way: $j_x^1 \pi_{k+1} F$ and $\lambda^1 F(x)$ define invertible linear maps from $T_x Q_{k+1}(V)$ onto $T_{\pi_{k+1} F(x)} Q_{k+1}(V)$, so $(\lambda^1 F(x))^{-1} \cdot j_x^1 \pi_{k+1} F$ is an endomorphism of $T_x Q_{k+1}(V)$ which induces the identity on $T_x M$; thus for $v \in T_x M$ we have

$$i(v)(\mathscr{D}F)_x \in VQ_{k+1}(V)_x \cong J_{k+1} V_x,$$

i.e.,

$$i(v)(\mathscr{D}F)_x = (\lambda^1 F(x))^{-1} \cdot j_x^1 \pi_{k+1} F \cdot v - v. \tag{4}$$

The following formulas hold for $\mathscr{D}$ ([5], [9]):

$$\mathscr{D}(G \circ F) = \mathscr{D}F + F^{-1}(\mathscr{D}G), \qquad F, G \in \tilde{\mathscr{Q}}_{k+2}(\mathscr{V}), \tag{5}$$

$$D\xi = [\mathscr{D}F, \xi] + (\pi_{k+1} F)^{-1}(DF(\xi)), \qquad \xi \in \mathscr{J}_{k+1}\mathscr{V}, \tag{6}$$

$$D\mathscr{D}F - \tfrac{1}{2}[\mathscr{D}F, \mathscr{D}F] = 0, \tag{7}$$

where $F(\ )$ denotes the action of F on $\bigwedge \mathscr{T}^* \otimes \mathscr{J}_{k+1}\mathscr{V}$. If

$$\mathscr{D}_1 : \mathscr{T}^* \otimes \mathscr{J}_{k+1}\mathscr{V} \to \textstyle\bigwedge^2 \mathscr{T}^* \otimes \mathscr{J}_k \mathscr{V}$$

is the operator defined by

$$\mathscr{D}_1 u = Du - \frac{1}{2}[u, u] \tag{8}$$

for $u \in \mathscr{T}^* \otimes \mathscr{J}_{k+1}(\mathscr{V})$, then it follows from (7) that $\mathscr{D}_1 \mathscr{D} F = 0$, so we get the first nonlinear Spencer complex

$$\tilde{\mathscr{Q}}_{k+2}(\mathscr{V}) \xrightarrow{\mathscr{D}} (\mathscr{T}^* \otimes \mathscr{J}_{k+1}\mathscr{V})^\wedge \xrightarrow{\mathscr{D}_1} \textstyle\bigwedge^2 \mathscr{T}^* \otimes \mathscr{J}_k \mathscr{V}, \tag{9}$$

which is exact ([9], [5]), where

$$(T^* \otimes J_{k+1} V)^\wedge = \{u \subset T^* \otimes J_{k+1} V : \pi_0 u + \mathrm{id}_T \in T^* \otimes T \text{ is invertible}\}.$$

The operator $\mathscr{D}$ induces a surjective morphism

$$p(\mathscr{D}) : Q_{(1, k+2)}(V) \to (T^* \otimes J_{k+1} V)^\wedge,$$

where $Q_{(1, k+2)}(V)$ stands for the 1-jets of elements of $\tilde{\mathscr{Q}}_{k+2}(\mathscr{V})$. It follows from (3) that

$$p(\mathscr{D})X = (\lambda^1 \pi_{0, k+2} X)^{-1} \circ (\pi_{1, k+1} X) - j_{\pi(X)}^1 I_{k+1}. \tag{10}$$

The symbol of $\mathscr{D}$ is a mapping

$$\sigma(\mathscr{D})\colon T^* \otimes VQ_{k+2}(V) \to T^* \otimes J_{k+1}V .$$

Lemma 1. *If* $\alpha \otimes \xi \in T^*_x \otimes V_Y Q_{k+2}(V)$, *then*

$$\sigma(\mathscr{D})(\alpha \otimes \xi) = \alpha \otimes (Y^{-1} \cdot \pi_{k+1*}\xi), \tag{11}$$

where $Y \in Q_{k+2}(V)$, $\alpha \in T^*_x$, $\xi \in V_Y Q_{k+2}(V)$, *and* $\pi(Y) = x$.

