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GAUSSIAN MAPS ON ALGEBRAIC CURVES

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0. Introduction

Let C be a complete nonsingular curve over $\mathbf{C}$ and let L be a line bundle of positive degree. We have previously considered [8] the natural map

$$\Phi_L : \Lambda^2 H^0(L) \to H^0(\Omega^1_C \otimes L^2),$$

defined essentially by

$$f \wedge g \mapsto f\,dg - g\,df.$$

If $C \subset \mathbf{P}^n$ is an embedding and $L = \mathcal{O}_C(1)$, one may consider the Gauss mapping

$$C \to \mathrm{Grass}(1, n),$$

associating to each point its tangent line in $\mathbf{P}^n$. Composing with the Plucker embedding of the Grassmannian into $\mathbf{P}^N$ gives the "associated curve" $\psi : C \to \mathbf{P}^N$. One checks that restriction of the hyperplane section $\psi^* : H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(1)) \to H^0(C, \psi^*\mathcal{O}(1))$ for this map gives Φ_L (note $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}}(1)) \simeq \wedge^2 H^0(\mathbf{P}^n, \mathcal{O}(1)) \simeq \wedge^2 H^0(C, L))$. For this reason we call Φ_L or its generalization a *Gaussian* map, and its image the Gaussian linear series.

The original interest in these maps arose from studying Φ_K, where K is the canonical bundle on a smooth curve (Φ_K has been named the "Wahl map" by certain authors [3]).

Theorem 1 [8]. *If the smooth curve C lies on a K-3 surface, then Φ_K is not surjective.*

Theorem 2 [8]. *If $C \subset \mathbf{P}^n$ is a complete intersection, with multidegrees $d_1 \leq d_2 \leq \cdots \leq d_{n-1}$ $(d_1 \geq 2)$, then Φ_K is surjective if $d_1 + \cdots + d_{n-2} > n + 1$.*

Theorem 3 [3]. *For a general curve C of genus 10 or ≥ 12, Φ_K is surjective.*

Theorem 1 gives the only known intrinsic property which a curve must satisfy in order to sit on a K-3 surface. Our original proof involved the

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interpretation of $(\operatorname{Coker}\Phi_K)^*$ in the deformation theory of the cone over the canonical curve; a more geometric proof, involving the nonsplitting of the normal bundle sequence, was given by Beauville and Mérindol [2]. It is known that a general curve of genus g sits on a K-3 surface iff $g \leq 9$ or $g = 11$; Theorem 3 naturally complements this result, and reproves the difficult $g = 10$ case (due to Mukai [6]). Theorem 3 was proved by smoothing a certain singular stable curve, so explicit examples with Φ_K surjective are not produced. For certain genera such curves can be constructed from Theorem 2 above or Theorem 4.8 of this paper. We mention more about Φ_K below.

Now let X be any smooth projective complex variety, and let L and M be two line bundles. One can again consider Φ_L as defined above. More generally, consider the multiplication map

$$\mu_{L,M}: H^0(L) \otimes H^0(M) \to H^0(L \otimes M),$$

and denote by $\mathscr{R}(L, M) = \operatorname{Ker}\mu_{L,M}$; one may construct another "Gaussian" map

$$\Phi_{L,M}: \mathscr{R}(L, M) \to H^0(\Omega^1_X \otimes L \otimes M).$$

If $L = M$, $\Phi_{L,L}$ is essentially equal to Φ_L. These maps may also be constructed via the sheaf of principal parts (or "Atiyah classes"), or from restriction to the diagonal on $X \times X$ (see [9], or §1 below). In fact, one can construct a natural filtration $\{\mathscr{R}_i(L, M)\}$ of $H^0(L) \otimes H^0(M)$ via "order of vanishing along the diagonal" ((1.3) below). Here, $\mathscr{R}_1 = \mathscr{R}$, $\mathscr{R}_2 = \operatorname{Ker}\Phi_{L,M}$, and $\mathscr{R}_i/\mathscr{R}_{i+1}$ embeds naturally in

$$H^0(X, \operatorname{Sym}^i\Omega^1_X \otimes L \otimes M).$$

For $X = C$ a curve, $L = M$, and $i = 2$, one recovers the dual of the "second fundamental form" of [5, p. 366]; our map is a morphism

$$\operatorname{Ker}(S^2H^0(L) \to H^0(L^2)) \to H^0(C, K^2 \otimes L^2).$$

In a forthcoming paper [10], we will examine these maps and this filtration when $X = G/P$, a complex projective homogeneous space; note that in this case, one gets a natural G-filtration of the tensor product of two irreducible G-modules. However, in the present paper we shall restrict ourselves to studying $\Phi_{L,M}$ when $X = C$ is a curve.

Mumford [7] proved that $\mu_{L,M}$ is surjective if $\deg L \geq 2g$ and $\deg M \geq 2g+1$; he used the "base-point-free pencil trick" and Castelnuovo's Lemma. In §§2 and 3 below, we adapt his method, with an "immersive net trick" and a generalized Castelnuovo Lemma, to prove our two main results:

Theorem 3.3. *Let C be a projective nonsingular curve, and let L and M be line bundles, with* $\deg L \geq 5g+1$ *and* $\deg M \geq 2g+2$. *Then* $\Phi_{L,M}$ *is surjective.*

Theorem 3.8. *Let C be a nonhyperelliptic curve, and let L be a line bundle of degree* $\geq 5g+2$. *Then* $\Phi_{L,K}$ *is surjective.*

These theorems have been applied in [9] to give information about the space T^1 of first-order deformations of the cone over the embedding of C into $\mathbf{P}^n$ by some very ample and projectively normal L. For instance, it is proved there using Theorem 3.8 that the cone over a nonhyperelliptic curve, embedded by a line bundle of degree $\geq 5g+2$, has only conical deformations, even infinitesimally. (We asserted this result in [9] when $\deg \geq 5g+1$; this is false for $g=3$, but J. Harris has shown that $5g+1$ works if $g \geq 4$.) We remark that a short proof is given in [3, Appendix], that $\Phi_{L,M}$ is surjective for $\deg L$ and $\deg M \geq 4g+5$.

In §4, we calculate $\Phi_{L,M}$ for special curves. Theorem 4.2 gives a sharp result on the surjectivity of $\Phi_{L,M}$ for an elliptic curve; one needs $\deg L \geq 3$ and $\deg M \geq 7$, or $\deg L \geq 4$ and $\deg M \geq 5$. One can compute Φ_L explicitly when one has an explicit basis of functions for $H^0(L)$; in particular, if C is hyperelliptic and L is multiple of the hyperelliptic pencil g_2^1, it is easy to compute the image of Φ_L (Theorem 4.4). We summarize other results in §4 in terms of the corank of the fundamental map Φ_K:

(1) $\operatorname{cork} \Phi_K \leq 3g-2$, with equality iff C is hyperelliptic or $g=3$.
(2) If C lies on $\mathbf{P}^2$ or a rational ruled $\mathbf{F}_n$, and $g \geq 4$, then $\operatorname{cork} \Phi_K \geq 9$.
(3) If C is trigonal (i.e., has a g_3^1), with $g \geq 4$, then $\operatorname{cork} \Phi_K \geq 9$.
(4) There are curves of arbitrarily high genus possessing a g_6^1, and with Φ_K surjective.

For (1), use that $\operatorname{rk} \Phi_L \geq 2h^0(L) - 3$ for any Gaussian (1.3.4), plus an easy calculation on hyperelliptic curves; for the converse, one uses a local calculation at a Weierstrass point (Theorem 4.6), which in general provides an upper bound for the corank of Φ_K from the corresponding gap sequence. The key to (2) is the study of the cohomology of the restriction map $\Omega_X^1 \to \Omega_C^1$ (Theorem 4.8). It is well known that trigonal curves lie on normal scrolls (e.g., [1]), so (2) implies (3). The examples in (4) are constructed in Theorem 4.11 by studying curves on surfaces $C_1 \times C_2$, where C_i are curves, and $g(C_2) \geq 2$.

