J. DIFFERENTIAL GEOMETRY
31 (1990) 763-769

A STRUCTURE THEOREM FOR HOLOMORPHIC CURVES IN $\mathrm{Gr}(3, \mathbf{C}^6)$

QING LIN

Abstract

A holomorphic curve f in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is called generic if the curvature of the canonical connection of $f^*(S(n, \mathbf{C}^{2n}))$ has distinct eigenvalues, where $S(n, \mathbf{C}^{2n})$ is the universal subbundle over $\mathrm{Gr}(n, \mathbf{C}^{2n})$. A holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is completely split if it is the orthogonal direct sum of n holomorphic curves in the projective plane. These two types of curves are both relatively simple. In this paper, we prove that a 1-nondegenerated holomorphic curve in $\mathrm{Gr}(3, \mathbf{C}^6)$ is either generic or completely split.

Introduction

Denote the Grassmannian of n-dimensional subspaces of $\mathbf{C}^{2n}$ by $\mathrm{Gr}(n, \mathbf{C}^{2n})$. A holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is locally a holomorphic mapping of some open disk in $\mathbf{C}$ into $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Because of the analytic structure, we can restrict ourselves to the local holomorphic curves only.

Let $f: \Omega \to \mathrm{Gr}(n, \mathbf{C}^{2n})$ be a holomorphic curve. For each z in Ω, we define $(f(z), f'(z)) = \mathrm{span}\{\gamma_1(z), \cdots, \gamma_n(z), \gamma_1'(z), \cdots, \gamma_n'(z)\}$, where $\gamma_j: \Omega \to \mathbf{C}^{2n}$ is holomorphic and $\mathrm{span}\{\gamma_1(z), \cdots, \gamma_n(z)\} = f(z)$. Clearly, (f, f') is independent of the choice of $\gamma_1, \cdots, \gamma_n$. We say f is 1-nondegenerated if $(f(z), f'(z)) = \mathbf{C}^{2n}$ for each $z \in \Omega$.

Throughout this paper, by "holomorphic curve" we mean "1-nondegenerated holomorphic curve". Let f be a 1-nondegenerated holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Then the holomorphic Hermitian vector bundle

$$E_f: \begin{array}{c} \text{the space } f(z) \\ \downarrow \\ z \end{array}$$

is a completely unitary invariant of f by the Calabi rigidity theorem. By

Received August 15, 1988.

the unitary equivalence of f_1 and f_2 we mean that there is a unitary transformation U of $\mathbf{C}^{2n}$ making $U \cdot f_1 = f_2$. We shall name the canonical connection of E_f and its curvature the connection of f and the curvature of f, respectively.

Definition 1. A holomorphic curve is called *generic* if its curvature has distinct eigenvalues at some point.

In [3] and [1], it was proved that a second order contact of two generic curves implies unitary equivalence. In this paper we shall prove that any holomorphic curve in $\mathrm{Gr}(3, \mathbf{C}^6)$ is either generic or an orthogonal direct sum of three holomorphic curves in the projective plane. Thus in $\mathrm{Gr}(3, \mathbf{C}^6)$, two holomorphic curves having second order contact must be unitarily equivalent, which answers the so-called Griffiths' conjecture in the simplest nontrivial case.

During the course of this work, the author benefitted from the discussions with Professor M. J. Cowen, and has also been inspired by an unpublished idea of Professor P. A. Griffiths. The author would like to thank both of them.

Main results

Let f be a holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Using the canonical coordinate of the Grassmannian, we see locally E_f has the columns of $\binom{I}{P}$ as a holomorphic frame, where I is the $n \times n$ identity matrix and P is an $n \times n$ matrix of analytic function entries. Over this holomorphic frame, the matrix of the curvature bundle map K_f ($K_f dz d\overline{z}$ is the curvature tensor) is

$$-(I + P^*P)^{-1} P'^*(I + PP^*)^{-1} P'.$$

A quick consequence of this expression is that the eigenvalues of K_f are all strictly negative.

From the above expression, it follows that

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log \det(-K_f) = -2 \frac{\partial^2}{\partial z \partial \overline{z}} \log \det(I + P^*P).$$

By a well-known lemma of S. S. Chern, $\operatorname{tr} K_f = K_{\bigwedge^n(E_f)}$. Noting that

$$K_{\bigwedge^n(E_f)} = -\frac{\partial^2}{\partial z \partial \overline{z}} \log \det(I + P^*P) = \frac{1}{2} \frac{\partial^2}{\partial z \partial \overline{z}} \log \det(-K_f),$$

we thus have shown

Lemma 1. *Let $\lambda_1, \cdots, \lambda_n$ be the smooth eigenfunctions of K_f and define*

$$f(\lambda_i) = 2\lambda_i^2 - \lambda_i \frac{\partial^2}{\partial z \partial \overline{z}} \log(-\lambda_i).$$

Then

$$\sum_{i=1}^{n} F(\lambda_i)/\lambda_i = 0.$$

Definition 2. We say a holomorphic curve f is completely split if E_f is an orthogonal direct sum of n holomorphic line bundles. Equivalently, f is an orthogonal direct sum of n holomorphic curves in the projective plane.