Proof. Let X be an element of $Q_{(1,k+2)}(V)$ such that $\pi_{0,k+2}X = Y$, and $u \in T^*_x \otimes V_Y Q_{k=2}(V)$. There exists a curve X_t in $Q_{1,k+2}(V)_x$ such that $X_0 = X$,

$$\pi_{0,k+2}X_t = Y, \qquad \frac{d}{dt}X_t|_{t=0} = u.$$

If

$$Y^{-1}\colon T_{\pi_{k+1}(Y)}Q_{k+1}(V) \to T_{I_{k+1}(x)}Q_{k+1}(V),$$

we have

$$\begin{aligned}\sigma(\mathscr{D})u &= \frac{d}{dt}p(\mathscr{D})X_t|_{t=0} = \frac{d}{dt}(\lambda^1\pi_{0,k+2}X_t)^{-1} \circ \pi_{1,k+1}X_t|_{t=0} \\ &= (\lambda^1 Y)^{-1} \cdot \frac{d}{dt}\pi_{1,k+1}X_t|_{t=0} = Y^{-1} \cdot \pi_{1,k+1*}u.\end{aligned}$$

As a consequence of this lemma, we see that

$$\sigma_1(\mathscr{D})\colon S^2T^* \otimes VQ_{k+2}(V) \to T^* \otimes T^* \otimes J_{k+1}V$$

is determined by

$$\sigma_1(\mathscr{D})(\alpha \cdot \beta \otimes \xi) = \alpha \cdot \beta \otimes Y^{-1}(\pi_{k+1*}\xi), \tag{12}$$

where $\alpha, \beta \in T^*_x$, $\xi \in V_Y Q_{k+2}(V)$, $Y \in Q_{k+2}(V)$, and $\pi(Y) = x$. We associate to $\mathscr{D}_1$ the morphism

$$p(\mathscr{D}_1)\colon J_1(T^* \otimes J_{k+1}V)^\wedge \to \textstyle\bigwedge^2 T^* \otimes J_k V$$

whose symbol

$$\sigma(\mathscr{D}_1)\colon J_1(T^* \otimes J_{k+1}V) \to \textstyle\bigwedge^2 T^* \otimes J_k V$$

is equal to $\sigma(D)$ and is given by

$$\sigma(\mathscr{D}_1)(\alpha \otimes \beta \otimes \xi) = \alpha \wedge \beta \otimes \pi_k \xi, \tag{13}$$

where $\alpha, \beta \in T^*$ and $\xi \in J_{k+1}V$.

The following lemma is easily verified.

Lemma 2. *If* $X \in J_1(T^* \otimes J_{k=1}V)^\wedge$ *and* $z \in T^* \otimes T^* \otimes J_{k+1}V$, *then*

$$p(\mathscr{D}_1)(X + z) = p(\mathscr{D}_1)X + \sigma(\mathscr{D}_1)z. \tag{14}$$

Main theorem

Theorem. *Suppose that L_{k+2} is a vector sub-bundle of $(J_{k+2}V)|_N$, satisfying*:

(a) $\pi_0 L_{k+2} = V|_N$;

(b) $L_{k+l} = \pi_{k+l}(L_{k+2})$ *is a vector sub-bundle of* $(J_{k+l}V)|_N$ *for* $l = 0, 1$;

(c) $[L_{k+2}, L_{k=2}] \subset L_{k+1}$;

(d) $D_H\colon \mathscr{L}_{k+2} \to \mathscr{H}^*|_{\mathscr{N}} \otimes \mathscr{L}_{k+1}$.

Then there exists a vector sub-bundle $R'_{k+1} \subset J_{k+1}$ such that:

(i) $R'_k = \pi_k(R'_{k+1})$ *is a vector sub-bundle of* $J_k V$;

(ii) $[R'_{k+1}, R'_{k+1}] \subset R'_k$;

(iii) $R'_{k+1}|_N = L_{k+1}$;

(iv) $R'_{k+1} \subset (R'_k)_{+1}$.