The reader should notice the use of four "concrete" methods to compute Gaussians on curves. First, as mentioned above, explicit calculation is possible for Φ_L when one has explicit functions in $H^0(L)$. Second, surjectivity of $\Phi_{L,M}$ may be deduced from cohomological vanishing on $C \times C$ ([3, Appendix], and (4.2) below). Third, if C is a smooth curve on a surface X, and L is a line bundle on X, one can deduce information about $\Phi_{\overline{L}}$ $(\overline{L} = L \otimes \mathscr{O}_C)$, via the diagram:

$$\begin{array}{ccccc} \wedge^2 H^0(X, L) & \longrightarrow & H^0(X, \Omega^1_X \otimes L^2) & \searrow & \\ \downarrow & & \downarrow & & H^0(C, \Omega^1_X \otimes L^2 \otimes \mathscr{O}_C) \\ \wedge^2 H^0(C, \overline{L}) & \longrightarrow & H^0(C, \Omega^1_C \otimes \overline{L}^2) & \swarrow & \end{array}$$

(This method is used in Theorems 4.8 and 4.11.) Finally, information about Gaussians on C_1 and C_2 gives information about Gaussians on curves C on $C_1 \times C_2$ which are linearly equivalent to sums of fibers (Theorem 4.11 and Corollary 4.13).

We mention two natural questions suggested by our work. First, one should try to imitate the Koszul methods of Mark Green [4] to find a uniform treatment of the surjectivity questions for $\Phi_{L,M}$ and $\Phi_{K,L}$ in §3, without referring to [7]; in particular, as suggested in [9, (7.11.2)], one should prove directly that $\Phi_{L,K\otimes L}$ is surjective when $\deg L \geq 2g+3$, without using deformation theory. Second, one should study the stratification of the moduli space of curves $\mathscr{M}_g$ by the corank of the map Φ_K. In this direction we offer a Conjecture (which perhaps needs fine tuning), which is true for X a K-3 surface (Theorem 1 above), rational ruled (Theorem 4.8), or $\mathbf{P}^2$ (Remark 4.9):

Conjecture. Let X be a regular surface $(H^1(\mathscr{O}_X) = 0)$. Then there is a g_0 so that for every smooth C on X of genus $\geq g_0$, one has

$$\operatorname{corank} \Phi_K \geq h^0(X, K_X^{-1}).$$

The alert reader will notice which arguments carry over immediately to characteristic $p \neq 2$. It is a pleasure to thank Arnaud Beauville and Joe Harris for numerous suggestions. Our research was partially supported by National Science Foundation Grant DMS-8601544.

Added in proof. An idea of R. Lazarsfeld provides the surjectivity of $\Phi_{L,M}$ when $\deg L \geq 4g+1$ and $\deg M \geq 2g+2$, and the surjectivity of $\Phi_{K,L}$ when $\deg L \geq 4g-3$ and C is neither trigonal nor a plane quintic (see the author's forthcoming article in the Proceedings of the Trieste Conference on Projective Varieties, 1989). Some deeper work of Lazarsfeld and L. Ein shows $\Phi_{L,M}$ is surjective when $\deg L$, $\deg M \geq 2g+4$ and $\deg L + \deg M \geq 6g+3$. In a recent preprint, C. Ciliberto and R. Miranda

(*Gaussian maps for certain families of canonical curves*) show the corank of Φ_K for a general trigonal curve is $g+5$.

1. Basics on Gaussian maps

(1.1) For convenience, we always assume X is a smooth, projective complex variety. If L is a line bundle and M is a coherent sheaf on X, we define the kernel of the multiplication map as

$$\mathscr{R}(L, M) = \operatorname{Ker}(\mu_{L,M}\colon \Gamma(L)\otimes\Gamma(M)\to\Gamma(L\otimes M)).$$

Consider $\alpha=\sum l_i\otimes m_i\in\mathscr{R}(L, M)$. On an open affine $U\subset X$, let $L|U$ have T as generator, and write $l_i=f_iT(f_i\in\Gamma(\mathscr{O}_U))$ locally on U. Then define

$$\Phi_{L,M}\colon \mathscr{R}(L, M)\to H^0(\Omega^1_X\otimes L\otimes M)$$

by

$$\Phi(\alpha)=-2\sum df_i\otimes T\otimes m_i.$$

It is straightforward to verify that Φ is well defined and **C**-linear [9, 7.2]. In case M is also invertible, writing m_i locally as g_iS one has (more symmetrically)

$$\Phi(\alpha)=\sum(f_idg_i-g_idf_i)\otimes T\otimes S.$$

Since $\bigwedge^2 H^0(L)\subset\mathscr{R}(L, L)$ (via $l_1\wedge l_2\mapsto\frac{1}{2}(l_1\otimes l_2-l_2\otimes l_1)$), the map $\Phi_{L,L}$ gives rise to

$$\Phi_L\colon \textstyle\bigwedge^2 H^0(L)\to H^0(\Omega^1\otimes L^2),$$

where

$$\Phi_L(l_1\wedge l_2)=\text{“}l_1dl_2-l_2dl_1\text{”}.$$

As remarked in [9], $\operatorname{Im}\Phi_{L,L}=\operatorname{Im}\Phi_L$, since $\Phi_{L,L}$ vanishes on $\operatorname{Ker}(S^2H^0(L)\to H^0(L^2))$.

(1.2) For a second description, let $p_i\colon X\times X\to X$ be the natural projections, $\Delta\subset X\times X$ the diagonal, $I=\mathscr{O}(-\Delta)\subset\mathscr{O}_{X\times X}$ the ideal sheaf of Δ, and $n\Delta$ the subscheme defined by $I^n=\mathscr{O}(-n\Delta)$. Let L and M be invertible on X. From the exact sequence

$$0\to I\to\mathscr{O}_{X\times X}\to\mathscr{O}_\Delta\to 0,$$

we deduce

$$\begin{array}{c} 0\to I\otimes p_1^*L\otimes p_2^*M\to p_1^*L\otimes p_2^*M\to p_1^*L\otimes p_2^*M\otimes\mathscr{O}_\Delta\to 0.\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \wr\| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad L\otimes M\end{array}\tag{1.2.1}$$

Then $H^0(X \times X, p_1^* L \otimes p_2^* M) \simeq H^0(X, L) \otimes H^0(X, M)$, and the multiplication $\mu_{L,M}$ equals H^0 of the right-hand map in (1.2.1); thus,

$$\mathscr{R}(L, M) = H^0(X \times X, I \otimes p_1^* L \otimes p_2^* M).$$

The restriction map $I \to I/I^2 \simeq \Omega_X^1$ gives

$$(1.2.2) \qquad \begin{array}{ccc} H^0(I \otimes p_1^* L \otimes p_2^* M) & \to & H^0(I/I^2 \otimes p_1^* L \otimes p_2^* M) \\ \| & & \| \\ \mathscr{R}(L, M) & & H^0(\Omega_X^1 \otimes L \otimes M) \end{array}$$

and this map gives again $\Phi_{L,M}$. From this description of $\Phi_{L,M}$ as a restriction, we have the criterion

(1.2.3) $\Phi_{L,M}$ is surjective if $H^1(X \times X, p_1^* L \otimes p_2^* M(-2\Delta)) = 0$. Therefore, usual vanishing theorems give

(1.2.4) If L and M are sufficiently ample line bundles, then $\Phi_{L,M}$ is surjective.

(1.3) We collect some general results about the maps $\Phi_{L,M}$.

(1.3.1) [9, 7.5] $\Phi_{L,M}$ may be defined via the "Atiyah class", i.e., via the sequence $0 \to \Omega^1 \to P \to \mathscr{O} \to 0$ defined by the class of L.