Our first aim is to show:

"f is completely split $\Leftrightarrow F(\lambda_i) = 0$ for all i".

In order to do this, we need to look back at the differential structure on E_f.

Recall that a bundle map of E to E is a C^∞ map which maps each fiber linearly to itself. Let φ be a bundle map of E_f to E_f, where f is a holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Then we define

$$[D, \varphi] = D \circ \varphi - (\varphi \otimes \mathrm{id}) \circ D = \varphi_z dz + \varphi_{\overline{z}} d\overline{z}.$$

Although D is not a bundle map, a quick check gives that φ_z and $\varphi_{\overline{z}}$ are all bundle maps of E_f to E_f. We call them the first covariant derivatives of φ. So $\varphi_{z\overline{z}}$ would be one of the first covariant derivatives of $\varphi_{\overline{z}}$.

If over an orthonormal frame S the connection matrix is $\Theta dz - \Theta^* d\overline{z}$, then

$$\varphi_z(S) = [\Theta, \varphi(S)] + \frac{\partial \varphi(S)}{\partial z},$$
$$\varphi_{\overline{z}}(S) = [-\Theta^*, \varphi(S)] + \frac{\partial \varphi(S)}{\partial \overline{z}}.$$

For details, we refer the reader to [2]. Also in [2] it was proved that $K_{\overline{z}z} = K_{z\overline{z}}$ (write K_f as K), although $K_{z^2\overline{z}} \neq K_{\overline{z}z^2}$ in general.

In [1], the following was proved:

(*) "an n-dimensional Hermitian holomorphic vector bundle is equivalent to some E_f with f a holomorphic curve in $\mathrm{Gr}(n, \mathbf{C}^{2n}) \Leftrightarrow 2K^2 + K_{\overline{z}} K^{-1} K_z = K_{z\overline{z}}$."

Definition 3. An orthonormal frame is called a first adapted frame if, over it, the matrix of K_f is smoothly diagonalized.

From now on K will stand for the matrix of the curvature.

Now we are ready to show

Theorem 1. *A holomorphic curve f in* $\mathrm{Gr}(n, \mathbf{C}^{2n})$ *is completely split* $\Leftrightarrow F(\lambda_i) = 0$ *for all i, where* $\lambda_1, \cdots, \lambda_n$ *are the smooth eigenfunctions of* K_f.

Proof. The forward direction is trivial. For the backward direction, we need the following fact from [2] to reduce the problem: a holomorphic curve is completely split iff over some first adapted frame $[\Theta, K] \equiv 0$ where $(\Theta dz - \Theta^* d\overline{z}$ is the connection matrix. Then, take any first adapted frame and write K_z, $K_{\overline{z}}$, $K_{z\overline{z}}$ in matrix form:

$$K_z = [\Theta, K] + \frac{\partial K}{\partial \overline{z}}, \qquad K_{\overline{z}} =]\Theta^*, K] + \frac{\partial K}{\partial \overline{z}},$$

$$K_{z\overline{z}} = [-\Theta^*, [\Theta, K]] + \left[-\Theta^*, \frac{\partial K}{\partial z}\right] + \frac{\partial}{\partial \overline{z}}[\Theta, K] + \frac{\partial^2}{\partial z \partial \overline{z}} K.$$

Substituting them into $(*)$ and taking the trace on both sides, we have

$$\sum_{i=1}^{n} F(\lambda_i) + \mathrm{tr}[\Theta, K]^* K^{-1} [\Theta, K] \equiv 0,$$

i.e., $\mathrm{tr}[\Theta, K]^* K^{-1}[\Theta, K] \equiv 0$. Since K^{-1} is negative definite, we obtain $[\Theta, K] \equiv 0$.

Now, we can direct our attention to our final aim. We assume there is a nongeneric curve f, which is not completely split, and fix it once and for all. We shall then use the following three steps to obtain a contradiction.

Let us assume that over some first adapted frame the curvature matrix is

$$K = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix}$$

with $\lambda \neq \mu$. By Theorem 1 and Lemma 1 above, we may assume $F(\lambda) \neq 0$ and $F(\mu) \neq 0$. Let

$$\Theta = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

and assume

$$[\Theta, K] = \begin{pmatrix} 0 & a_{12}(\mu - \lambda) & a_{13}(\mu - \lambda) \\ a_{21}(\lambda - \mu) & 0 & 0 \\ a_{31}(\lambda - \mu) & 0 & 0 \end{pmatrix} \neq 0.$$

Without loss of generality, we take $|a_{12}|^2 + |a_{13}|^2 \neq 0$.