Proof. We set

$$\omega = \sum dy^j \otimes j^{k+3} \frac{\partial}{\partial y^j} \in \mathscr{T}^* \otimes \mathscr{J}_{k+3}\mathscr{V},$$

and we define the following (partial) flat connection (see [4,§3])

$$\nabla\colon \mathscr{J}_{k+2}\mathscr{V} \to \mathscr{V}^* \otimes \mathscr{J}_{k+2}\mathscr{V}$$

by

$$\nabla\xi = [\tilde{\omega}, \xi] \tag{15}$$

for $\xi \in \mathscr{J}_{k+2}(\mathscr{V})$, where the bracket

$$[\ ,\ ]\colon \tilde{\mathscr{J}}_{k+3}\mathscr{V} \times \mathscr{J}_{k+2}\mathscr{V} \to \mathscr{J}_{k+2}\mathscr{V}$$

is given by [9, (2.3)]. If $\bar{\xi}$ is a section of $\mathscr{J}_{k+3}\mathscr{V}$ such that $\pi_{k+2}(\bar{\xi}) = \xi$, then

$$\nabla\xi = D_V\bar{\xi} + [\omega, \bar{\xi}]. \tag{16}$$

We have

$$\nabla(\nabla\xi) = [\tilde{\omega}, [\tilde{\omega}, \xi]] = [[\tilde{\omega}, \tilde{\omega}], \xi] - [\tilde{\omega}, [\tilde{\omega}, \xi]];$$

since $[\tilde{\omega}, \tilde{\omega}] = 0$, we see that ∇ is flat. In the same way, we can define connections ∇_{k+l} on $J_{k+l}V$ in terms of $\omega_{k+l+1} = \pi_{k+l+1}(\omega)$ for $l = 0, 1$.

It follows from Jacobi's identity that

$$\nabla_{k+1}[\xi, \eta] = [\nabla\xi, \eta] + [\xi, \nabla\eta], \tag{17}$$

where $\xi, \eta \in \mathscr{J}_{k+2}\mathscr{V}$. Let ξ_i, $1 \le i \le r$, be a basis of sections of L_{k+2}, and let ξ'_i, $1 \le i \le r$, be sections of $\mathscr{J}_{k+2}\mathscr{V}$ such that

$$\xi'_i|_N = \xi_i, \qquad \nabla\xi'_i = 0.$$

Let R_{k+2} be the sub-bundle of $J^{k+2}V$ generated by the ξ_i', $1 \leq i \leq r$, and set $R_{k+l} = \pi_{k+l}(R_{k+2})$ for $l = 0, 1$. Then by (b), R_{k+l} is a sub-bundle of $J_{k+l}V$ for $l = 0, 1$; also, we have

$$\nabla(\mathscr{R}_{k+2}) \subset \mathscr{T}^* \otimes \mathscr{R}_{k+1}. \tag{18}$$

Furthermore, we obtain from (17)

$$\nabla_{k+1}[\xi_i', \xi_j'] = 0,$$

and from (c),

$$[R_{k+2}, R_{k+2}] \subset R_{k+1}. \tag{19}$$

Lemma 3. *Let u be an element $\bigwedge \mathscr{H}^* \otimes \mathscr{J}_{k+2}\mathscr{V}$ satisfying*

$$u|_N \in \left(\bigwedge \mathscr{H}^* \otimes \mathscr{R}_{k+2}\right)|_{\mathscr{N}}, \quad \nabla u \in \mathscr{V}^* \wedge \left(\bigwedge \mathscr{H}^*\right) \otimes \mathscr{R}_{k+2}.$$

Then u belongs to $\bigwedge \mathscr{H}^* \otimes \mathscr{R}_{k+2}$.

Proof. Let ξ_i', $1 \leq i \leq s$, be a basis of sections of $\mathscr{J}_{k+2}\mathscr{V}$, such that ξ_i', $1 \leq i \leq r$, is a basis of $\mathscr{R}_{k+2}$, and $\nabla \xi_i' = 0$ for $1 \leq i \leq s$. Then

$$u = \sum_{i=1}^{s} \alpha_i \otimes \xi_i',$$

with $\alpha_i = \sum f_\beta^i \, dx^\beta \in \bigwedge \mathscr{H}^*$, and $f_\beta^i(x, 0) = 0$ for $r < i \leq s$. Therefore

$$\nabla u = \sum_{i=1}^{s} d_V \alpha_i \otimes \xi_i'$$

and by hypothesis

$$d_V \alpha_i = 0, \qquad r < i \leq s.$$

This implies that

$$\frac{\partial f_\beta^i}{\partial y^j} = 0, \qquad r < i \leq s.$$

Hence $f_\beta^i(x, y) = f_\beta^i(x, 0) = 0$, $r < i \leq s$, and $u \in \bigwedge \mathscr{H}^* \otimes \mathscr{R}_{k+2}$. q.e.d.