(1.3.2) [9, 7.9] On $\mathbf{P}^n$, $\Phi_{\mathscr{O}(r), \mathscr{O}(s)}$ is surjective for $r, s \geq 0$.

(1.3.3) (Cf. [8, 6.4].) If $X \subset \mathbf{P}^n$ is a linearly normal embedding, $L = \mathscr{O}_X(1)$, then Φ_L is essentially the restriction

$$H^0(\Omega_{\mathbf{P}^n}^1(2)) \to H^0(\Omega_X^1(2)).$$

(1.3.4) $\Phi_L \colon \wedge^2 H^0(L) \to H^0(\Omega^1 \otimes L^2)$ is injective on decomposable vectors $l_1 \wedge l_2$, hence

$$\operatorname{rank} \Phi_L \geq 2 \dim h^0(L) - 3.$$

(1.3.5) (Follows from [9, 7.5–7.7].) If $X \subset \mathbf{P}^n$ is a projectively normal embedding, $L = \mathscr{O}_X(1)$ and $A = \bigoplus_{m=0}^{\infty} \Gamma(X, L^m)$ is the cone, then the ith graded piece of the local cohomology module $H^1_{\{m\}}(\Omega_A^1)$ is $\operatorname{Coker} \Phi_{L, L^{i-1}}$, for all integers i.

(1.3.6) [9, 7.7] Let $C \subset \mathbf{P}^n$ be a projectively normal embedding of a smooth curve, $L = \mathscr{O}_C(1)$, and let $A = \bigoplus_{m=0}^{\infty} \Gamma(C, L^m)$ be the cone. Let

$$T_A^1 = \text{module of first-order deformations of } A,$$

$$\omega_A = \bigoplus_{-\infty}^{\infty} \Gamma(K \otimes L^m), \quad \text{the dualizing module of } A,$$

$$V_j = \operatorname{Coker} \Phi_{K \otimes L^j, L}.$$

Then for all $j \in Z$, the dual of the i th graded piece of T_A^1 is

$$V_{-(j+1)} \simeq T_j^{1*}, \quad A \text{ Gorenstein},$$
$$0 \to V_{-(j+1)} \to T_j^{1*} \to (\omega_A/m\omega_A)_{-j} \to 0, \quad A \text{ not Gorenstein}.$$

For the proof of (1.3.4), note $\Phi(f \wedge g) = 0$ implies $d(f/g) = 0$, whence $f = cg$, so $f \wedge g = 0$; Φ_L is therefore injective on decomposables. As the decomposables in $\bigwedge^2 V$ form a subvariety of dimension $2\dim V - 3$ (the cone over the Grassmannian), the codimension of the kernel of Φ ($=\operatorname{rank}\Phi$) must be at least $2\dim V - 3$.

(1.4) Returning to the description in (1.2), let

$$\mathscr{R}_j(L, M) = H^0(X \times X, p_1^* L \otimes p_2^* M \otimes I^j), \qquad j \geq 0.$$

Then $\{\mathscr{R}_j\}$ gives a filtration of $\mathscr{R}_0(L, M) \simeq H^0(L) \otimes H^0(M)$. Further, since $I^j \otimes \mathscr{O}_\Delta = I^j/I^{j+1}$ is isomorphic to $\operatorname{Sym}^j \Omega_X^1$, there are natural restriction maps

$$\Phi_j : \mathscr{R}_j(L, M) \to H^0(X, L \otimes M \otimes \operatorname{Sym}^j \Omega_X^1).$$

Note $\operatorname{Ker}\Phi_j = \mathscr{R}_{j+1}$. If $M = L$, then the involution of the factors of $X \times X$ gives an involution on the $\mathscr{R}_i$'s; further, Φ_j vanishes on symmetric forms for j odd, and on alternating forms for j even. This gives a filtration for $S^2 H^0(L)$ and $\bigwedge^2 H^0(L)$. In particular, when $X \subset \mathbf{P}^n$ is linearly normal, $L = \mathscr{O}_X(1)$, Φ_2 induces a map

$$I_2 \to H^0(X, L^2 \otimes \operatorname{Sym}^2 \Omega^1),$$

where $I_2 = \operatorname{Ker}(S^2 H^0(L) \to H^0(L^2))$ is the space of quadratic forms on $\mathbf{P}^n$ vanishing on X; this should be compared with the "second fundamental form" (e.g. [5]). We summarize by

Proposition 1.5. *Let X be nonsingular and projective, and L and M line bundles on X. Then there exists a natural filtration $\{\mathscr{R}_j\}$ of $H^0(X, L) \otimes H^0(X, M)$, by "order of vanishing along the diagonal", whose subquotients $\mathscr{R}_j/\mathscr{R}_{j+1}$ are contained in $H^0(X, L \otimes M \otimes \operatorname{Sym}^j \Omega_X^1)$. If $M = L$, the restriction of the filtration to $S^2 H^0(L)$ (respectively $\bigwedge^2 H^0(L)$) gives a filtration whose subquotients are contained in $H^0(L^2 \otimes \operatorname{Sym}^j \Omega_X^1)$, with j even (respectively, j odd).*

Remarks. (1.6.1) If a group G acts on X such that L and M are G-bundles, then the spaces $\mathscr{R}_i(L, M)$ are G-modules, and the maps Φ_i are G-equivariant.

(1.6.2) The maps Φ_L on $\bigwedge^2 H^0(L)$ may be generalized in other natural ways, e.g., to

$$\bigwedge^k H^0(L) \to H^0(X, \Omega^{k-1} \otimes L^k),$$

defined essentially by

$$l_1 \wedge \cdots \wedge l_k \mapsto \sum_{j=1}^{k} (-1)^{j+1} l_j dl_1 \wedge \cdots \wedge \widehat{dl_j} \wedge \cdots \wedge dl_k.$$

Comparing with (1.3.3), for $X \subset \mathbf{P}^n$ linearly normal, $L = \mathcal{O}_X(1)$, this map is the restriction

$$H^0(\Omega^{k-1}_{\mathbf{P}^n}(k)) \to H^0(\Omega^{k-1}_X(k)).$$

2. Generalized Castelnuovo's Lemma

(2.1) For the rest of the paper, C will denote a complete nonsingular curve, L and M line bundles on C, and F a coherent sheaf on C. We shall say a subspace of $H^0(L)$ is base-point free (or immersive) if the corresponding linear system is base-point free (or defines an immersion). We start with the "base-point free pencil trick".

Castelnuovo's Lemma (e.g., [1, p. 151]). *Suppose $H^0(L)$ is base-point free, and F is such that $H^1(F \otimes L^{-1}) = 0$. Then*

$$\mu: \Gamma(L) \otimes \Gamma(F) \to \Gamma(L \otimes F)$$

is surjective.

Proof. One may choose a two-dimensional subspace $V \subset H^0(L)$ defining a base-point free pencil. Therefore, there is a surjection $V \otimes \mathcal{O}_C \to L$. The kernel is invertible, and hence is isomorphic to L^{-1}, whence

$$0 \to L^{-1} \to V \otimes \mathcal{O}_C \to L \to 0$$

is exact. Tensoring with F and taking global sections give that

$$V \otimes \Gamma(F) \to \Gamma(L \otimes F)$$

is surjective, since $H^1(F \otimes L^{-1}) = 0$, whence the result.