Step 1. There is a first adapted frame such that $a_{12} > 0$, $a_{31} > 0$ and $a_{21} \equiv a_{13} \equiv 0$.

Observe that if S is a first adapted frame and

$$K(S) = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix},$$

then for any $\left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right)$, where U is a 2×2 C^∞ unitary matrix, $S\left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right)$ is again a first adapted frame. But Θ changes to

$$\begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} \Theta \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & U^* \partial U/\partial z \end{pmatrix},$$

so we may choose U smoothly, such that $a_{12} > 0$, $a_{13} \equiv 0$ and $a_{31} \geq 0$. Now

$$[\Theta, K] = \begin{pmatrix} 0 & a_{12}(\mu - \lambda) & 0 \\ a_{21}(\lambda - \mu) & 0 & 0 \\ a_{31}(\lambda - \mu) & 0 & 0 \end{pmatrix}.$$

Equation $(*)$ can be rewritten as

$$(**) \qquad \begin{aligned} &\begin{pmatrix} F(\lambda) & & \\ & F(\mu) & \\ & & F(\mu) \end{pmatrix} + [\Theta, K]^* K^{-1} [\Theta, K] \\ &+ [\Theta, K]^* \begin{pmatrix} \frac{\partial \log \lambda}{\partial z} & & \\ & \frac{\partial \log \mu}{\partial z} & \\ & & \frac{\partial \log \mu}{\partial \mu} \end{pmatrix} \\ &+ \begin{pmatrix} \frac{\partial \log \lambda}{\partial \bar{z}} & & \\ & \frac{\partial \log \mu}{\partial \bar{z}} & \\ & & \frac{\partial \log \mu}{\partial \bar{z}} \end{pmatrix} [\Theta, K] \\ &= [-\Theta^*, [\Theta, K]] + \left[-\Theta^*, \frac{\partial K}{\partial z} \right] + \frac{\partial}{\partial \bar{z}} [\Theta, K], \end{aligned}$$

where

$$[\Theta, K]^* K^{-1} [\Theta, K] = \begin{bmatrix} \frac{(\lambda-\mu)^2}{\mu}(|a_{21}|^2 + |a_{31}|^2) & 0 & 0 \\ 0 & a_{12}^2 \frac{(\lambda-\mu)^2}{\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[-\Theta^*, [\Theta, K]] =$$

$$\begin{bmatrix} (a_{12}^2 + |a_{21}|^2 + a_{31}^2)(\mu - \lambda) & a_{12}(\mu - \lambda)(\bar{a}_{22} - \bar{a}_{11}) & a_{12}\bar{a}_{32}(\mu - \lambda) \\ a_{21}(\bar{a}_{22} - \bar{a}_{11})(\mu - \lambda) + \bar{a}_{32} a_{31}(\mu - \lambda) & (|a_{21}|^2 + a_{12}^2)(\lambda - \mu) & a_{21}\bar{a}_{31}(\lambda - \mu) \\ a_{31}(\bar{a}_{33} - \bar{a}_{11})(\mu - \lambda) + a_{21}\bar{a}_{23}(\mu - \lambda) & a_{31}\bar{a}_{21}(\lambda - \mu) & a_{31}^2(\lambda - \mu) \end{bmatrix}.$$

Considering the $(2, 3)$ entry of $(**)$, we have

$$\frac{F(\mu)}{\lambda-\mu}=a_{31}^2.$$

Therefore $a_{31}>0$ and $a_{21}\equiv 0$.

Step 2. λ/μ =constant.

Fix the first adapted frame from Step 1 and consider the $(1, 3)$ and $(2, 1)$ entries of $(**)$. Then we get

$$\overline{a}_{32}=\frac{a_{31}}{a_{12}}\frac{\partial\log((\lambda-\mu)/\mu)}{\partial z}, \qquad \overline{a}_{32}=\frac{a_{12}}{a_{31}}\frac{\partial\log(\lambda/(\lambda-\mu))}{\partial z},$$

which are combined to give

$$(\dagger) \qquad \frac{a_{31}^2}{a_{12}^2}\frac{\partial\log((\mu-\lambda)/\mu)}{\partial z}=\frac{\partial\log(\lambda/(\mu-\lambda))}{\partial z}.$$

Now from the $(2, 2)$ entry of $(**)$ it follows that

$$a_{12}^2=\frac{F(\mu)}{\lambda-\mu}\cdot\frac{\lambda}{\mu},$$

so that $a_{31}^2/a_{12}^2=\mu/\lambda$. Substituting it into $(\dagger)$, we get

$$\frac{\partial\log(\lambda/(\mu-\lambda))}{\partial z}=\frac{\mu}{\lambda}\frac{\partial\log((\mu-\lambda)/\mu)}{\partial z},$$

i.e., $\partial\frac{\lambda}{\mu}/\partial z=0$, or $\lambda/\mu=$ constant.