On account of the equalities $[\psi_H, \psi_V] = [\psi_H, \omega] = 0$ we obtain

$$\begin{aligned} \nabla_{k+1}(D_H \xi_i') &= [\psi_V + \omega_{k+2}, [\psi_H, \xi_i']] \\ &= [[\psi_V + \omega, \psi_H], \xi_i'] - [\psi_H, [\psi_V + \omega, \xi_i']] \\ &= -D_H(\nabla \xi_i') = 0. \end{aligned}$$

It follows from hypothesis (d) that $(D_H \xi_i')|_N \in (\mathscr{H}^* \otimes \mathscr{R}_{k+1})|_{\mathscr{N}}$, and from Lemma 3 that $D_H \xi_i' \in \mathscr{H}^* \otimes \mathscr{R}_{k+1}$ for $1 \leq i \leq r$. Thus

$$D_H(\mathscr{R}_{k+2}) \subset \mathscr{H}^* \otimes \mathscr{R}_{k+1}. \tag{20}$$

We have finished the first step of the proof of the theorem, namely constructing the vector bundle R_{k+1} satisfying properties (i), (ii), (iii), and (20). Now, we are going to twist equation R_{k+1} by a section of $\widetilde{\mathscr{Q}}_{k+2}(\mathscr{V})$ such that (iv) holds for the twisted equation. If $\xi \in \mathscr{R}_{k+1}$, and $\phi \in \widetilde{\mathscr{Q}}_{k+2}(\mathscr{V})$, it follows from (6) that

$$D\phi(\xi) \in \mathscr{T}^* \otimes \pi_{k+1}\phi(\mathscr{R}_k)$$

if and only if

$$D\xi - [\mathscr{D}\phi, \xi] \in \mathscr{T}^* \otimes \mathscr{R}_k. \tag{21}$$

If ϕ is an element of $\widetilde{\mathscr{Q}}_{k+2}(\mathscr{V})$, with $\phi|_N = j^{k+2}\,\mathrm{id}$, for which (21) holds for all $\xi \in \mathscr{R}_{k+1}$, then $R'_{k+1} = \phi(R_{k+1})$ is a sub-bundle of $J_{k+1}(V)$ satisfying the condition of the theorem. For $\xi \in \mathscr{R}_{k+1}$, we have

$$D\xi = D_V\xi + D_H\xi = D_H\xi + \pi_k(\nabla_{k+1}\xi) - [\omega_{k+1}, \xi];$$

thus, by (18) and (20), we see that (21) is equivalent to

$$[\mathscr{D}\phi + \omega_{k+1}, \xi] = 0 \mod T^* \otimes R_k. \tag{22}$$

It follows from (19) that (22) holds for all $\xi \in \mathscr{R}_{k+1}$ if

$$\mathscr{D}\phi = -\omega_{k+1} \mod T^* \otimes R_{k+1}. \tag{23}$$

Thus it suffices to solve (23) for an element ϕ of $\widetilde{\mathscr{Q}}_{k+2}(\mathscr{V})$, with $\phi|_N = j^{k+2}\,\mathrm{id}$.

Set

$$A^{k+l} = (-\omega_{k+l} + T^* \otimes R_{k+l}) \cap (T^* \otimes J_{k+l}V)^\wedge, \qquad l = 1, 2. \tag{24}$$

We have $-\pi_0\omega + \mathrm{id} = \sum dx^i \otimes \partial/\partial x^i$, and by hypothesis (a), $\pi_0(R_{k+l}) = V$ and $A_x^{k+l} \neq \varnothing$ for every $x \in M$. Furthermore, since $(T^* \otimes J_{k+l}V)^\wedge$ is open in $T^* \otimes J_{k+l}V$, we see that A^{k+l} is open in $-\omega_{k+l} + T^* \otimes R_{k+l}$. This implies that $VA^{k+l} \cong T^* \otimes R_{k+l}$.

Define

$$S^{k+l+1} = \{X \in Q_{(1, k+l+1)}V \mid p(\mathscr{D})X \in A^{k+l}\}, \qquad l = 1, 2. \tag{25}$$

Then S^{k+2} is the partial differential equation associated with the relation (23). We will show the following:

(e) $S^{k+l+1} \to Q_{k+l+1}(V)$ is surjective, for $l = 1, 2$;

(f) $\pi_{1, k+2}\colon S^{k+3} \to S^{k+2}$ is surjective;

(g) $(S^{k+2})_{+1} \to S^{k+2}$ is surjective;

(h) If g^1_X is the symbol of S^{k+2} at the point $X \in S^{k+2}$, with $\pi(X) = x$, then

$$g^1_X = T^*_x \otimes ((\pi^{k+2}_{k+1})^{-1}_* \cdot (\pi_{0,k+2}X) \cdot R_{k+1,x}).$$

From (g) and (h) and by the theorem of the Appendix, there is a $\phi \in \widetilde{\mathscr{Q}}_{k+2}(\mathscr{V})$ such that $\phi|_N = j^{k+2}\,\mathrm{id}|_N$, and $j^1\phi \in \mathscr{S}^{k+2}$. Then $R'_{k+1} = \phi(R_{k+1})$ satisfies the conditions of the theorem.