(2.2) In dealing with maps $\Phi_{L,F}$ of §1, it is natural to deal not with base-point free pencils, but immersive nets; i.e., we consider $V \subset H^0(L)$ of dimension 3 defining a local immersion $C \to \mathbf{P}^2$. The map Φ_L induces a homomorphism $\bigwedge^2 V \to H^0(\Omega^1 \otimes L^2)$, hence a sheaf map

(2.2.1) $$\bigwedge^2 V \otimes \mathcal{O}_C \to \Omega^1 \otimes L^2.$$

Lemma 2.3. *$V \subset H^0(L)$ gives an immersive net iff the map* (2.2.1) *is surjective.*

Proof. Choose a point $P \in C$, a local coordinate t at P, and a local generator T for the sections of L. Then a basis f, g, h of V may be expanded formally near P as

$$f = (a_0 + a_1 t + \cdots)T,$$
$$g = (b_0 + b_1 t + \cdots)T,$$
$$h = (c_0 + c_1 t + \cdots)T,$$

with $a_i, b_j, c_k \in \mathbf{C}$. The local immersion condition for V at P is exactly the statement

$$\text{(2.3.1)} \qquad \operatorname{rk}\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \end{pmatrix} = 2.$$

But by definition one has

$$\Phi_L(f \wedge g) = ((a_0 b_1 - a_1 b_0)dt + 2(a_0 b_2 - a_2 b_0)t\, dt + \cdots) \otimes T^2,$$

from which the lemma is easily deduced.

Lemma 2.4. *Suppose $V \subset H^0(L)$ gives an immersive net. Define the sheaf Q by*

$$\text{(2.4.1)} \qquad 0 \to Q \to \bigwedge^2 V \otimes \mathcal{O}_C \to \Omega^1 \otimes L^2 \to 0.$$

Then Q is locally free of rank 2, *and sits naturally in a short exact sequence*

$$\text{(2.4.2)} \qquad 0 \to L^{-1} \to Q \to (K \otimes L)^{-1} \to 0.$$

Proof. Since Q is torsion-free on a smooth curve, it is locally free (of rank 2). One checks by computing Φ_L that a global section of

$$H^0(Q \otimes L) = \operatorname{Ker}\left(\bigwedge^2 V \otimes H^0(L) \to H^0(\Omega^1 \otimes L^3)\right)$$

is given by

$$\text{(2.4.3)} \qquad f \wedge g \otimes h + g \wedge h \otimes f + h \wedge f \otimes g.$$

We claim that (2.4.3) is a nowhere 0 section of $Q \otimes L$; this will provide a short exact sequence

$$0 \to \mathcal{O} \to Q \otimes L \to \bigwedge^2 (Q \otimes L) \to 0.$$

But $\bigwedge^2 (Q \otimes L) = \bigwedge^2 Q \otimes L^2$. Via (2.4.1), $\bigwedge^2 Q \simeq (K \otimes L^2)^{-1}$, whence one has

$$0 \to \mathcal{O} \to Q \otimes L \to K^{-1} \to 0.$$

Tensoring with L^{-1} gives (2.4.2).

We return to (2.4.3), and show it is a nowhere 0 section of $Q \otimes L$. One first computes that a change of basis of V, via an invertible $3 \otimes 3$ matrix A, changes (2.4.3) by $\det A$. (In fact (2.4.3) is the image of an element in $\bigwedge^3 V$.) Now let $P \in C$ be a point, t a local coordinate, and choose a local generator T for L and a basis f, g, h for V so that $f = T$, $g = tT$, $h = t^2 p(t)T$. A local section of Q is given by $\alpha f \wedge g + \beta g \wedge h + \gamma h \wedge f$ (α, β, γ local functions) for which Φ_L is 0; computing gives the condition

$$\{\alpha + \beta t^2(tp' + p) - \gamma t(2p + tp')\}\, dt \otimes T \otimes T = 0.$$

Thus, Q is locally generated by

$$\delta_1 = -t^2(p + tp')f \wedge g + g \wedge h,$$
$$\delta_2 = t(2p + tp')f \wedge g + h \wedge f,$$

whence $Q \otimes L$ is generated locally by $\delta_1 \otimes T$ and $\delta_2 \otimes T$. We now compute the section in (2.4.3):

$$f \wedge g \otimes t^2 pT + (\delta_1 + t^2(p + tp')f \wedge g) \otimes T + (\delta_2 - t(2p + tp')f \wedge g) \otimes tT$$
$$= (\delta_1 + t\delta_2) \otimes T.$$

This section of $Q \otimes L$ is nonzero at P, whence the claim.

Remark 2.5. It is clear that a base-point free linear system contains a base-point free pencil. Similarly (by projection), an immersive linear system contains an immersive net.

Theorem 2.6 (Generalized Castelnuovo's Lemma). *Let C be a complete nonsingular curve and L a line bundle so that $H^0(L)$ defines an immersion of C. If F is a coherent sheaf so that*

$$H^1(F \otimes L^{-2}) = H^1(F \otimes L^{-2} \otimes K^{-1}) = 0,$$

then

$$\Phi_{L,F} : \mathscr{R}(L, F) \to H^0(K \otimes L \otimes F)$$

is surjective.

Proof. Let $V \subset H^0(L)$ define an immersive net, as in (2.2). This gives the exact sequences (2.4.1) and (2.4.2). Tensoring with $L^{-1} \otimes F$ and using the hypotheses, one sees Φ_L induces a surjection

$$\textstyle\bigwedge^2 V \otimes H^0(L^{-1} \otimes F) \to H^0(K \otimes L \otimes F).$$

But by naturality, this map factors

$$\textstyle\bigwedge^2 V \otimes H^0(L^{-1} \otimes F) \to \mathscr{R}(L, L) \otimes H^0(L^{-1} \otimes F)$$
$$\to \mathscr{R}(L, F) \to H^0(K \otimes L \otimes F).$$

Thus, the last map is surjective.

Example 2.7. Let $C \subset \mathbf{P}^2$ be nonsingular of degree $d \geq 3$, and let $L = \mathscr{O}_C(1)$. For which $M = \mathscr{O}_C(k)$ does one have $\Phi_{L,M}$ surjective? Since $K_C = \mathscr{O}_C(d-3)$, Theorem 2.6 gives that $\Phi_{L,M}$ is surjective if

$$H^1(\mathscr{O}_C(k-2-d+3)) = 0,$$

or $k > 2d-4$. On the other hand, the commutative diagram

$$\begin{array}{ccc} \mathscr{R}(\mathscr{O}_{\mathbf{P}^2}(1), \mathscr{O}_{\mathbf{P}^2}(k)) & \longrightarrow & H^0(\Omega^1_{\mathbf{P}^2}(k+1)) \\ \downarrow & & \downarrow r \\ \mathscr{R}(L, M) & \longrightarrow & H^0(\Omega^1_C(k+1)) \end{array}$$

has a surjective left vertical map and top horizontal map (by (1.3.2)); so the surjectivity of $\Phi_{L,M}$ is equivalent to that of the restriction r. By the usual calculation [8, 6.6], this again occurs exactly when $H^1(\mathscr{O}_C(-d+k+1)) = 0$, or $k > 2d-4$. In this sense, Theorem 2.6 is sharp.

(2.8) In order to apply Theorem 2.6, we will need

Lemma 2.9. *Let L, M be line bundles on C, and D an effective divisor. Suppose*

(i) $\Phi_{L,M}$ *is surjective;*
(ii) $H^0(L)$ *is immersive;*
(iii) $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$ *is surjective;*
(iv) $H^1(M) = 0$.