Let $\lambda/\mu=c$; then $c\neq 1$, $c>0$. Thus $F(\lambda)/\lambda+2F(\mu)/\mu=0$ becomes

$$2\left(\frac{c+2}{3}\right)\mu=\frac{\partial^2\log\mu}{\partial z\partial\overline{z}}.$$

Moreover, $a_{32}=0$, $a_{31}^2=-\frac{2}{3}\mu$, $a_{12}^2=-\frac{2}{3}c\mu$.

Step 3. $a_{23}=0$ and $a_{23}\neq 0$ (a contradiction).

Considering the $(3, 1)$ and $(1, 2)$ entries of $(**)$, we get

$$a_{33}-a_{11}=\frac{1}{2}\frac{\partial}{\partial z}\log\mu, \qquad a_{11}-a_{22}=\frac{1}{2}\frac{\partial}{\partial z}\log\mu,$$

addition of which gives $a_{33}-a_{22}=(\partial/\partial z)\log\mu$. Recall that K and Θ have to satisfy the connection-curvature equation:

$$(\Delta) \qquad -K=[\Theta, \Theta^*]+\frac{\partial\Theta}{\partial\overline{z}}+\frac{\partial\Theta^*}{\partial z},$$

where

$$[\Theta, \Theta^*] = \left[\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}, \begin{pmatrix} \overline{a}_{11} & 0 & a_{31} \\ a_{12} & \overline{a}_{22} & 0 \\ 0 & \overline{a}_{23} & \overline{a}_{33} \end{pmatrix}\right]$$
$$= \begin{pmatrix} a_{12}^2 - a_{31}^2 & a_{12}(\overline{a}_{22} - \overline{a}_{11}) & a_{31}(a_{11} - a_{33}) \\ a_{12}(a_{22} - a_{11}) & |a_{23}|^2 - a_{12}^2 & a_{23}(\overline{a}_{33} - \overline{a}_{22}) \\ a_{31}(\overline{a}_{11} - \overline{a}_{33}) & \overline{a}_{23}(a_{33} - a_{22}) & a_{31}^2 - |a_{23}^2| \end{pmatrix}.$$

Considering the $(2, 2)$ and $(3, 3)$ entries of $(\triangle)$, we have

$$-\mu = |a_{23}|^2 - a_{12}^2 + 2\operatorname{Re}\frac{\partial a_{22}}{\partial \overline{z}}, \qquad -\mu = a_{31}^2 - |a_{23}^2| + 2\operatorname{Re}\frac{\partial a_{33}}{\partial \overline{z}},$$

subtraction of which yields

$$2|a_{23}|^2 = a_{12}^2 + a_{31}^2 + 2\operatorname{Re}\frac{\partial(a_{33} - a_{22})}{\partial \overline{z}}$$
$$= \left(-\frac{2c}{3} - \frac{2}{3}\right)\mu + 2\frac{\partial^2 \log\mu}{\partial z \partial \overline{z}} = \frac{2}{3}(c+3)\mu.$$

Thus $a_{23} \neq 0$, since $c > 0$.

Next, consider the $(2, 3)$ entry of $(\triangle)$:

$$0 = a_{23}(\overline{a}_{33} - \overline{a}_{22}) + \partial a_{23}/\partial \overline{z}.$$

Since $a_{23} \neq 0$, if we write $a_{23} = |a_{23}|e^{i\theta}$, then

$$\frac{\partial(\log\mu + i\theta)}{\partial \overline{z}} = -\frac{\partial \log|a_{23}|}{\partial \overline{z}} = -\frac{1}{2}\frac{\partial \log\mu}{\partial \overline{z}},$$

or $(\partial/\partial\overline{z})[\log(-\mu)^{3/2} + i\theta] \equiv 0$, which implies

$$\frac{\partial^2}{\partial z \partial \overline{z}}\log(-\mu) = 2\left(\frac{c+2}{3}\right)\mu \equiv 0,$$

a contradiction.

Because of this contradiction, we have reached

Theorem 2. *A holomorphic curve in* $\mathrm{Gr}(3, \mathbf{C}^6)$ *is either generic or completely split.*

References

[1] M. J. Cowen & R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978) 187–261.

[2] ——, *Equivalence of connections*, Advances in Math. **56** (1985) 39–91.

[3] P. A. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. **41** (1974) 775–814.

ACADEMIA SINICA, BEIJING