In the proof of (e)–(h), the following diagram will be useful; the dotted vertical arrows represent affine actions:

$$\begin{tikzcd}
 & S^2T^* \otimes VQ_{k+2}(V) \arrow[rrr, "\sigma_1(\mathscr{D})"] \arrow[dd, dotted] & & & T^* \otimes T^* \otimes J_{k+1}V & & \\
 & & & S^2T^* \otimes VQ_{k+3}(V) \arrow[ull] \arrow[rr, "\sigma_1(\mathscr{D})"] \arrow[dd, dotted] & & T^* \otimes T^* \otimes J_{k+2}V \arrow[ul] \arrow[r, "\sigma(\mathscr{D}_1)"] \arrow[dd] & \bigwedge^2 T^* \otimes J_{k+1}V \arrow[dd] \\
(S^{k+2})_{+1} \arrow[r, hook] \arrow[dd] & Q_{(2,k+2)}(V) \arrow[dd] & & & & & \\
 & & & Q_{(2,k+3)}(V) \arrow[ull] \arrow[rr, "p_1(\mathscr{D})"] \arrow[dd] & & J_1(T^* \otimes J_{k+2}V)^\wedge \arrow[r, "p(\mathscr{D}_1)"] \arrow[dd] & \bigwedge^2 T^* \otimes J_{k+1}V \arrow[dd] \\
S^{k+2} \arrow[r, hook] & Q_{(1,k+2)}(V) & & & & & \\
 & & S^{k+3} \arrow[ull] \arrow[r, hook] & Q_{(1,k+3)}(V) \arrow[ull] \arrow[rr, "p(\mathscr{D})"] & & (T^* \otimes J_{k+2}V)^\wedge \arrow[r] & 0
\end{tikzcd}$$

Proof of (e). The morphism

$$p(\mathscr{D})\colon Q_{(1,k+l+1)}(V) \to (T^* \otimes J_{k+l+1}V)^\wedge$$

is surjective and has constant rank, and A^{k+l} is a submanifold of $(T^* \otimes J_{k+k+1}V)^\wedge$. Hence

$$S^{k+l+1} = p(\mathscr{D})^{-1}(A^{k+l}) \tag{26}$$

is a submanifold of $Q_{(1,k+l+1)}(V)$. From (5), we see that

$$p(\mathscr{D})(H \circ X) = p(\mathscr{D})X + (\lambda^1 X)^{-1}(p(\mathscr{D})H).$$

If $p(\mathscr{D})X = h = p(\mathscr{D})(H \circ X)$, then $p(\mathscr{D})H = 0$, so

$$p(\mathscr{D})^{-1}(h) = Q_{k+l+2}(V)_{\beta(X)} \circ X, \tag{27}$$

where $\beta\colon Q_{k+l+2}(X) \to X$ is the "target" projection. When $X \in S^{k+l+1}$, we have $Q_{k+l+2}(V)_{\beta(X)} \circ X \subset S^{k+l+1}$ which implies (e).

Proof of (f). If $X \in S^{k+2}$, then $h = p(\mathscr{D})X \in A^{k+1}$. Let $\tilde{h}$ be an element of A^{k+2} such that $\pi_{k+1}(\tilde{h}) = h$. Then there is an $\tilde{X} \in S^{k+3}$ such that $p(\mathscr{D})\tilde{X} = \tilde{h}$, so that $p(\mathscr{D})^{-1}(\tilde{h}) = Q_{k+4}(V)_{\beta(X)} \circ X$. Hence we have

$$\pi_{1,k+2}(Q_{k+4}(V)_{\beta(\tilde{X})} \circ \tilde{X}) = \pi_{1,k+2}(p(\mathscr{D})^{-1}\tilde{h}) = p(\mathscr{D})^{-1}h = Q_{k+3}(V)_{\beta(X)} \circ X$$

which implies (f).