Then $\Phi_{L,M(D)}$ is surjective. Further, the same conclusion holds if (iii) *and* (iv) *are replaced by* :

(iii)′ $\mathscr{R}(L, M(D)) \to \mathscr{R}(L, M(D) \otimes \mathscr{O}_D)$ *is surjective.*

Proof. First, consider (cf. [7, p. 46])

$$(2.9.1)\quad \begin{array}{ccccc} 0 \to H^0(L) \otimes H^0(M) & \to & H^0(L) \otimes H^0(M(D)) & \to & H^0(L) \otimes H^0(M(D) \otimes \mathscr{O}_D) \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to H^0(L \otimes M) & \to & H^0(L \otimes M(D)) & \to & H^0(L \otimes M(D) \otimes \mathscr{O}_D). \end{array}$$

Considering the snake diagram, it is clear that (iii) and (iv), or (iii)′, imply the exactness of

$$0 \to \mathscr{R}(L, M) \to \mathscr{R}(L, M(D)) \to \mathscr{R}(L, M(D) \otimes \mathscr{O}_D) \to 0.$$

Since $\deg L > 0$ and $\deg M \geq 0$, one has $H^1(K \otimes L \otimes M) = 0$, hence an exact diagram with vertical maps Gaussians:

$$\begin{array}{ccccc} 0 \to \mathscr{R}(L, M) & \to & \mathscr{R}(L, M(D)) & \to & \mathscr{R}(L, M(D) \otimes \mathscr{O}_D) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to H^0(K \otimes L \otimes M) & \to & H^0(K \otimes L \otimes M(D)) & \to & H^0(K \otimes L \otimes M(D) \otimes \mathscr{O}_D) \to 0 \end{array}$$

The first vertical map, $\Phi_{L,M}$, is surjective by (i); the third is surjective by Theorem 2.6 applied to the finite support sheaf $F = M(D) \otimes \mathscr{O}_D$. Therefore, $\Phi_{L,M(D)}$ (the middle map) is surjective.

3. Surjectivity of $\Phi_{L,M}$ for curves

(3.1) Our main results, Theorems 3.3 and 3.8 below, are derived from Theorem 2.6 in the same way that Mumford's Theorems 6 and 7 of [7] are deduced from the usual Castelnuovo Lemma. We imitate Mumford's dimension-counting arguments, although we could instead quote Brill-Noether results. We start with

Proposition 3.2. *Suppose L is a line bundle of degree $d \geq 5g+1$. Then there is a divisor A of degree $g+2$ such that*

(i) $h^0(A) = 3$, $h^1(A) = 0$, *and* $|A|$ *is immersive.*

(ii) $H^1(L \otimes K^{-1}(-2A)) = 0$.

In particular, $\Phi_{L,\mathscr{O}(A)}$ *is surjective.*

Proof. To verify (i), it suffices to produce an A of degree $g+2$ so that for all $P \in C$,

$$h^0(A) = 3, \quad h^0(A(-P)) = 2, \quad h^0(A(-2P)) = 1. \tag{3.2.1}$$

(This says $|A|$ has no base-points and separates tangent vectors.) So, let A be any divisor of degree $g+2$. Suppose that for some $P \in C$, $h^0(A(-2P)) > 1$. By Riemann-Roch, $h^1(A(-2P)) > 0$, so $h^0(K+2P-A) > 0$, and

$$K + 2P - A \equiv P_1 + \cdots + P_{g-2},$$

or

$$A \equiv K + 2P - P_1 - \cdots - P_{g-2}.$$

Such an A involves at most $g-1$ parameters $(P, P_1, \cdots, P_{g-2})$; so, for general A one has that for all $P \in C$,

$$h^0(A(-2P)) \leq 1.$$

Next, if $h^0(A(-P)) > 2$, then $h^1(A(-P)) > 0$, so $h^0(K+P-A) > 0$, and

$$K + P - A \equiv P_1 + \cdots + P_{g-3}.$$

Again, A depends on at most $g-2$ parameters; so, for general A, one has that for all $P \in C$,

$$h^0(A(-P)) \leq 2.$$

As $\deg A = g+2$, then $h^0(A) \geq 3$; as $h^0(A(-P)) \geq h^0(A) - 1$ and $h^0(A(-2P)) \geq h^0(A(-P)) - 1$, for general A one has (3.2.1).

Finally, $h^1(L \otimes K^{-1}(-2A)) \neq 0$ implies $h^0(L^{-1} \otimes K^2(2A)) \neq 0$. As the degree of the last bundle is $6g-d$, which is $\leq g-1$ by hypothesis, one again has that for general A such an H^0 is 0. This yields (i) and (ii). The surjectivity of $\Phi_{L,\mathscr{O}(A)}$ is a consequence of Theorem 2.6.

Theorem 3.3. *Let C be a projective nonsingular curve, and let L and M be line bundles with $\deg L \geq 5g+1$ and $\deg M \geq 2g+2$. Then $\Phi_{L,M}$ is surjective.*

Proof. Given L, find A as in Proposition 3.2, with $\Phi_{L,\mathscr{O}(A)}$ surjective. As $\deg M(-A) \geq g$, we may write $M(-A) \simeq \mathscr{O}(D)$, with D effective. We use Lemma 2.9 to deduce the surjectivity of $\Phi_{L,\mathscr{O}(A+D)} = \Phi_{L,M}$. First, $H^0(L)$ is immersive (L is even very ample). Next,

$$H^0(L) \otimes H^0(\mathscr{O}(A)) \to H^0(L(A))$$

is surjective, by the usual form of Castelnuovo's Lemma (cf. (2.1), since $H^1(L(-A)) = 0$, by degrees). Finally, $H^1(\mathscr{O}(A)) = 0$, by the choice of A. This completes the proof.

Corollary 3.4. *If $\deg L \geq 5g+1$, then $\Phi_L\colon \bigwedge^2 H^0(L) \to H^0(K \otimes L^2)$ is surjective.*

Remark (3.5). In the Appendix of [3], the surjectivity of Φ_L is proved for $\deg L \geq 4g+5$, and $\Phi_{L,M}$ is surjective for $\deg L$, $\deg M \geq 4g+5$.

(3.6) We now turn to the surjectivity of $\Phi_{L,K}$, when C is nonhyperelliptic. Again, we need first a particular divisor A with $\Phi_{L,\mathscr{O}(A)}$ surjective.

Proposition 3.7. *Let C be a nonhyperelliptic curve and L a line bundle of degree $\geq 5g+2$. Then there is a divisor A of degree $g+1$ such that*

(i) *$h^0(A) = 3$, $h^0(A) = 1$, $|A|$ is immersive.*

(ii) *$H^1(L \otimes K^{-1}(-2A)) = 0$.*

In particular, $\Phi_{L,\mathscr{O}(A)}$ is surjective.

Proof. Suppose an A as in (i) and (ii) is constructed. Then $H^1(L(-2A)) = 0$, by degrees; combining with (ii) and Theorem 2.6 gives the surjectivity of $\Phi_{L,\mathscr{O}(A)}$.

Let B be a divisor linearly equivalent to $K(-A)$. Thus, $\deg B = g-3$, and we will require $h^0(B) = h^1(A) = 1$. Writing $B = P_1 + \cdots + P_{g-3}$, and comparing with (3.2.1), we rewrite the conditions in (i) as: for all $P \in C$,

$$h^0\left(\sum P_i\right) = h^0\left(P + \sum P_i\right) = h^0\left(2P + \sum P_i\right) = 1.$$

We show that for general $P_1, \cdots, P_{g-3}$, the above equalities are satisfied. It clearly suffices to check the third condition. Suppose that for all $P_1, \cdots, P_{g-3}$, there is a P with $h^0(2P+\sum P_i) \geq 2$. Then one has a $(g-3)$-dimensional family of divisors of degree $g-1$ and dimension ≥ 2. In the notation of [1] (e.g., p. 176), one has $\dim W^1_{g-1} \geq g-3$. But by Martens' Theorem [1, p. 191], this can happen only if C is hyperelliptic, contrary to our assumption.

Next, the condition $h^1(L \otimes K^{-1}(-2A)) = 0$ becomes

$$h^0\left(L^{-1} \otimes K^4\left(-2\sum P_i\right)\right) = 0.$$

The line bundles considered have degree $d \leq 8g-8-(5g+2)-2(g-3) = g-4$. If for all P_i one had $h^0 > 0$, one would have

$$\dim W^0_d \geq g-3.$$

But clearly d copies of C map onto W^0_d; as $d \leq g-4$, one has a contradiction.