Proof of (g). Take $X \in \mathscr{S}^{k+2}$. We must show there exists $\tilde{X} \in (\mathscr{S}^{k+2})_{+1}$ with $\pi_{1,k+2}(\tilde{X}) = X$. It follows from (f) that there is an element F of $\mathscr{S}^{k+3}$ such that $\pi_{1,k+2}F = X$, hence $p(\mathscr{D})F = -\omega_{k+2} + \theta$, with $\theta \in \mathscr{T}^* \otimes \mathscr{R}_{k+2}$. Choose $\tilde{F} \in \mathscr{Q}_{(2,k+3)}(\mathscr{V})$ satisfying $\pi_{1,k+3}\tilde{F} = F$. Then

$$\pi_0(p_1(\mathscr{D})\tilde{F}) = p(\mathscr{D})F = -\omega_{k+2} + \theta .$$

If $z = p_1(\mathscr{D})\tilde{F} - j^1(-\pi_{k+2}\omega + \theta)$, then $z \in \mathscr{T}^* \otimes \mathscr{T}^* \otimes \mathscr{J}_{k+2}\mathscr{V}$ and

$$\begin{aligned}\sigma(\mathscr{D}_1)z &= p(\mathscr{D}_1)(p_1(\mathscr{D})\tilde{F}) - p(\mathscr{D}_1)(j^1(-\omega_{k+2} + \theta)) = -\mathscr{D}_1(-\omega_{k+2} + \theta) \\ &= D\omega_{k+2} + \tfrac{1}{2}[\omega_{k+2}, \omega_{k+2}] - (D\theta + [\omega_{k+2}, \theta]) + \tfrac{1}{2}[\theta, \theta],\end{aligned}$$

by (14). By the choice of ω, we have

$$D\omega_{k+2} = \tfrac{1}{2}[\omega_{k+2}, \omega_{k+2}] = 0 .$$

It follows from (16), (18), and (20) that

$$D\theta + [\omega_{k+2}, \theta] = D_H\theta + \pi_{k+1}(\nabla\theta) \in \textstyle\bigwedge^2 \mathscr{T}^* \otimes \mathscr{R}_{k+1},$$

and from (19) that

$$\tfrac{1}{2}[\theta, \theta] \in \textstyle\bigwedge^2 \mathscr{T}^* \otimes \mathscr{R}_{k+1} .$$

Thus

$$\sigma(\mathscr{D}_1)z \in \textstyle\bigwedge^2 \mathscr{T}^* \otimes \mathscr{R}_{k+1} .$$

By (13) we see that $\sigma(\mathscr{D}_1)\colon \mathscr{T}^* \otimes \mathscr{T}^* \otimes \mathscr{R}_{k+2} \to \bigwedge^2 \mathscr{T}^* \otimes \mathscr{R}_{k+1}$ is surjective, and so there exists $y \in \mathscr{T}^* \otimes \mathscr{T}^* \otimes \mathscr{R}_{k+2}$ such that

$$\sigma(\mathscr{D}_1)y = \sigma(\mathscr{D}_1)z \qquad \text{or} \qquad \sigma(\mathscr{D}_1)(y - z) = 0 .$$

The sequence

$$(28) \quad S^2T^* \otimes VQ_{k+3}(V) \xrightarrow{\sigma_1(\mathscr{D})} T^* \otimes T^* \otimes J_{k+2}V \xrightarrow{\sigma(\mathscr{D}_1)} \textstyle\bigwedge^2 T^* \otimes J_{k+1}V$$

is not exact. From (13), it follows that

$$\ker \sigma(\mathscr{D}_1) = (S^2T^* \otimes J_{k+2}V) + (T^* \otimes T^* \otimes S^{k+2}T^* \otimes V),$$

so that

$$\pi_{k+1}(y - z) \in \mathscr{S}^2\mathscr{T}^* \otimes \mathscr{J}_{k+1}\mathscr{V} .$$

Using (12) we obtain that

$$\sigma_1(\mathscr{D})(S^2T^* \otimes VQ_{k+2}) = S^2T^* \otimes J_{k+1}V ;$$

hence there exists $h \in \mathscr{S}^2\mathscr{T}^* \otimes \mathscr{V}\mathscr{Q}_{k+2}(\mathscr{V})$, with

$$h(x) \in S^2T_x^* \otimes V_{\pi_{0,k+2}X(x)}Q_{k+2}(V)$$

for all $x \in M$, such that $\sigma(\mathscr{D}_1)h = \pi_{k+1}(y - z)$. Set $\tilde{X} \in \pi_{2,k+2}\tilde{F} + h$. Then $\pi_{1,k+2}(\tilde{X}) = \pi_{1,k+2}(\tilde{F}) = X$, and

$$\begin{aligned} p_1(\mathscr{D})\tilde{X} &= p_1(\mathscr{D})(\pi_{2,k+2}\tilde{F}) + \sigma_1(\mathscr{D})h \\ &= \pi_{1,k+1}(p_1(\mathscr{D})\tilde{F}) + \pi_{k+1}(y - z) \\ &= \pi_{1,k+1}(j^1(-\omega_{k+2} + \theta) + z) + \pi_{k+2}(y - z) \\ &= -j^1\omega_{k+1} + j^1\pi_{k+1}\theta + y; \end{aligned}$$

hence

$$p_1(\mathscr{D})\tilde{X} = -j^1\omega_{k+1} \quad \operatorname{mod} \mathscr{J}_1(\mathscr{T}^* \otimes \mathscr{R}_{k+1})$$

and $\tilde{X} \in (\mathscr{S}^{k+2})_{+1}$, which proves (g).

Proof of (h). Denote the canonical projection by

$$\rho\colon T^* \otimes J_{k+1}V \to (T^* \otimes J_{k+1}V)/(T^* \otimes R_{k+1}).$$

Then

$$S^{k+2} = [\rho \circ p(\mathscr{D})]^{-1}(\rho(-\omega_{k+1})),$$

and therefore

$$g^1 = \ker \rho \circ \sigma(\mathscr{D})$$

(cf. [3]), i.e., if $X \in S_x^{k+2}$, then

$$g_X^1 = \{h \in T_x^* \otimes V_{\pi_{0,k+2}(X)}Q_{k+2}(V) | \sigma(\mathscr{D})h \in T_x^* \otimes R_{k+1,x}\}.$$

From (11) it thus follows that

$$g_X^1 = T_x^* \otimes (\pi_{k+1}^{k+2})_*^{-1}((\pi_{0,k+2}X) \circ R_{k+1,x}).$$

Corollary. *In the hypothesis of the theorem, suppose furthermore that* $h_k = \{\xi \in L_k | \pi_{k-1}\xi = 0\}$ *is 2-acyclic at every point* $x \in N$. *Then* R_k' *is formally integrable.*

Proof. We must show that $g_k = \{\xi \in R_k' | \pi_{k-1}\xi = 0\}$ is 2-acyclic. We know $g_k|_N = h_k$. Applying an argument of [4] (cf. Remarque after Proposition 5.3), adapted to the intransitive case, we get

$$H_{k+l,j}(g_k)_{(x,y)} \simeq H_{k+l,j}(h_k)_{(x,0)}.$$

Hence g_k is 2-acyclic. Now, from Theorem 4.1 of [2], it follows that R_k is formally integrable.

Appendix

We prove here a generalization of Theorem 5.1 of [8] which we state in a simplified form.

Let $\pi\colon E \to M$ be a fibered manifold, where $\dim M = m$ and $\dim E = m+n$. The manifold J_1E of 1-jets of sections of (E, M, π) has dimension $m+n+mn$. If (x^i, y^j) is a fibered chart of E, then (x^i, y^j, p_i^j) is a chart for J_1E, where

$$p_i^j(j_a^1 f) = \frac{\partial f^j}{\partial x^i}(a),$$

and $f = (f^1, \cdots, f^n)$ is a section of (E, M, π). We denote $x = (x^1, \cdots, x^m)$, $y = (y^1, \cdots, y^n)$, and $p^j = (p_1^j, \cdots, p_m^j)$.

If we denote $V_0J_1E = \ker(\pi_0)_*$, then it is well known ([8]) that

$$\begin{array}{ccc} V_0J_1E & \tilde{\to} & T^* \otimes VE, \\ \frac{\partial}{\partial p_i^j} & \mapsto & dx^i \otimes \frac{\partial}{\partial y^j}. \end{array}$$

Theorem. *Suppose $R_1 \subset J_1E$ is a system of partial differential equations such that:*

(1) $(R_1)_{+1} \xrightarrow{\pi_1} R_1$ *is surjective;*

(2) $\pi_0(R_1) = E$;

(3) *the symbol $g_1 = (V_0J_1E) \cap TR_1$ of R_1 is equal to $T^* \otimes F$, where F is a vector sub-bundle of VE.*

Then, for every $X \in R_{1,a}$, $a \in M$, there exists a solution f of R_1 such that $j_a^1 f = X$, and this solution depends arbitrarily on r functions, where r is the dimension of F.