Theorem 3.8. *Let C be a nonhyperelliptic curve and L a line bundle of degree $\geq 5g+2$. Then $\Phi_{L,K}$ is surjective.*

Proof. Given L, choose A as in Proposition 3.7; then $\Phi_{L,A}$ is surjective. Now, $h^0(K(-A)) = h^1(A) \neq 0$, so there is an effective $D \equiv K - A$. We will use Lemma 2.9 to conclude $\Phi_{L,A(D)} = \Phi_{L,K}$ is surjective. It suffices to check (iii)$'$, i.e., the surjectivity of

$$\mathscr{R}(L, K) \to \mathscr{R}(L, K \otimes \mathscr{O}_D).$$

Via the diagram (2.9.1), with $M = K(-D) = A$, it suffices to show the surjectivity of $H^0(K) \to H^0(K \otimes \mathscr{O}_D)$ and $H^0(L) \otimes H^0(A) \to H^0(L \otimes A)$. The first map is surjective because $H^1(K(-D)) = H^1(A) \to H^1(K)$ is an isomorphism. The second is surjective by the usual form of Castelnuovo's Lemma (2.1), because $H^1(L(-A)) = 0$ (by degrees).

4. Gaussians for special curves

(4.1) We start with C an elliptic curve; in this case, one has

$$\Phi_L \colon \bigwedge^2 H^0(L) \to H^0(L^2).$$

Theorem 4.2. *Let C be an elliptic curve, and L and M line bundles. Suppose either*

(i) $\deg L \geq 3$ *and* $\deg M \geq 7$, *or*
(ii) $\deg L \geq 4$ *and* $\deg M \geq 5$.

Then $\Phi_{L,M}$ is surjective.

Proof. Let $(l, m) = (\deg L, \deg M)$. By Theorem 3.3, one has surjectivity if $l \geq 4$ and $m \geq 6$. By Theorem 2.6, $\Phi_{L,M}$ is surjective if $l = 3$ and $m \geq 7$. It remains to verify the cases $(l, m) = (4, 5)$ and $(5, 5)$.

Assume $\deg L = 4$; write $L = N^{\otimes 2}$. On $C \times C$,

$$H^0(p_1^* N \otimes p_2^* N(-\Delta)) = \operatorname{Ker}(H^0(N) \otimes H^0(N) \to H^0(N^2))$$

has dimension 1, yielding a smooth curve E on $C \times C$. In fact, $E = \{(P, Q) \in C \times C | N \simeq \mathscr{O}(P + Q)\}$. Note $E \cdot E = 0$, and E is nef (E is isomorphic to C). If F is any fiber of p_2, then F intersects positively any irreducible curve not a fiber. Therefore, $2E+F$ has positive self-intersection and positive intersection with any irreducible curve; by Nakai's criterion, it is ample. Since $K_{C\times C} = \mathscr{O}_{C\times C}$, the Kodaira vanishing theorem gives

$$H^1(C \times C, \mathscr{O}(2E + F)) = 0.$$

If $\deg M = 5$, choose the fiber F over the unique point represented by $M \otimes L^{-1}$; one then has

$$\mathscr{O}(2E + F) \simeq p_1^* L \otimes p_2^* M(-2\Delta),$$

and H^1 vanishes. By (1.2.3), $\Phi_{L,M}$ is surjective. The remaining case $(l, m) = (5, 5)$ is similar.

Remarks. (4.3.1) The proof using $C \times C$ is very close to that of [3, Appendix]. The cases $(4, 5)$ and $(5, 5)$ can also be derived from a direct calculation, doing first the case of $\Phi_{\mathscr{O}(5P)}$, using the Weierstrass $\wp$-function to write down an explicit 10×10 matrix.

(4.3.2) Note $\mathscr{R}(L, M) = 0$ if $l \leq 1$. If $l, m \geq 2$, for $\Phi_{L,M}$ to be surjective requires $\dim \mathscr{R}(L, M) \geq \dim H^0(K \otimes L \otimes M)$; this happens only in the cases of Theorem 4.2, and (3.6), and Theorem 4.4. By Theorem 2.6, the (3.6) case has $\Phi_{L,M}$ surjective unless $M = L^{\otimes 2}$; Φ_{L,L^2} is not surjective because for a plane cubic

$$H^0(\Omega^1_{\mathbf{P}^2} \otimes \mathscr{O}_C(3)) \to H^0(\Omega^1_C(3))$$

is not surjective. If $L = M$ has degree 4, clearly $\Phi_{L,L}$ is not surjective, as Φ_L is not $(\dim \bigwedge^2 H^0(L) < \dim H^0(K \otimes L^2))$.

Theorem 4.4. *Let C be a hyperelliptic curve of genus $g \geq 2$, with g_2^1 a divisor defining the hyperelliptic pencil. Let $L = \mathscr{O}(n g_2^1)$, $n \geq 1$.*

(4.4.1) *If $1 \leq n \leq g$, then* $\operatorname{Im} \Phi_L = 2n - 1$*; in particular,* $\operatorname{cork} \Phi_K = 3g - 2$.

(4.4.2) *If $g + 1 \leq n \leq \frac{3}{2} g + 1$, then* $\operatorname{cork} \Phi_L = 3g + 2 - 2n$.

(4.4.3) *If $n \geq \frac{3}{2} g + 1$, then Φ_L is surjective.*

Proof. We may choose functions $x \in H^0(\mathscr{O}(g_2^1))$ and $y \in H^0(\mathscr{O}(g+1)g_2^1)$, such that $y^2 = p(x)$ $(\deg p(x) = 2g+2)$, and

(i) $n \le g$ implies $H^0(L) = \{1, x, \cdots, x^n\}$,

(ii) $n \ge g+1$ implies $H^0(L) = \{1, x, \cdots, x^n, y, xy, \cdots, x^{n-g-1}y\}$.

Then we have

$$\Phi(x^i \wedge x^j) = (j-i)x^{i+j-1}\,dx, \qquad 0 \le i < j \le n, \tag{4.4.4}$$

$$\Phi(x^i \wedge x^k y) = (k-i)x^{i+k-1}y\,dx + x^{i+k}\,dy, \quad 0 \le i \le n,\ 0 \le k \le n-g-1, \tag{4.4.5}$$

$$(x^k y \wedge x^l y) = (l-k)x^{k+l-1}y^2\,dx, \quad 0 \le k < l \le n-g-1. \tag{4.4.6}$$

So, (4.4.1) is a simple consequence of (i) and (4.4.4) (note $K = (g-1)g_2^1$). We can count independent elements of $\operatorname{Im}\Phi_L$ because dx, $y dx$, and dy are independent over polynomials in x, modulo relations derived from

$$2p(x)\,dy = p'(x)y\,dx. \tag{4.4.7}$$

Assume now $n \ge g+1$. By (4.4.4) and (ii), $\operatorname{Im}\Phi_L$ contains

$$x^i\,dx, \qquad 0 \le i \le 2n-2. \tag{4.4.8}$$

Note that elements of type (4.4.6) are of this class, since $x^{k+l-1}y^2$ is a polynomial of degree $\le (n-g-1)+(n-g-2)-1+2g+2 = 2n-2$. If $n = g+1$, it is clear there are $n+1$ independent elements of type (4.4.5); combining with (4.4.3) gives $\dim \operatorname{Im}\Phi_L = 3n = 3g+3$, from which (4.4.2) is easily divided. So, assume $n \ge g+2$. If two different pairs (i, k) and (i', k'), allowed by the condition of (4.4.5), satisfy $i+k = i'+k'$, we deduce that $x^{i+k-1}y\,dx$ and $x^{i+k}\,dy$ are in $\operatorname{Im}\Phi_L$. Such an (i', k') can be found whenever $1 \le i+k \le 2n-g-2$, producing elements

$$x^j y\,dx, \qquad 0 \le j \le 2n-g-3, \tag{4.4.9}$$

$$x^j dy, \qquad 1 \le j \le 2n-g-2. \tag{4.4.10}$$

The only excluded pairs (i, k) are $(0, 0)$ and $(n, n-g-1)$; these give

$$dy, \tag{4.4.11}$$

$$-(g-1)x^{2n-g-2}y\,dx + x^{2n-g-1}\,dy. \tag{4.4.12}$$

If $2n-g-1 < 2g+2$ (i.e., $2n \le 3g+2$), then (4.4.7) imposes no relations on (4.4.9)–(4.4.12), and one has

$$(2n-g-2)+(2n-g-2)+1+1 = 4n-2g-2$$

independent elements from type (4.4.5). Combining with the $(2n-1)$ elements from (4.4.8) gives

$$\dim \operatorname{Im} \Phi_L = 6n - 2g - 3.$$

But $\dim H^0(K \otimes L^2) = g + 4n - 1$ by Riemann-Roch, so the corank of Φ_L is as asserted in (4.4.3).