Proof. Choose a chart on E such that F_a is generated by

$$\frac{\partial}{\partial y^{n-r+1}}(a), \cdots, \frac{\partial}{\partial y^n}(a).$$

Choose $\{\phi_\sigma | \sigma \in \Sigma, \phi_\sigma\colon J_1E \to \mathbf{R}\}$, with $d\phi_\sigma$ linearly independent, such that

$$R_1 = \{X \in J_1E | \phi_\sigma(X) = 0, \sigma \in \Sigma\}.$$

Clearly, Σ has $m(n-r)$ elements. Let

$$v = \sum_{j=1}^{n-r}\sum_{i=1}^{m} a_i^j \frac{\partial}{\partial p_i^j}(a)$$

be an element of V_0J_1E. Then $v \in V_0R_1$ if and only if the linear system

$$\sum_{j=1}^{n-r}\sum_{i=1}^{m} a_i^j \frac{\partial}{\partial p_i^j}(a) = 0$$

has only the trivial solution $a_i^j = 0$; thus

$$\left(\frac{\partial \phi_\sigma}{\partial p_i^j}(a)\right),$$

$\sigma \in \Sigma$, $1 \leq i \leq m$, $1 \leq j \leq n-r$, is an invertible matrix. The implicit function theorem allows us to replace $\{\phi_\sigma, \sigma \in \Sigma\}$ by

$$\{\phi_i^j = p_i^j - \psi_i^j(x, y, p^{n-r+1}, \cdots, p^n),\ 1 \leq i \leq m,\ 1 \leq j \leq n-r\}.$$

For every $X \in R_{1,a}$, we choose r functions $f^{n-r+1}(x), \cdots, f^n(x)$ such that $y^k(X) = f^k(a)$ and $p_i^k(X) = (\partial f^k/\partial x^i)(a)$ for $1 \leq i \leq m$, $n-r < k \leq n$. Set

$$\tilde{\phi}_i^j = p_i^j - \psi_i^j\left(x, y^1, \cdots, y^{n-r}, f^{n-r+1}(x), \cdots, f^n(x), \right.$$
$$\left. \frac{\partial f^{n-r+1}}{\partial x}(x), \cdots, \frac{\partial f^n}{\partial x}(x)\right),$$

$1 \leq i \leq m$, $1 \leq j \leq n-r$.

This is a Frobenius system and its integrability conditions are a consequence of hypothesis (1) (cf. the proof of Theorem 5.1 of [8]). If $(f^1(x), \cdots, f^{n-r}(x))$ is a solution of $\tilde{\phi}_i^j = 0$, such that $y^j(X) = f^j(a)$ and $p_i^j(X) = (\partial f^j/\partial x^i)(a)$, then $(f^1, \cdots, f^n)$ is a solution of R^1. The same proof works when the initial data is well posed on a submanifold of M.

References

[1] E. Cartan, *Sur le structure des groupes infinis de transformations*, Ann. Sci. École Norm. Sup. **21** (1904) 153–206; **22** (1905) 219–308.

[2] H. Goldschmidt, *Existence theorems for analytic linear partial differential equations*, Ann. of Math. (2) **86** (1967) 246–270.

[3] ——, *Integrability criteria for systems of non-linear partial differential equations*, J. Differential Geometry **1** (1967) 269–307.

[4] ——, *Sur la structure des équations de Lie*: I. *Le troisiéme théorème fondamental*, J. Differential Geometry **6** (1972) 357–373; II. *Équations formellement transitives*, J. Differential Geometry **7** (1972) 67–95.

[5] H. Goldschmidt & D. Spencer, *On the non-linear cohomology of Lie equations*. I, II, Acta Math. **136** (1976) 103–239.

[6] V. W. Guillemin & S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964) 16–47.

[7] I. Hayashi, *Embedding and existence theorems of infinite Lie algebras*, J. Math. Soc. Japan **22** (1970) 1–14.
[8] M. Kuranishi, *Lectures on involutive systems of partial differential equations*, Publ. Soc. Mat. São Paulo, 1967.
[9] B. Malgrange, *Équations de Lie*, I, II, J. Differential Geometry **6** (1972) 503–522, **7** (1972) 117–141.

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