Finally, suppose $2n \geq 3g+3$. Then considering (4.4.7) one still has independent elements

$$\begin{aligned} &x^j y\,dx, && 0 \leq j \leq 2n-g-3, \\ &x^j\,dy, && 0 \leq j \leq 2g+1, \\ &x^i\,dx, && 0 \leq i \leq 2n-2. \end{aligned}$$

But a count shows these span $H^0(K \otimes L^2)$.

(4.5) We now turn to the question of the surjectivity of Φ_K, or rather its corank. Theorem 4.4 included the computation for hyperelliptic curves. More generally, we have

Theorem 4.6. *If $g \geq 4$, then*

(4.6.1) $$\operatorname{corank} \Phi_K \leq 3g-2$$

with equality if and only if C is hyperelliptic.

Proof. The inequality is given by (1.3.4), and the hyperelliptic case has been computed in Theorem 4.4. Let $P \in C$ be arbitrary, let $\omega \in \Gamma(C, K)$ be nonzero at P, and let t be a local coordinate at P. Every global 1-form may be written about P as $f_i\omega = (a_i t^{b_i} + \cdots)\omega$, where $a_i \neq 0$; and one may choose a basis as above, with $0 = b_1 < b_2 < \cdots < b_g$. Since

$$\Phi(f_i\omega \wedge f_j\omega) = a_i a_j (b_j - b_i)(t^{b_i+b_j-1} + \cdots)\omega \otimes \omega,$$

we see

(4.6.2) $$\operatorname{rank} \Phi_K \geq \#\{b_i + b_j \mid 1 \leq i < j \leq g\}.$$

We have a chain of length $2g-3$ (reproving (4.6.1)):

(4.6.3) $$b_1 + b_2 < b_1 + b_3 < \cdots < b_1 + b_g < b_2 + b_g < \cdots < b_{g-1} + b_g.$$

We show below that $\#\{b_i + b_j \mid i < j\}$ equals $2g-3$ if and only if either

(i) $g = 3$,
(ii) $g = 4$, $b_1 + b_4 = b_2 + b_3$, or
(iii) $g \geq 5$, $b_i = (i-1)b_2 - (i-2)b_1$, all i.

Assuming this for a moment, let C be a nonhyperelliptic curve of genus $g \geq 5$, and let $P \in C$ be a Weierstrass point. Thus, $h^0(\mathcal{O}(sP)) = 2$, for some s with $3 \leq s \leq g$, and $h^0(\mathcal{O}(iP)) = 1$, $0 \leq i < s$. By Riemann-Roch we conclude

$$h^0(K(-iP)) = g - i, \qquad 0 \leq i < s,$$
$$h^0(K(-sP)) = g - s + 1 = h^0(K(-s-1)P)).$$

Therefore, in the notation above, $b_1 = 0$, $b_2 = 1$, but $b_s \geq s$, since the sth function is in $H^0(K(-(s-1)P)) = H^0(K(-sP))$. This contradicts (iii) above, settling the $g \geq 5$ case. For $g = 4$, a nonhyperelliptic curve is a complete intersection of a quadric and a smooth cubic in $\mathbf{P}^3$; the map Φ_K can then be shown to be injective by a calculation as in [8, 6.6].

Returning to the claim, choose $2 \leq i < j \leq g - 1$, and replace part of the chain in (4.6.3) from $b_1 + b_j$ to $b_i + b_g$ (which has $g + i - j$ terms) by

$$b_1 + b_j < b_2 + b_j < \cdots < b_i + b_j < b_i + b_{j+1} < \cdots < b_i + b_g.$$

The new chain has the same length $2g - 3$ as before, hence must be identical to it. From this, assertions (i)–(iii) follow easily.

(4.7) Because every trigonal curve sits on a rational normal scroll, such curves are governed by the following theorem.

Theorem 4.8. *Let $X = \mathbf{F}_n = \mathrm{Proj}(\mathcal{O} \oplus \mathcal{O}(n)) \to \mathbf{P}^1$ be a rational ruled surface and $C \subset X$ a smooth curve of genus $g \geq 5$. Then*

$$\operatorname{cork} \Phi_K \geq h^0(X, K_X^{-1}) \geq 9.$$

Proof. $\mathrm{Pic}\, X$ is spanned by the section H (with $H^2 = n$) and a fiber F. The line bundle $\mathcal{O}(aH + bF)$ has a section iff $a \geq 0$ and $an + b \geq 0$. Suppose $C \equiv aH + bF$. Since C is smooth and irrational, $a = \mathrm{degree}(C \to \mathbf{P}^1) \geq 2$. The curve of self-intersection $-n$ belongs to $|H - nF|$; so, $C \cdot (H - nF) = b \geq 0$. One has an exact sequence

$$0 \to \mathcal{O}_X(-2F) \to \Omega_X^1 \to \mathcal{O}_X(-2H + nF) \to 0, \tag{4.8.1}$$

hence

$$K_X \equiv -2H + (n-2)F.$$

Thus, standard cohomology yields

$$h^0(K_X^{-1}) = \max(n + 6, 9). \tag{4.8.2}$$

We also deduce

$$2g(C) - 2 = na(a-1) + 2(ab - a - b). \tag{4.8.3}$$

If $a = 2$, then C is hyperelliptic, of genus $g = n+b-1$. Thus $\operatorname{cork}\Phi_K$ by (4.4.1) is $3g-2 = 3n+3b-5$, which is easily checked to be at least $h^0(K_X^{-1})$ if $g \geq 5$.

Assume $a \geq 3$. We assert that if $g(C) \geq 5$, then with few exceptions

$$H^1(\Omega_X^1(2K_X + 2C) \otimes \mathscr{O}_C) = 0. \tag{4.8.4}$$

In fact, tensoring (4.6.1) with $2K_X + 2C$ and restricting to C, one finds $H^1(\mathscr{O}_C(2K_X+2C-F) = H^1(\mathscr{O}_C(2K_X+2C-2H+nF)) = 0$ if $2a < 2g-2$ and $na+2b < 2g-2$. Comparing with (4.8.3), one finds after simple but messy calculation that the desired inequalities are fulfilled for all $g \geq 5$ unless: $n = 0$, $b = 2$, $g = a-1$; or, $n = 1$, $b = 0$, $a = 5$ (so $g = 6$). The first case is hyperelliptic, so $\operatorname{cork}\Phi_K = 3g-2 \geq h^0(K_X^{-1}) = 9$. In the second case, $\operatorname{cork}\Phi_K \geq 5\cdot 6 - 5 - \binom{6}{2} = 10 > h^0(K_X^{-1}) = 9$.

We may therefore assume (4.8.4), whence

$$H^0(\Omega_X^1(2K+2C)\otimes\mathscr{O}_C) \xrightarrow{\alpha} H^0(K_C^{\otimes 3}) \to H^1(K_C^{\otimes 2}(-C)) \to 0$$

is exact. Since $H^0(X, K_X(C)) \to H^0(C, K_C)$ ($p_g = q = 0$), by (0.1) we see Φ_K factors via the map α above. So,

$$\operatorname{cork}\Phi_K \geq h^1(K_C^{\otimes 2}(-C)) = h^0(K_X^{-1}|_C).$$

As $h^0(K_X^{-1}(-C)) = 0$ (as $a \geq 3$), one has $h^0(K_X^{-1}|_C) \geq h^0(K_X^{-1})$. This proves the theorem.

Remark (4.9). One can likewise easily compute that for a smooth plane curve in $\mathbf{P}^2$ of degree $d \geq 5$, $\operatorname{cork}\Phi_K = h^0(\mathbf{P}^2, K_{\mathbf{P}}^{-1}) = 10$.

(4.10) We next show how to construct a class of curves for which Φ_K is surjective.

Theorem 4.11. *Let C_i be a complete nonsingular curve of genus g_i $(i = 1, 2)$, K_i the canonical line bundle on C_i, and D_i a divisor on C_i of degree d_i. Suppose*

(a) *D_i is very ample on C_i.*

(b) *$d_i > \max(0, 4-4g_i)$.*

(c) *On C_i, $K(D_i)$ is normally generated and $\Phi_{K(D_i)}$ is onto.*

(d) *$g_2 \geq 2$.*

Let $X = C_1 \times C_2$. Then the general element of the complete linear system $|p_1^*D_1 + p_2^*D_2|$ on X is a nonsingular curve C for which Φ_K is surjective, and

$$2g(C) - 2 = d_1(2g_1-2) + d_2(2g_2-2) + 2d_1d_2. \tag{4.11.1}$$

Proof. Since each D_i is very ample, so is $p_1^*D_1 + p_2^*D_2 \subset X$, hence the general C in the linear system is nonsingular. As $K_X = p_1^*K_1 \otimes p_2^*K_2$, the adjunction formula gives (4.11.1).

Consider as in (0.1) the commutative diagram:

$$\begin{array}{ccc} \bigwedge^2 H^0(X, K_X(C)) & \longrightarrow & H^0(X, \Omega_X^1 \otimes K_X^2(2C)) \\ \downarrow & & \downarrow \\ \bigwedge^2 H^0(C, K_C) & \xrightarrow{\Phi_K} & H^0(C, K_C^3) \end{array}$$

The top map is onto by assumption (c) and Lemma 4.12 below. We show Φ_K onto by showing the surjectivity of the right vertical map, which factors as

$$H^0(\Omega_X^1 \otimes K_X^2(2C)) \xrightarrow{\alpha} H^0(\Omega_X^1 \otimes K_X^2(2C) \otimes \mathcal{O}_C) \xrightarrow{\beta} H^0(C, K_C^3).$$

The cokernel of α is contained in

$$H^1(\Omega_X^1 \otimes K_X^2(C)), \tag{4.11.2}$$

while the cokernel of β is contained in

$$H^1(C, K_C^2(-C)); \tag{4.11.3}$$

we show both spaces are zero.

Since $\Omega_X^1 = p_1^*K_1 \oplus p_2^*K_2$, (4.11.2) is equal to

$$H^1(C_1 \times C_2, (p_1^*K_1^2(D_1) \otimes p_2^*K_2(D_2)) \oplus (p_1^*K_1(D_1) \otimes p_2^*K_2^2(D_2))),$$

which is 0 by the Kunneth formula and assumption (b).

Next, (4.11.3) is dual to

$$H^0(C, K_C^{-1}(-C)) = H^0(C, K_X^{-1} \otimes \mathcal{O}_C).$$

From

$$0 \to K_X^{-1}(-C) \to K_X^{-1} \to K_X^{-1} \otimes \mathcal{O}_C \to 0,$$

vanishing follows once we know $h^0(K_X^{-1}) = h^1(K_X^{-1}(-C)) = 0$. That $h^0(K_X^{-1}) = 0$ follows easily from assumption (d). The other vanishing follows from Künneth and assumption (b).

Lemma 4.12. *Let X_i be a smooth projective variety $(i = 1, 2)$ and let L_i be a line bundle on X_i, with L_i normally generated and Φ_{L_i} : $\bigwedge^2 H^0(X_i, L_i) \to H^0(X_i, \Omega_{X_i}^1 \otimes L_i^2)$ surjective. Then the Gaussian of $p_1^*L_1 \otimes p_2^*L_2$ on $X_1 \times X_2$ is surjective.*

Proof. Let $\Phi_i = \Phi_{L_i}$ and $\Omega_i^1 = \Omega_{X_i}^1$. One has a sequence of maps and isomorphisms:

$$\begin{aligned}
&\bigwedge^2 H(X_1 \times X_2, p_1^* L_1 \otimes p_2^* L_2) \xrightarrow{\sim} \bigwedge^2 (H^0(X_1, L_1) \otimes H^0(X_2, L_2)) \\
&\quad \xrightarrow{\sim} \bigwedge^2 H^0(X_1, L_1) \otimes S^2 H^0(X_2, L_2) \oplus S^2 H^0(X_1, L_1) \otimes \bigwedge^2 H^0(X_2, L_2) \\
&\quad \to H^0(X_1, \Omega_1^1 \otimes L_1^2) \otimes S^2 H^0(X_2, L_2) \\
&\qquad \oplus S^2 H^0(X_1, L_1) \otimes H^0(X_2, \Omega_2^1 \otimes L_2^2) \\
&\quad \to H^0(X_1, \Omega_1^1 \otimes L_1^2) \otimes H^0(X_2, L_2^2) \oplus H^0(X_1, L_1^2) \otimes H^0(X_2, \Omega_2^1 \otimes L_2^2) \\
&\quad \xrightarrow{\sim} H^0(X_1 \times X_2, p_1^*(\Omega_1^1 \otimes L_1^2) \otimes p_2^* L_2^2) \\
&\qquad \oplus H^0(X_1 \times X_2, p_1^* L_1^2 \otimes p_2^*(\Omega_2^1 \otimes L_2^2)) \\
&\quad = H^0(X_1 \times X_2, \Omega^1_{X_1 \times X_2} \otimes (p_1^* L_1 \otimes p_2^* L_2)^2).
\end{aligned}$$

The second map above sends

$$(f_1 \otimes g_1) \wedge (f_2 \otimes g_2) \to f_1 \wedge f_2 \otimes (g_1 \cdot g_2) + (f_1 \cdot f_2) \otimes g_1 \wedge g_2.$$

It is easy to check (using (1.1)) that the composed map is the Gaussian of $p_1^* L_1 \otimes p_2^* L_2$. The lemma follows easily.

Corollary 4.13. *There are smooth curves C of every genus $g = 5e + 2$ $(e \geq 6)$ for which Φ_K is surjective, but which possess a g_6^1.*

Proof. In Theorem 4.11, let $C_1 = \mathbf{P}^1$ and $d_1 = e - 1 \geq 5$; and let C_2 have genus 2, and $D_2 = 3g_2^1$ (so, $d_2 = 6$). Clearly, assumptions (a), (b), and (d) are fulfilled, and normal generation is satisfied for D_1 and D_2 (one needs merely $\deg \geq 2g + 1$). Any nonzero Gaussian on $\mathbf{P}^1$ is surjective (1.3.2); and the surjectivity on a genus 2 curve of the Gaussian of $K(D_2) = \mathscr{O}(4g_2^1)$ is given by (4.4.3). Therefore, Theorem 4.11 provides a smooth C of genus $5e + 2$ with Φ_K surjective. On $X = C_1 \times C_2$, the curve C has degree $d_2 = 6$ over $C_1 = \mathbf{P}^1$, and hence possesses a g_6^1.

Remark (4.14). One can of course produce other examples in this way of curves with Φ_K surjective. Using, e.g., $g_1 = 1, d_1 \geq 5$ and $g_2 = 2, d_2 = 6$, one gets $g(C) = 6e + 1$, $e \geq 6$. Similar constructions easily produce examples for all genera g between 42 and 100, with eight exceptions.

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